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A Fixed-Point Algorithm for the AC Power Flow Problem

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Background

Fixed-point power flow algorithms

- Given generation and transmission network topology data, solve for bus voltages $\xi = (V, \theta)$ via fixed-point iterations $\xi_{k+1} = f(\xi_k)$, and terminate until $\|\xi_{k+1} - \xi_k\|_\infty \le \epsilon$
- Convergence analysis via the Banach fixed-point theorem

Relevant Literature

- Lossy DC Power Flow (L-DCPF) [JWSP '17]: decoupled active power flow equation
- Fixed-Point Power Flow (FPPF) [JWSP '18]: lossless power flow equations

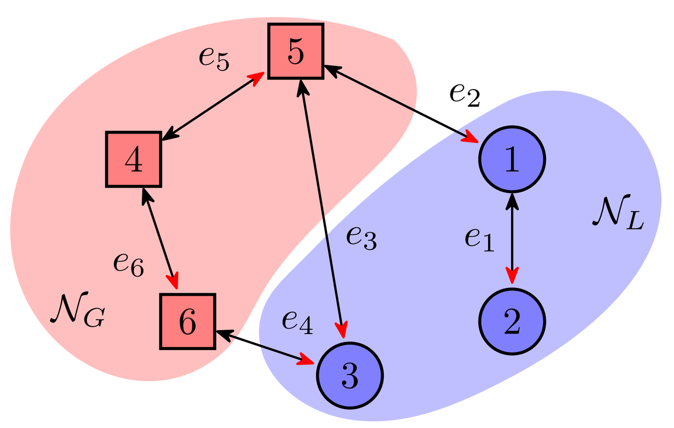
Goal: Extend L-DCPF & FPPF to

- solve the AC power flow problem with *coupled* and *lossy* power flow equations,
- accommodate phase-shifting transformers and the distributed slack bus model, and
- derive sufficient solvability conditions for the 2-bus model.

Graph modelling of an AC transmission system

Graph structure: weakly connected bidirected graph

- Each node of $\mathcal{G}$ is a load/generator bus (in the set $\mathcal{N}_L/\mathcal{N}_G$), total of $n+m$ buses
- Each edge is a branch: transmission line/transformer in standard Π-model, total of $|\mathcal{E}|$ branches



Graph matrices

- Node-edge incidence matrix A, cycle matrix C, and asymmetrically weighted incidence matrix Γ

Vectorized power flow equations

$$\bar{P}_i + \alpha_i P_{\text{slack}} = \sum_{j=1}^{n+m} V_i V_j G_{ij} \cos(\theta_i - \theta_j) + V_i V_j B_{ij} \sin(\theta_i - \theta_j), \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = \sum_{j=1}^{n+m} V_i V_j G_{ij} \sin(\theta_i - \theta_j) - V_i V_j B_{ij} \cos(\theta_i - \theta_j), \qquad i \in \mathcal{N}_L$$

$$\Downarrow \quad \text{change of variables } \psi := \mathbf{sin}\left(A^\top \theta\right)$$

$$R^\top \bar{P} = R^\top \left([V^\circ][g(v)][G_{ii}][V^\circ]g(v) + |\Gamma_G|[h(v)]\sqrt{\mathbb{1} - [\psi]\psi}\right) + M_B[h(v)]\psi$$

$$Q_L = -[V_L^\circ][v][B_{ii}]_L[V_L^\circ]v + \Gamma_{G_L}[h(v)]\psi - |\Gamma_{B_L}|[h(v)]\sqrt{\mathbb{1} - [\psi]\psi}$$

Key variables:

- *Normalized* load bus voltage magnitude $v \in \mathbb{R}^n$, open-circuit voltage magnitude $V^\circ \in \mathbb{R}^{n+m}$
- Γ_B, Γ_G, M_B encode network connections weighted by conductance and susceptance values
- Generator participation factor $0 \le \alpha_i \le 1$ s.t. $\sum_i \alpha_i = 1$, and R matrix s.t. $R^\top \alpha = \mathbb{0}$

Proposed fixed-point algorithm

Equivalent fixed-point form + loop flow constraint

$$v = \mathbb{1}_n - \frac{1}{4} S^{-1}[v]^{-1}\left(Q_L - \Gamma_{G_L}[h(v)]\psi - |\Gamma_{B_L}|[h(v)]\left(\mathbb{1}_{|\mathcal{E}|} - \sqrt{\mathbb{1} - [\psi]\psi}\right)\right) \tag{1}$$

$$\psi = [h(v)]^{-1} M_B^\dagger R^\top \left(\bar{P} - [V^\circ][g(v)][G_{ii}][V^\circ]g(v) - |\Gamma_G|[h(v)]\sqrt{\mathbb{1} - [\psi]\psi}\right) + [h(v)]^{-1} K x_c \tag{2}$$

$$\mathbb{0}_{n_c} = C^\top \mathbf{arcsin}(\psi) \bmod 2\pi \tag{3}$$

Algorithm: The extended fixed-point power flow algorithm

Require: Power flow data, power balance mismatch tolerance ϵ, maximum iteration limit

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v[0] ← V_L / V°_L , ψ[0] ← sin(Aᵀθ) , x_c[0] ← 0_{n_c}, k ← 0
while power balance mismatch > ε AND k < maximum iteration limit do
    update v[k+1] using (1) with v[k], ψ[k], x_c[k]
    if G is not radial then
        update x_c[k+1] using (3) with a Newton step
    update ψ[k+1] using (3) with v[k+1], ψ[k], x_c[k+1]
    k ← k+1
return ψ[k+1], v[k+1]
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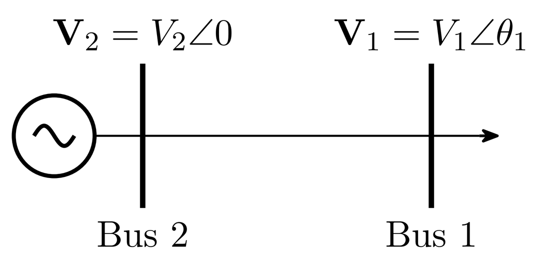
Algorithm convergence analysis: two-bus system

Update rule

Define state variable $\xi = (\psi, v-1) = (\psi, x)$, then

$$\xi_{k+1} = F_\mu(\xi_k)$$



- F_μ denotes the update rule of the extended FPPF algorithm
- μ is the vector of "perturbations": conductance, transformer parameters, and line-charging susceptance in the Π-model of the branch; $\mu = \mathbb{0}$ is the nominal case studied in [JWSP '18]

Key constants

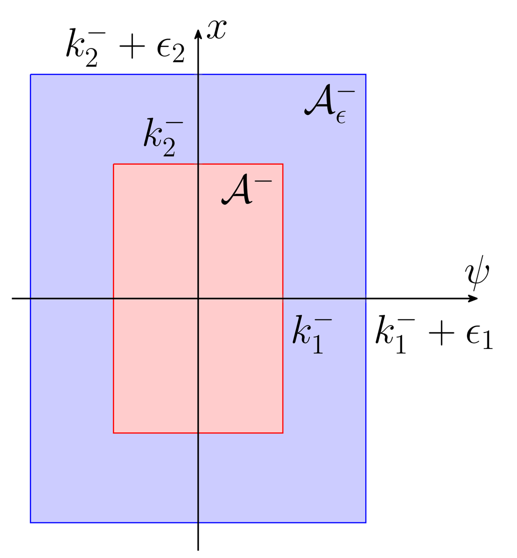
- Active/reactive power loading margins:

$$\gamma_P := \bar{P}_1/(bV_1^\circ V_2^\circ), \ \gamma_Q := Q_1/(bV_1^\circ V_2^\circ)$$

- Contraction region boundaries

$$k_2^- := 1 - \sqrt{\tfrac{1}{2} + \gamma_Q + \sqrt{\tfrac{1}{4} + \gamma_Q - \gamma_P^2}}, \ k_1^- := -\frac{\gamma_P}{1 - k_2^-}$$

- Assumption: $0 < \gamma_P^2 - \gamma_Q < \frac{1}{4}$



Main Theoretical Result

When $\mu = \mathbb{0}$

- $\mathcal{A}^- := \{\xi : |\psi| \le k_1^-, |x| \le k_2^-\}$ is F_μ-invariant; F_μ is a contraction on $\mathcal{A}^-$ in the ℓ_∞ norm
- Unique high-voltage soln. is $(k_1^-, -k_2^-)$

When $\mu \ne \mathbb{0}$ and is sufficiently small, $\exists\, \epsilon_1, \epsilon_2 > 0$ such that

- $\mathcal{A}_\epsilon^- := \{\xi : |\psi| \le k_1^- + \epsilon_1, |x| \le k_2^- + \epsilon_2\}$ is F_μ-invariant; F_μ is a contraction on $\mathcal{A}_\epsilon^-$ in the ℓ_∞ norm
- FPPF always converges to the unique high-voltage soln. in the set $\mathcal{A}_\epsilon^-$, from any $\xi_0 \in \mathcal{A}_\epsilon^-$

Numerical results: convergence

Linear convergence; comparable to fast-decoupled load flow (FDLF)

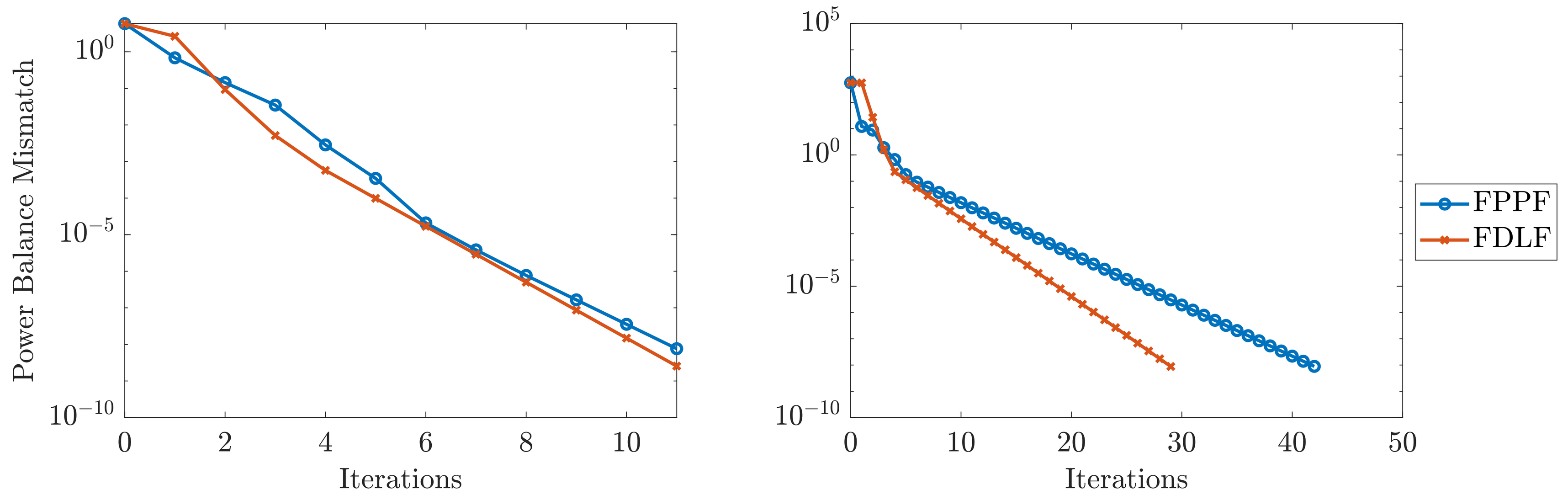


Figure 1. Left: IEEE 118-bus system, base load; Right: PEGASE 2869-bus system, high load

Numerical results: algorithm robustness

- Test algorithm success rate (%) under random initial load bus voltage magnitudes, generated uniformly from $[1-\delta, 1+\delta]$
- Higher success rate as δ increases $\Longrightarrow$ more robust

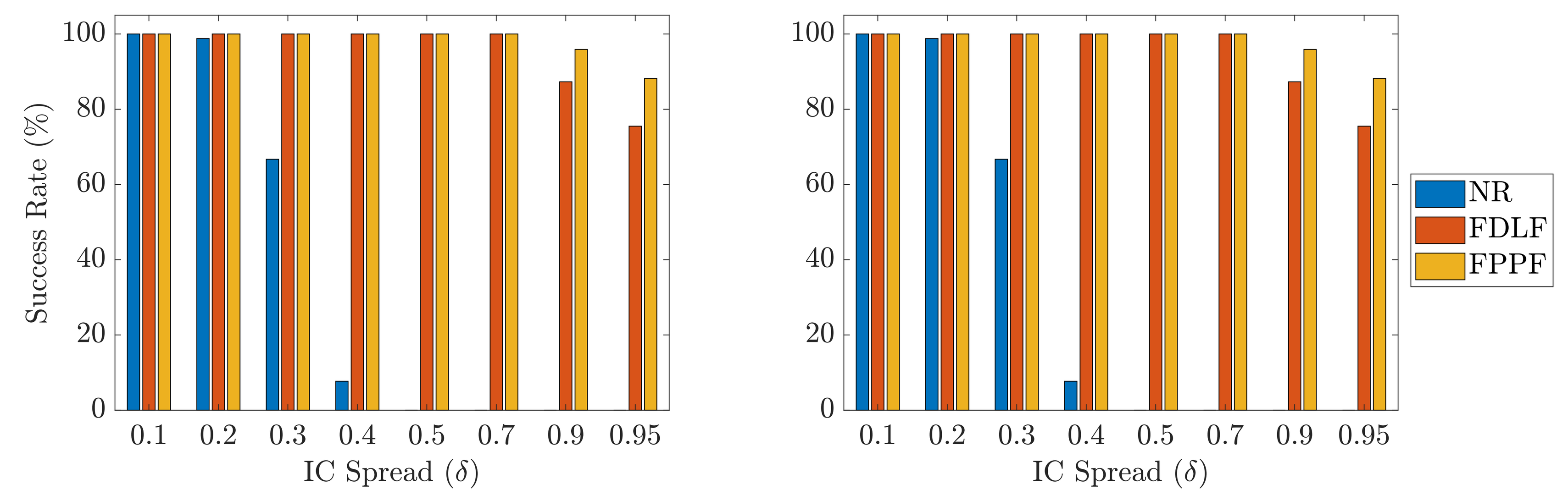


Figure 2. Left: IEEE 30-bus system, high load; Right: IEEE 118-bus system, high load

Conclusion

Summary

- A new fixed-point algorithm for the AC power flow problem with higher robustness than NR and linear convergence rate comparable to FDLF
- The algorithm accounts for resistive losses and active/reactive power flow coupling, and accommodates phase-shifting generators and distributed slack bus model in the network
- A framework to study the solvability of the full power flow equations

Future Work

- Improve algorithm robustness for networks with high R/X ratio branches and large variations in initial voltage angles
- Develop algorithm convergence conditions for general radial and meshed systems