Spring 2022

Quantitative Analysis
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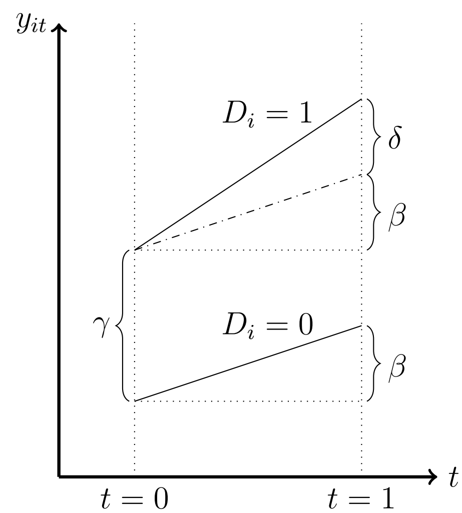
Problem Set 4: Solution

Part One: Hand-Written Exercise

1. .



2. (a)

$$\tilde{\beta}_1 = \frac{\sum (x_{i1} - \bar{x}_1) y_i}{\sum (x_{i1} - \bar{x}_1)^2} = \frac{\sum (x_{i1} - \bar{x}_1)(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + u_i)}{\sum (x_{i1} - \bar{x}_1)^2}$$
$$= \beta_1 + \beta_2 \frac{\sum (x_{i1} - \bar{x}_1) x_{2i}}{\sum (x_{i1} - \bar{x}_1)^2} + \frac{\sum (x_{i1} - \bar{x}_1) u_i}{\sum (x_{i1} - \bar{x}_1)^2}.$$

By the given assumptions, we have:

$$\frac{1}{n} \sum (x_{i1} - \bar{x}_1) u_i \xrightarrow{p} \mathbb{E}(x_1 u) - \mu_{x1} \mathbb{E}(u) = 0$$
$$\frac{1}{n} \sum (x_{i1} - \bar{x}_1)^2 \xrightarrow{p} \sigma^2_{x_1}$$
$$\frac{1}{n} \sum (x_{i1} - \bar{x}_1) x_{i2} = \frac{1}{n} \sum (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) \xrightarrow{p} \sigma_{x_1 x_2}.$$

Therefore, we have:

$$\tilde{\beta}_1 \xrightarrow{p} \beta_1 + \beta_2 \frac{\sigma_{x_1 x_2}}{\sigma^2_{x_1}}.$$

$\tilde{\beta}_1$ is consistent for β_1 only when $\sigma_{x_1 x_2} = 0$, otherwise $\tilde{\beta}_1$ is not consistent.

(b) Since $\sigma_{x_1 x_2} > 0$, then if $\beta_2 > 0$, $\tilde{\beta}_1$ overestimates β_1 by $\beta_2 \frac{\sigma_{x_1 x_2}}{\sigma^2_{x_1}}$ as $n \to \infty$. On the other hand, if $\beta_2 < 0$, then $\tilde{\beta}_1$ underestimates β_1 by $-\beta_2 \frac{\sigma_{x_1 x_2}}{\sigma^2_{x_1}}$ as $n \to \infty$.

(c) $\sqrt{n}(\tilde{\beta}_1 - \beta_1)$ does not follow any distributions. Since $\sigma_{x_1 x_2} > 0$, thus

$$\tilde{\beta} - \beta_1 \xrightarrow{p} \beta_2 \frac{\sigma_{x_1 x_2}}{\sigma_{x_1}^2} \neq 0.$$

So $\sqrt{n}(\tilde{\beta}_1 - \beta_1)$ clearly diverges as $n \to \infty$.

3. (a) False. Recall that $\hat{\sigma}^2_{OLS} = \frac{1}{n-k-1} \sum \hat{e}_i^2$ is an unbiased estimator of σ^2. Hence, for $s \in \mathbb{R}$, $\hat{\sigma}^2(s) = \frac{1}{n-s} \sum \hat{e}_i^2$ is a biased estimator as long as $s \neq k+1$. However, $\forall s \in \mathbb{R}$, $\hat{\sigma}^2(s)$ is consistent for σ^2.

   (b) False. Consider a case where we have the data $x_i,\ i = 1, ..., n$, and the true population mean $\mu_x = 0$. Our estimator is designed as:

$$\hat{\mu}_n = \begin{cases} -1 & \text{with } p = 1/2 \\ 1 & \text{with } p = 1/2 \end{cases}.$$

   This estimator, although completely disregards the data x_i, is still unbiased. It is, however, not consistent ($\lim_{n\to\infty} \hat{\mu}_n \neq 0$).

   (c) False. $\hat{\boldsymbol{\beta}}$ could be consistent even when $\mathbf{V}$ doesn't exist.