Quantitative Analysis
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Problem Set 2: Solution

Part One: Hand-Written Exercise

1. Since $\hat{r}_{i,1}$ is the OLS residual of regressing x_1 on the constant one and $x_2, ..., x_k$, we can write:

$$x_{i,1} = \hat{c}_0 + \hat{c}_2 x_{i,2} + ... + \hat{c}_k x_{i,k} + \hat{r}_{i,1},$$

where $\hat{c}_j$ are the OLS estimates. We now continue with the proof:

$$\begin{aligned}\sum \hat{r}_{i,1}^2 &= \sum \hat{r}_{i,1}\left(x_{i,1} - \hat{c}_0 - \hat{c}_2 x_{i,2} - ... - \hat{c}_k x_{i,k}\right) \\ &= \sum \hat{r}_{i,1} x_{i,1} - \hat{c}_0 \sum \hat{r}_{i,1} - \hat{c}_2 \sum \hat{r}_{i,1} x_{i,2} - ... - \hat{c}_k \sum \hat{r}_{i,1} x_{i,k} \\ &= \sum \hat{r}_{i,1} x_{i,1},\end{aligned}$$

by the fact that $\sum \hat{r}_{i,1} = \sum \hat{r}_{i,1} x_{i,2} = ... = \sum \hat{r}_{i,1} x_{i,k} = 0$. ■

2. (a) By $\beta = (\mathbf{X'X})^{-1}\mathbf{X'y}$, we can get $\hat{\beta}_0$ = -0.25, $\hat{\beta}_1$ =2 and $\hat{\beta}_2$ = 1.25.

   (b) By $\tilde{\beta}_1 = \dfrac{\sum_{i=1}^n (y_i - \bar{y})(x_{i1} - \bar{x}_1)}{\sum_{i=1}^n (x_{i1} - \bar{x}_1)^2}$ and $\breve{\beta}_2 = \dfrac{\sum_{i=1}^n (y_i - \bar{y})(x_{i1} - \bar{x}_2)}{\sum_{i=1}^n (x_{i2} - \bar{x}_2)^2}$, we can get $\tilde{\beta}_1$ = 2 and $\breve{\beta}_2$ = 1.25 .Thus, $\hat{\beta}_1$ - $\tilde{\beta}_1$ = $\hat{\beta}_2$ - $\breve{\beta}_2$ = 0.

   (c) x_1 and x_2 should be uncorrelated, which means $Cov(x_1, x_2)$ is equal to 0. ■

3. (a)

$$\begin{aligned}\tilde{\beta}_1 &= \frac{\sum x_i y_i}{\sum x_i^2} \\ \mathbb{E}(\tilde{\beta}_1) &= \frac{\sum x_i \mathbb{E}(y_i)}{\sum x_i^2} = \beta_0 \cdot \frac{\sum x_i}{\sum x_i^2} + \beta_1.\end{aligned}$$

   (b) If $\beta_0 \neq 0$, then $\tilde{\beta}_1$ is biased iff $\sum x_i \neq 0$.

   (c)

$$\mathrm{Var}(\tilde{\beta}_1) = \mathrm{Var}\left(\frac{\sum x_i y_i}{\sum x_i^2}\right) = \mathrm{Var}\left(\frac{\sum x_i u_i}{\sum x_i^2}\right) = \frac{\sigma^2}{\sum x_i^2}$$

   (d)

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}.$$

   As $\sum x_i^2 \geq \sum (x_i - \bar{x})^2$ for any sample of data, $\mathrm{Var}(\hat{\beta}_1) \geq \mathrm{Var}(\tilde{\beta}_1)$ in general.

(e) No, since $\tilde{\beta}_1$ is NOT unbiased in general. In the case where $\tilde{\beta}_1$ is unbiased, then we have $\bar{x} = 0$, causing $\mathrm{Var}(\hat{\beta}_1) = \mathrm{Var}(\tilde{\beta}_1)$. ■