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Quantitative Analysis
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Problem Set 5: Solution

Part One: Hand-Written Exercise

1.

$$\mathbf{R}\tilde{\mathbf{D}}\mathbf{R}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1(k+1)} \\ d_{21} & d_{22} & \cdots & d_{2(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{(k+1)1} & d_{(k+1)2} & \cdots & d_{(k+1)(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} d_{21} & d_{22} & \cdots & d_{2(k+1)} \\ d_{31} & d_{32} & \cdots & d_{3(k+1)} \\ d_{41} & d_{42} & \cdots & d_{4(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} d_{22} & d_{23} & d_{24} \\ d_{32} & d_{33} & d_{34} \\ d_{42} & d_{43} & d_{44} \end{bmatrix}. \quad \blacksquare$$

2. From slide 47, Lecture 4, we have known that

$$\hat{\beta}_{\mathbf{IV}} = (\mathbf{Z}'\mathbf{X})^{-\mathbf{1}}\mathbf{Z}'\mathbf{y} = \left(\sum_{i=1}^{n} \mathbf{z_i}\mathbf{x_i}'\right)^{-1} \left(\sum_{i=1}^{n} \mathbf{z_i} y_i\right).$$

Letting

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

and

$$\mathbf{Z} = \begin{bmatrix} \mathbf{1} & \mathbf{z} \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 1 & z_2 \\ \vdots & \vdots \\ 1 & z_n \end{bmatrix},$$

we have

$$\begin{aligned}
\hat{\beta}_{\mathbf{IV}} &= \left(\sum_{i=1}^{n} \mathbf{z_i x_i^{'}}\right)^{-1} \left(\sum_{i=1}^{n} \mathbf{z_i} y_i\right) \\
&= \left(\sum_{i=1}^{n} \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 & x_i \end{bmatrix}\right)^{-1} \sum_{i=1}^{n} \begin{bmatrix} 1 \\ z_i \end{bmatrix} y_i \\
&= \begin{bmatrix} n & n\bar{x} \\ n\bar{z} & \sum_{i=1}^{n} z_i x_i \end{bmatrix}^{-1} \begin{bmatrix} n\bar{y} \\ \sum_{i=1}^{n} z_i y_i \end{bmatrix} \\
&= \frac{1}{n\sum_{i=1}^{n} z_i x_i - n^2 \bar{x}\bar{z}} \begin{bmatrix} \sum_{i=1}^{n} z_i x_i & -n\bar{x} \\ -n\bar{z} & n \end{bmatrix} \begin{bmatrix} n\bar{y} \\ \sum_{i=1}^{n} z_i y_i \end{bmatrix} \\
&= \frac{1}{n\sum_{i=1}^{n} z_i x_i - n^2 \bar{x}\bar{z}} \begin{bmatrix} n\bar{y}\sum_{i=1}^{n} z_i x_i - n\bar{x}\sum_{i=1}^{n} z_i y_i \\ -n^2\bar{z}\bar{y} + n\sum_{i=1}^{n} z_i y_i \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\bar{y}\sum_{i=1}^{n} z_i x_i - \bar{x}\sum_{i=1}^{n} z_i y_i}{\sum_{i=1}^{n} z_i x_i - n\bar{x}\bar{z}} \\ \dfrac{\sum_{i=1}^{n} z_i y_i - n\bar{z}\bar{y}}{\sum_{i=1}^{n} z_i x_i - n\bar{x}\bar{z}} \end{bmatrix}
\end{aligned}$$

Note that

$$\begin{aligned}
\hat{\beta}_{1,IV} &= \frac{\sum_{i=1}^{n} z_i y_i - n\bar{z}\bar{y}}{\sum_{i=1}^{n} z_i x_i - n\bar{x}\bar{z}} \\
&= \frac{\sum_{i=1}^{n}(z_i - \bar{z})(y_i - \bar{y})}{\sum_{i=1}^{n}(z_i - \bar{z})(x_i - \bar{x})}. \quad \blacksquare
\end{aligned}$$

3. Recall that from slide 55, Lecture 4, we have

$$\hat{\boldsymbol{\beta}}_{2\text{SLS}} = \boldsymbol{\beta}_o + \left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}(\boldsymbol{Z}'\boldsymbol{X}/n)\right]^{-1}\left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}(\boldsymbol{Z}'\mathbf{u}/n)\right].$$

Thus,

$$\hat{\boldsymbol{\beta}}_{\text{2SLS}} - \boldsymbol{\beta}_o = \left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}(\boldsymbol{Z}'\boldsymbol{X}/n)\right]^{-1}\left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}(\boldsymbol{Z}'\boldsymbol{u}/n)\right]$$

$$\Rightarrow \sqrt{n}(\hat{\boldsymbol{\beta}}_{\text{2SLS}} - \boldsymbol{\beta}_o) = \left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}(\boldsymbol{Z}'\boldsymbol{X}/n)\right]^{-1}\left[(\boldsymbol{X}'\boldsymbol{Z}/n)(\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1}\boldsymbol{V}^{\frac{1}{2}}\underbrace{\boldsymbol{V}^{-\frac{1}{2}}(\boldsymbol{Z}'\boldsymbol{u}/\sqrt{n})}_{\xrightarrow{D}\mathcal{N}(\mathbf{0},\mathbf{I})}\right]$$

$$\Rightarrow \sqrt{n}(\hat{\boldsymbol{\beta}}_{\text{2SLS}} - \boldsymbol{\beta}_o) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{D_o}). \quad \blacksquare$$

4. (a) False. OLS method makes sure its estimated $\hat{\mathbf{u}}$ is orthogonal to $\mathbf{X}$. Thus, if $\mathbf{X}$ is endogenous, $\hat{\mathbf{u}}$ can't correctly reflect that $\mathbb{E}(\mathbf{x_i u_i}) \neq \mathbf{0}$.

5. We can rewrite $\tilde{\beta}_1$ and $\hat{\beta}_1$ as follows:

$$\tilde{\beta}_1 = \beta_1 + \frac{Cov(z_i, u_i)}{Cov(z_i, x_i)} = \beta_1 + \frac{\rho_{zu}\sigma_z\sigma_u}{\rho_{zx}\sigma_z\sigma_x} = \beta_1 + \frac{\rho_{zu}}{\rho_{zx}}\frac{\sigma_u}{\sigma_x}$$

$$\hat{\beta}_1 = \beta_1 + \frac{Cov(x_i, u_i)}{Cov(x_i, x_i)} = \beta_1 + \frac{\rho_{xu}\sigma_x\sigma_u}{\rho_{xx}\sigma_x\sigma_x} = \beta_1 + \rho_{xu}\frac{\sigma_u}{\sigma_x}$$

   (a) From the above, we can see that $\tilde{\beta}_1$ is consistent for β_1 when $\rho_{zu} = 0$.

   (b) The bias of $\hat{\beta}_1$ will be less than $\tilde{\beta}_1$ when $|\rho_{xu}|$ is less than $|\frac{\rho_{zu}}{\rho_{zx}}|$.