Quantitative Analysis HW2

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(1)

Let $\hat{r}_{i,1}$ be the OLS residual of regressing x_1 on the constant 1 and $x_2, x_3, ...x_k$

We can write the Regression model as:

$$x_{i,1} = \beta_0 + \beta_2 x_{i,2} + \beta_3 x_{i,3} + ...\beta_k x_{i,k} + r_{i,1}$$

let $\hat{\beta}_0, \hat{\beta}_1, ...\hat{\beta}_k$ and $\hat{r}_{i,1}$ be the OLS estimates of this model.

By the FOCs of OLS algebraic properties

$$\begin{gathered}
\sum \hat{r}_{i,1}^2 = \sum \hat{r}_{i,1}.\hat{r}_{i,1} \\
= \sum \hat{r}_{i,1}(x_{i,1} - \hat{\beta}_0.1 - \hat{\beta}_2 x_2 - ...\hat{\beta}_k x_k) \\
= \sum \hat{r}_{i,1} x_{i,1} - \hat{\beta}_0 \sum \hat{r}_{i,1} - \hat{\beta}_2 \sum \hat{r}_{i,1} x_2 - ...\hat{\beta}_k \sum \hat{r}_{i,1} x_k \\
= \sum \hat{r}_{i,1} x_{i,1} - 0 - 0 - ...0 \\
= \sum_{i,1} x_{i,1}
\end{gathered}$$

(2)

If we observe the first two columns of this matrix, it is obvious that there exist multicollinearity among regressors. That is,

$$\mathbf{X}_{i,2} = 2\mathbf{X}_{i,1} \ , \quad \forall i \in 1, 2, 3, 4$$

Since $\mathbf{X}$ is not full rank, we cannot solve the inverse matrix of $(\mathbf{X}'\mathbf{X})$ and thus we can not calculate the OLS estimators.

(3)

(a)

From the first order condition of OLS esitmators with special condition $\tilde{\beta}_0 = 0$

$$
\begin{aligned}
\frac{\partial \sum (y_i - \tilde{\beta}_1 x_i)^2}{\partial \tilde{\beta}_1} &= 0 \\
\implies \sum (y_i - \tilde{\beta}_1 x_i) x_i &= 0 \\
\implies \sum y_i x_i - \tilde{\beta}_1 \sum x_i^2 &= 0 \\
\implies \tilde{\beta}_1 &= \frac{\sum y_i x_i}{\sum x_i^2} \\
\implies \tilde{\beta}_1 &= \frac{\sum (\beta_0 + \beta_1 x_i + u_i) x_i}{\sum x_i^2} \\
\implies \mathbb{E}[\tilde{\beta}_1] &= \frac{\sum (\beta_0 + \beta_1 x_i) x_i}{\sum x_i^2} \\
\implies \mathbb{E}[\tilde{\beta}_1] &= \frac{\sum \beta_0 x_i}{\sum x_i^2} + \beta_1
\end{aligned}
$$

(b)

Obviously, if $\beta_0 \neq 0$, $\mathbb{E}[\tilde{\beta}_1] \neq \beta_1$. Therefore, $\tilde{\beta}_1$ is not an unbiased estimator.

(c)

$$
\begin{aligned}
Var(\tilde{\beta}_1) &= Var\left(\frac{\sum y_i x_i}{\sum x_i^2}\right) \\
&= \frac{\sum Var(y_i) x_i^2}{(\sum x_i^2)^2} \\
&= \sigma^2 \frac{\sum x_i^2}{(\sum x_i^2)^2} \\
&= \sigma^2 \frac{1}{\sum x_i^2}
\end{aligned}
$$

(d)

Yes, in general

$$
Var(\tilde{\beta}_1) \leq Var(\hat{\beta}_1)
$$

given that

$$
\frac{1}{\sum (x_i - \bar{x})^2} \geq \frac{1}{\sum x_i^2}
$$

and they only equals when $\bar{x} = 0$.

(e)

No, it doesn't violates the Gauss-Markov Theorem since the theorem states that $\hat{\beta}_1$ is the best linear unbiased estimator (BLUE), whereas $\tilde{\beta}_1$ is a biased estimator of β_1.