Week2 参考答案：

2、证明：对于任意 n 阶方阵 A，A=$\frac{A+A^T}{2}+\frac{A-A^T}{2}$。

其中，$(\frac{A+A^T}{2})^T=\frac{A^T+A}{2}=\frac{A+A^T}{2}$，为对称矩阵。

$(\frac{A-A^T}{2})^T=\frac{A^T-A}{2}=\frac{A-A^T}{-2}$，为反对称矩阵。

所以每个方阵都可以表示为一个对称矩阵一个反对称矩阵之和的形式，证毕。

3、AB=$\begin{pmatrix}-18 & -11 & 16\\ 30 & 11 & -26\end{pmatrix}$ AC=$\begin{pmatrix}3 & 0\\ -15 & -4\end{pmatrix}$ ABC=$\begin{pmatrix}10 & -61\\ 12 & 93\end{pmatrix}$

B^2=$\begin{pmatrix}4 & -4 & -6\\ -12 & -3 & 8\\ 8 & -4 & -11\end{pmatrix}$ BC=$\begin{pmatrix}-4 & 8\\ 12 & 11\\ -5 & 13\end{pmatrix}$ CA=$\begin{pmatrix}-2 & 2 & 2\\ 13 & 7 & 12\\ 1 & -5 & -6\end{pmatrix}$

5、$(x_1 \quad x_2 \quad \cdots \quad x_m)\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}=(\sum_{i=1}^m x_i a_{i1} \quad \sum_{i=1}^m x_i a_{i2} \quad \cdots \quad \sum_{i=1}^m x_i a_{im})$

$$(x_1 \quad x_2 \quad \cdots \quad x_m)\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}$$

$$=\left(\sum_{i=1}^m x_i a_{i1} \quad \sum_{i=1}^m x_i a_{i2} \quad \cdots \quad \sum_{i=1}^m x_i a_{im}\right)\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}=\sum_{i=1}^m x_i \sum_{j=1}^n a_{ij}\, y_j$$

6、（1）满足题意的矩阵不存在。（需要简单论证，此处略）

（2）A=$\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$或 A=$\begin{pmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}$

（3）满足题意的矩阵有无穷多个，举出一个即可。

7、（1）$\because \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^2=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$

$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^3=\begin{pmatrix}\cos 3\theta & \sin 3\theta\\ -\sin 3\theta & \cos 3\theta\end{pmatrix}$

猜想：$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^k=\begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}(*)(k\geq 1，k\in N)$

下用数学归纳法证明：

（i）当 k=1，由题意，(*)式成立。

（ii）假设当 k=n(n>1 且 n∈ N)时，$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^n=\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$,

则当 k=n+1 时，$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^{n+1}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^{n}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

$$=\begin{pmatrix}\cos n\theta\cos\theta-\sin n\theta\sin\theta & \cos n\theta\sin\theta+\sin n\theta\cos\theta\\ -\cos n\theta\sin\theta-\sin n\theta\cos\theta & \cos n\theta\cos\theta-\sin n\theta\sin\theta\end{pmatrix}$$

$=\begin{pmatrix}\cos(n+1)\theta & \sin(n+1)\theta\\ -\sin(n+1)\theta & \cos(n+1)\theta\end{pmatrix}$，(*)式成立。

综上，猜想成立。$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^{k}=\begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}$对任意 k≥ 1成立。

（2）当 a=b=0，$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^{k}=\mathbf{0}$

当 a、b 不全为 0，$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}=\sqrt{a^2+b^2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$，其中$\theta=\cos^{-1}\frac{b}{\sqrt{a^2+b^2}}$

根据（1）结论，此时$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^{k}=(a^2+b^2)^{\frac{k}{2}}\begin{pmatrix}\cos k\theta & \sin k\theta\\ \sin k\theta & \cos k\theta\end{pmatrix}$

综上，$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^{k}=\begin{cases}\mathbf{0} & ,a\text{、}b\text{全为 }0\\ (a^2+b^2)^{\frac{k}{2}}\begin{pmatrix}\cos k\theta & \sin k\theta\\ \sin k\theta & \cos k\theta\end{pmatrix} & ,a\text{、}b\text{不全为 }0\end{cases}$

（3）A=$\begin{pmatrix}1 & a & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & a\\ 0 & 0 & 0 & 1\end{pmatrix}=I+B$，B=$\begin{pmatrix}0 & a & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & a\\ 0 & 0 & 0 & 0\end{pmatrix}$

∴$A^k=(I+B)^k$

∵B=$\begin{pmatrix}0 & a & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & a\\ 0 & 0 & 0 & 0\end{pmatrix}$，$B^2$=$\begin{pmatrix}0 & 0 & 0 & 2a\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$，$B^3$=$\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$,

B^n=$\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ (n>3)

∴$A^k=(I+B)^k=\sum_{i=0}^{k}\binom{k}{i}B^iI^{k-i}=I^k+\binom{k}{1}B^1I^{k-1}+\binom{k}{2}B^2I^{k-2}$

$$=\begin{pmatrix}1 & ka & k & k(k-1)a\\ 0 & 1 & 0 & k\\ 0 & 0 & 1 & ka\\ 0 & 0 & 0 & 1\end{pmatrix}$$

（4）A=$\begin{pmatrix}1 & 1 & \dots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & 1 & 1\\ 0 & 0 & \cdots & 1\end{pmatrix}_{n*n}$

$$
猜想：A^k=\begin{cases}\begin{pmatrix}1 & C_k^1 & C_k^2 & C_k^3 & \cdots & C_k^{n-1}\\ \vdots & 1 & C_k^1 & C_k^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_k^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_k^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_k^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}, & k\geq n-1\\ \begin{pmatrix}1 & C_k^1 & C_k^2 & C_k^3 & \cdots & C_k^k & 0 & \cdots & 0\\ \vdots & 1 & C_k^1 & C_k^2 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^k\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^1 & C_k^2 & C_k^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^1 & C_k^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & C_k^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{pmatrix}, & k<n-1\end{cases} \qquad （*）
$$

下用数学归纳法证明：

（i）当 k=1，（*）成立；

$$
（ii）假设当 k=m（m<n-2）时，A^m=\begin{pmatrix}1 & C_m^1 & C_m^2 & C_m^3 & \cdots & C_m^m & 0 & \cdots & 0\\ \vdots & 1 & C_m^1 & C_m^2 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^m\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^1 & C_m^2 & C_m^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^1 & C_m^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & C_m^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{pmatrix}
$$

$$
则 A^{m+1}= A^mA=\begin{pmatrix}1 & C_m^1 & C_m^2 & C_m^3 & \cdots & C_m^m & 0 & \cdots & 0\\ \vdots & 1 & C_m^1 & C_m^2 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^m\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^1 & C_m^2 & C_m^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_m^1 & C_m^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & C_m^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & \dots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & 1 & 1\\ 0 & 0 & \cdots & 1\end{pmatrix}
$$

$$
==\begin{pmatrix}1 & C_{m+1}^1 & C_{m+1}^2 & C_{m+1}^3 & \cdots & C_{m+1}^{m+1} & 0 & \cdots & 0\\ \vdots & 1 & C_{m+1}^1 & C_{m+1}^2 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{m+1}^{m+1}\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{m+1}^1 & C_{m+1}^2 & C_{m+1}^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{m+1}^1 & C_{m+1}^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & C_{m+1}^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{pmatrix}（*）式成立
$$

（iii）当 k=n-1，$A^{n-1}=A^{n-2}A=\begin{pmatrix}1 & C_{n-2}^1 & C_{n-2}^2 & C_{n-2}^3 & \cdots & C_{n-2}^{n-2} & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{n-2}^{n-2}\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{n-2}^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{n-2}^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_{n-2}^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & \dots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & 1 & 1\\ 0 & 0 & \cdots & 1\end{pmatrix}$

$$=\begin{pmatrix}1 & C_{n-1}^1 & C_{n-1}^2 & C_{n-1}^3 & \cdots & C_{n-1}^{n-1}\\ \vdots & 1 & C_{n-1}^1 & C_{n-1}^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_{n-1}^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_{n-1}^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_{n-1}^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$ (*)式成立

（iv）假设当 k=p（l>n-1），$A^p=\begin{pmatrix}1 & C_p^1 & C_p^2 & C_p^3 & \cdots & C_p^{n-1}\\ \vdots & 1 & C_p^1 & C_p^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_p^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_p^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_p^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}$

$A^{p+1}=\begin{pmatrix}1 & C_p^1 & C_p^2 & C_p^3 & \cdots & C_p^{n-1}\\ \vdots & 1 & C_p^1 & C_p^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_p^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_p^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_p^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}\begin{pmatrix}1 & 1 & \dots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & 1 & 1\\ 0 & 0 & \cdots & 1\end{pmatrix}$

$=\begin{pmatrix}1 & C_{p+1}^1 & C_{p+1}^2 & C_{p+1}^3 & \cdots & C_{p+1}^{n-1}\\ \vdots & 1 & C_{p+1}^1 & C_{p+1}^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_{p+1}^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_{p+1}^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_{p+1}^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}$ (*)式成立

综上，猜想成立，$A^k=\begin{cases}\begin{pmatrix}1 & C_k^1 & C_k^2 & C_k^3 & \cdots & C_k^{n-1}\\ \vdots & 1 & C_k^1 & C_k^2 & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & C_k^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_k^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & C_k^1\\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}, & \mathrm{k}\geq \mathrm{n}-1\\ \begin{pmatrix}1 & C_k^1 & C_k^2 & C_k^3 & \cdots & C_k^k & 0 & \cdots & 0\\ \vdots & 1 & C_k^1 & C_k^2 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^k\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^1 & C_k^2 & C_k^3\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & C_k^1 & C_k^2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & C_k^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{pmatrix}, & \mathrm{k}<\mathrm{n}-1\end{cases}$

（5）$A=\begin{pmatrix} a_1b_1 & a_2b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}(b_1 \quad b_2 \quad \cdots \quad b_n)$

将$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$记为 α，$(b_1 \quad b_2 \quad \cdots \quad b_n)$记为 β，βα=$\sum_{i=1}^{n} a_ib_i$

A^k=αβαβαβ……αβ=α(βα) (βα) (βα)……β=$(\sum_{i=1}^{n} a_ib_i)^{k-1}\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}$

10、证明：设 Eij 为 $c_{ij}=1$，其余为 0 的 n 阶方阵。

因为方阵 A 与任意 n 阶方阵乘法可交换，所以 EijA=AEij

其中，EijA 为第 i 行为 a_{j1}、a_{j2}……a_{in}，其余行为 0 的 n 阶方阵，

AEij 为第 j 列为 a_{1i}、a_{2i}……a_{ni}，其余列为 0 的 n 阶方阵。

所以当 i≠j，$a_{ij}=0$，A 为对角阵。

Sij 为交换 i，j 两行元素的初等矩阵，SijA=ASij，则 $a_{ii}=a_{jj}$。

综上，与任意 n 阶方阵乘法可交换的方阵一定为数量矩阵。

13、证明：$A^k=\boldsymbol{0}$，所以 $I+A^k=I$。

$(I+A)[I-A+A^2-A^3+\cdots\cdots+(-1)^{k-1}A^{k-1}]=[I-A+A^2-A^3+\cdots\cdots+(-1)^{k-1}A^{k-1}](I+A)=I+A^k=I$

故 I+A 可逆，又因为逆矩阵唯一，$(I+A)^{-1}=I-A+A^2-A^3+\cdots\cdots+(-1)^{k-1}A^{k-1}$

14、证明：$I-2A-3A^2+4A^3+5A^4-6A^5=\boldsymbol{0}$，所以 $I+I-2A-3A^2+4A^3+5A^4-6A^5=I$

$(I-A)(2I--3A^2+A^3+6A^4)=(2I--3A^2+A^3+6A^4)(I-A)=2I-2A-3A^2+4A^3+5A^4-6A^5=I$

故 I-A 可逆，又因为逆矩阵唯一，$(I-A)^{-1}=2I-3A^2+A^3+6A^4$

本周作业存在问题总结：

①**过程不全，今后若存在同样问题将着重扣分。**

②部分同学书写潦草。

③矩阵格式不规范

另，希望同学们都能独立完成作业。加油！