AER210 Problem Set 2

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1 Question 1

a) Prove

$$\int_{\Omega} \nabla\phi \; d\Omega = \oint_{\partial\Omega} \phi\vec{n} \; d\Gamma$$

Recall, divergence theorem states that

$$\int_{\Omega} \nabla \cdot \mathbf{F} d\Omega = \oint_{\partial\Omega} \mathbf{F} \cdot \hat{n} d\Gamma$$

Let $\mathbf{F} = \phi\mathbf{c}$. Choose s.t. $\mathbf{c} = 1\hat{e_k}$, where $\hat{e_k}$ is the k-th standard basis vector of $\mathbb{R}^n$ $(1 \leq k \leq n)$. The vector function becomes $\mathbf{F} = \phi\hat{e_k}$. Let $\hat{n} = n_1\hat{e_1} + n_2\hat{e_2} + ... + n_k\hat{e_k} + ... + n_n\hat{e_n}$, where $\hat{n}$ is the normal vector. By applying the divergence theorem to this particular vector function and evaluating the dot product:

$$\int_{\Omega} \nabla \cdot (\phi\hat{e}_k) d\Omega = \oint_{\partial\Omega} (\phi\hat{e}_k) \cdot \hat{n} d\Gamma \Rightarrow \int_{\Omega} \frac{\partial\phi}{\partial x_k} d\Omega = \oint_{\partial\Omega} \phi n_k d\Gamma$$

Now, we can extend the same logic to all n terms and take their sum:

$$\int_{\Omega} \frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + ... + \frac{\partial\phi}{\partial x_k} + ... + \frac{\partial\phi}{\partial x_n} \; d\Omega = \oint_{\partial\Omega} \phi\hat{e}_1 n_1 + \phi\hat{e}_2 n_2 + ... + \phi\hat{e}_k n_k + ... + \phi\hat{e}_n n_n \; d\Gamma$$

$$\sum_{i=1}^{n} \int_{\Omega} \hat{e}_i \frac{\partial\phi}{\partial x_i} \; d\Omega = \sum_{i=1}^{n} \oint_{\partial\Omega} \hat{e}_i \phi n_i \; d\Gamma \rightarrow \int_{\Omega} \nabla\phi \; d\Omega = \oint_{\partial\Omega} \phi\vec{n} \; d\Gamma$$

The identity is proven.

b) Apply the above identity with $\phi = p$, where p is the pressure experienced at any point in $\mathbb{R}^{\nVdash}$.

$$\int_{V} \nabla p \; dV = \oint_{\partial V} p\vec{n} \; d\Gamma$$

Now, since we are given that the "object is submersed in a static fluid in a uniform gravitational field", the acceleration of the fluid $\vec{a} = 0$:

$$\rho\vec{a} = -\nabla p + \rho\vec{g} \rightarrow \nabla p = \rho\vec{g}$$

By plugging this expression for the pressure gradient, we get that the integral evaluates to the specific gravity multiplied with the volume:

$$\int_{V} \nabla p dV = \int_{V} \rho g dV = \rho g \int_{V} 1 dV = \rho g V$$

Now on the RHS, we take the surface integral of the volume of our object, and sum up the pressure components that act normally outward$(P\vec{n})$ to our infinitesimal surface element $d\Gamma$.

This surface integral computes the net pressure force of the fluid acting on the submersed object. Thus,

$$\oint_{\partial V} p\vec{n}\, d\Gamma = \text{Net Pressure Force}$$

Therefore, we have that

$$\rho g V = \text{Net Pressure Force}$$

as required.

c) No net pressure $\rightarrow$ homogeneous pressure: The pressure gradient is equal to $\nabla p = \frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}$. Let some $\vec{f} = \nabla p$. Through dimensional analysis, the units of f are $[\frac{Pa}{m}] = [\frac{N}{m^3}]$. $\vec{f}$ can be thought of as the pressure force exerted on the fluid element per unit volume (the specific pressure force). Given that there is no net pressure force, then there is also no specific force either, $\vec{f} = \vec{0}$. By equality, $\nabla p = \vec{0}$ as well; thus there is no change in pressure in all of $\mathbb{R}^{\nVdash}$. Therefore the pressure is homogeneous.

Pressure gradient $\rightarrow$ acceleration: Conversely, if there is a pressure gradient, then the specific force $\vec{f} \neq \vec{0}$; thus there must be a net acceleration, as $\vec{f} \propto \vec{F} = m\vec{a}$.

d) Requirements:

- $f(\mathbf{x}, t)$ represents a C^1 scalar function that maps $\mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$
  - Thus, we may choose $f(\mathbf{x}, t) = \rho(\mathbf{x}, t)$, as the density in compressible flow is also a function of position and time. Note that $\rho(\mathbf{x}, t) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$. We can also assume that the density throughout the fluid is continuous (i.e. no sudden boundary changes).
- The region over which this occurs is some $\Omega(t) \subset \mathbb{R}^n$ with a piece-wise smooth boundary $\Gamma(t) = \partial\Omega(t)$
  - We can assume that the region or volume in question V has a piece-wise smooth boundary with time. So $\Omega(t) = V(t) \subset \mathbb{R}^3$ and $\Gamma(t) = \partial V(t)$

Conservation of mass is expressed as the following ODE

$$\frac{dm}{dt} = 0$$

Plug in our function $\rho(\mathbf{x}, t)$ into Reynolds Transport Theorem:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x}, t) dV = \int_{\partial V(t)} \rho(\mathbf{x}, t)\mathbf{v}_\Gamma \cdot \hat{n} d\Gamma + \int_{V(t)} \frac{\partial}{\partial t}\rho(\mathbf{x}, t) dV$$

Looking at the LHS, the volume integral of the density function represents the total mass of the element, and by taking the derivative of the mass, the same differential equation as the assumption appears.

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x}, t) dV = \frac{dm}{dt} = 0$$

Apply the divergence theorem to the first surface integral term:

$$\int_{\partial V(t)} \rho(\mathbf{x}, t)\mathbf{v}_\Gamma \cdot \hat{n} d\Gamma = \int_{V(t)} \nabla \cdot (\rho(\mathbf{x}, t)\mathbf{v}_\Gamma) dV$$

Plugging everything back into Reynold's Transport theorem:

$$0 = \int_{V(t)} \nabla \cdot (\rho(\mathbf{x}, t)\mathbf{v}_\Gamma) dV + \int_{V(t)} \frac{\partial}{\partial t}\rho(\mathbf{x}, t) dV$$

Notice now that the bounds of integration are the same for both terms. By differentiating both sides with respect to volume we get the differential form of the 1st compressible Euler and Navier-Stokes equation (since we are given that $\mathbf{v_\Gamma} = \mathbf{v}$)

$$\nabla \cdot \rho(\mathbf{x}, t)\mathbf{v_\Gamma} + \frac{\partial}{\partial t}\rho(\mathbf{x}, t) \rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

e) Express the change of momentum as the sum of all forces acting on an object:

$$\vec{F}_{net} = \frac{d}{dt}(m\vec{v}) = m\vec{a}$$

In this case, the net force is equal to the negative pressure force plus the gravitational force:

$$\vec{F}_{net} = -\oint_{\partial V} \rho \hat{n}\ d\Gamma + \int_V \rho \vec{g}\ dV$$

Considering momentum alone, it can be expressed as the velocity vector times the density scalar function:

$$\vec{p} = \int \rho \vec{v} dV$$

Now, in taking the first time derivative of the above expression, we get the change in momentum, which is equal to the net force: $\frac{d}{dt}\vec{p} = F_{net}$.

$$\frac{d}{dt}\int_V \rho \vec{v}\ dV = -\oint_{\partial V} \rho \hat{n}\ d\Gamma + \int_V \rho \vec{g}\ dV$$

Apply the outer product on the left hand side along with Reynold's Theorem:

$$\frac{d}{dt}\int_V \rho \vec{v} dV \Rightarrow \oint_{\partial V} \rho \vec{v} \otimes \vec{v} \cdot \hat{n}\ d\Gamma + \int_V \frac{\partial}{\partial t}(\rho \vec{v})\ dV$$

Apply divergence theorem to the first term on the right. Notice that the bounds of integration are the same. Due to the linearity of integral operations, combine all terms into one integral.

$$\int_V \nabla \cdot (\rho \vec{v} \otimes \vec{v})\ dV + \int_V \frac{\partial}{\partial t}(\rho \vec{v})\ dV \rightarrow \int_V \nabla \cdot (\rho \vec{v} \otimes \vec{v}) + \frac{\partial}{\partial t}(\rho \vec{v})\ dV$$

Expand and factor terms:

$$\int_V (\nabla \cdot \rho \vec{v})\vec{v} + \rho \vec{v} \cdot \nabla \vec{v}\ + (\rho \vec{v}' + \rho' \vec{v})\ dV \rightarrow \int_V \rho(\vec{v} \cdot \nabla \vec{v} + \vec{v}') + \vec{v}(\rho' + \nabla \cdot \rho \vec{v})\ dV$$

Recall the acceleration $\vec{a}$ is given by $\vec{a} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$. Which is exactly the term in the first bracket. Additionally, the second bracket term is the first compressible Euler equation, which is equal to 0:

$$\int_V \rho \vec{a} + \vec{v}(0)\ dV$$

Now, on the right hand side, apply the identity found in 1a to the first term:

$$-\oint_{\partial V} \rho \hat{n}\ d\Gamma + \int_V \rho \vec{g}\ dV \rightarrow -\int_V \nabla p\ dV + \int_V \rho \vec{g}\ dV$$

Since LHS = RHS:

$$\int_V \rho \vec{a}\ dV = -\int_V \nabla p\ dV + \int_V \rho \vec{g}\ dV$$

Once again, the bounds of integration are all the same so the final result becomes

$$\rho \vec{a} = -\nabla p + \rho \vec{g}$$

as required.

2 Question 2

First calculate the Reynold's number:

$$Re = \frac{\rho v_\infty L}{\mu}$$

At 10 km in the air, we know the following properties:

- $T = -50 \deg C = 223\ K$
- $P = 16000\ Pa$
- $L = 10\ m$
- $c = 299.5\ m/s$ (Engineering Toolbox)
- $\mu = 14.56 \times 10^{-6}\ Pa\ s$

We are missing the speed of the aircraft relative to the air. Since we are given the Mach number and we know the speed of sound at an altitude of 10 km, we can calculate the speed of the aircraft to be:

$$Ma = \frac{v_\infty}{c} \to v_\infty = Ma \times c \to v_\infty = 299.5\ m/s \times 0.8 = 239.6\ m/s$$

We must also calculate the density of air at the given altitude, which is found through the ideal gas law:

$$\rho = \frac{m}{V} = \frac{P}{RT} = \frac{16000}{287.05 \times 223} = 0.250\ kg/m^3$$

Thus, the Reynold's number is

$$Re = \frac{0.250 \times 239.6 \times 10}{14.56 \times 10^{-6}} = 41.1 \times 10^6$$

Similar to the demo session, let us start by considering air at standard conditions. From there we can see what properties need to be satisfied in order to approximate the actual conditions:

2.1 Iteration 1: air

- Reynold's number:

$$41.1 \times 10^6 = \frac{1.2\ kg/m^3\ \times V_\infty \times 0.1\ m}{1.8 \times 10^{-5}\ Pa\ s} \to v_\infty = 6169\ m/s$$

- Mach number:

$$Ma = \frac{6169}{343} = 17.99$$

These conditions are simply unachievable, as air at standard temperatures and pressures in the tunnel would have to be going at almost 18 times the speed of sound.

From this first attempt we know the following:

- Decrease v_∞
  - This can be done by decreasing μ,
  - Increasing ρ
  - Increasing L (it shouldn't be increased by much though)

2.2 Iteration 2: Water

It is more important to match the Mach number than Re, so let us consider a fluid with a higher speed of sound like water. For our purposes, let us assume that the density of water stays constant at around $\rho = 1000\ kg/m^3$

$$41.1 \times 10^6 = \frac{1000 \times v_\infty \times 0.1}{\mu} \rightarrow v_\infty = 41.1 \times 10^4 \mu(T)$$

The speed of sound in water can be approximated as 1500 m/s, so

$$Ma = \frac{v_\infty}{1500} \rightarrow v_\infty = 0.8 \times 1500$$

Setting the speeds equal to each other, we require $\mu(t) = 0.00289 Pa\ s$. This value for dynamic viscosity is achievable at a temperature of 100 C. Thus, at a temperature of around 100 C and a pressure higher than atmospheric $120kPa$, we are able to maintain liquid water by increasing its boiling point. We would require the water to be flowing at a speed of 1200 m/s

3 Question 3

a) To derive the partial derivatives of pressure in spherical coordinates, we make use of euler's momentum equation:

$$\rho\vec{a} = -\nabla\vec{p} + \rho\vec{g}$$

We plug in the vector expression for $\vec{a}$ and apply the gradient operator to the pressure vector. NOTE, we must also consider the gradient function in spherical coordinates:

$$\nabla F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial F}{\partial \theta}\hat{\theta}$$

$$\rho(\omega\hat{z} \times (\omega\hat{z} \times \mathbf{r})) = -\begin{bmatrix} \frac{\partial p}{\partial r} \\ \frac{\partial p}{\frac{1}{r}\partial\phi} \\ \frac{\partial p}{\frac{1}{r\sin\phi}\partial\theta} \end{bmatrix} + \rho\begin{bmatrix} -\frac{GM(r)}{r^2} \\ 0 \\ 0 \end{bmatrix}$$

Apply the following coordinate transformation for $\hat{z} = \cos(\phi)\hat{r} - \sin(\phi)\hat{\phi}$. In column vector form, we have:

$$\rho(\omega\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix} \times (\omega\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix} \times \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix})) = -\begin{bmatrix} \frac{\partial p}{\partial r} \\ \frac{\partial p}{\frac{1}{r}\partial\phi} \\ \frac{\partial p}{\frac{1}{r\sin\phi}\partial\theta} \end{bmatrix} + \rho\begin{bmatrix} -\frac{GM(r)}{r^2} \\ 0 \\ 0 \end{bmatrix}$$

The double cross product on the left becomes:

$$\begin{bmatrix} -\rho\omega^2 r\sin^2(\phi) \\ -\rho\omega^2 r\cos(\phi)\sin(\phi) \\ 0 \end{bmatrix} = -\begin{bmatrix} \frac{\partial p}{\partial r} \\ \frac{\partial p}{\frac{1}{r}\partial\phi} \\ \frac{\partial p}{\frac{1}{r\sin\phi}\partial\theta} \end{bmatrix} + \rho\begin{bmatrix} -\frac{GM(r)}{r^2} \\ 0 \\ 0 \end{bmatrix}$$

We now have an expression for each component in spherical coordinates:

$$\frac{\partial p}{\partial r} = \rho\omega^2 r\sin^2(\phi) - \rho\frac{GM(r)}{r^2}$$

We can express $M(r)$ as the volume of the sphere at radius r times the density: $M(r) = \frac{4}{3}\pi r^3 \rho$

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \sin^2(\phi) - \frac{4}{3}\rho^2 G\pi r$$

$$\frac{1}{r}\frac{\partial p}{\partial \phi} = \rho\omega^2 r\cos(\phi)\sin(\phi) \rightarrow \frac{\partial p}{\partial \phi} = \rho\omega^2 r^2 \cos(\phi)\sin(\phi)$$

$$\frac{1}{r\sin\phi}\frac{\partial p}{\partial \theta} = 0 \rightarrow \frac{\partial p}{\partial \theta} = 0$$

b) Reconstructing a function from its gradient:

$$\frac{\partial p}{\partial r} = \rho r \omega^2 \sin^2\phi - \frac{4}{3}\rho^2 G\pi r$$

$$\frac{\partial p}{\partial \phi} = \rho r^2 \omega^2 \sin\phi\cos\phi$$

$$p(r,\phi) = \int \frac{\partial p}{\partial r}\, dr + f(\phi)$$

$$\frac{\partial p(r,\phi)}{\partial \phi} = \frac{d}{d\phi}\left[\int \frac{\partial p}{\partial r}\, dr + f(\phi)\right] = \frac{\partial p}{\partial \phi}$$

$$p(r,\phi) = \int \rho r \omega^2 \sin^2\phi - \frac{4}{3}\rho^2 G\pi r \, dr + f(\phi)$$

$$p(r,\phi) = \frac{1}{2}\rho r^2 \omega^2 \sin^2\phi - \frac{2}{3}\rho^2 G\pi r^2 + f(\phi)$$

$$\frac{\partial p}{\partial \phi} = \rho r^2 \omega^2 \sin\phi\cos\phi + f'(\phi) =$$

$$\cancel{\rho^2\omega^2 \sin\phi\cos\phi} + f'(\phi) = \rho r^2 \cancel{\omega^2 \sin\phi\cos\phi}$$

$$f'(\phi) = 0$$

$$\int f'(\phi)\, d\phi = C$$

$$p(r,\phi) = \frac{1}{2}\rho r^2 \omega^2 \sin^2\phi - \frac{2}{3}\rho^2 G\pi r^2 + C$$

$$p(r,\phi) = \rho r^2 \left(\frac{1}{2}\omega^2 \sin^2\phi - \frac{2}{3}\rho G\pi\right) + C$$

boundary: $p(R,0) = P_{atm}$

$$P_{atm} = \frac{1}{2}\rho R^2 \omega^2 \sin^2(0) - \frac{2}{3}\rho^2 G\pi R^2 + C$$

$$C = P_{atm} - \underbrace{\cancel{\frac{1}{2}\rho R^2 \omega^2 \sin^2(0)}}_{0} + \frac{2}{3}\rho^2 G\pi R^2$$

$$C = P_{atm} + \frac{2}{3}\rho^2 G\pi R^2$$

Figure 1: Step by Step derivation.

$$p(r,\theta) = \frac{1}{2}\rho r^2 \omega^2 \sin^2(\phi) - \frac{2}{3}\rho^2 G\pi r^2 + C$$

Where the constant is equal to the following. (R is the radius at the poles of the earth)

$$C = P_{atm} + \frac{2}{3}\rho^2 G\pi R^2$$

The final relation becomes

$$p(r,\theta) = \frac{1}{2}\rho r^2 \omega^2 \sin^2(\phi) - \frac{2}{3}\rho^2 G\pi r^2 + p_{atm} + \frac{2}{3}\rho^2 G\pi R^2$$

c) Rearrange for the radius r:

$$r_{eq} = \sqrt{\frac{P_{eq} - P_{atm} - \frac{2}{3}\rho^2 G\pi R^2}{\frac{1}{2}\rho\omega^2 \sin^2(\phi) - \frac{2}{3}\rho^2 G\pi}}$$

Plugging in the given values for R, ρ, ω and G and taking $P_{atm} = 101325\ Pa = P_{eq}$ and $\phi = \frac{\pi}{2}$, we receive a radius of $r_{eq} = 6.4175 \times 10^6\ m$. The percentage error of this calculation is $\frac{|6.4175\times10^6 - 6.378\times10^6|}{6.378\times10^6} = 100\% = 0.619\%$. We have an overestimation of the value for the radius.

1. ρ is not constant, throughout the earth. The earth is not composed of purely incompressible water, but different layers and crusts all at different radii from the center of the earth.

2. Treating water as incompressible is not accurate, especially at the scale of the earth. Density would increase as we go deeper into the "water earth"

3. Gravitational acceleration is not accurate, as the earth is not perfectly spherical, and has imperfections along its surface, where g is not constant. The relation for g therefore does not purely depend on the radius.

d) If the partial derivative of pressure w.r.t radius at $\phi = \frac{\pi}{2}$ is greater than 0, then the fluid must necessarily be escaping the earth's gravitational pull:

$$\frac{\partial p}{\partial r} = \rho r^2 \omega^2 \sin^2(\phi) - \frac{4}{3}\rho^2 G\pi r^2$$

We can now rearrange for ω

$$\omega = \sqrt{\frac{4G\pi\rho}{3}}$$

By plugging in values, we have that $\omega = 0.53 \times 10^{-4}\ rad/s$. This estimate is off by an order of magnitude; however, note that this calculated value is a minimum value. So the larger, theoretical value still holds within the equality of $\omega > 0.53 \times 10^{-4}\ rad/s$.