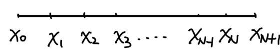
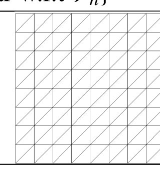
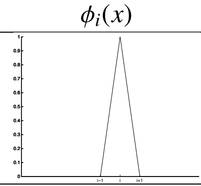
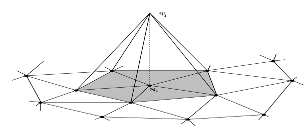
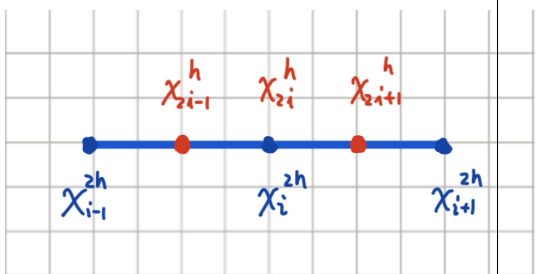
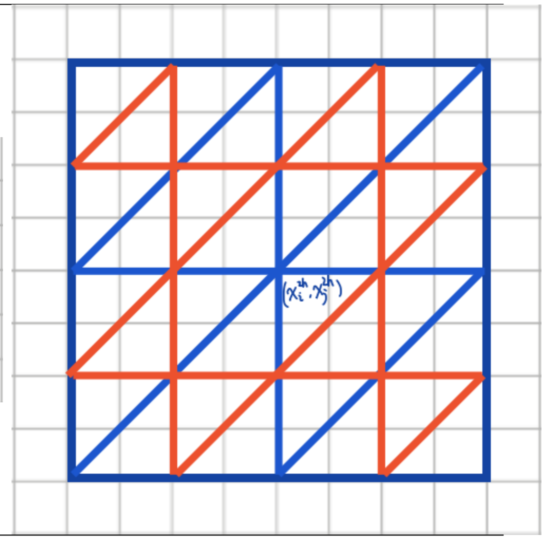
8.2 1D and 2D Finite Element and Multigrid

1D and 2D Comparison for Finite Element and Multigrid

$1D : \Omega = (0,1)$	$2D : \Omega = (0,1) \times (0,1)$
$\begin{cases} -u'' = f & x \in \Omega \\ u(0) = u(1) = 0 \end{cases}$	$\begin{cases} -\triangle u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$
$u = \underset{v \in V}{\arg\min}\, J(v)$	
$J(v) = \dfrac{1}{2}\displaystyle\int_0^1 \lvert v' \rvert^2 dx - \int_0^1 f v dx$	$J(v) = \dfrac{1}{2}\displaystyle\int_\Omega \lvert \nabla v \rvert^2 dx - \int_\Omega f v dx$
$V = \{v : \Omega \to \mathbb{R}^1$ is continuous and piecewise smooth, $v\vert_{\partial\Omega} = 0\}$	
FE space: $V_h = \{v_h \in V, v_h$ is piecewise linear w.r.t $\mathcal{T}_h\}$	
	
$\phi_i(x)$	$\phi_{ij}(x, y)$
	
Find $u_h \in V_h$ s.t. $J(u_h) = \underset{v_h \in V_h}{\min} J(v_h)$	
$u_h = \sum_{i=1}^n \mu_i \phi_i(x)$	$u_h = \sum_{i,j=1}^n \mu_{ij} \phi_{ij}(x, y)$
Find $\mu \in \mathbb{R}^n$ s.t. $I(\mu) = \min_{v \in \mathbb{R}^n} I(v)$	Find $\mu \in \mathbb{R}^{n \times n}$ s.t. $I(\mu) = \min_{v \in \mathbb{R}^{n \times n}} I(v)$
$I(v) = \frac{1}{2}(A * v, v)_{\ell^2} - (b, v)_{\ell^2}$	
$A = \frac{1}{h}(-1, 2, -1)$	$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\mu = \arg\min I(v) \Longleftrightarrow \nabla J(\mu) = A * \mu - b = 0$	
GD Method: $\mu^{m+1} = \mu^m - \eta(A * \mu^m - b)$	
$\eta = \frac{h}{4}$	$\eta = \frac{1}{8}$

Basic multigrid components

	
$\phi^{2h}_i = \frac{1}{2}\phi^h_{2i-1} + \phi^h_{2i} + \frac{1}{2}\phi^h_{2i+1}$	$\phi^{2h}_{i,j} = \phi^h_{2i,2j} + \frac{1}{2}\left(\phi^h_{2i-1,2j-1} + \phi^h_{2i+1,2j+1}\right) + \frac{1}{2}\left(\phi^h_{2i-1,2j} + \phi^h_{2i,2j-1} + \phi^h_{2i+1,2j} + \phi^{2h}_{2i,2j+1}\right)$
$\Phi^{2h} = R *_2 \Phi^h$	
$R = (\frac{1}{2}, 1, \frac{1}{2})$	$R = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

8.2.1 Multigrid algorithm for $A * \mu = f$

Algorithm 11 A multigrid algorithm $\mu = \mathrm{MG1}(f; \mu^0; J, \nu_1, \cdots, \nu_J)$

Set up

$$f^1 = f, \quad \mu^1 = \mu^0.$$

Smoothing and restriction from fine to coarse level (nested)
for $\ell = 1 : J$ **do**
  for $i = 1 : \nu_\ell$ **do**

$$\mu^\ell \leftarrow \mu^\ell + S^\ell * (f^\ell - A_\ell * \mu^\ell). \tag{8.33}$$

  end for
  Form restricted residual and set initial guess:

$$\mu^{\ell+1} \leftarrow \Pi_\ell^{\ell+1} \mu^\ell, \quad f^{\ell+1} \leftarrow R *_2 (f^\ell - A_\ell * \mu^\ell) + A_{\ell+1} * \mu^{\ell+1},$$

end for
Prolongation and restriction from coarse to fine level
for $\ell = J - 1 : 1$ **do**

$$\mu^\ell \leftarrow \mu^\ell + R *_2^\top (\mu^{\ell+1} - \Pi_\ell^{\ell+1} \mu^\ell).$$

end for

$$\mu \leftarrow \mu^1.$$

Remark 9. The above multigrid method for the linear problem $A * \mu = b$ is independent of the choice of the interpolation operation $\Pi_\ell^{\ell+1} : \mathbb{R}^{n_\ell \times n_\ell} \mapsto \mathbb{R}^{n_{\ell+1} \times n_{\ell+1}}$ and in particular, we could take $\Pi_\ell^{\ell+1} := 0$. But such an operation is critical for nonlinear problems.

8.2.2 MgNet

Algorithm 12 μ^J = MgNet1$(f;\mu^0;J,\nu_1,\cdots,\nu_J)$

Set up

$$f^1=\theta * f,\quad \mu^1=\mu^0.$$

Smoothing and restriction from fine to coarse level (nested)

for $\ell = 1 : J$ **do**

for $i = 1 : \nu_\ell$ **do**

$$\mu^\ell \leftarrow \mu^\ell + \sigma\circ S^\ell * \sigma\circ(f^\ell - A_\ell * \mu^\ell). \tag{8.34}$$

end for

Form restricted residual and set initial guess:

$$\mu^{\ell+1} \leftarrow \Pi_\ell^{\ell+1}\mu^\ell,\quad f^{\ell+1} \leftarrow R *_2 (f^\ell - A_\ell * \mu^\ell) + A_{\ell+1} * \mu^{\ell+1},$$

end for