MA672 - (Topics) Number Theory and Cryptography

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These are the course notes for Topics in Advanced Mathematics (**MA672**) at Hotchkiss taught by Dr. Weiss. These notes were last updated January 9, 2020. Any sections denoted with asterisks (***) are currently incomplete, and I will update them when I get to those.

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January 7, 2020

Course Overview

- Abstract algebra: groups, rings, fields.
- Number Theory, arbitrary precision integer arithmetics.
- Cryptographic algorithms

1 Groups

We first define the group, which we will be using extensively.

Definition 1.1. (Group). A group is a set G with a binary operation "$\circ$" such that

- G is closed under $\circ$.
- G is associative.
- There is an Identity Element: $\exists e \in G \mid x \circ e = e \circ x - x \; \forall x \in G$.
- Inverses: $\forall x \in G \; \exists y \in G \mid x \circ y = y \circ x = e$.

Definition 1.2. (Abelian Group). If $\circ$ is commutative in group G, we call G Abelian. In that case, G is often written additively; i.e. we use "+" for "$\circ$".

(If $\circ$ is not commutative, we often write G multiplicatively.)

Definition 1.3. (Subgroup). Let G be a group, and $\emptyset \neq H \subseteq G$. Then H is called a subgroup of G if H is also a group.

A small proof to begin. . .

Proposition 1.4. Let G be a group and $x \in G$. Then x has a unique inverse y, so we can write $y = x^{-1}$.

Proof. Assume y and z are both inverses of x.

$$y = y \circ (x \circ z) = (y \circ x) \circ z = z$$

□

Proposition 1.5. A non-empty subset $H \subseteq G$ is a subgroup of G iff $xy^{-1} \in H \; \forall x, y \in H$.

Proof. ($\Rightarrow$)

- Identity: Pick $x \in H$. Then $xx^{-1} = e \in H$.
- Inverse: If $y \in H$, $ey^{-1} = y^{-1} \in H$

- Closure: If $x, y \in H$, $y^{-1} \in H$, so $x(y^{-1})^{-1} = xy \in H$.

($\Leftarrow$) If H is a group, then $y^{-1} \in H$ (existence of unverse) and $xy^{-1} \in H$ (closure of $\circ$). $\square$

Example:

- Every vector space (without the scalars) is an Abelian group with identity $\vec{0}$.
- Modular arithmetic:
$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$
  $\mathbb{Z}_n$ is an Abelian group under modular addition.
$$\mathbb{Z}_n^* = \{k \in \mathbb{Z}_n \mid \gcd(k, n) = 1\}$$
  $\mathbb{Z}_n^*$ is an Abelian group under modular multiplication (this is sometimes also $\mathbb{U}_n$).

  Let's take $\mathbb{Z}_4 - \{0\}$ and why it's not a group under multiplication. We can create a multiplication table:

$\circ$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

  However there is no such problem with $\mathbb{Z}_4^*$:

$\circ$	1	3
1	1	3
3	3	1

Definition 1.6. (Cyclic). A group G is called cyclic if $\exists g \in G$ (called generator) such that $G = \{g^n \mid n \in Z\}$.

Example: Z_n are cyclic grous with generator 1.

$\mathbb{Z}_4^*$ is cyclic with generator 3.

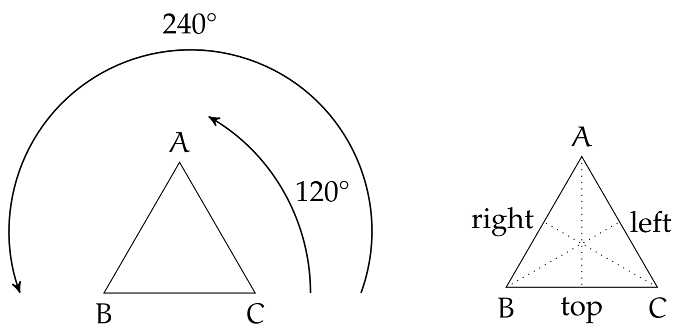
Example: The Klein 4-group is not cyclic:
$$K = \{(0,0), (1,0), (0,1), (1,1)\}$$
with componentwise addition mod 2.
$$K = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(k, l) \mid k \in \mathbb{Z}_2, l \in \mathbb{Z}_2\}$$

Proposition 1.7. Every cyclic group is Abelian.

Proof. Let $x, y \in G$, a cyclic Abelian group. Let g be the generator in G. We write $x = g^a$ and $y = g^b$. Then $xy = g^a g^b = \underbrace{g \circ g \circ \cdots \circ g}_{k+l \text{ times}} = g^b g^a = yx$. $\square$

Example: The symmetry transformation of an equilateral $\triangle$ form a group under composition.

$$D_3 = \{\mathrm{id}, 120^\circ, 240^\circ, \mathrm{top}, \mathrm{left}, \mathrm{right}\}$$



$\circ$	id	120°	240°	top	left	right
id	id	120°	240°	top	left	right
120°	120°	240°	id	left	right	top
240°	240°	id	120°	right	top	left
top	top	right	left	id	240°	120°
left	left	top	right	120°	id	240°
right	right	left	top	240°	120°	id

Definition 1.8. (Equivalence). Let G be a group and H a subgroup. Define the relation $x \sim y$ if $xy^{-1} \in H$.

Proposition 1.9. $\sim$ is an equivalence relation on G.

If $H = \{e\}$, then $\sim$ is $=$.

If $H = G$, then $\sim$ is trivial.

Proof. We need to show that $\sim$ is

- reflexive: $x \sim x$ for all $x \in G$
$$xx^{-1} = e \in H.$$
- symmetric: $x \sim y \iff y \sim x$ for all $x, y \in G$

  Suppose $x \sim y$. Then $xy^{-1} \in H$. So $(xy^{-1})^{-1} = yx^{-1} \in H \Rightarrow y \sim x$.
- transitive: If $x \sim y, y \sim z$ then $x \sim z$ for all $x, y, z \in G$.

  Suppose $x \sim y, y \sim z$. Then $xy^{-1} \in H, yz^{-1} \in H$. Then $(xy^{-1})(yz^{-1}) = x(y^{-1}y)z^{-1} = xz^{-1} \in H \Rightarrow x \sim z$.

$\square$

If $\sim$ is an equivalence relation on any set X, then $\sim$ partitions X into equivalence classes: If $y \in X, [y] = \{x \in X \mid x \sim y\}$.

Every element of X is in some equivalence class because $\sim$ is reflexive and no two equivalence classes intersect. Consider $[y_1], [y_2]$ and $z \in [y_1] \cap [y_2]$. Then $z \sim y_1$ and $z \sim y_2$ and $y_1 \sim y_2$. Hence, $[y_1] = [y_2]$.

Theorem 1.10. (Lagrange's Theorem). Let G be a finite group of order $|G|$ and H a subgroup of G. Then $|H|$ divides $|G|$.

Proof. We show that the above equivalence relation partitions G into equivalence classes of equal cardinality.

First, notice that H is an equivalence class by itself: $H = [e]$.

Let $[x]$ be another equivalence class. Then $[x] = Hx$: Let $y \in Hx$. Then $\exists a \in H$ such that $y = ax$. Then $\exists a \in H$ such that $y = ax$. But then $yx^{-1} = (ax)x^{-1} = a \in H$, so $y \sim x$.

We need to find a bijection between H and Hx for $x \in G$. Let $f : H \to Hx$, $f(a) = ax$. We ened to show that f is one-to-one and onto:

$$\begin{aligned} \text{1-1: If } f(a_1) &= f(a_2) \\ a_1 x &= a_2 x \\ a_1 x x^{-1} &= a_2 x x^{-1} \\ a_1 &= a_2. \end{aligned}$$

Onto: Let $y \in Hx$. Then $\exists a \mid y = ax \Rightarrow y = f(a)$.

$\Rightarrow H \simeq Hx \Rightarrow$ they have the same cardinality.

$\square$

Definition 1.11. (Index). $[G : H]$ is the number of equivalence relations, which is called the <u>index</u> of H in G.

$$[G : H] = \frac{|G|}{|H|}.$$

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Quick Summary

We were talking about <u>groups</u>, which are

- associative,
- have an identity element,
- and have inverses.

There can be subgroups in groups, a subset of a group that is also a group.

An Abelian group is a commutative group.

A cyclic group is generated by an element g, they're always Abelian.

Lagrange's Theorem: the order of a subgroup divides the order of the group.

Index of H in G: $[G : H] = \frac{|G|}{|H|}$.

Example:

- $(\mathbb{R}^n, +), (\mathbb{C}^n, +)$.
- $(\mathbb{Z}_n, +), (\mathbb{Z}_n^*, \cdot), (\mathbb{R}, +), (\mathbb{R}^*, \cdot), (\mathbb{C}, +), (\mathbb{C}^*, \cdot)$.
- $GL_n(\mathbb{R})$: Invertible $n \times n$ matrices.
- $SL_n(\mathbb{R})$: Subgroup of $GL_n(\mathbb{R})$ with determinant 1.
- $GL_n(\mathbb{C})$: Invertible $n \times n$ complex matrices.
- $U(n)$: Unitary group – determinant of absolute value 1.
- Symmetry groups of geometric shapes. (Dihedral groups)
- Frieze groups.
- Wallpaper groups.
- Crystallographic groups.
- Permutation groups of $\{1, \ldots, n\}$ under composition (S_n), and its subgroups.

We will be focusing mainly on $(\mathbb{Z}_n, +), (\mathbb{Z}_n^*, \cdot)$.

Definition 1.12. (Order of an element). Let G be a group and $g \in G$. Then the order of $o(g)$ is the smallest positive integer n such that $g^n = e$. (May be infinite)

Proposition 1.13. Every group of prime order is cyclic.

Proof. Let $e \neq g \in G$. Consider the subgroup $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ generated by g. Then $|\langle g \rangle|$ divides $|G|$, which is prime. Hence $\langle g \rangle = G$. □

Exercise 1. Show that $\{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a subgroup of S_4.

Proof. We can construct a Cayley table for the subgroup. (Top row is evaluated first).

$\circ$	(1)	$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$
(1)	(1)	$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$
$(1,2)(3,4)$	$(1,2)(3,4)$	(1)	$(1,4)(2,3)$	$(1,3)(2,4)$
$(1,3)(2,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$	(1)	$(1,2)(3,4)$
$(1,4)(2,3)$	$(1,4)(2,3)$	$(1,3)(2,4)$	$(1,2)(3,4)$	(1)

Every element is its own inverse, and (1) is the identity element. It is evident from the table that we also have closure. Hence, the set is a valid subgroup under composition. □

Exercise 2. Let G be an Abelian group. Show that the set of all elements of G of finite order forms a subgroup of G.

Proof. Let H be the subset of G with elements of finite order. We want to show that $\forall x, y \in H, xy^{-1} \in H$. In other words, xy^{-1} also has finite order. Well, xy^{-1} has order at most $k = \text{lcm}(o(x), o(y))$ such that

$$\begin{aligned}(xy^{-1})^k &= x^k(y^{-1})^k \\ &= x^k(y^k)^{-1} \\ &= e \cdot e^{-1} \\ &= e\end{aligned}$$

Thus, H is a subgroup of G. □

Exercise 3. Let G be a group. Define the set $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$ of all elements that commute with every other element of G is called the <u>center</u> of G.

(a) Show that $Z(G)$ is a subgroup of G.

(b) Show that $Z(G) = \cap_{a \in G} C(a)$.

(c) Compute the center of S_3.

Proof. (a) e by definition commutes with every other element. $\forall x, y \in Z(G)$, $xy \in Z(G)$ as x, y commutes with every element.

$$(xy)a = x(ya) = x(ay) = (ay)x = a(yx) = a(xy)$$

Inverses also exist as

$$x^{-1}a = (a^{-1}x)^{-1} = (xa^{-1})^{-1} = ax^{-1}$$

(b) If an element is in the intersection of all those sets, then it commutes with every element.

(c) Realize this is simply D_3, where only the identity commutes with one another. *Answer:* $\boxed{\{e\}}$

□