## **Quantitative Analysis HW5**

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1.

$$\begin{split} \boldsymbol{R}\tilde{\boldsymbol{D}}\boldsymbol{R}' = \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1(k+1)} \\ d_{21} & d_{22} & \dots & d_{1(k+1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{(k+1)1} & d_{(k+1)2} & \dots & d_{(k+1)(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \dots & d_{2(k+1)} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & \dots & d_{3(k+1)} \\ d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & \dots & d_{4(k+1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} d_{22} & d_{23} & d_{24} \\ d_{32} & d_{33} & d_{34} \\ d_{42} & d_{43} & d_{44} \end{bmatrix}$$

Obviously,  $m{R} ilde{m{D}} m{R}$  is a 3 imes 3 matrix but not a diagonal matrix.

## 2.

In a two-stage least squares (2SLS) method estimating  $\beta_1$  using instrumental variable  $z_i$ , we first regress  $x_i$  on  $z_i$  with the following regression.

$$x_i = \alpha_0 + \alpha_1 z_i + v_i$$

We can easily solve  $\hat{\alpha}_1$  that minimize the residual sum of squares.

$$\hat{lpha}_1 = rac{\sum (x_i - ar{x})(z_i - ar{z})}{\sum (z_i - ar{z})^2}$$

Let the estimated value of  $x_i$  using  $z_i$  be  $\hat{x}_i$ . We can know that

$$\hat{x}_i = \hat{lpha}_0 + \hat{lpha}_1 z_i$$

we then plug  $\hat{x}_i$  into the original simple regression.

$$y_i = \beta_0 + \beta_1 \hat{x}_i + \varepsilon_i$$

 $\hat{eta}_{1,IV}$  is then given by

$$\hat{eta}_{1,IV} = rac{\sum (y_i - ar{y})(\hat{lpha}_0 + \hat{lpha}_1 z_i - \hat{lpha}_0 - \hat{lpha}_1 ar{z})}{\sum \sum (\hat{lpha}_0 + \hat{lpha}_1 z_i - \hat{lpha}_0 - \hat{lpha}_1 ar{z})^2} \ = rac{\hat{lpha}_1 \sum (y_i - ar{y})(z_i - ar{z})}{\hat{lpha}_1^2 \sum (z_i - ar{z})^2} \ = rac{\sum (y_i - ar{y})(z_i - ar{z})}{\hat{lpha}_1 \sum (z_i - ar{z})^2}$$

and the denominator of this fraction can be simplified to:

$$\hat{lpha}_1 \sum_{} (z_i - ar{z})^2 \ = rac{\sum_{} (x_i - ar{x})(z_i - ar{z})}{\sum_{} (z_i - ar{z})^2} imes \sum_{} (z_i - ar{z})^2 \ = \sum_{} (x_i - ar{x})(z_i - ar{z})$$

we can get

$$\hat{eta}_{1,IV} = rac{\sum (y_i - ar{y})(z_i - ar{z})}{\sum (x_i - ar{x})(z_i - ar{z})}$$

3.

By WLLN:

$$egin{aligned} oldsymbol{X'}oldsymbol{Z}/n & \stackrel{oldsymbol{P}}{
ightarrow} \mathbb{E}(oldsymbol{x}_ioldsymbol{z}_i') =: oldsymbol{M}_{xz} \ oldsymbol{Z'}oldsymbol{Z}/n & \stackrel{oldsymbol{P}}{
ightarrow} \mathbb{E}(oldsymbol{z}_ioldsymbol{z}_i') =: oldsymbol{M}_{zz} \ oldsymbol{Z'}oldsymbol{arepsilon}/n & \stackrel{oldsymbol{P}}{
ightarrow} \mathbb{E}(oldsymbol{z}_ioldsymbol{arepsilon}_i') = oldsymbol{0} \end{aligned}$$

And assuming CLT,

$$oldsymbol{V}^{-1/2}rac{1}{\sqrt{n}}\sum_i z_i arepsilon_i \stackrel{D}{
ightarrow} \mathcal{N}(oldsymbol{0},oldsymbol{I})$$

We can know that as  $n \to \infty$ ,

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_{GMM}-\boldsymbol{b}_o)=[(\boldsymbol{X'Z}/n)\widehat{W}(\boldsymbol{Z'X}/n)]^{-1}[(\boldsymbol{X'Z}/n)\widehat{W}(\boldsymbol{Z'\varepsilon}/\sqrt{n})]$$

$$\stackrel{D}{ o} (oldsymbol{M}_{xz} oldsymbol{W} oldsymbol{M}_{zx})^{-1} oldsymbol{M}_{xz} oldsymbol{W} oldsymbol{V}^{1/2} \mathcal{N}(oldsymbol{0}, oldsymbol{I}) \ \stackrel{d}{=} \mathcal{N}(oldsymbol{0}, oldsymbol{D}_o)$$

where

$$oldsymbol{D}_o = (oldsymbol{M}_{xz} oldsymbol{W} oldsymbol{M}_{zx})^{-1} (oldsymbol{M}_{xz} oldsymbol{W} oldsymbol{V} oldsymbol{W} oldsymbol{M}_{zx}) (oldsymbol{M}_{xz} oldsymbol{W} oldsymbol{M}_{zx})^{-1}$$