Problem Set 9

Due: 5/18

Part One: Hand-Written Exercise

1. Let
$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)_+^3,$$
where $(x-\xi)_+^3 = (x-\xi)^3$ if $x > \xi$ and equals 0 otherwise. We will now show that $f(x)$ is a polynomial continuous at ξ, up to second derivatives, regardless of the values of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$.

   (a) Find a cubic polynomial $f_1(x)$ such that $f(x) = f_1(x)$ for all $x \le \xi$. Find another cubic polynomial $f_2(x)$ such that $f(x) = f_2(x)$ for all $x > \xi$. We have now established that $f(x)$ is a piecewise polynomial.

$$\begin{aligned}
&f(x) = f_1(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3, \quad x \le \xi;\\
&f(x) = f_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)^3\\
&= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^3 - 3x^2\xi + 3x\xi^2 - \xi^3)\\
&= (\beta_0 - \beta_4\xi^3) + (\beta_1 + 3\beta_4\xi^2)x + (\beta_2 - 3\beta_4\xi)x^2 + (\beta_3 + \beta_4)x^3, \quad x > \xi.
\end{aligned}$$

   (b) Show that $f_1(\xi) = f_2(\xi)$, $f_1'(\xi) = f_2'(\xi)$, and $f_1''(\xi) = f_2''(\xi)$. Therefore, $f(x)$, $f'(x)$, and $f''(x)$ are all continuous at ξ.

      i.
$$\begin{aligned}
&f_1(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3\\
&f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 + \beta_4(\xi-\xi)^3 = f_1(\xi)
\end{aligned}$$

      ii.
$$\begin{aligned}
&f_1^{'}(\xi) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2|_{x=\xi} = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2\\
&f_2^{'}(\xi) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2 + 3\beta_4(x-\xi)^2|_{x=\xi}\\
&= \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 + 3\beta_4(\xi-\xi)^2\\
&= f_1^{'}(\xi)
\end{aligned}$$

      iii.
$$\begin{aligned}
&f_1^{''}(\xi) = 2\beta_2 + 6\beta_3 x|_{x=\xi} = 2\beta_2 + 6\beta_3\xi\\
&f_2^{''}(\xi) = 2\beta_2 + 6\beta_3 x + 6\beta_4(x-\xi)|_{x=\xi} = 2\beta_2 + 6\beta_3\xi + 6\beta_4(\xi-\xi) = f_1^{''}(\xi)
\end{aligned}$$

(c) Show that $\frac{\partial^3 f_1(\xi)}{\partial x} \neq \frac{\partial^3 f_2(\xi)}{\partial x}$. Therefore, $\frac{\partial^3 f(x)}{\partial x}$ is not continuous at ξ.

$$
\begin{aligned}
f_1^{'''}(\xi) &= 6\beta_3 \\
f_2^{'''}(\xi) &= 6\beta_3 + 6\beta_4 \neq f_1^{'''}(\xi)
\end{aligned}
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