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Lecture 2

Riemann Integrability

Definition: A function $f(x) : [a, b] \to R$ is Riemann integrable if as $||P|| \to 0$ Any Riemann sum (right, left, min, max) converges to the same value:

$$\int_a^b f(x)dx = \lim_{||P||\to 0} \sum^n f(t_i)(x_i - x_{i-1}) \tag{1}$$

Theorem: If $f(x) : [a, b] \to R$ is **bounded** and **continuous everywhere on [a,b]** except at a **finite** number of points, then f is Riemann Integrable on [a,b]. Note that a function may be bounded and continuous, but not integrable. In this case, it could be improperly integrable

Lecture 3

Theorem: Rules for Riemann Integrable functions:

1. Linearity (linear combinations)
2. Additivity
3. Monotonicity $F_1 \le F_2 \Rightarrow \int_\Omega F_1 d\Omega \le \int_\Omega F_2 d\Omega$
4. Subnormality $| \int_\Omega f d\Omega | \le \int_\Omega | f | d\Omega$

Theorem: Let $\Omega \subset R^n$ be enclosed within a finite (n-1) dimensional boundary. If $f : \Omega \to \mathbf{R}$ is bounded and continuous everywhere except at a finite number of (n-1) dimensional surfaces, then f is Riemann integrable on its domain

Theorem: Let $\Omega = [a, b] \times [c, d] \subset R^n$ if $f(x, y)$ is integrable on Ω and every iterated integral is integrable.

- If $f(x, y)$ is integrable on [a,b] i.e. $\int_a^b f(x, y)dx$ exists for every y
- $g(y) = \int_a^b f(x, y)dx$ is integrable on [c,d] i.e. $\int_c^d [g(y)] = \int_c^d \int_a^b f(x, y)dxdy$ also exists. then

$$\int_\Omega f(x, y)d\Omega = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

Lecture 7

Application of Integrals (formulae)

Mass: where rho of x is a scalar function that takes in a vector. Density * volume element summed up = mass

$$\text{mass} = \int_\Omega \rho(\vec{x})d\Omega$$

Volume: Where the function is 1

$$\text{volume} = \int_\Omega 1 d\Omega$$

Moment about x axis

$$M_x = \iint_D y\rho(x, y)dA$$

Moment about y axis

$$M_y = \iint_D x\rho(x,y)dA$$

Center of Masses

$$\bar{x} = \frac{M_y}{m} \; \bar{y} = \frac{M_x}{m}$$

Lecture 13

Gradient Theorem (FTC for line integrals)

Theorem (FTC line integrals): let $\gamma \subset R^n$ be a C^1 (once diff, cont) curve which starts at a and ends at b. Let $\vec{F} : R^n \to R^n$ be some vector field. If $\vec{F} = \nabla f$ for some C1 scalar function $f : R^n \to R$, then

$$\int_\gamma \vec{F} \cdot \vec{dl} = f(\vec{b}) - f(\vec{a})$$

This tells us that if there exists a scalar potential f of F, then the path integral only depends on the endpoints .

Path Independent: the line integral $\int_\gamma \vec{F} \cdot d\vec{l}$ is path independent if $\int_\gamma \vec{F} \cdot d\vec{l} = \int_{\tilde{\gamma}} \vec{F} \cdot d\vec{l}$ for any other curve, so long as the endpoints are the same.

Path Connected Region: a region Ω is path connected if every pair of points $\vec{a}, \vec{b} \in \Omega$ can be joined by some curve $\gamma \subset \Omega$

Closed Curve: a curve γ is closed if it begins and ends at the same point. We use $\oint_\gamma d\vec{l}$ to denote a closed path integral.

Theorem: Path Independence: let γ be a curve in $\Omega \subset R^n$ and $\vec{F}$ be a C_1 vector field in Ω .

$$\int_\gamma \vec{F} \cdot d\vec{(l)} \text{ is path independent } \textbf{if and only if} \oint_{\tilde{\gamma}} \vec{F} \cdot d\vec{l} = 0 \text{ for any closed curve } \tilde{\gamma}$$

Open Region: an open region $\Omega \subset R^n$ is called open if it consists only of interior points.

Conservative Vector Field: A vector field $F : \Omega \to R^n$ on an open set $\Omega \subset R^n$ is conservative if there exists a *C1* function $\phi : \Omega \to R$ such that $\vec{F} = \nabla\phi$.

Theorem: if $\vec{F}$ is C1 vector field on an open, path connected region $\Omega \subseteq R^n$ then the following statements are equivalent:

1. $\vec{F}$ is conservative, meaning that there exists a scalar potential function f s.t.$\nabla f = \vec{F}$
2. $\int_\gamma \vec{F} \cdot d\vec{l}$ is path independent, $\int_{\gamma 1} \vec{F} \cdot d\vec{l} = \int_{\gamma 2} \vec{F} \cdot d\vec{l}$
3. for a closed curve γ $\oint_\gamma \vec{F} \cdot d\vec{l} = 0$

Theorem: if $\vec{F} : \Omega \to R^n$ is a conservative vector field then it's **cross partials** commute.

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

Where $0 \leq i, j < n$. As a concrete example, in the 2D (x, y) case, we have that:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

The proof is intuitive:

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\frac{\partial \phi}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial \phi}{\partial y}) = \frac{\partial F_y}{\partial x}$$

Since, F is conservative, there exists some scalar function phi. By Clairaut's Theorem, we rearranged the partials.

Result: the gradient of the gradient of the potential scalar function of a conservative vector field $\vec{F}$ is 0.

$$\nabla \times \nabla f = 0$$

If we think about what this means, we see that $\nabla f = \vec{F}$, and we know that $\nabla \times \vec{F}$ is the curl of the function. For curl the cross partials $\pm$ each other are components, but since these partials commute, the components of the curl becomes zero. Thus, we have that a conservative vector field has no curl.

Simple Curve: a simple curve $y = \vec{r}(t) : I \to R^n$ is injective on the interior of an interval I. So every t, gets mapped to a unique point in R^n **except on endpoints**. i.e. the curve does not intersect itself (it could still be closed though)

Simply-Connected Region: a region Ω is simply-connected if every simple closed curve in Ω encloses only points in Ω.

Theorem: if $\vec{F}$ is a C1 vector field on a simply connected region $\Omega \subseteq R^n$ and $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ on Ω then $\vec{F}$ is conservative.

Recap: Given $\vec{F}$ is conservative, $\implies \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ because there exists a scalar potential function ϕ or f and $\oint_\gamma \vec{F} \cdot d\vec{l} = 0$ because path independence: starting and ending at 0 is a closed path and $f(r(t = 0)) - f(r(t = 0))$. BUT, we can't know if Ω is simply connected or not (donut shape).

Given that the partials commute, but the region over which this occurs is not simply connected, then F is not conservative. Just because there exists a scalar potential function ϕ or f whose gradient equals F $\nabla \phi = \vec{F}$ does not mean F is conservative.

Lecture 15

Regular Region: $\Omega \subseteq R^n$ is a regular region if it is closed, bounded and contains no isolated points. Consider the following examples:

- $R^2/\{0, 0\}$: is not a regular region, because it is not closed (does not include the origin), it is not bounded. It does satisfy the condition of no isolated points.
- $\{x^2 + y^2 < 1\}$: is not a regular region, because it is not closed (does not include 1). It is, however, bounded and contains no isolated points
- $\{x^2 + y^2 \leq 1\}$: is a regular region, because it is closed
- $[0, 1] \cup 2$: is not a regular region. Although it is closed, and bounded, it contains isolated points
- Note that any donut shape is regular if it is closed on its inner and outer radii.

Right Hand Rule: The stokes positive orientation is dictated by the right hand rule. Moreover, if your regular region of interest it kept to the left, as you walk around the path, it is positive. (CCW +ve, CW -ve)

Green's Theorem: given $\Omega \subseteq R^2$ is a *regular region* and the path $\gamma = \partial\Omega \subset R$ is the piece-wise smooth boundary and $\vec{F} : R^2 \to R^2$ is a C1 vector field, then:

$$\oint_{\gamma=\partial\Omega} \vec{F} \cdot d\vec{l} = \iint_A (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})dxdy$$

Note that if $\vec{F}$ is a conservative vector field, we have that the partials commute, and so the integral is 0. This makes sense intuitively, as the curl of a conservative vector field is 0 i.e.$\nabla \times \nabla\phi = 0$

Splitting up Regions: let $\Omega \subset R^2$ be a regular region (closed, bounded, no isolated points), then if $\Omega = \sum^n \Omega_i$ where Ω_i are sub-regular regions we have that:

$$\oint_\gamma \vec{F} \cdot d\vec{l} = \sum^n \oint_{\gamma_i}$$

Where γ_i is the smooth boundary that encircles the regular sub region Ω_i . Intuitively this makes sense as the inner boundaries cancel each other out by virtue of the RHR. This leaves us only with the curl along the boundary.