Quantitative Analysis

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Part one:

P1.

Define the sum of square error of given β_0 and β_1, $Q_n(\beta_0, \beta_1) = \sum_{i=1}^n u_i^2$

We want to minimize Q_n. Therefore, β_0 and β_1 must satisfiy the following FOCs.

$$\frac{\partial Q_n}{\partial \beta_0} = -2\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{1}$$

$$\frac{\partial Q_n}{\partial \beta_1} = -2\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)x_i = 0 \tag{2}$$

Simplifying the first derivative:

$$\begin{aligned} -2\sum_{i=0}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \\ \Rightarrow -2\sum_{i=1}^n y_i + 2n\beta_0 + 2\beta_1 \sum_{i=0}^n x_i = 0 \end{aligned}$$

Solve for β_0

$$\beta_0 = \frac{1}{n}\left(\sum_{i=1}^n y_i - \beta_1 \sum_{i=1}^n x_i\right)$$

Now simplify (2)

$$
\begin{aligned}
& -2\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)x_i=0 \\
\Rightarrow & -2\sum_{i=1}^{n}(x_iy_i-\beta_0x_i-\beta_1x_i^2)=0 \\
\Rightarrow & \sum_{i=1}^{n}x_iy_i-\beta_0\sum_{i=1}^{n}x_i-\beta_1\sum_{i=1}^{n}x_i^2=0
\end{aligned}
$$

Substituting the expression for β_0 obtained earlier, we get:

$$
\sum_{i=1}^{n}x_iy_i-\left(\frac{1}{n}\sum_{i=1}^{n}y_i-\beta_1\frac{1}{n}\sum_{i=1}^{n}x_i\right)\sum_{i=1}^{n}x_i-\beta_1\sum_{i=1}^{n}x_i^2=0
$$

Simplifying the above expression and solving for β_1:

$$
\begin{aligned}
\beta_1 &= \frac{\sum\limits_{i=1}^{n}x_iy_i-\frac{1}{n}\sum\limits_{i=1}^{n}x_i\sum\limits_{i=1}^{n}y_i}{\sum\limits_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum\limits_{i=1}^{n}x_i\right)^2} \\
&= \frac{\sum\limits_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})}{\sum\limits_{i=1}^{n}(xi-\bar{x})^2}
\end{aligned}
$$

This gives us the expression for the OLS estimator of β_1. Finally, we can use the expression for β_0 obtained earlier to get the OLS estimator for β_0:

$$
\begin{aligned}
\beta_0 &= \frac{1}{n}\sum_{i=1}^{n}y_i-\beta_1\frac{1}{n}\sum_{i=1}^{n}x_i \\
&= \bar{y}-\beta_1\bar{x}
\end{aligned}
$$

P2.

From `P1` we know that:

$$
\begin{aligned}
\beta_1 &= \frac{\sum\limits_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})}{\sum\limits_{i=1}^{n}(xi-\bar{x})^2} \\
\beta_0 &= \bar{y}-\beta_1\bar{x}
\end{aligned}
$$

And for α_0 and α_1 , we need to solve the following partial derivatives.

$$\frac{\partial}{\partial \alpha_0} \sum_{i=1}^{n} (y_i - \alpha_0 - \alpha_1(x_i - \bar{x}))^2 = -2 \sum_{i=1}^{n} (y_i - \alpha_0 - \alpha_1(x_i - \bar{x})) = 0$$
$$\frac{\partial}{\partial \alpha_1} \sum_{i=1}^{n} (y_i - \alpha_0 - \alpha_1(x_i - \bar{x}))^2 = -2 \sum_{i=1}^{n} (y_i - \alpha_0 - \alpha_1(x_i - \bar{x}))(x_i - \bar{x}) = 0$$

Simplify the first equation we get:

$$\sum_{i=1}^{n} y_i = n\alpha_0 + \alpha_1 \sum_{i=1}^{n} (x_i - \bar{x})$$
$$\implies \alpha_0 = \bar{y}$$

Now the second equation,

$$\sum_{i=1}^{n} y_i(x_i - \bar{x}) = \alpha_0 \sum_{i=1}^{n} (x_i - \bar{x}) + \alpha_1 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Since $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$. The first term on the RHS could be eliminated.

$$\sum_{i=1}^{n} y_i(x_i - \bar{x}) = \alpha_1 \sum_{i=1}^{n} (x_i - \bar{x})^2$$
$$\implies \alpha_1 = \frac{\sum_{i=1}^{n} y_i x_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$
$$= \frac{\sum\limits_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}$$

So the estimators of α_0 and β_0 are not identical. But the estimators of α_1 and β_1 are identical.

Comparing the variances:

$$Var(\beta_0) = \sigma^2 \frac{\sum_{i=1}^{n} x_i^2 / n}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and for α_0

$$Var(\alpha_0) = Var(\bar{y})$$

under classical assumption,

$$Var(\alpha_0) = \frac{\sigma^2}{n}$$

Since

$$\frac{\sum x_i^2}{\sum(x_i - \bar{x})^2} > 1$$
$$\implies Var(\alpha_0) < Var(\beta_0)$$

For β_1 and α_1 , since they are identical, their variances are also identical.

P3

(a)

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$$
$$SSR = \sum_{i=1}^{n} u_i^2$$
$$SSE = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$$

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(u_i + \hat{y}_i - \bar{y})62$$
$$= \sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} u_i(\hat{y}_i - \bar{y}) + \sum_{i=1}^{n}(\hat{y} - \bar{y})^2$$

For the second term, under classical assumption

$$\sum_{i=1}^{n} u_i(\hat{y}_i - \bar{y}) = \sum_{i=1}^{n} u_i\hat{y}_i - \sum_{i=1}^{n} u_i\bar{y}$$
$$= \beta_0 \sum_{i=1}^{n} u_i + \beta_1 \sum_{i=1}^{n} u_i x_i - \bar{y}\sum_{i=1}^{n} u_i$$
$$= 0$$

Therfore,

$$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 = SSR + SSE$$

(b)

Since we know that

$$\sum_{i=1}^{n} u_i x_i$$

need not to be zero if the model has no intersept term. The above term cannot be eliminated. Therefore,

$$SST \neq SSR + SSE$$