MA672 - (Topics) Number Theory and Cryptography

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These are the course notes for Topics in Advanced Mathematics (**MA672**) at Hotchkiss taught by Dr. Weiss. These notes were last updated January 10, 2020. Any sections denoted with asterisks (***) are currently incomplete, and I will update them when I get to those.

Contents

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Course Overview

- Abstract algebra: groups, rings, fields.
- Number Theory, arbitrary precision integer arithmetics.
- Cryptographic algorithms

1 Groups

We first define the group, which we will be using extensively.

Definition 1.1. (Group). A group is a set G with a binary operation "$\circ$" such that

- G is closed under $\circ$.
- G is associative.
- There is an Identity Element: $\exists e \in G \mid x \circ e = e \circ x - x \; \forall x \in G$.
- Inverses: $\forall x \in G \; \exists y \in G \mid x \circ y = y \circ x = e$.

Definition 1.2. (Abelian Group). If $\circ$ is commutative in group G, we call G Abelian. In that case, G is often written additively; i.e. we use "$+$" for "$\circ$".

(If $\circ$ is not commutative, we often write G multiplicatively.)

Definition 1.3. (Subgroup). Let G be a group, and $\emptyset \neq H \subseteq G$. Then H is called a subgroup of G if H is also a group.

A small proof to begin. . .

Proposition 1.4. Let G be a group and $x \in G$. Then x has a unique inverse y, so we can write $y = x^{-1}$.

Proof. Assume y and z are both inverses of x.

$$y = y \circ (x \circ z) = (y \circ x) \circ z = z$$

□

Proposition 1.5. A non-empty subset $H \subseteq G$ is a subgroup of G iff $xy^{-1} \in H \; \forall x, y \in H$.

Proof. ($\Rightarrow$)

- Identity: Pick $x \in H$. Then $xx^{-1} = e \in H$.
- Inverse: If $y \in H$, $ey^{-1} = y^{-1} \in H$

- Closure: If $x, y \in H$, $y^{-1} \in H$, so $x(y^{-1})^{-1} = xy \in H$.

($\Leftarrow$) If H is a group, then $y^{-1} \in H$ (existence of unverse) and $xy^{-1} \in H$ (closure of $\circ$). □

Example:

- Every vector space (without the scalars) is an Abelian group with identity $\vec{0}$.
- Modular arithmetic:
$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$
$\mathbb{Z}_n$ is an Abelian group under modular addition.
$$\mathbb{Z}_n^* = \{k \in \mathbb{Z}_n \mid gcd(k, n) = 1\}$$
$\mathbb{Z}_n^*$ is an Abelian group under modular multiplication (this is sometimes also $\mathbb{U}_n$).
Let's take $\mathbb{Z}_4 - \{0\}$ and why it's not a group under multiplication. We can create a multiplication table:

$\circ$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

However there is no such problem with $\mathbb{Z}_4^*$:

$\circ$	1	3
1	1	3
3	3	1

Definition 1.6. (Cyclic). A group G is called cyclic if $\exists g \in G$ (called generator) such that $G = \{g^n \mid n \in Z\}$.

Example: Z_n are cyclic grous with generator 1.

$\mathbb{Z}_4^*$ is cyclic with generator 3.

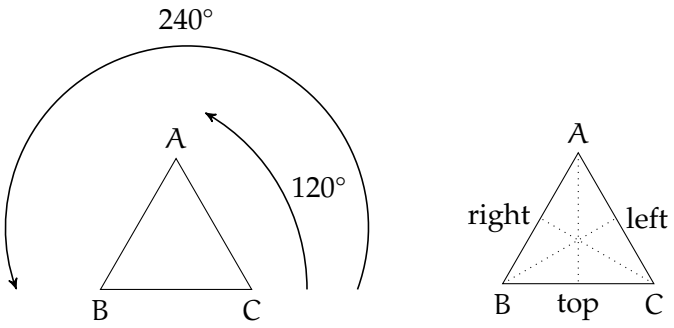
Example: The Klein 4-group is not cyclic:
$$K = \{(0,0), (1,0), (0,1), (1,1)\}$$
with componentwise addition mod 2.
$$K = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(k, l) \mid k \in \mathbb{Z}_2, l \in \mathbb{Z}_2\}$$

Proposition 1.7. Every cyclic group is Abelian.

Proof. Let $x, y \in G$, a cyclic Abelian group. Let g be the generator in G. We write $x = g^a$ and $y = g^b$. Then $xy = g^a g^b = \underbrace{g \circ g \circ \cdots \circ g}_{k+l \text{ times}} = g^b g^a = yx$. □

Example: The symmetry transformation of an equilateral $\triangle$ form a group under composition.

$$D_3 = \{\mathtt{id}, 120°, 240°, \mathtt{top}, \mathtt{left}, \mathtt{right}\}$$



$\circ$	id	120°	240°	top	left	right
id	id	120°	240°	top	left	right
120°	120°	240°	id	left	right	top
240°	240°	id	120°	right	top	left
top	top	right	left	id	240°	120°
left	left	top	right	120°	id	240°
right	right	left	top	240°	120°	id

Definition 1.8. (Equivalence). Let G be a group and H a subgroup. Define the relation $x \sim y$ if $xy^{-1} \in H$.

Proposition 1.9. $\sim$ is an equivalence relation on G.

If $H = \{e\}$, then $\sim$ is $=$.

If $H = G$, then $\sim$ is trivial.

Proof. We need to show that $\sim$ is

- reflexive: $x \sim x$ for all $x \in G$
$$xx^{-1} = e \in H.$$
- symmetric: $x \sim y \iff y \sim x$ for all $x, y \in G$

  Suppose $x \sim y$. Then $xy^{-1} \in H$. So $(xy^{-1})^{-1} = yx^{-1} \in H \Rightarrow y \sim x$.
- transitive: If $x \sim y, y \sim z$ then $x \sim z$ for all $x, y, z \in G$.

  Suppose $x \sim y, y \sim z$. Then $xy^{-1} \in H, yz^{-1} \in H$. Then $(xy^{-1})(yz^{-1}) = x(y^{-1}y)z^{-1} = xz^{-1} \in H \Rightarrow x \sim z$.

□

If $\sim$ is an equivalence relation on any set X, then $\sim$ partitions X into equivalence classes: If $y \in X, [y] = \{x \in X \mid x \sim y\}$.

Every element of X is in some equivalence class because $\sim$ is reflexive and no two equivalence classes intersect. Consider $[y_1], [y_2]$ and $z \in [y_1] \cap [y_2]$. Then $z \sim y_1$ and $z \sim y_2$ and $y_1 \sim y_2$. Hence, $[y_1] = [y_2]$.

Theorem 1.10. (Lagrange's Theorem). Let G be a finite group of order $|G|$ and H a subgroup of G. Then $|H|$ divides $|G|$.

Proof. We show that the above equivalence relation partitions G into equivalence classes of equal cardinality.

First, notice that H is an equivalence class by itself: $H = [e]$.

Let $[x]$ be another equivalence class. Then $[x] = Hx$: Let $y \in Hx$. Then $\exists a \in H$ such that $y = ax$. Then $\exists a \in H$ such that $y = ax$. But then $yx^{-1} = (ax)x^{-1} = a \in H$, so $y \sim x$.

We need to find a bijection between H and Hx for $x \in G$. Let $f : H \to Hx$, $f(a) = ax$. We ened to show that f is one-to-one and onto:

$$\begin{aligned} \text{1-1: If } f(a_1) &= f(a_2) \\ a_1x &= a_2x \\ a_1xx^{-1} &= a_2xx^{-1} \\ a_1 &= a_2. \end{aligned}$$

Onto: Let $y \in Hx$. Then $\exists a \mid y = ax \Rightarrow y = f(a)$.

$\Rightarrow H \simeq Hx \Rightarrow$ they have the same cardinality.

□

Definition 1.11. (Index). $[G : H]$ is the number of equivalence relations, which is called the index of H in G.

$$[G : H] = \frac{|G|}{|H|}.$$

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Quick Summary

We were talking about groups, which are

- associative,
- have an identity element,
- and have inverses.

There can be subgroups in groups, a subset of a group that is also a group.

An Abelian group is a commutative group.

A cyclic group is generated by an element g, they're always Abelian.

Lagrange's Theorem: the order of a subgroup divides the order of the group.

Index of H in G: $[G : H] = \frac{|G|}{|H|}$.

Example:

- $(\mathbb{R}^n, +), (\mathbb{C}^n, +)$.
- $(\mathbb{Z}_n, +), (\mathbb{Z}_n^*, \cdot), (\mathbb{R}, +), (\mathbb{R}^*, \cdot), (\mathbb{C}, +), (\mathbb{C}^*, \cdot)$.
- $GL_n(\mathbb{R})$: Invertible $n \times n$ matrices.
- $SL_n(\mathbb{R})$: Subgroup of $GL_n(\mathbb{R})$ with determinant 1.
- $GL_n(\mathbb{C})$: Invertible $n \times n$ complex matrices.
- $U(n)$: Unitary group – determinant of absolute value 1.
- Symmetry groups of geometric shapes. (Dihedral groups)
- Frieze groups.
- Wallpaper groups.
- Crystallographic groups.
- Permutation groups of $\{1, \ldots, n\}$ under composition (S_n), and its subgroups.

We will be focusing mainly on $(\mathbb{Z}_n, +), (\mathbb{Z}_n^*, \cdot)$.

Definition 1.12. (Order of an element). Let G be a group and $g \in G$. Then the order of $o(g)$ is the smallest positive integer n such that $g^n = e$. (May be infinite)

Proposition 1.13. Every group of prime order is cyclic.

Proof. Let $e \neq g \in G$. Consider the subgroup $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ generated by g. Then $|\langle g \rangle|$ divides $|G|$, which is prime. Hence $\langle g \rangle = G$. □

Exercise 1. Show that $\{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a subgroup of S_4.

Proof. We can construct a Cayley table for the subgroup. (Top row is evaluated first).

$\circ$	(1)	$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$
(1)	(1)	$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$
$(1,2)(3,4)$	$(1,2)(3,4)$	(1)	$(1,4)(2,3)$	$(1,3)(2,4)$
$(1,3)(2,4)$	$(1,3)(2,4)$	$(1,4)(2,3)$	(1)	$(1,2)(3,4)$
$(1,4)(2,3)$	$(1,4)(2,3)$	$(1,3)(2,4)$	$(1,2)(3,4)$	(1)

Every element is its own inverse, and (1) is the identity element. It is evident from the table that we also have closure. Hence, the set is a valid subgroup under composition. □

Exercise 2. Let G be an Abelian group. Show that the set of all elements of G of finite order forms a subgroup of G.

Proof. Let H be the subset of G with elements of finite order. We want to show that $\forall x, y \in H$, $xy^{-1} \in H$. In other words, xy^{-1} also has finite order. Well, xy^{-1} has order at most $k = \text{lcm}(o(x), o(y))$ such that

$$\begin{aligned}(xy^{-1})^k &= x^k(y^{-1})^k \\ &= x^k(y^k)^{-1} \\ &= e \cdot e^{-1} \\ &= e\end{aligned}$$

Thus, H is a subgroup of G. □

Exercise 3. Let G be a group. Define the set $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$ of all elements that commute with every other element of G is called the center of G.

(a) Show that Z(G) is a subgroup of G.

(b) Show that $Z(G) = \cap_{a \in G} C(a)$.

(c) Compute the center of S_3.

Proof. (a) e by definition commutes with every other element. $\forall x, y \in Z(G)$, $xy \in Z(G)$ as x, y commutes with every element.

$$(xy)a = x(ya) = x(ay) = (ay)x = a(yx) = a(xy)$$

Inverses also exist as

$$x^{-1}a = (a^{-1}x)^{-1} = (xa^{-1})^{-1} = ax^{-1}$$

(b) If an element is in the intersection of all those sets, then it commutes with every element.

(c) Realize this is simply D_3, where only the identity commutes with one another. *Answer:* $\boxed{\{e\}}$

□

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Exercise 4. Show that a non-Abelian group must have at least five distinct elements.

Proof. 1 element is trivial. 2 and 3 are primes so all groups of order 2 or 3 are cyclic. So we turn to groups of order 4. Every element cannot have order 1, and shown previously the orders have to divide 4. Additionally, if an element has order 4, then it is a generator and the group is cyclic. So all non-trivial elements must have order 2. We can construct a Cayley table for this specific group.

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Which gives us an Abelian group. □

Exercise 5. Let G be a group. Prove that $(ab)^n = a^n b^n$ for all $a, b \in G$ and all $n \in \mathbb{Z}$ if an only if G is Abelian.

Proof. The left implication is trivial (rearrange). We focus on the right implication.

We let $n = 2$. Then

$$\begin{aligned}(ab)^2 &= a^2b^2\\ abab &= aabb\\ a^{-1}babb^{-1} &= a^{-1}aabbb^{-1}\\ ba &= ab\end{aligned}$$

For all $a, b \in G$. □

Exercise 6. Let G be a group. Prove that G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

Proof.

$$\begin{aligned}(ab)^{-1} &= a^{-1}b^{-1}\\ (b^{-1}a^{-1})^{-1} &= (a^{-1}b^{-1})^{-1}\\ ab &= ba\end{aligned}$$

We can travel in both directions in this proof. □

Exercise 7. Let G be a group. Prove that if $x^2 = e$ for all $x \in G$, then G is Abelian.

Proof. We use the fact that $x^{-1} = e$ for all $x \in G$. Then we use the conclusion arrived at *Exercise 6* to our advantage.

$$ab = ab$$
$$(ab)^{-1} = a^{-1}b^{-1}$$

Which is as desired. □

Exercise 8. Show that if G is a finite group with an even number of elements, then there must exist an element $a \in G$ with $a \neq e$ such that $a^2 = e$.

Proof. Assume otherwise, that except for the identity, we can pair elements off such that their inverse isn't themselves. This gives us pairs and the identity, which means the group has an odd number of elements. So there has to be an element whose inverse is itself. □

Exercise 9. Let G be a group, and let $a \in G$. The set $C(a) = \{x \in G \mid xa = ax\}$ of all elements of G is called the centralizer of a.

(a) Show that $C(a)$ is a subgroup of G.

Let $x, y \in C(a)$. Consider

$$\begin{aligned}(xy)a &= x(ya)\\ &= x(ay)\\ &= (ax)y\\ &= a(xy)\end{aligned}$$

So $C(a)$ is closed. We now show that inverses exist:

$$x^{-1}a = x^{-1}(ax)x^{-1} = x^{-1}(xa)x^{-1} = ax^{-1}$$

So $xx^{-1} \in C(a)$ so $C(a)$ is a group.

(b) Show that $\langle a \rangle \subseteq C(a)$.

$\langle a \rangle$ is cyclic and therefore Abelian, so all elements commute with a.

(c) Compute $C(a)$ if $G = S_3$ and $a = (1, 2, 3)$.

(d) Compute $C(a)$ if $G = S_3$ and $a = (1, 2)$.