Quantitative Analysis HW6

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1.

(a)

Given PDF of exponential distribution and $\{y_i\}_{i=1}^n$ independent variables.

The joint log-likelihood function is then given by:

$$
\begin{aligned}
L_n(\theta) = \ln\left(\prod_{i=1}^n \frac{1}{\theta} e^{-\frac{1}{\theta} y_i}\right) &= \sum_{i=1}^n \ln\left(\frac{1}{\theta} e^{-\frac{1}{\theta} y_i}\right) \\
&= \sum_{i=1}^n - \ln(\theta) - \left(\frac{1}{\theta} y_i\right) \\
&= -n \ln\theta - \frac{1}{\theta} \sum_{i=1}^n y_i
\end{aligned}
$$

To obtain $\tilde{\theta}$, we need to solve θ that maximize the log-likelihood function. Thus it must satisfy the FOC given by:

$$
\begin{aligned}
\frac{\partial L_n(\theta)}{\partial \theta} = 0 &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n y_i \\
\implies \theta n &= \sum_{i=1}^n y_i \\
\implies \theta &= \frac{1}{n} \sum_{i=1}^n y_i
\end{aligned}
$$

Finally, the maximum likelihood estimator $\tilde{\theta} = \frac{1}{n} \sum_{i=1}^n y_i$

which is the mean of y_i .

(b)

Let

$$
H(\theta) = \mathbb{E}\left[\frac{\partial^2 L_n(\theta)}{\partial \theta^2}\right]
$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} y_i\right] \\
&= \frac{n}{\theta^2} - \frac{2n}{\theta^2} \\
&= \frac{-n}{\theta^2}
\end{aligned}
$$

if the information equality holds,

$$
\begin{aligned}
&\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{D} \mathcal{N}(0, -H(\theta)^{-1}) \\
\implies &\lim_{n\to\infty} var(\sqrt{n}(\tilde{\theta} - \theta)) \xrightarrow{p} -H(\theta)^{-1} = \frac{\theta^2}{n} \to 0
\end{aligned}
$$