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Quantitative Analysis
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Problem Set 3: Solution

Part One: Hand-Written Exercise

1. (a) Let

$$\hat{\sigma}_r^2 = \frac{1}{n-(k+1-q)}\sum_{1=1}^{n}\hat{e}_{i,r}^2 = \frac{SSR_r}{n-(k+1-q)}$$
$$\hat{\sigma}_{ur}^2 = \frac{1}{n-(k+1)}\sum_{1=1}^{n}\hat{e}_{i,ur}^2 = \frac{SSR_{ur}}{n-(k+1)},$$

where $\hat{e}_{i,r}^2$ and $\hat{e}_{i,ur}^2$ are the residuals from restricted and unrestricted models respectively. The fact that

$$\frac{(n-k-1+q)\hat{\sigma}_r^2}{\sigma^2} - \frac{(n-k-1)\hat{\sigma}_{ur}^2}{\sigma^2} \sim \chi^2(q)$$

implies

$$\frac{SSR_r - SSR_{ur}}{\sigma^2} \sim \chi^2(q).$$

Moreover, we know that

$$\frac{(n-k-1)\hat{\sigma}_{ur}^2}{\sigma^2} = \frac{SSR_{ur}}{\sigma^2} \sim \chi^2(n-k-1).$$

Finally, since $(\mathrm{SSR}_r - \mathrm{SSR}_{ur})/q$ and $\mathrm{SSR}_{ur}/(n-k-1)$ are independent of each other (proof omitted), we have the following result:

$$\frac{(\mathrm{SSR}_r - \mathrm{SSR}_{ur})/q}{\mathrm{SSR}_{ur}/(n-k-1)} = \frac{SSR_r - SSR_{ur}}{\sigma^2 q}\Big/\frac{SSR_{ur}}{\sigma^2(n-k-1)}$$
$$\sim \frac{\chi^2(q)/q}{\chi^2(n-k-1)/(n-k-1)} \sim F(q, n-k-1). \quad \blacksquare$$

(b)

$$F = \frac{(\mathrm{SSR}_r - \mathrm{SSR}_{ur})/q}{\mathrm{SSR}_{ur}/(n-k-1)} = \frac{(\frac{\mathrm{SSR}_r}{\mathrm{SST}} - \frac{\mathrm{SSR}_{ur}}{\mathrm{SST}})/q}{\frac{\mathrm{SSR}_{ur}}{\mathrm{SST}}/(n-k-1)}$$
$$= \frac{(1-R_r^2-1+R_{ur}^2)/q}{(1-R_{ur}^2)/(n-k-1)} = \frac{(R_{ur}^2-R_r^2)/q}{(1-R_{ur}^2)/(n-k-1)}. \quad \blacksquare$$

2. (a) $y_i = \alpha_0 + \alpha_1 D_{i1} + \alpha_2 D_{i2} + \beta_0 x_i + \beta_1 (x_i D_{i1}) + \beta_2 (x_i D_{i2}) + u_i$

   (b) i. When $D_{i1} = 1, D_{i2} = 0$: $y_i = (\alpha_0 + \alpha_1) + (\beta_0 + \beta_1) x_i + u_i$

   ii. When $D_{i1} = 0, D_{i2} = 1$: $y_i = (\alpha_0 + \alpha_2) + (\beta_0 + \beta_2) x_i + u_i$

   iii. When $D_{i1} = 0, D_{i2} = 0$: $y_i = \alpha_0 + \beta_0 x_i + u_i$

3. (a) An 1% increase in "expendA" will lead to an $0.01\beta_1$ unit increase for "voteA".

   (b) $H_0: \ \beta_1 + \beta_2 = 0.$

   (c) Let $\mathbf{R} = (0, 1, 1, 0)$, and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. Then under the null hypothesis, our test statistic t and its distribution is then given by:

$$t = \frac{\mathbf{R}\hat{\boldsymbol{\beta}}}{\sqrt{\mathbf{R}\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}})\mathbf{R}'}} = \frac{\mathbf{R}\hat{\boldsymbol{\beta}}}{\hat{\sigma}\sqrt{\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'}} \sim t(n-4)$$

   (d) $H_0: \ \beta_1 = \beta_2 = 0.$

   (e) Let $\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. Then under the null hypothesis, our test statistic F and its distribution is then given by:

$$F = \frac{(\mathbf{R}\hat{\boldsymbol{\beta}})' \left[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right]^{-1} (\mathbf{R}\hat{\boldsymbol{\beta}})}{2\hat{\sigma}^2} \sim F(2, n-4).$$