Assignment 3

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Problem 1

a.

$$\begin{aligned}
f_Y(y) &= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx \\
&= \left[\frac{1}{\pi}x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \\
&= \frac{2}{\pi}\sqrt{1-y^2}
\end{aligned}$$

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-x^2}} \\
&= \frac{1}{2\sqrt{1-x^2}}
\end{aligned}$$

b.

$$\begin{aligned}
\mathbb{E}[Y|X=x] &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y}{2\sqrt{1-x^2}}\,dy \\
&= 0
\end{aligned}$$

Since

$$f(x,y) = \frac{y}{2\sqrt{1-x^2}}$$

is an odd function of y

Problem 2

a.

No it's not.

A valid CDF shoud have following property

$$\lim_{x,y\to 1} F(x,y) = 1 \quad (1)$$
$$\lim_{x,y\to 0} F(x,y) = 0 \quad (2)$$

And by definition of marginal CDF

$$\lim_{x\to 0, y\to 1} F(x,y) = 0 \quad (3)$$
$$\lim_{x\to 1, y\to 0} F(x,y) = 0 \quad (4)$$

And F(x,y) does not satisfy property (3) and (4)

b.

By definition, we can derive marginal CDF by

$$\begin{aligned} F(1,y) &= F_Y(y) &= \frac{1+y}{2} \quad &, y \in [0,1] \\ F(x,1) &= F_X(x) &= \frac{1+x}{2} \quad &, x \in [0,1] \end{aligned}$$

By definition of marginal pdf

$$\frac{\partial F(x,y)}{\partial x} = f_X(x) = \frac{1}{2} \quad , x \in [0,1]$$
$$\frac{\partial F(x,y)}{\partial y} = f_Y(y) = \frac{1}{2} \quad , y \in [0,1]$$

c.

No, since

$$\frac{\partial^2 F(x,y)}{\partial y \partial x} = 0$$

which is not a valid joint PDF

Problem 3

a.

$$
\begin{aligned}
P(X > Y) &= P(X=2, Y=1) + P(X=3, Y \leq 2) + P(X=4, Y \leq 3) + \ldots + P(X=12, Y \leq 6) \\
&= \frac{1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 + 6 \times 6 + (6+5+4+3+2+1) \times 6}{6 \times 6 \times 6} \\
&= \frac{181}{216}
\end{aligned}
$$

b.

Let the sum of Ray's daughter throwing $2i$ dices be X_i and him throwing i dices Y_i

Let $\bar{W}_i = \bar{X}_i - \bar{Y}_i = \frac{X_i - Y_i}{i}$

Then we know that

$$
\begin{aligned}
\mathbb{E}[\bar{W}_i] &= \mathbb{E}[\bar{X}_i] - \mathbb{E}[\bar{Y}_i] \\
&= 7 - 3.5 \\
&= 3.5 \\
Var[\bar{W}_i] &= Var[\bar{X}_i] + Var[\bar{Y}_i] = \frac{35}{12} + \frac{70}{12} = \frac{35}{4}
\end{aligned}
$$

By Central limit theorem,

$$
\begin{aligned}
& \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \\
\Rightarrow \quad & \sqrt{n}(\bar{W}_n - 3.5) \xrightarrow{d} N(0, \frac{35}{4}) \\
\Rightarrow \quad & \frac{\sqrt{n}(\bar{W}_n - 3.5)}{\sqrt{\frac{35}{4}}} \xrightarrow{d} N(0, 1) \\
\Rightarrow \quad & P(\bar{W}_n > 0) = \Phi(\frac{-3.5\sqrt{n}}{\sqrt{\frac{35}{4}}})
\end{aligned}
$$

if n is large enough

Problem 4

a.

$$
\mathbb{E}[X] = \frac{1}{2} \times 3 + \frac{1}{2} \times \frac{1}{3} = \frac{5}{3}
$$

$$
\mathbb{E}[\ln(X)] = 1/2 \times \ln(3) + 1/2 \times \ln\left(\frac{1}{3}\right) = 0
$$

b.

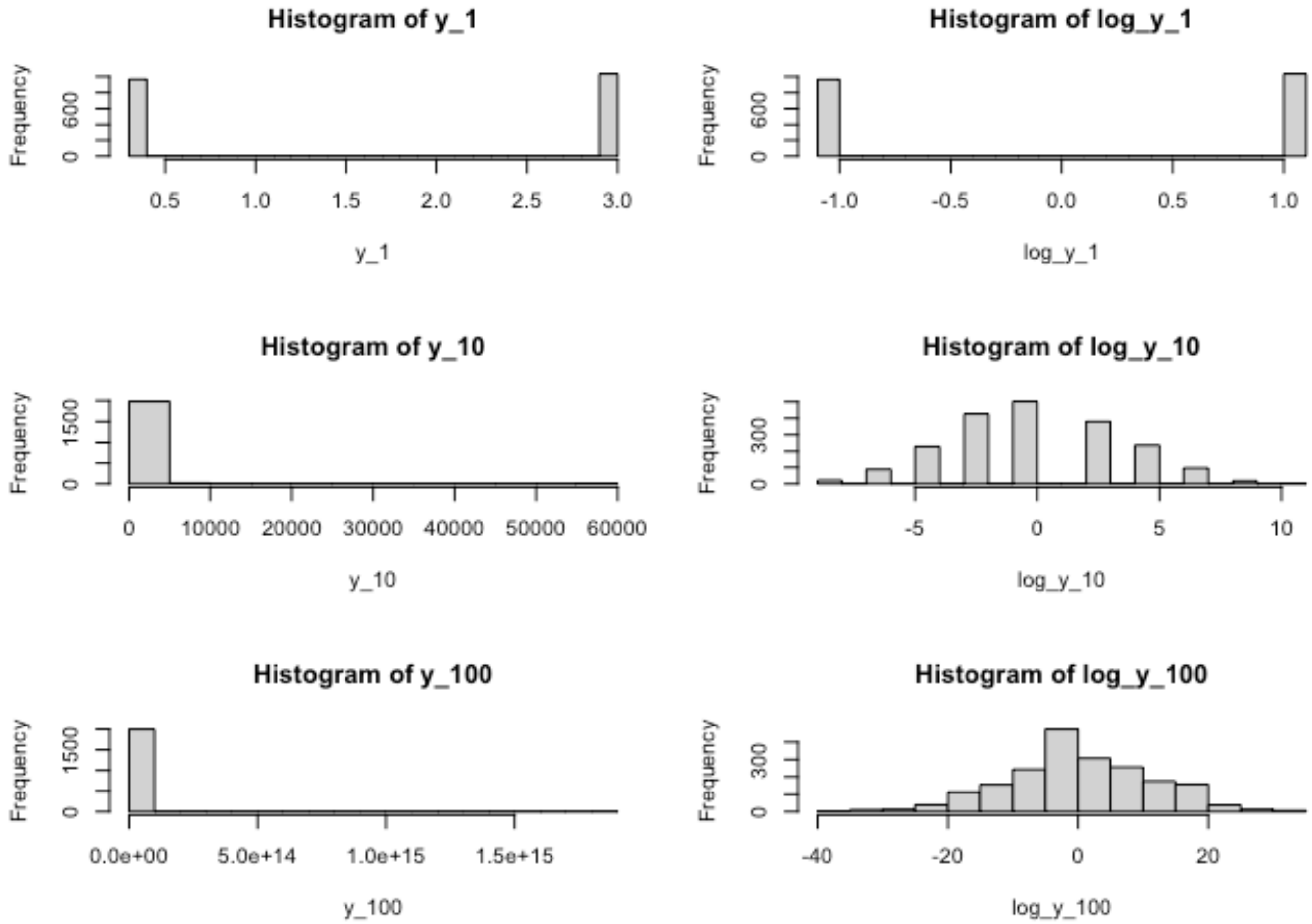
Since $\{X_i\}_{i=1}^{n}$ are i.i.d random variables

$$\mathbb{E}[Y_n] = \mathbb{E}[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} \mathbb{E}[X_i] = \left(\frac{5}{3}\right)^n$$

$$\mathbb{E}[\ln (Y_n)] = \mathbb{E}[\sum_{i=1}^{n} \ln X_i] = \sum_{i=1}^{n} \mathbb{E}[\ln X_i] = 0$$

c.

```
par(mfcol=c(3,2))
y_n <- function(n){
    # do n trials(Xn) with fliping only once and count success time (0 or 1)
    return(replicate(2000,prod(8/3*rbinom(n,1,0.5)+1/3), simplify = "array"))
}
log_y_n <- function(n){
    return(replicate(2000,sum(log(8/3*rbinom(n,1,0.5)+1/3)), simplify = "array"))
}
y_1 <- y_n(1)
y_10 <- y_n(10)
y_100 <- y_n(100)
log_y_1 <- log_y_n(1)
log_y_10 <- log_y_n(10)
log_y_100 <- log_y_n(100)
hist(y_1,breaks = 20)
hist(y_10,breaks= 20)
hist(y_100,breaks= 20)
hist(log_y_1,breaks = 20)
hist(log_y_10,breaks = 20)
hist(log_y_100,breaks = 20)
```



Problem 5

a.

Let $\mathbb{E}[X] = \mu$ and $\mathbb{E}[Y] = \nu$

$$\mathbb{E}\left[\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})\right] = \mathbb{E}\left[\sum_{i=1}^{n}\Big[(X_i - \mu) + (\mu - \bar{X})\Big]\Big[((Y_i - \nu) + (\nu - \bar{Y})\Big]\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\Big[(X_i - \mu)(Y_i - \mu) + (X_i - \mu)(\nu - \bar{Y}) + (\mu - \bar{X})(Y_i - \nu) + (\mu - \bar{X})(\nu - \bar{Y})\Big]\right]$$

$$= ncov(X, Y) - cov(X, Y) - cov(X, Y) + cov(X, Y)$$

$$= (n - 1)cov(X, Y)$$

Therfore $\hat{\sigma}_{XY} = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})\right] = \frac{n-1}{n}cov(X, Y)$

which is a downward biased estimator

b.

$$\begin{aligned}\hat{\sigma}_{XY} &= \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y}) \\ &= \frac{1}{n}\sum_{i=1}^{n}X_iY_i-\bar{X}\bar{Y}\end{aligned}$$

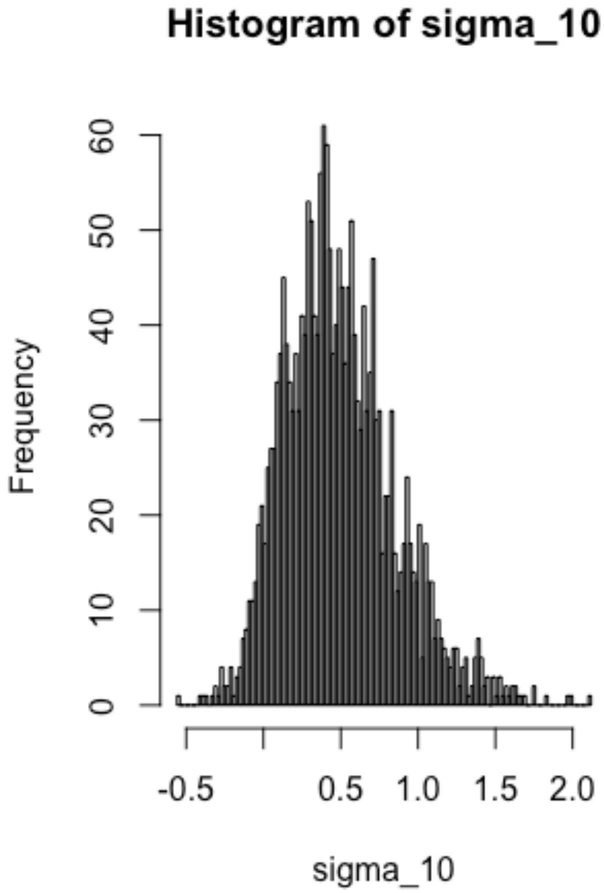
By WLLN

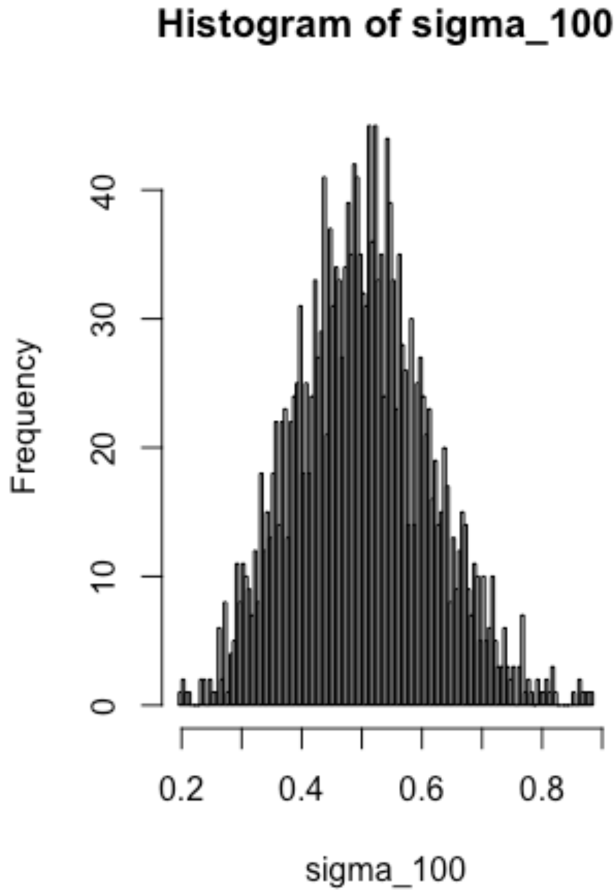
$$\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\bar{X}\bar{Y}\xrightarrow[W.L.L.N]{p}\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\mu\nu=cov(X,Y)$$

Therefore, $\sigma\hat{_{XY}}$ is a consistent estimator

c.

```
library("MASS")
cov_mat  <- matrix(c(1,0.5,0.5,1),2,2)
mu <- c(0,0)
sigma_n <- function(n){
    #cov return a covariance matrix
    return(replicate(2000,cov(mvrnorm(n,mu,cov_mat))[1,2],simplify = "array"))
}
sigma_10 <- sigma_n(10)
sigma_100 <- sigma_n(100)
par(mfrow=c(1,2))
hist(sigma_10,breaks=100)
hist(sigma_100,breaks=100)
```


Histogram of sigma_10
Frequency
0
10
20
30
40
50
60
-0.5
0.5
1.0
1.5
2.0
sigma_10


Histogram of sigma_100
Frequency
0
10
20
30
40
0.2
0.4
0.6
0.8
sigma_100