Quantitative Analysis HW5

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1.

$$
\boldsymbol{R}\tilde{\boldsymbol{D}}\boldsymbol{R}' =
$$

$$
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}
\begin{bmatrix} d_{11} & d_{12} & \dots & d_{1(k+1)} \\ d_{21} & d_{22} & \dots & d_{1(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{(k+1)1} & d_{(k+1)2} & \dots & d_{(k+1)(k+1)} \end{bmatrix}
\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \dots & d_{2(k+1)} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & \dots & d_{3(k+1)} \\ d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & \dots & d_{4(k+1)} \end{bmatrix}
\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} d_{22} & d_{23} & d_{24} \\ d_{32} & d_{33} & d_{34} \\ d_{42} & d_{43} & d_{44} \end{bmatrix}
$$

Obviously, $\boldsymbol{R}\tilde{\boldsymbol{D}}\boldsymbol{R}$ is a 3×3 matrix but not a diagonal matirx.

2.

In a two-stage least squares (2SLS) method estimating β_1 using instrumental variable z_i , we first regress x_i on z_i with the following regression.

$$
x_i = \alpha_0 + \alpha_1 z_i + v_i
$$

We can easily solve $\hat{\alpha}_1$ that minimize the residual sum of squares.

$$
\hat{\alpha}_1 = \frac{\sum(x_i - \bar{x})(z_i - \bar{z})}{\sum(z_i - \bar{z})^2}
$$

Let the estimated value of x_i using z_i be $\hat{x}_i$. We can know that

$$
\hat{x}_i = \hat{\alpha}_0 + \hat{\alpha}_1 z_i
$$

we then plug $\hat{x}_i$ into the original simple regression.

$$y_i = \beta_0 + \beta_1 \hat{x}_i + \varepsilon_i$$

$\hat{\beta}_{1,IV}$ is then given by

$$\begin{aligned}\hat{\beta}_{1,IV} = \frac{\sum(y_i - \bar{y})(\hat{\alpha}_0 + \hat{\alpha_1} z_i - \hat{\alpha}_0 - \hat{\alpha}_1 \bar{z})}{\sum(\hat{\alpha}_0 + \hat{\alpha_1} z_i - \hat{\alpha}_0 - \hat{\alpha}_1 \bar{z})^2} \\ = \frac{\hat{\alpha}_1 \sum(y_i - \bar{y})(z_i - \bar{z})}{\hat{\alpha}_1{}^2 \sum(z_i - \bar{z})^2} \\ = \frac{\sum(y_i - \bar{y})(z_i - \bar{z})}{\hat{\alpha}_1 \sum(z_i - \bar{z})^2}\end{aligned}$$

and the denominator of this fraction can be simplified to:

$$\begin{aligned}\hat{\alpha}_1 \sum(z_i - \bar{z})^2 \\ = \frac{\sum(x_i - \bar{x})(z_i - \bar{z})}{\sum(z_i - \bar{z})^2} \times \sum(z_i - \bar{z})^2 \\ = \sum(x_i - \bar{x})(z_i - \bar{z})\end{aligned}$$

we can get

$$\hat{\beta}_{1,IV} = \frac{\sum(y_i - \bar{y})(z_i - \bar{z})}{\sum(x_i - \bar{x})(z_i - \bar{z})}$$

3.

By WLLN:

$$\begin{aligned}\boldsymbol{X'Z}/n \xrightarrow{\boldsymbol{P}} \mathbb{E}(\boldsymbol{x}_i \boldsymbol{z}_i') =: \boldsymbol{M}_{xz} \\ \boldsymbol{Z'X}/n \xrightarrow{\boldsymbol{P}} \mathbb{E}(\boldsymbol{z}_i \boldsymbol{x}_i') =: \boldsymbol{M}_{zx} \\ \boldsymbol{Z'Z}/n \xrightarrow{\boldsymbol{P}} \mathbb{E}(\boldsymbol{z}_i \boldsymbol{z}_i') =: \boldsymbol{M}_{zz} \\ \boldsymbol{Z'}\varepsilon/n \xrightarrow{\boldsymbol{P}} \mathbb{E}(\boldsymbol{z}_i \varepsilon_i') = \boldsymbol{0}\end{aligned}$$

And assuming CLT,

$$\boldsymbol{V}^{-1/2} \frac{1}{\sqrt{n}} \sum_i z_i \varepsilon_i \xrightarrow{D} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$$

We can know that as $n \to \infty$,

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_{GMM} - \boldsymbol{b}_o) = [(\boldsymbol{X'Z}/n)\widehat{W}(\boldsymbol{Z'X}/n)]^{-1}[(\boldsymbol{X'Z}/n)\widehat{W}(\boldsymbol{Z'}\varepsilon/\sqrt{n})]$$

$$
\xrightarrow{D} (\boldsymbol{M}_{xz}\boldsymbol{W}\boldsymbol{M}_{zx})^{-1}\boldsymbol{M}_{xz}\boldsymbol{W}\boldsymbol{V}^{1/2}\mathcal{N}(\mathbf{0},\boldsymbol{I})
\overset{d}{=} \mathcal{N}(\mathbf{0},\boldsymbol{D}_o)
$$

where

$$
\boldsymbol{D}_o = (\boldsymbol{M}_{xz}\boldsymbol{W}\boldsymbol{M}_{zx})^{-1}(\boldsymbol{M}_{xz}\boldsymbol{W}\boldsymbol{V}\boldsymbol{W}\boldsymbol{M}_{zx})(\boldsymbol{M}_{xz}\boldsymbol{W}\boldsymbol{M}_{zx})^{-1}
$$