PHY293 Oscillations and Waves

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Lecture 2: September 11, 2023

Simple Harmonic Motion

- Periodic motion about an equilibrium requires a restoring force
- The governing equation is $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$
- The natural frequency of the oscillation is $\omega^2 = \frac{k}{m}$
- SHM: general solution $x(t) = X_0 + A\cos(\omega t + \phi_0)$
  - X_o is an offset
  - ω is the angular frequency which relates to frequency through $\omega = 2\pi f$
  - Velocity and Acceleration can be found: $v_{max} = -A\omega$ and $a_{max} = -A\omega^2$.
  - What is also notable is that $a(t) = -\omega^2 x(t)$
  - Sum identity: $x(t) = A\cos(\omega t + \phi_0) = A\cos(\omega t)\cos(\phi_0) - A\sin(\omega t)\sin(\phi_0) = a\cos(\omega t) + b\sin(\omega t)$

SHM Energy

- Spring-mass system has energy $KE + U = \frac{1}{2}m(v(t))^2 + \frac{1}{2}k(x(t))^2$
- At equilibrium, KE is maximized. At max displacements, U is maximized: $E_{tot} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}kx_{max}^2$. Note that we can plug the kinematic values into the energy equations.

Lecture 3: September 12, 2023

Damped Harmonic Oscillator

- Linear drag force is proportional to velocity: $F_d = -bv$
- Governing equation is $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \Rightarrow \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$

Light Damping

- Light damping is a oscillatory system that gradually reaches equilibrium. Subsequent oscillations become smaller and smaller
- The condition for light damping is $\omega^2 = \omega_o^2 - \frac{\gamma^2}{4}$. We can assume that $\omega \approx \omega_o$, where ω_o is the frequency otherwise experience without any losses. Note that the period increases with an increase in damping.
- The general solution is $x(t) = A_o \exp(-\frac{\gamma t}{2})\cos(\omega t + \phi_0)$
- We get that the amplitude is: $A(t) = A_o \exp(-\frac{\gamma t}{2})$

Heavy Damping

- Heavy damping is sluggish to return to equilibrium
- We guess the solution $x(t) = \exp - (\beta t) f(t)$
- Back in the DE, we get that $\frac{d^2 f}{dt^2} = \alpha^2 f, \alpha^2 = \frac{\gamma^2}{4} - \omega_o^2$. Note that we use α because there is no angular frequency in heavy damping
- The general solution is $x(t) = \exp(-\frac{\gamma t}{2})(Ae^{\alpha t} + Be^{\beta t})$

Critical Damping

- Critical damping is when the system returns to equilibrium in the most efficient way
- When $\omega_0^2 - \frac{\gamma^2}{4} = 0$
- The general solution is $x(t) = A\exp(-\frac{\gamma t}{2}) + Bt\exp(-\frac{\gamma t}{2})$

Lecture 4: September 14, 2023

Energy of DHO

- **When studying the energy, we only look at the underdamped case where $\frac{\gamma^2}{4} \ll \omega_0^2 \Rightarrow \omega \approx \omega_0$**
- Use the energy equation, find the velocity of the underdamped system. When *massaging* you should eventually get the trig pythag identity leading to: $E(t) = \frac{1}{2}kA_0^2\exp(-\gamma t)$
- **It should be noted that energy decays twice as fast as amplitude** (this is because in the energy equations, both position and velocity are squared, and as they are both functions of amplitude, energy decays twice as fast)
- We also define our lifetime or time constant as $\tau = \frac{1}{\gamma}$ thus, $E(t) = E_0\exp(-\frac{t}{\tau})$

Rate of Energy Loss of DHO

- We find the time derivative of energy: $\frac{dE}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = mv\frac{dv}{dt} + kx\frac{dx}{dt} = v(ma + kx)$
- Now recall that $ma = -kx - bv \rightarrow ma + kx = bv$ leading us to $\frac{dE}{dt} = -bv(v) = -bv^2$

Quality Factor Q of a DHO

- The quality factor measures how *good* an oscillator is, which consists of its ability to maintain its energy. It is defined as $Q = \dfrac{\omega_0}{\gamma}$
- We also define the quality factor by looking at the energy dissipated from the system. We consider the underdamped oscillator
- In between an arbitrary cycle, we calculate the ratio of energies $\dfrac{E_{n+1}}{E_n} = \dfrac{E_0 exp(-\gamma(t_n + T)}{E_0 exp(-\gamma t_n)} = exp(-\gamma T)$
- Now, we can apply the maclaurin expansion of e^x
- $\dfrac{E(t_{n+1} = t_n + T)}{E(t_n = t_n)} = 1 - \gamma T$
- If we look at the relative change: $\dfrac{\Delta E}{E(t_n)} = \dfrac{E(t_{n+1}) - E(t_n)}{E(t_n)} = \dfrac{E(t_{n+1})}{E(t_1)} - 1 = 1 - \gamma T - 1 = -\gamma T$
- $-\dfrac{E(t_{n+1}) - E(t_n)}{E(t_n)} \approx \gamma T \approx \dfrac{2\pi\gamma}{\omega} = \dfrac{2\pi}{Q}$
- $\dfrac{2\pi E(t_n)}{E(t_n) - E(t_{n+1})} = Q = \dfrac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$

Application: Damped Electrical Oscillator

- Applying KVL around the loop of an RLC circuit: $RI + L\frac{dI}{dt} + +\frac{q}{C} = 0$
- We can express everything in terms of charge: $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$. Notice the similarities between the DHO DE: $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^2 = 0$
- By inspection, $\omega^2 = (\dfrac{1}{LC} - \dfrac{R^2}{4L^2})$
- Meaning that $\omega_0 = \sqrt{\dfrac{1}{LC}}$ and $\gamma = \frac{R}{L}$
- Finally, the q-factor of the circuit: $Q = \dfrac{\omega_0}{\gamma} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

Lecture 5: September 18, 2023

Forced/Driven Harmonic Motion

- Maximum amplitude of oscillation occurs when the driving frequency is equal to that of the natural oscillator, when $\gamma = 0$
- With an increase in damping γ the maximum amplitude is reached when the driving frequency is smaller than the natural

Undampened Force Oscillations

- We consider $F(t) = F_0\cos(\omega t)$ as the driving force
- From N2L: $m\dfrac{d^2x}{dt^2} + kx = F_0\cos(\omega t)$
- The solution ends up being a period function with an amplitude dependent on the angular frequency of the driver: $x(t) = A(\omega)\cos(\omega t - \delta)$
- δ is the phase difference between the driving force and the resultant displacement. It shows that Displacement Lags behind the driving force (there is some delay in reaction)
- Note that $F_0 = \zeta_0 k$
- We receive two relations once we plug in our solution:
  1. $A(\omega)[1 - \dfrac{\omega^2}{\omega_0^2}]\sin(\delta) = 0$
  2. $A(\omega)[1 - \dfrac{\omega^2}{\omega_0^2}]\cos(\delta) = 1$
- If we take the ratio if 1 over 2, we get that $\tan(\delta) = 0$
  1. $\delta = 0$: then $A(\omega) = +\dfrac{\zeta_0}{1 - \frac{\omega^2}{\omega_0^2}}$ (for $\omega < \omega_0$) This is because, the denominator must remain **positive**
  2. $\delta = \pi$: then $A(\omega) = -\dfrac{\zeta_0}{1 - \frac{\omega^2}{\omega_0^2}}$ (for $\omega > \omega_0$) This is because, the denominator must be **negative** such that the amplitude is **positive**.
- Thus, in a simple system it can either be out of phase or in phase.
- In an undamped system, amplitude goes to infinity as $\omega \to \omega_0$. *Practically, however, the system would break at a certain point.*

Dampened Force Oscillations

- From N2L: $m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = F_0\cos(\omega t)$
- Eventually we receive the following ratio: $\tan(\delta) = \dfrac{\omega\gamma}{\omega_0^2 - \omega^2}$
  - **Note that ω_0 is the natural frequency and ω_0 is the frequency of the driving force**
- Solving for sin and cos from basic trig ratios:
  - $\sin(\delta) = \dfrac{\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$
  - $\cos(\delta) = \dfrac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$
- From sin and cos: $A(\omega) = \dfrac{\zeta\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$
- Behaviour of amplitude
  1. For $\omega \to 0,\ A(\omega) \to \zeta_0 = \dfrac{F_0}{k}$
  2. For $\omega \to \omega_0\ A(\omega) \to \dfrac{\zeta_0\omega_0}{\gamma}$
  3. For $\omega \to \infty\ A(\omega) \to 0$

Max Amplitude

- In order to find the maximum amplitude, we want to minimize the denominator term of $A(\omega)$
- When $\dfrac{d}{d\omega}\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} = 0$
- Thus, $A_{max} \to \omega = \omega_0(1 - \dfrac{\gamma^2}{2\omega_0^2})^{\frac{1}{2}}$
- In terms of q factor $\omega = \omega_0(1 - \dfrac{1}{2Q^2})^{\frac{1}{2}}$

Power Absorbed During Forced Oscillations

- Since the instantaneous power fluctuates rapidly, we talk about the average power of one cycle
- $\bar{P}(\omega) = \dfrac{1}{T}\displaystyle\int_{t_0}^{t_0+T} P(t)dt$
- $\bar{P} = \dfrac{b[v_o(\omega]^2}{2} = \dfrac{\omega^2 F_0^2\gamma}{2m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}$
  1. $\omega \to 0, \bar{P} \to 0$
  2. $\omega \to \infty, \bar{P} \to 0$

Lecture 6: September 21, 2023

Average Power Curve

- When the difference between the driving and natural frequency of the oscillations is minimized $\Delta\omega = \omega_0 - \omega = 0$, the power is maximimzed
- $\bar{P}_{maximum} = \dfrac{F_0^2}{2m\gamma}$
- Given the average power curve as a function of ω (which looks like a bell curve with a peak at ω_0, we define the full width at half height of the curve as $\Delta\omega_{fwhh} = 2\Delta\omega = \gamma\dfrac{\omega_0}{Q}$

Driving AC power supply in RLC

- Similarly to the damped harmonic spring system, we can also add a driving force to the RLC circuit, this electromotive force comprises an AC power source.

Transient Phenomena

- When a driving force is first applied, the system will be inclined to oscillate at its free oscillation frequency
- So, we will in fact see both frequencies during this *transient state*
- In this case of a dampened system, the oscillation at the free frequency ω_0 will die down at a rate dependent on γ

Mathematically:

- $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F_0\cos(\omega t)$
- if $x_1(t)$ is a solution to the driven DE, then
- $\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2 x_1 = F_0\cos(\omega t)$
- The equation for a dampened free oscillator is:
- $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$
- If we consider x_2 to be a solution then $\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2 x_2 = 0$
- Thus, taking the sum of both x_1 and x_2 then we can rewrite the dependent variable as $x_1 + x_2$:
- $\frac{d^2(x_1+x_2)}{dt^2} + \gamma\frac{d(x_1+x_2)}{dt} + \omega_0^2(x_1+x_2) = F_0\cos(\omega t)$

Lecture 7: September 21, 2023

Simple Pendulum

- $-g\sin(\theta) = L\frac{d^2\theta}{dt^2}$
- For small enough angular displacements, $\theta \approx \sin(\theta)$ for small theta
- $\theta(t) = \theta_{maximum}\sin(\omega t + \theta_0)$ where $\omega_0^2 = \frac{g}{l}$

Physical Pendulum

- For a physical pendulum, we consider torque: $\tau = I\alpha = I\frac{d^2\theta}{dt^2}$, which leads to $\omega^2 = \frac{mgd}{I_pivot}$

Coupled Oscillators

- Consider two pendulums with masses ma and mb connected by a spring constant with constant k. When both pendulums are displaced in the same direction at the same distance, the spring is unstretched and both masses move together.
- Thus, when they oscillate in phase with each other, with the same amplitude A and frequency, $\omega_1 = \sqrt{\frac{g}{L}}$
- The second case we can consider is when the pendulums are displaced away from each other. Approximating the angular displacement $x \approx s = L\theta$, we can write the differential equation for one of the masses:

- $\dfrac{d^2x_A}{dt^2} = -(\dfrac{g}{L} + \dfrac{2k}{m})x_A = -\omega_2^2 x_A$
- Where the second angular frequency: $\omega_2 = \sqrt{\dfrac{g}{L} + \dfrac{2k}{m}}$. The masses oscillate out of phase with each other with the same amplitude B and frequency ω_2
- Now, we consider the general case where the masses A and B are displaced by an arbitrary amount. In any case, the resulting displacement will be a linear combination of the above normal modes where $x_A = x_B$ and $x_A = -x_B$.
- Thus, the restoring force on mass A is $m\dfrac{d^2x_A}{dt^2} = -\dfrac{mg}{L}x_A - k(x_A - x_B)$
- And the restoring force on mass B is $m\dfrac{d^2x_B}{dt^2} = -\dfrac{mg}{L}x_B + k(x_A - x_B)$
- If we add the equations together: $m\dfrac{d^2(x_A + x_B)}{dt^2} = \dfrac{g}{L}(x_A + x_B)$
- If we subtract the equations: $m\dfrac{d^2(x_A - x_B)}{dt^2} = (\dfrac{g}{L} + \dfrac{2k}{m})(x_A + x_B)$
- We introduce two variables $q_1 = x_A + x_B$ and $x_A - x_B = q_2$
- $q_1 = C_1 cos(\omega_1 t + \phi_1)$ and $q_2 = C_2 cos(\omega_2 t + \phi_2)$
- Where C_1 is the amplitude of mode q_1 and C_2 is the amplitude of mode q_2
- Rearranging, we get that $x_A = \dfrac{1}{2}(q_1 + q_2)$ and $x_A = \dfrac{1}{2}(q_1 + q_2)$

Lecture 10:

Traveling Pulse:

- A fixed pulse moving at a velocity v

$$y(x,t) = f(x \pm vt)$$

  - +: pulse moves to left (f(0), requires a negative x)
  - -: pulse moves to right (f(0), requires positive x)
-

Traveling Waves:

- Traveling wave: organized disturbance traveling at a speed
- Transverse wave: displacement is perpendicular to motion
- Longitudinal: displacement is parallel to motion

Sinusoidal Wave:

- $$y(x,t) = A\sin(kx \pm \omega t + \phi_0) = A\sin(\frac{2\pi}{\lambda}(x \pm vt) + \phi_o)$$
- Where k is called he angular wave number $k = \frac{2\pi}{\lambda}$
  - λ : **the distance between repetitions**
  - We can plot $y(x, t = c)$, which is an instant in time: SNAPSHOT
- Where ω is the angular frequency $\omega = \frac{2\pi}{T} = 2\pi f$
  - T: **the time between repetitions**
  - We can plot $y(x = c, t)$, which holds displacement constant: HISTORY GRAPH

Note that the amplitude will be the same in either graph.

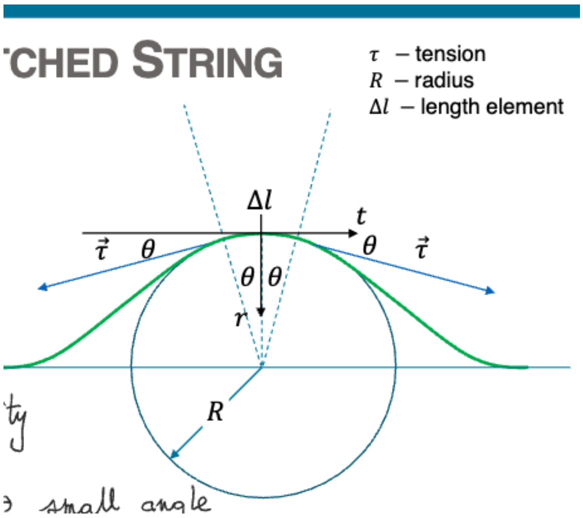

Figure 1: Enter Caption

Wave Equation:

- Each particle undergoes SHM w.r.t time and position, so we can take the partials w.r.t both variables:
- For time:
$$y(x,t) = -\frac{\partial^2 y}{\partial t^2}\frac{1}{\omega^2}$$
- For position:
$$y(x,t) = -\frac{\partial^2 y}{\partial x^2}\frac{1}{k^2}$$
- By setting these PDEs equal to each other, we get the wave equation:
$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

Wave Speed of Stretched String:

- To solve for the speed, go through the following steps:
  - Find centripetal acceleration of length element:
  $$\frac{mv^2}{R} = 2\tau\sin(\theta)$$
  - Express mass in terms of mass density of string: $m = \Delta l\mu$
  - Assume small angle approx $\theta \approx \sin(\theta)$
  - $$v = \sqrt{\frac{\tau}{\mu}}$$
  where μ is linear mass density: m/L

Mechanical Impedance:

- Property of a medium that relates velocity to the driving force (recall electrical impedance)
- $$Z = \frac{\tau_y(x,t)}{v_y(x,t)}$$
- For a sine wave, $Z = \frac{\tau}{v} = \sqrt{\mu\tau}$ (according to speed of string equation)

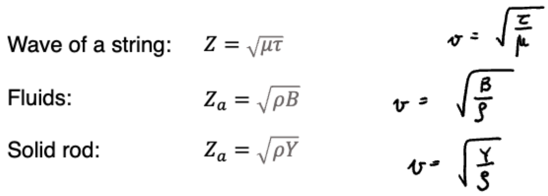

Figure 2: Enter Caption

Lecture 11:

Wave Boundary Conditions:

- When a wave crosses one medium to another, there will be an incident, reflected and transmitted wave.
- Boundary conditions:
  - Displacement are continuous at boundary for all t
  - $\frac{dy}{dx}$ is continuous at boundary for all t
    - RECALL SAME CONDITIONS AS TUNNELLING
- Note that $k : (\lambda)$ changes through the medium, but $\omega : (f)$ remains the same
- Equation 1:
$$A_i + A_r = A_t$$
- Equation 2:
$$Z_1 A_i - Z_1 A_r = A_t Z_2$$
- Define Amplitude reflection and transmission coefficients:
- $R \equiv \frac{A_r}{A_i}$ and $T \equiv \frac{A_t}{A_i}$
- From here we have that
$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$
- And that
$$T = \frac{2Z_1}{Z_1 + Z_2}$$
- Such that $1 + R = T$

Standing Waves:

- **Assume each particle starts at a maximum or minimum displacement:** $y(x,t) = f(x)\cos(\omega t)$
- The above statement imposes
$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$
- Substituting these conditions into the wave equation we get the following PDE
$$\frac{\partial^2 f(x)}{\partial x^2} = -\frac{\omega^2}{v^2} f(x)$$
- Which has a general solution:
$$f(x) = A\sin(\frac{\omega}{v}x) + B\cos(\frac{\omega}{v}x)$$

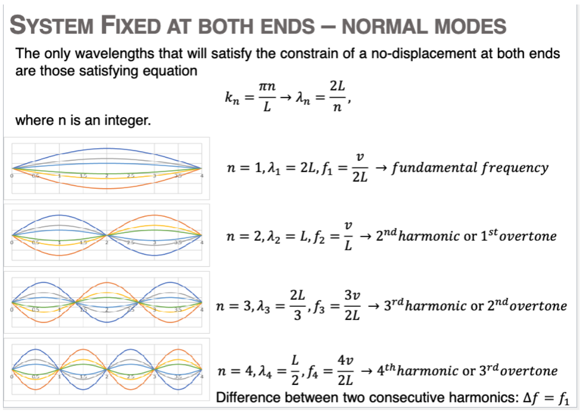

Figure 3: Enter Caption

- By imposing boundary conditions, A and B are determined. $f(x = 0) = 0$ and $f(x = L) = 0$ for a standing wave on a string of length L fixed at both sides
- First condition imposes $B = 0$
- Second condition imposes $A \sin(\frac{\omega L}{v}) = 0$, which means that $\frac{\omega L}{v} = n\pi$ where $n = Z^+$
- Means that there are discrete solutions:

$$\omega_n = \frac{n\pi v}{L} \to f = \frac{nv}{2L}$$

$$y(x,t) = A_n \sin(\frac{n\pi}{L}x)\cos(\omega_n t)$$

Useful Information

- $f_n = \frac{nv}{2L}$: it can be derived from ω_n which is the argument of the cosine
- $k_n =$ the argument of the sin function $= \frac{n\pi}{L}$
- $\lambda_n = \frac{2\pi}{k_n}$
- NOTE: you don't need wave number to find λ_n. Recall that $v = f_n\lambda_n$, so $\lambda_n = \frac{v}{f_n} = \frac{2L}{n}$

System fixed @ both ends

- $\Delta f = f_{n+1} - f_n = f_1 = \frac{v}{2L}$
- Note that for a string, we could replace the velocity with $v = \sqrt{\frac{F}{\mu}}$
- Position of nodes and antinodes
  - Position of nodes: $\sin(kx) = 0$
  - Position of antinodes: $\sin(kx) = \pm 1$

System Open @ both ends

- Basically the same thing as closed ends except boundary conditions change to $f(x) = y_0$ and $f(L) = y_0$
- Intuitively we expect a cosine function (@0, antinode) $\cos(0) = 1$
- $$y(x,t) = A_n \cos(\frac{n\pi}{L}x)\cos(\omega_n t)$$
- Naturally, the position of nodes and antinodes will differ
  - Position of nodes: $\cos(kx) = 0$
  - Position of antinodes: $\cos(kx) = \pm 1$

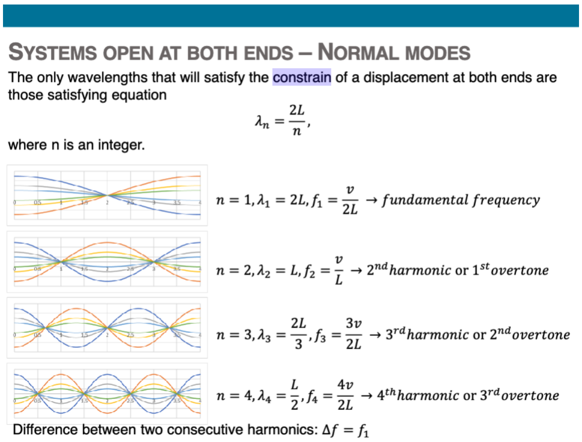

Figure 4: Enter Caption

System Open @ one end closed at other

- The only way to satisfy the mixed boundary conditions requires $\lambda_n = \frac{4L}{n}$, where n is an odd whole number
- Thus, $f_n = \frac{nv}{4L}$, meaning that $\omega_n = \frac{\pi n v}{2L}$ again for $n = 1, 3, 5$
- Only odd harmonics will be present
- Thus, any change in subsequent harmonic is a change equal to twice the fundamental frequency $n = 1$: $\Delta f = f_{2n+1} - f_{2n-1} = 2f_1 = \frac{v}{2L}$

Lecture 12+13:

Standing Waves as Normal Modes of a Vibrating String:

- By superposition if $y_1(x,t)$ and $y_2(x,t)$ are solutions, then so is any lin. comb.
- So, the superposition of all standing waves is a solution to the wave equation:

$$Y(x,t) = \sum_n \cos(\omega_n t)(A_n \sin(k_n x) + B_n \cos(k_n x))$$

-

Standing Waves Fixed at both ends:

- Recall: a string fixed at both ends with harmonic n is described by: $y(x,t) = A_n \sin(\frac{n\pi}{L}x)\cos(\omega_n t)$
- Thus, any pattern on a fixed string can be represented as a linear combination of any normal modes:

$$y(x,0) = \sum_n A_n \sin(\frac{n\pi x}{L}) = f(x)$$

- The issue is that we need a way to find the constant factors A_n
-

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

  - A_n is the coefficient of the n-th mode
  - L is the length of the string
  - $f(x)$ is the function that describes the shape of the wave:
    * Square: $f(x) = 1$ $\{0 < x < L\}$
    * Saw-tooth
    * Triangle
  - n is the mode number

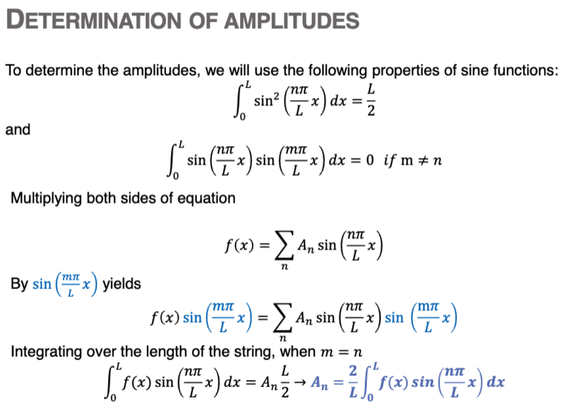

Figure 5: Derivation of the amplitude of the n-th normal mode

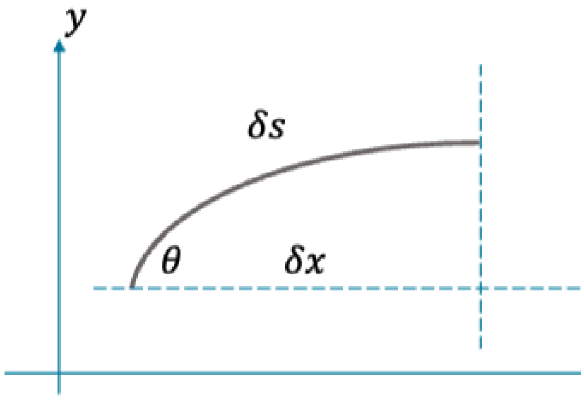

Figure 6: Enter Caption

Lecture 14:

Energy of a Wave:

- Find dK and dU, which will be proportional to dx, then integrate over the entire string length L
- Waves carry two types of mechanical energy with them:
  - Kinetic (related to the motion of medium element):
    - $dK = \frac{1}{2}dm\ v_y^2$
  - Elastic Potential (related to the displacement of the medium elements from equilibrium)
    - A string element of length $dl = \sqrt{dx^2 + dy^2}$ under some tension τ will experience $dU = \frac{1}{2}\tau\ dl\ (\frac{dy}{dx})^2$

Power of a Wave:

- Take $E_n = K_n + U_n$, and then integrate over **one wavelength** $\frac{2L}{n}$

$$E_n = \frac{1}{4}\mu(\omega_n A_n)^2\lambda_n$$

- Recall: $\omega_n = \frac{n\pi v}{L}$
- For power:

$$P = \frac{dE}{dt} \to \frac{E_n}{T_n} = \frac{1}{4}\mu(\omega_n A_n)^2 v$$

Power Analysis for change of medium:

- For a standing wave on a string:

$$v = \sqrt{\frac{\tau}{\mu}} \to \mu v = \sqrt{\tau\mu} = Z$$

- **Incident wave:**
  - Average power: $c\ Z_1 A_i^2 \omega^2$
- **Reflected wave:**
  - Average power: $c\ Z_1 A_r^2 \omega^2$
  - $R \equiv \frac{A_r}{A_i}$,
  - So Average power: $c\ Z_1 (RA_i)^2 \omega^2$
- **Reflected Power Ratio:**
  -
$$\frac{\text{Reflected Power}}{\text{Incident Power}} = R^2 = Re$$
- **Transmitted Power:**
  - Average power: $c\ Z_2 A_t^2 \omega^2$
  - $T \equiv \frac{A_t}{A_i}$,
  - Average power: $c\ Z_2 (A_i T)^2 \omega^2$
- **Transmitted Power Ratio:**
  -
$$\frac{\text{Transmitted Power}}{\text{Incident Power}} = \frac{Z_2}{Z_1} T^2 = Te$$
  - Note, that since we are dealing with transmission, the impedance coefficients are not necessarily equal.

Conservation of Energy:

-
$$P_i = P_r + P_t$$
- The reflection transmission coefficients for energy are not the same as those for amplitude:
-
$$R_e = \frac{\text{Reflected Energy}}{\text{Incident Energy}} = R^2$$
-
$$T_e = \frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_2}{Z_1} T^2 = 1 - R_e$$
- Therefore, $R_e + T_e = 1$ or $R^2 + \frac{Z_2}{Z_1} T^2 = 1$

Lecture 15:

Traveling Waves

Velocity and Acceleration of a Medium Particle

- As the wave travels through the medium, particles of the medium must undergo simple harmonic motion according to $y(x,t) = A\sin(\pm\omega t + kx + \phi_o)$
- **Note that the particles is fixed in space, so** $kx + \phi_o = C$

Amplitude Reflection and Transmission Coefficients

Reflection Coefficient: $R \equiv \frac{A_r}{A_i}$

Transmission Coefficient: $T \equiv \frac{A_t}{A_i}$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$T = \frac{2Z_1}{Z_1 + Z_2}$$

Power Reflection and Transmission Coefficients

Reflection Coefficient: $R_e \equiv R^2$

Transmission Coefficient: $T_e \equiv \frac{Z_2}{Z_1} T^2$

$$R_e + T_e = 1$$

$$R^2 + \frac{Z_2}{Z_1} T^2 = 1$$

Figure 7: Enter Caption

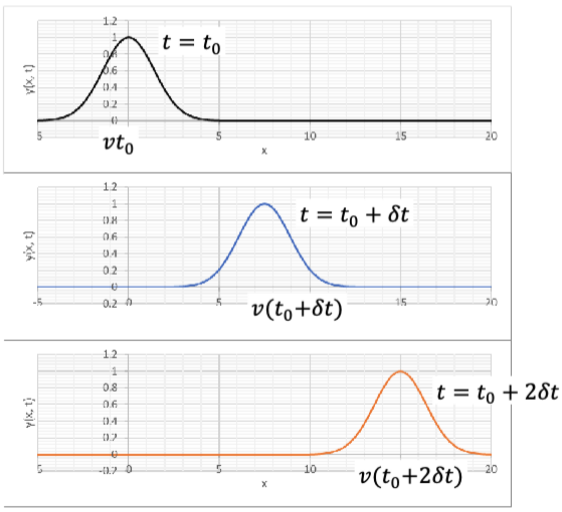

Figure 8: Enter Caption

Velocity and Acceleration of a Medium Particle

Lecture 16:

Traveling Waves

- Power carried by transverse wave on a piece of string is $P = \tau_y v_y$
- $P(x,t) = -\tau \frac{dy}{dx}\frac{dy}{dt}$
- Take the above derivatives and we end up with

$$P(x,t) = \sqrt{\mu\tau}A^2\omega^2 \sin^2(kx - \omega t + \phi_o)$$

  - Maximum Power: $P_{max} = \sqrt{\mu\tau}A^2\omega^2$
  - Average Power: $P_{avg} = \frac{1}{2}\sqrt{\mu\tau}A^2\omega^2$

- Note that we can also consider fluids and solids (rods):
  - Mechanical impedance: $Z_{solid} = \sqrt{\rho Y}$
  - Acoustical impedance: $Z_{fluid} = \sqrt{B\rho}$

Power in Sound Waves (spreading)

- **Define intensity** as the average power transported by a wave per unit area $I = \frac{P_{avg}}{S}$
- **1 Dimension:** Energy transported uniformly, so the intensity is the same at every position $I = C$
- **2 Dimensions:** Intensity changes inversely proportional to the distance from the source $I \propto \frac{1}{r}$
- **3 Dimensions:** If the sound radiates in all directions, then dispersion is spherical

$$I = \frac{P}{4\pi r^2}$$

Attenuation

- Wave media absorbs energy as wave passes through it. Atoms and molecules collide with each other, transforming power of wave into heat.
- Rate of absorption is proportional to the wave intensity:

$$\frac{dI}{dx} = -\alpha I$$

- Solving the ODE yields $I(x) = I(x_o)e^{-\alpha(x-x_o)}$

Attenuation and Spreading:

- 3D (point source):

$$I(r) = I(r_o)[e^{-\alpha(r-r_o)}]\frac{r_o^2}{r^2}$$

- 2D (surface):

$$I(r) = I(r_o)[e^{-\alpha(r-r_o)}]\frac{r_o}{r}$$

- 1D is defined as above.

Intensity and Intensity Levels (dB scale)

- Threshold of hearing $I_o = 10^{-12}\ \frac{W}{m^2}$
- Threshold of pain $I = 10^{1}\ \frac{W}{m^2}$
- Sound intensity level is defined as

$$\beta = (10\ dB)\log(\frac{I}{I_o})$$

Lecture 17:

Superposition in Non-Dispersive Media

- The travelling wave $\psi = A\cos(kx - \omega t)$ is monochromatic because it has a unique frequency, and wavelength
- The simplest superposition is that which has two monochromatic waves that have the same amplitude and velocity
- $$\Psi = \psi_1 + \psi_2 = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$
- By applying some trig identities:
$$\Psi = \psi_1 + \psi_2 = 2A\cos(\frac{1}{2}(\omega_1 + \omega_2)t)\cos(\frac{1}{2}(\omega_2 - \omega_1)t)$$

Superposition in Dispersive Media

- In a non-dispersive medium, the velocity of a wave is independent of its wave number
- In a dispersive medium, the velocity and frequencies are functions of k
  - $v = v(k)$
  - $\omega = \omega(k)$
- look at two monochromatic waves again: $\psi_1 = A\cos(k_1 x - \omega_1 t)$ and $\psi_2 = A\cos(k_2 x - \omega_2 t)$
- Find average frequency and wave number: $\omega_o = \frac{\omega_1 + \omega_2}{2}$ and $k_o = \frac{k_1 + k_2}{2}$
- Since the differences between frequencies and wave numbers are small, define $\Delta k = \frac{k_2 - k_1}{2}$ and $\Delta w = \frac{w_2 - w_1}{2}$
- Combining the two waves, we have
$$\psi_1 + \psi_2 = A(x,t)\cos(k_o x - \omega_o t)$$
- Where the amplitude contains the difference between the two k and ω
$$A(x,t) = 2A\cos(\Delta k x - \Delta\omega t)$$
- Thus, in this case, we have that the phase velocity is equal to
$$v = \frac{\omega_o}{k_o}$$
- The envelope will travel forward with the wave, but it does so with a velocity that is different from the phase velocity.
- The amplitude of the crest remains constant, so $A(x,t) = C \rightarrow x\Delta k - t\Delta\omega = C'$
- Differentiating the above expression $w.r.t$ time $\frac{dx}{dt}\Delta k - \Delta\omega \rightarrow \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} = v_g$
- $$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\omega(k_2) - \omega(k_1)}{k_2 - k_1} = v_g$$

Taylor Expansion for Group Velocities

- Recall, in a dispersive medium, frequencies are functions of k
- Looking at the angular frequency we have that

$$\omega(k_o \pm \Delta k) = \omega(k_o) \pm \Delta k \, (\frac{d\omega}{dk})\Big|_{k=k_o} + \ ...$$

  through a first order Taylor Expansion about $k = k_o$. From here, we can do some algebraic manipulation:

$$\omega(k_2) - \omega(k_1) = (k_2 - k_1) \, (\frac{d\omega}{dk})\Big|_{k=k_o} \to v_g = \frac{d\omega}{dk}\Big|_{k=k_o}$$

- v_g is rewritten as

$$\frac{d\omega}{dk} = \frac{d(kv)}{dk} = v + k\frac{dv}{dk} = v + k\frac{dv}{d\lambda}\frac{d\lambda}{dk}$$

- Since $k = \frac{2\pi}{\lambda} \to \frac{d\lambda}{dk} = -\frac{2\pi}{k^2} = -\frac{\lambda}{k}$
- $$v_g = v - k\frac{dv}{d\lambda}\frac{\lambda}{k} = v - \lambda\frac{dv}{d\lambda}$$
  - Normal dispersion: $\frac{dv}{d\lambda} > 0 \implies v_g < v$
  - Anomalous dispersion: $\frac{dv}{d\lambda} < 0 \implies v_g > v$
  - No dispersion: $\frac{dv}{d\lambda} = 0$

Dispersion Relation

- The dispersion relation for a medium describes how the frequency of a wave ω depends on the wavenumber k
- For an ideal string we saw that $v_o = \sqrt{\frac{\tau}{\mu}} \to \omega = k\sqrt{\frac{\tau}{\mu}}$
- However, a non-ideal string will have an inherent stiffness to it

$$\omega = \sqrt{\frac{k^2\tau}{\mu} + \alpha k^4}$$

- What is important is that under non-idealized circumstances, the relation between ω and k will not be linear.

Wave Packets

- Formulation of a wave packet is a sum of plane waves:

$$\psi = \sum_n a_n \cos(k_n x - \omega_n t)$$

- The sum and constant term can be rearranged to form the following:

$$\psi = A(x,t)\cos(k_o x - \omega_o t)$$

  where k_o and ω_o are the average values of angular frequency and wave number, and the phase velocity is $v = \frac{\omega_o}{k_o}$
- Distributions of wave numbers in a packet (some spread Δk) results in a physical spread of the wave (Δx, width of packet at time t).
- Thus, $\Delta x \Delta k = C$

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon_0 \mu_0}{\varepsilon \mu}} = \sqrt{\varepsilon_r \mu_r} \rightarrow \frac{c}{v} = \sqrt{\varepsilon_r \mu_r} \rightarrow v = \frac{c}{\sqrt{\mu_r}} \cdot \frac{1}{\sqrt{\varepsilon_r}}$$

$$v_g = v - \lambda \frac{dv}{d\lambda} = v - \lambda \left(\frac{dv}{d\varepsilon_r}\right)\left(\frac{d\varepsilon_r}{d\lambda}\right) = v - \lambda \left(-\frac{1}{2}\frac{v}{\varepsilon_r}\right)\left(\frac{d\varepsilon_r}{d\lambda}\right)$$

$$v_g = v\left(1 + \frac{\lambda}{2\varepsilon_r}\frac{d\varepsilon_r}{d\lambda}\right)$$

Figure 9: Enter Caption

EM Waves: Dispersion Relation

- In a vacuum, the speed of light is constant:

$$c = \sqrt{\frac{1}{\epsilon_o \mu_o}}$$

- The speed of an EM wave in a given medium is

$$v = \sqrt{\frac{1}{\epsilon \mu}}$$