The Honeycomb Problem

Let $\mathcal{H} = \mathbb{Z}^2$ be a discrete hexagonal vector space, spanned by $\vec{d}_L = \begin{bmatrix}-1 & 1\end{bmatrix}^T$ and $\vec{d}_R = \begin{bmatrix}1 & 1\end{bmatrix}^T$, endowed with a super-support vector $\vec{d}_U = \vec{d}_R + \vec{d}_L = \begin{bmatrix}0 & 2\end{bmatrix}^T$, which adds an additional direction as a linear combination of the other directions and allows us to compute the shortest path. The question is: what is the minimum number of steps required to go from $\vec{p}$ to $\vec{q}$ in $\mathcal{H}$ space?

The set of ordered translation units, $\mathcal{M} \subset \mathbb{Z}/6\mathbb{Z} \times \mathcal{H}$, is defined as

$$\mathcal{M} = \{(0, \vec{d}_R), (1, -\vec{d}_L), (2, -\vec{d}_U), (3, -\vec{d}_R), (4, \vec{d}_L), (5, \vec{d}_U)\} \tag{1}$$

which can also be used a bijective access function, e.g., $\mathcal{M}(0) = \vec{d}_R$. Then, as a result of the ordering, the rotation $R_\theta : \mathcal{H} \to \mathcal{H}$ function (where the parameter θ is the amount of discrete counter-clockwise rotations) can be defined as

$$R_\theta(\vec{d}) = \mathcal{M}((\mathcal{M}^{-1}(\vec{d}) + \theta) \mod 6) \tag{2}$$

The correctness of this formula can be verified by checking $R_\theta(\vec{d}) = -\vec{d}$ if $\theta \in 3\mathbb{Z}$.

Any position $\vec{p} = \begin{bmatrix}x & y\end{bmatrix}^T \in \mathcal{H}$ can be written as a linear combination:

$$\vec{p} = p_R \cdot \vec{d}_R + p_L \cdot \vec{d}_L \tag{3}$$

By solving this as a system of equations, p_R and p_L can be inferred:

$$p_R = \frac{y - x}{2} \tag{4}$$

$$p_L = \frac{y + x}{2} \tag{5}$$

Since we want to find the shortest path and we have the extra super-support vector $\vec{d}_U$, we can rewrite $\vec{p}$ to maximize $|p_U|$ in

$$\vec{p} = p'_R \cdot \vec{d}_R + p'_L \cdot \vec{d}_L + p_U \cdot \vec{d}_U \tag{6}$$

We can do this because $\vec{d}_U$ is a linear combination of $\vec{d}_L + \vec{d}_R$, i.e., one step in the top-left direction ($\vec{d}_L$) and one step in the top-right direction ($\vec{d}_R$). Given the structure of the grid, the same position can be reached by one step in the $\vec{d}_U$ direction. So basically we are substituting a scaling of $\lambda \cdot (\vec{d}_L + \vec{d}_R)$ with $\lambda \cdot \vec{d}_U$. We are interested in the value of λ. It is easy to see that $\lambda \cdot \vec{d}_U = p_U \cdot \vec{d}_U$, so that $\lambda = p_U$. Now we can compute p_U as follows:

$$p_U = \begin{cases} \operatorname{sgn}(p_R) \cdot \min(|p_R|, |p_L|) & \text{if } \operatorname{sgn}(p_R) = \operatorname{sgn}(p_L) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

As a consequence p'_R and p'_L can be computed by subtracting p_U from p_R and p_L respectively:

$$p'_R = p_R - p_U \tag{8}$$

$$p'_L = p_L - p_U \tag{9}$$

The *minimum number of steps* required to go from $\vec{0}$ to $\vec{p}$ is a mapping $L : \mathcal{H} \rightarrow \mathbb{Z}$ and can be counted as a sum:

$$L(\vec{p}) = |p_R| + |p_L| - \min(|p_R|, |p_L|) \tag{10}$$

From that we can derive the minimum number of steps needed to go from $\vec{p}$ to $\vec{q}$, since this is the same as $L(\vec{p} - \vec{q})$.