Spring 2022

Quantitative Analysis
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Problem Set 1: Solution

Part One: Hand-Written Exercise

1. The least-squares(LS) criterion function is

$$Q_n(\beta_0, \beta_1) := \textstyle\sum_{i=1}^n u_i^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

The first order conditions(FOCs) are

$$\frac{\partial Q_n(\beta_0, \beta_1)}{\partial \beta_0} = -2 \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{1}$$

$$\frac{\partial Q_n(\beta_0, \beta_1)}{\partial \beta_1} = -2 \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) x_i = 0 \tag{2}$$

By (1)(2),

$$n\beta_0 + \sum_{i=1}^n x_i \beta_1 = \sum_{i=1}^n y_i \tag{3}$$

$$\sum_{i=1}^n x_i \beta_0 + \sum_{i=1}^n x_i^2 \beta_1 = \sum_{i=1}^n x_i y_i \tag{4}$$

By (3)(4), we can get

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}. \blacksquare$$

2. (a)

$$
\begin{aligned}
\because \sum_{i=1}^{n} \hat{u}_i(\hat{y}_i - \bar{y}) &= \sum_{i=1}^{n} \hat{u}_i \hat{y}_i - \sum_{i=1}^{n} \hat{u}_i \bar{y} \\
&= \sum_{i=1}^{n} \hat{u}_i(\hat{\beta}_0 + \hat{\beta}_1 x_i) - \sum_{i=1}^{n} \hat{u}_i \bar{y} \\
&= \hat{\beta}_0 \underbrace{\sum_{i=1}^{n} \hat{u}_i}_{=0} + \hat{\beta}_1 \underbrace{\sum_{i=1}^{n} \hat{u}_i x_i}_{=0} - \bar{y} \underbrace{\sum_{i=1}^{n} \hat{u}_i}_{=0} \\
&= 0 \\
\therefore \sum_{i=1}^{n} (\hat{u}_i + \hat{y}_i - \bar{y})^2 &= \sum_{i=1}^{n} \hat{u}_i^2 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2. \blacksquare
\end{aligned}
$$

(b)

$$
\begin{aligned}
\because \sum_{i=1}^{n} \hat{u}_i(\hat{y}_i - \bar{y}) &= \sum_{i=1}^{n} \hat{u}_i \hat{y}_i - \sum_{i=1}^{n} \hat{u}_i \bar{y} \\
&= \sum_{i=1}^{n} \hat{u}_i(\hat{\beta}_1 x_i) - \sum_{i=1}^{n} \hat{u}_i \bar{y} \\
&= \hat{\beta}_1 \underbrace{\sum_{i=1}^{n} \hat{u}_i x_i}_{=0} - \bar{y} \underbrace{\sum_{i=1}^{n} \hat{u}_i}_{\neq 0} \\
&\neq 0 \\
\therefore \sum_{i=1}^{n} (\hat{u}_i + \hat{y}_i - \bar{y})^2 &\neq \sum_{i=1}^{n} \hat{u}_i^2 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2. \blacksquare
\end{aligned}
$$

3.

$$
\begin{aligned}
\because Cov(\bar{y}, \hat{\beta}_1) &= Cov\left(\frac{\sum_{i=1}^n y_i}{n}, \frac{\sum_{i=1}^n y_i(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right) \\
&= \frac{1}{n\sum_{i=1}^n (x_i - \bar{x})^2} Cov\left(\sum_{i=1}^n y_i, \sum_{i=1}^n y_i(x_i - \bar{x})\right) \\
&= \frac{1}{n\sum_{i=1}^n (x_i - \bar{x})^2} \sum_{i=1}^n (x_i - \bar{x}) Cov(\sum_{j=1}^n y_j, y_i) \\
&= \frac{1}{n\sum_{i=1}^n (x_i - \bar{x})^2} \underbrace{\sum_{i=1}^n (x_i - \bar{x})}_{=0} \sigma^2 \\
&= 0 \\
\therefore Var(\hat{\beta}_0) &= Var(\bar{y} - \hat{\beta}_1 \bar{x}) \\
&= Var(\bar{y}) + Var(\hat{\beta}_1 \bar{x}) \\
&= \sigma_0^2 \frac{\sum_{i=1}^n x_i^2 / n}{\sum_{i=1}^n (x_i - \bar{x})^2}. \blacksquare
\end{aligned}
$$

4. We already know that for model A:

$$
\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
\hat{\beta}_1 &= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}
\end{aligned}
$$

Now for model B, from the F.O.C. of $\sum \left(y_i - \alpha_0 - \alpha_1(x_i - \bar{x})\right)^2$ we have:

$$
\begin{cases}
-2\sum \left(y_i - \hat{\alpha}_0 - \hat{\alpha}_1(x_i - \bar{x})\right) = 0 \\
-2\sum \left(y_i - \hat{\alpha}_0 - \hat{\alpha}_1(x_i - \bar{x})\right)(x_i - \bar{x}) = 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\sum y_i = n\hat{\alpha}_0 + \hat{\alpha}_1 \sum (x_i - \bar{x}) \\
\sum y_i (x_i - \bar{x}) = \hat{\alpha}_0 \sum (x_i - \bar{x}) + \hat{\alpha}_1 \sum (x_i - \bar{x})^2
\end{cases}
$$

$\hat{\alpha}_0$ and $\hat{\alpha}_1$ is therefore given by:

$$
\begin{aligned}
\hat{\alpha}_0 &= \bar{y} \\
\hat{\alpha}_1 &= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}
\end{aligned}
$$

(a) $\hat{\alpha}_0$ and $\hat{\beta}_0$ are not identical, and their variance is given by:

$$\begin{aligned}
\text{Var}(\hat{\alpha}_0) &= \text{Var}(\bar{y}) = \frac{\sigma^2}{n} \\
\text{Var}(\hat{\beta}_0) &= \frac{\sigma^2}{n} \cdot \frac{\sum x_i^2}{\sum (x_i - \bar{x})^2}.
\end{aligned}$$

Since for any sample of data, $\sum x_i^2 \geq \sum (x_i - \bar{x})^2$ (please verify), hence $\text{Var}(\hat{\beta}_0) \geq \text{Var}(\hat{\alpha}_0)$.

(b) $\hat{\alpha}_1$ and $\hat{\beta}_1$ are identical. ■