PHY293 Modern Physics

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Lecture 1

Lecture 2

Photoelectric effect

- UV causes metals to release electrons; electron energy does not depend on the light intensity, only on the frequency of light.
- Einstein proposes the quantization of light $E_{max} = \frac{1}{2}mv_{max}^2 = hf - \phi$. Where ϕ is the work function of the specific metal, it is the energy needed to remove an electron from a specific metal.

Compton Effect

- This experiment shows what happens when light is shined at electrons
- Since light can act as a particle, when it collides with the electron both the photon and electron will change course.
- Thus, the when the photon collides with the electron, energy is conserved, and the wavelength of the photon will change from λ to λ' since energy is lost by the photon and transferred to the electron.
- The energy lost increases with the angle at which the photon scatters: $\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$

Lecture 3

Bohr's Model of the Atom

- Bohr's Model had 3 assumptions:
  1. Electrons are in a circular orbit
  2. Electrons in stationary states do not radiate energy
  3. Radiation is emitted when electrons change orbits
- Bohr assumed that angular momentum is quantized by $L = mvr = n\hbar$, where n is the state and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant
- To solve the energy of the electron we equate the centripetal force (assumption 1) to the Coulomb force. We get the following results:

$$v = \frac{ke^2}{n\hbar}$$

$$r = \frac{(n\hbar)^2}{kme^2}$$

- From the above, we can calculate the total energy of an electron occupying the nth state:

$$E_{tot} = KE + PE = \frac{1}{2}mv^2 + -k\frac{e^2}{r} = \frac{m}{2}(\frac{ke^2}{n\hbar})^2 - ke^2\frac{kme^2}{(n\hbar)^2} = -\frac{13.6}{n^2}$$

- For hydrogen, radiation emitted, is equal to $hf = E_n - E_m$

- For an atom with atomic number Z:

$$E_n = \frac{m(kZe^2)^2}{2\hbar^2}\frac{1}{n^2}$$

Lecture 4

Franck Hertz Experiment

- Confirmed energy quantization of atoms
- Electrons lose discrete quantities of energy periodically

Matter Waves (de Broglie)

- At the time, it was known that light exhibits matter properties, but what about the other way, can matter exhibit wave properties?

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

- Note that we only observe Matter Waves when the order of magnitude of the wavelength is around the same as that of the particle in movement.
- For an electron
$$\lambda = \frac{h}{\sqrt{2mE_k}} = 1.23A$$
- Thus, electrons exhibit wavelike properties, when excited through a potential of around $V = 100V$. Thus, they can be diffracted, given that the grating is around the same size as λ . Which leads to the X-ray diffraction from crystals (recall electron diffraction lab).
- Consider electrons orbiting a state in a Bohr atom, de Broglie's idea was that electrons occupied a standing wave with closed ends: $\lambda = 2l$, $\lambda = l$ $\lambda = l/2 ... \lambda = 2l/n$
  - Where l is simply the circumference of the circular orbital $l = 2\pi r$
  - Thus, $\lambda = \frac{2\pi r}{n}$

Bragg Diffraction

- The condition needed to have constructive interference is $\lambda = 2d\sin\theta$, where d is the inter-planar spacing.

Lecture 5

Tonomura (Hitachi) experiment

- Wavelike nature of the electron shown as they pass through double slit setup
- Electrons are sent one at a time and show a wavelike distribution
- Matter behaves like a wave, but what governs this distribution?

Wave Equation for a Matter Wave

- We make the analogy with E+M wave: $A\sin(kx - \omega t)$
  - Intensity is amplitude squared: $I \propto A^2$
- Matter wave is interpreted as a wave function $\Psi(x, t)$
  - intensity is interpreted as a probability distribution: $I \propto |\Psi(x, t)|^2$

Properties of a wave function

1. $|\Psi(x,t)|^2$ is a probability density
   (a) The probability of finding the particle in the interval $[x, x + dx]$ at some time t is $|\Psi(x,t)^2|dx$
2. $\Psi(x,t)$ is square-integrable
3. Superposition applies
4. $\Psi(x,t)$ is a continuous function
5. Normalization is conserved in time (total probability remains 1 through time)
   (a)
$$\frac{d}{dt}[\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx]$$

Superposition of States

- An electron's quantum state is described by a wave function: $\Psi = \psi_L + \psi_R$
- In the Tonomura Experiment, the electron is in the above superposition state:
  - where ψ_L and ψ_R are the electron states going through the left and right slit
- Consequences of superposition:
  - If we measure which slit the electron goes through, the wave function collapses onto either $|\psi_L|^2$ or $|\psi_R|^2$
  - However, if we don't measure the wave function arriving on screen, the probability function of the electron is in a superposition of states of $|\psi_L + \psi_R|^2$

Lecture 6

Fourier Series

- Any periodic function is expressed as a fourier expansion:
$$f(x) = \sum_{0}^{\infty} A_n sin(\frac{2\pi n}{\lambda}x)$$
- Different terms correspond to fundamental and higher harmonic in wavelengths: $\frac{\lambda}{n}$
- The coefficient of the expansion are given by
$$A_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(\frac{2\pi nx}{\lambda})dx$$

Hilbert Space

- A possible analogy for Hilbert space is the usage of basis vectors in R^3 . We use basis vectors $\hat{x},\ \hat{y},\ \hat{z}$, which are equivalent to the set of $n = 3$ basis vectors e_1, e_2, e_3
- Determining the coefficient in the nth direction is given as the dot product between the basis vector in that direction and the vector: $A_n = e_n \cdot \vec{r}$
- Hilbert space is an infinite dimensional vector space given by:
$$\sin(\frac{n\pi x}{\lambda})$$
- Thus, when considering the fourier expansion coefficient A_n, we take the dot product of the nth basis vector, which is $\sin(\frac{n\pi x}{\lambda})$ thus,
$$A_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(\frac{2\pi nx}{\lambda})dx$$

Uncertainty of a Gaussian wave packet

- For a **free particle** or a **travelling wave**(more on this in lecture 10) a plane wave $e^{i(kx-\omega t)}$ cannot be used as a wave function
- However, a wave packet is an infinite summation of plane waves that can become a wave function itself.
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega t) dk}$$
- Where $\phi(k)$ is a coefficient that corresponds to the $k-th$ wave in the wave packet:
- Plugging in t = 0 to the above:
$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk$$
- Conversely $\Psi(x,0)$ is the inverse Fourier transform of $\phi(k)$

Heisenberg Uncertainty Principle:

- For a wave packet, we see that σ_x and σ_k are inversely proportional. Taking σ_x and σ_k as uncertainties Δx and Δk:
$$(\Delta x)(\Delta k) \sim 1$$
- We wish to express these values in terms of momentum: we know that $p = h/\lambda$; and $k = 2\pi/\lambda$ (the wave number). Thus we receive $p = hk/2\pi = \hbar k$. Thus, the relations becomes:
$$(\Delta x)(\Delta p) \geq \hbar/2$$
- The factor of 2 is unimportant since we are working with an inequality, whose uncertainty could always be larger.
- What is Important is that this is the fundamental limit.

Examples:

- Particle in a small box of size a
- Uncertainty in position is small so the uncertainty in momentum becomes large $\Delta p \geq \hbar/a$
- So, we know that the particle cannot be at rest (have $\Delta p = 0$) otherwise the Heisenberg Uncertainty Principle is violated
- Thus, a particle's Kinetic Energy cannot be zero when confined in finite space. Its energy can be related through the size of the box as $\frac{\hbar^2}{2ma^2}$
- Time-Energy Uncertainty
- $$(\Delta t)(\Delta E) \geq \frac{\hbar}{2}$$
- So ΔE means a stationary state, and so Δt must become infinitely large so its lifetime is infinite.
- We can observe this relationship from X-ray spectroscopy. The uncertainty in energy ΔE is given by the Half Width at Half Maximum, from this value, we can find Δt and the lifetime of electrons in different orbitals.

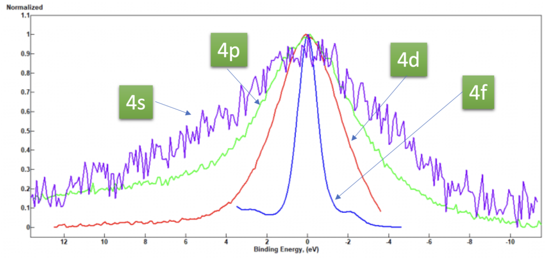

Figure 1:

Lecture 7

Schrodinger Equation

- For photons and matter waves we know that $E = \hbar\omega$ and $p = \hbar k$
- However, we note that the relation between ω and k is different (the dispersion relation is different)
  - For a photon (classical wave) $\omega = ck$
  - But for a matter wave

$$\omega = \frac{E}{\hbar} = \frac{1}{2\hbar}mv^2 = \frac{p^2}{2m\hbar} = \frac{(\hbar k)^2}{2m\hbar} = \frac{\hbar}{2m}k^2$$

    Thus, for a matter wave angular frequency is quadratic in wave number.

- Recall that classical waves $y(x,t)$ need $\omega = ck$ to satisfy the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

- We know that the sinusoidal $y(x,t) = Asin(kx - \omega t)$ satisfies the above
- However, the sine function would not satisfy the wave equation with a dispersion of $\omega = \frac{\hbar}{2m}k^2$
- So we need to come up with another function to describe matter waves.
- Try the complex exponential: $\Psi(x,t) = Ae^{i(kx-\omega t)}$, which satisfies the wave equation and we get the following form of Schrodinger's equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}^2 + U(x)\Psi(x,t)$$

Separation of Variables

- Let us assume that the wave function $\Psi(x,t) = \psi(x)\phi(t)$ is separable
- By substituting the above variable into the Schrodinger equation, we end up with terms as a function of t and x on opposite sides of the equation:

$$i\hbar\frac{1}{\phi}\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + U$$

- The only possibility for this to occur is if LHS = RHS = const = E
  - RHS: **Time Independent Schrodinger Equation**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

  - LHS:

$$\phi(t) = e^{-i\frac{E}{\hbar}t}$$

- Thus, we get the following relation:

$$\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$$

Finding the Wave Function

1. Solve T.I.S.E for a given potential function $U(x)$
2. Find E and $\psi(x)$ (from T.I.S.E)
3. Multiply the time-dependence part $\phi(t)$ in order to find the wave function

$$\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$$

Lecture 8

Infinite Square Well

- 1D box, rigid 1D box

$$\begin{cases} 0 & \text{for } 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

- The infinite potential on regions outside of the box simply represent the fact that the particle cannot exist outside of this region.
- For the region $x < 0$ and $x > a$, $\psi(x) = 0$ (zero probability outside of the box)
- For $0 \le x \le a$, the potential is $U = 0$, so our time independent Schrodinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \to \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

- This is a second order differential equation. with two boundary conditions (needed for continuity) $\psi(0) = 0$ and $\psi(a) = 0$
- Now we consider two cases, one with $E < 0$ and one with $E > 0$. The negative energy case can be refuted as we receive that $\psi(x) = 0$ for all x which isn't the case, since it cannot be normalized (probability of finding the particle somewhere must be 1).
- So we consider only the positive energy case $E > 0$
  - We start by defining $k \equiv \frac{2mE}{\hbar^2}$ and by solving the second order DE, we get that

$$\psi(x) = Asin(kx) + Bcos(kx)$$

  - For boundary conditions:
    - $\psi(0) = B = 0$
    - $\psi(a) = Asin(ka) = 0 \to ka = \pi, 2\pi, 3\pi, ..., n\pi \; k_n = \frac{\pi}{a}n$
  - Normalizing the function gives us the following

$$\psi_n(x) = A_n \sin(\frac{n\pi}{a}x) = \sqrt{\frac{2}{a}}\sin(k_n x)$$

Expectation Value of x

- Now that we derived our wave equation for an ISW, we can consider its expectation value $\langle x \rangle$ or where the probability of observing the particle is most likely:

$$\langle x_n \rangle = \int_{-\infty}^{\infty} x|\psi_n(x)|^2 dx$$

- that this integral technically takes into account a finite energy state n; however, we will see that this integral evaluates to the same value regardless of n:

$$\frac{2}{a}\int_0^a x\sin^2(\frac{n\pi x}{a})dx = \frac{a}{2}$$

- Since the well is square from 0 to a, this means that for any energy level (harmonic), the most likely location to find an electron is halfway between the walls of the ISW.

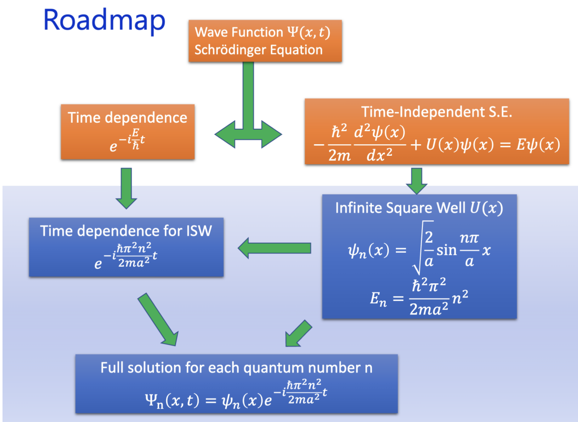

Figure 2:

Discussion about the Quantization of Energy Levels

- Recall, that we had defined k to be equal to $-\frac{2mE}{\hbar^2}$, by rearranging for E:

$$E_n = \frac{\hbar^2}{2m}k_n^2 = \frac{\hbar^2}{2m}(\frac{n\pi}{a})^2$$

- The energy increases quadratic in n
- Also an important question to consider is **why not n = 0?** , why do we start from n = 1?
- From the uncertainty principle, $\Delta x \sim a$

$$\Delta P =$$

  come back to this after tree is growing lol

Lecture 9

Infinite Square Well Recap

Stationary States

- Recall that our ISW wave function is

$$\Psi(x,t) = \psi_n(x)e^{-i\frac{\hbar\pi^2n^2}{2ma^2}t}$$

- Therefore, we note that the probability of the time dependent wave equation is equal to the time independent wave equation:

$$|\Psi(x,t)|^2 = |\psi_n(x)|^2$$

- Probability density is therefore independent of time
- This is possible because E_n is held constant, called an Energy Eigenstate or a Stationary State

Generalities in Superposition

- For the wave function, we know that for example, $\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ is a valid solution of the T.I.S.E
- And as we previously showed, the expectation value is always $\langle x\rangle = \frac{a}{2}$
- What about the expectation value for energy though?
- The general solution for the ISW $U(x)$ is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\Psi_n(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-i\frac{E_n}{\hbar}t}$$

- Where $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$ and $E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$
- Essentially, the total wave function is an infinite linear combination of superimposed states from $n = 1, 2, 3..\infty$.
- $|c_n|^2$ is the probability of finding the particle at the energy state $n = n$. In other words, it is the probability to find the particle in the energy eigenstate $\psi_n(x)$ with energy E_n
- Now let us observe the wave function for a superposition case with n = 1 and n = 2 where $c_1 = 2c_2$ and all other $c_n = 0$

$$\Psi(x,t) = c_1\psi_1 e^{-i\frac{E_1 t}{\hbar}} + c_2\psi_2 e^{-i\frac{E_1 t}{\hbar}}$$

- By normalizing $\Psi(x,t)$, we end up with $1 = c_1^2 + c_2^2$ and by applying our initial conditions, $c_1 = \frac{2}{\sqrt{5}}$ and $c_2 = \frac{1}{\sqrt{5}}$
- Thus, the wave function becomes

$$\Psi(x,t) = \frac{2}{\sqrt{5}}\psi_1 e^{-i\frac{E_1 t}{\hbar}} + \frac{1}{\sqrt{5}}\psi_2 e^{-i\frac{E_1 t}{\hbar}}$$

- By squaring $\Psi(x,t)$, we get the probability:

$$|\Psi(x,t)^2| = \frac{4}{5}|\psi_1|^2 + \frac{1}{5}|\psi_2|^2 + \frac{4}{\sqrt{5}}\psi_1\psi_2\cos(\frac{(E_1 - E_2)t}{\hbar})$$

- Recall that ψ_1 and ψ_2 are wave functions by themselves, and are thus normalized, so when we integrate our superimposed wave function, normalization is preserved only if the oscillating term is 0. Thus, the probability function is

$$|\Psi(x,t)|^2 = \frac{4}{5}|\psi_1|^2 + \frac{1}{5}|\psi_2|^2$$

- Integrating should yield 1 as required:

$$\int |\Psi(x,t)|^2 dx = \int \frac{4}{5}|\psi_1|^2 dx + \int \frac{1}{5}|\psi_2|^2 dx \rightarrow \int |\Psi(x,t)|^2 dx = \frac{4}{5}\int |\psi_1|^2 dx + \frac{1}{5}\int |\psi_2|^2 dx$$

$$\int |\Psi(x,t)|^2 dx = \frac{4}{5} + \frac{1}{5} = 1$$

Copenhagen Interpretation

- The energy is indeterminant until the measurement. Thus, the energy is a superposition of many wave function stationary states
- Measurements of the total energy can give the expectation value - statistically.
- Each individual measurement will give one discrete energy state whose likeliness can be determined by its respective probability
- When the measurement is made yielding some energy E_n, the wave function collapses to that stationary state $\Psi(x,t) = \psi_n e^{-iE_n t/\hbar}$
- On average, the experiments will give an expectation value. Note that this expectation value will not be one of the discrete energies E_n. Analogy with flipping a coin, state is either 0 or 1, expectation is 0.5 which is not equal to 0 or 1.
- Going back to the previous example,

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx = \frac{4}{5}\int x|\psi_1|^2 dx + \frac{1}{5}\int x|\psi_1|^2 dx + \frac{4}{\sqrt{5}}\cos(\frac{(E_1 - E_2)t}{\hbar})\int x\psi_1\psi_2 dx$$

- We see that there is some time dependent oscillation.

Lecture 10

Finite Square Well (FSW) vs Infinite Square Well (ISW)

- Similarities:
  - Quantized energy levels with quantum number n
  - n-1 nodes for the n-th level quantum states (don't count the boundaries as nodes)
- Differences;
  - Non-zero probability of finding the particle within the walls of the well (classically forbidden)
  - Finite number of bound states

Free Particles

- We are now in the scattering state, and are considering free particles (propagating waves), we know from the Heisenberg Uncertainty principle that since Δx is large, Δp is now small.
- $U(x) = 0$ for all x
- The Schrodinger Equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

- Solving the equation we get $\psi(x) = Ae^{-ikx} + Be^{-ikx}$
- Adding time dependence: $\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx-\omega t)}$
- In either case, we have a travelling wave, note that one is going to the right, and the other is going to the left, so we can choose the first term and vary k as desired:
- However, we note that we cannot normalize this function as the exponential term disappears when squared (conjugate).
- Thus, we require a wave packet since its probability is 0 at $\pm\infty$

Wave Packets Revisited

- So the plane wave cannot be used as a wave function, the a wave packet made up of a sum of plane waves can be a wave function.
- $$\begin{cases} A & \text{for } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$
- Consider the above example, normalizing we have that $A = \frac{1}{\sqrt{2a}}$
- We can find $\phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)e^{-ikx}dx = \sqrt{\frac{a}{\pi}}\frac{\sin(ka)}{ka}$
- Next, we can plug our expression for $\phi(k)$ back into the Fourier transform:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{a}{\pi}}\frac{\sin(ka)}{ka}e^{-i(kx-\omega t)}dk$$

Lecture 11

Free particle encountering a potential

- Work with plane wave: $Ae^{i(kx-\omega t)}$ because working with a packet is mathematically challenging (we will disregard the normalization)
- We are interested in the amplitude of these plane waves when they encounter a potential

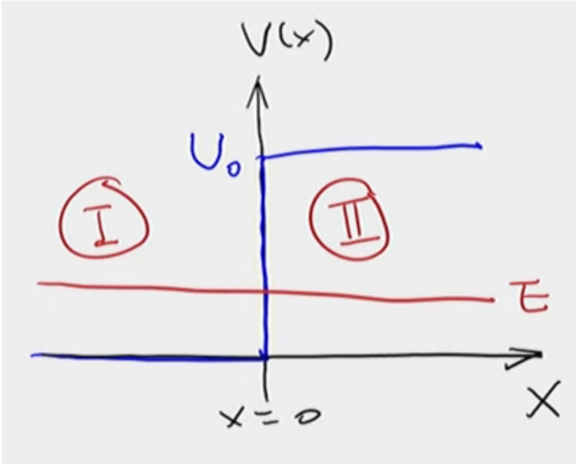

Figure 3:

Some examples of a free particle

- Classical Mechanics:
  - A particle rolling up a sort of bump or hill. The potential in this case is the height differential. In order to scale the bump, the particle's kinetic energy must be converted in to gravitational potential energy. The point at which the gravitational potential equates the kinetic energy, is called the classic turning point, where the particle will be turned back around.
- Quantum Mechanics:
  - Consider an electron with energy E s.t. $E < U_o$ where U_o is a potential step
  - Consider a potential barrier of an electron, it still has less energy than the potential; however, contrary to classical turning points, the electron will have a chance of tunnelling and passing through the barrier

Potential Step Problem

- Strategy:
  1. Divide and conquer by solving the T.I.S.E for regions I and II
  2. Use the boundary conditions
- Boundary Conditions:
  - Continuity of $\psi(x)$ at $x = 0$
  - Continuity of $\psi'(x)$ at $x = 0$

Potential Step: Solving for For $0 < E < U_0$

I:

- The particle is free the equation is $\psi_1 = Ae^{ikx} + Be^{-ikx}$

II:

- The energy in this case is $U_o - E$, so the Schrodinger equation changes to

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(U_o - E)\psi = 0$$

- By making $\alpha = \frac{2m}{\hbar^2}(U_o - E)$, the solution becomes:

$$\psi_2(x) = Ce^{\alpha x} + De^{-\alpha x}$$

• Potential Step (E < U_0)

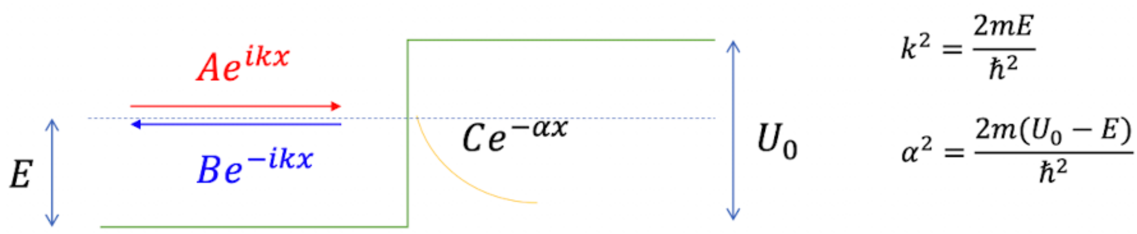

Figure 4:

- $C = 0$

Apply Boundary Conditions:

- BC1 : $\psi_1(0) = \psi_2(0) \rightarrow A + B = D$
- BC2: $\psi_1'(0) = \psi_2'(0) \rightarrow ikA - ikB = -\alpha D$

We have 3 unknowns and w equations (from the 2 boundary conditions). We are interested in the ratios B/A and D/A By adding equation 1 to equation 2, we get that

$$\frac{D}{A} = \frac{2}{1 + \frac{i\alpha}{k}}$$

By adding equation 1 times $\frac{\alpha}{ik}$ + equation 2, we get that $\frac{B}{A} = \frac{1 - i\alpha/k}{a + i\alpha/k}$

Physical Meaning:

- Recall that A is the amplitude of the plane wave moving towards the potential step Ae^{ikx}. Thus it is the incoming plane wave.
- B is the amplitude of the plane wave moving away from the step Be^{-ikx}. Thu, it is the reflected plane wave
- D is the amplitude of the wave that is transmitted through the step $De^{-\alpha x}$. Thus, we know that the probability of this occurring is non-zero, as $|\psi|^2 \propto e^{-2ax}$
- We can find the reflection probability which is defined as the probability of reflected, over the probability of incident
$$R = \frac{|B|^2}{|A|^2} = \frac{B * B}{A * A} = 1$$
- This means that we have total reflection. Even though the particles have a non-zero probability of going into the step, they eventually all come back out.

Potential Barrier: Solving for For $0 < E < U_0$

- Now we can consider a potential barrier (wall/speed bump)

$$\begin{cases} 0 & x < 0 \\ U_o & 0 < x \leq 0 \\ 0 & x > a \end{cases}$$

- Once again, we divide the region of space into 3 and solve the T.I.S.E for each:
  - I:
  $$\psi_1 = Ae^{ikx} + Be^{-ikx}$$
  - II:
  $$\psi_2 = Fe^{\alpha x} + Ge^{-\alpha x}$$

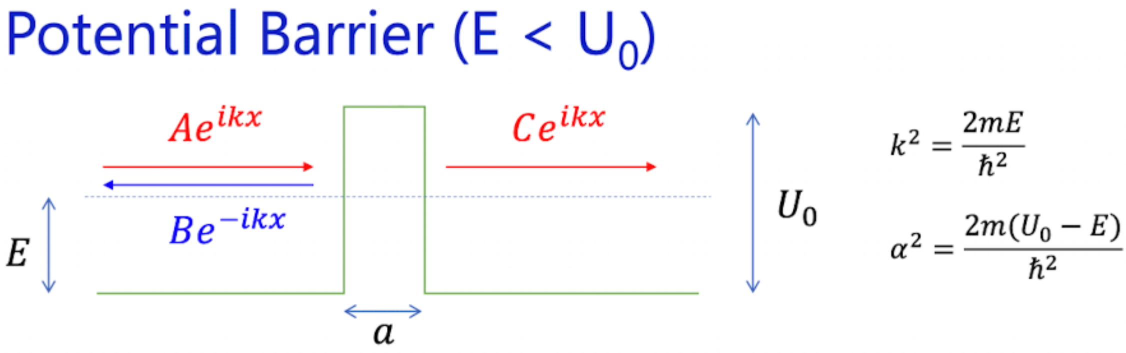

Figure 5:

– III:

$$\psi_3 = Ce^{ikx}$$

- Now we can find the probability of reflection and transmission:
  –
  $$R = \frac{|B|^2}{|A|^2}$$
  –
  $$T = \frac{|C|^2}{|A|^2} \simeq \frac{16E(U_o - E)}{U_o^2} e^{-2\alpha a}$$

Lecture 12

Tunnelling in the general case

- For an arbitrary geometry, we simply integrate the probabilities of small potentials (slivers)
- For a general n sliver, the transmission probability is $T_n \propto e^{-2\alpha_n(\Delta x)n}$, where $\alpha_n^2 = \frac{2m(U_n - E)}{\hbar^2}$, where the potential of the "shape" is a function of x $U(x)$
- WKB approximation:

$$T = \exp\{-2\int_{x_i}^{x_f} \sqrt{\frac{2m(U(x) - E)}{\hbar^2}} dx\}$$

Lecture 13

Relativity: Reference Frames

- a particular perspective from which the universe is observed. Imagine a set of axes from which an observer can measure the position and motion of all points in the system, and a clock
- Special relativity deals with only an **inertial reference frame**
- Any frame moving with constant velocity with respect to an inertial reference frame is also an i.r.f
- u is the velocity of a particle in a reference frame
- v is the velocity of the reference frame itself

Relativity in classical mechanics

- Consider a thought experiment, bicycle that is moving $0.5c$. You have a flashlight that is on, the light moves at c. You are observing the speed of light at a distance away from the bicycle. is the light detected $0.5c + c = 1.5c$? According to Galileo this is the case

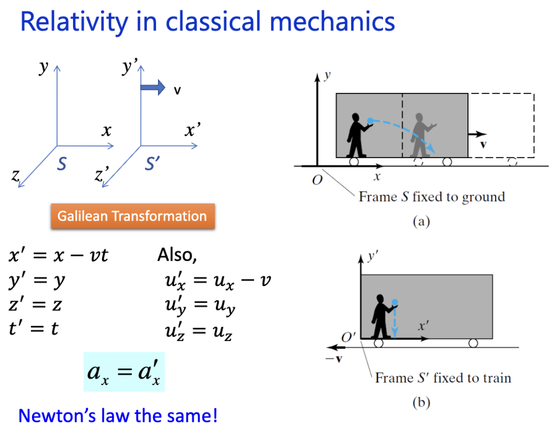

Figure 6:

Electromagnetic Waves (Maxwell)

- $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)$$
- But $c=\frac{1}{\sqrt{\mu_0\epsilon_0}}=3\times 10^8 m/s$
- Moving magnetic field generates electric field and vice versa, so no medium is required for an EM wave to propagate
- Speed of light can be different if reference frame is not moving

Michelson-Morley Experiment:

- Trying to demonstrate Aether (imaginary medium allowing light to propagate in space)
- It was believed that the Aether wind had a direction to it. If this were the case, the observed speed of light should vary with direction
- Intensity did not change with rotation, so the Aether does not exist

Einstein's Postulates:

- **Postulate 1:**
  - The laws of physics are the same in each inertial reference frame, independent of the motion of this reference frame. This does not mean that the measured values of quantities are the same throughout inertial reference frames.
- **Postulate 2:**
  - The speed of light, c, is the same in every inertial reference frame. The speed of light (in vacuum), c is the ultimate speed. Nothing can be faster, so in the bicycle experiment, you will still observe the velocity to be c.

Definitions:

- An event is something that happens, and every event can be assigned three space coordinates and one time coordinate
- The coordinates characterizing an event are called **space time coordinates** (x, y, z, t)
- Among possible events are collisions, flashes, explosions, lasers
- Different observers in different frames will assign different space time coordinates to an event.

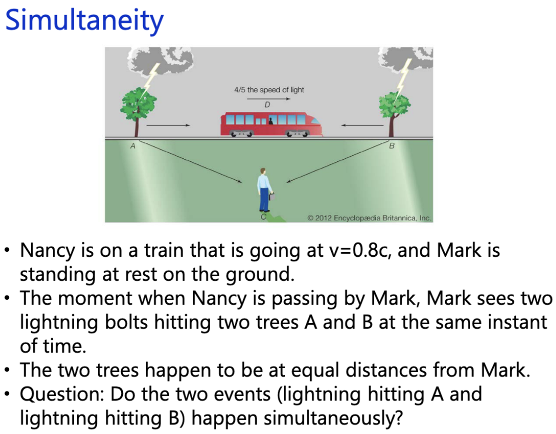

Figure 7:

- According to Nancy, she will see the lightning strike tree B before A, whereas, mark will see lightnight strike the trees at the same time
- Einstein argues that both Mark and Nancy are correct in their own inertial reference frames.

1. The laws of physics remain the same in inertial reference frames although quantities need not.
2. the speed of light is c in every inertail reference frame
3. Simuntaneity is relative

Lecture 14

Relativity: Beta and Gamma

- $$\beta = \frac{v}{c}$$
  - $$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$
- Taylor expansion for $\gamma = a + \frac{\beta^2}{2}$

Light Cones

- Usually light cones are shown with position on the x axis and time on the y axis
- Events at the origin can influence events in the positive light cone.
- Called light cones as we can add the 2nd dimension as a y-axis

World Lines and Space-time

- We draw a world like which is the trajectory of an object in space and time.

1. Vector 1 shows an object that is not moving
2. Vector 2: an object moving in the positive x direction
3. Vector 3: an object moving in the negative x direction
4. Vector 4: shows an object moving with light speed in the +ve x
5. Vector 5: shows an object moving with light speed in the -ve x

(slope of 1 = light speed)

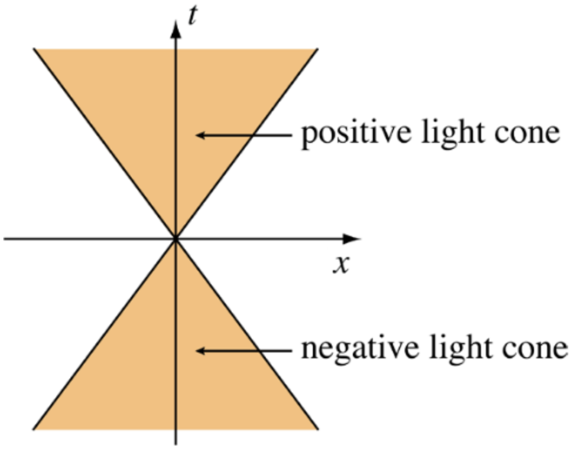

Figure 8:

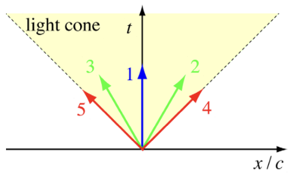

Figure 9:

- Going back to the train example: Nancy on train is going at $v = 0.8c$. Mark is at rest on ground
- Moment Nancy passes Mark, he sees two lighting bolts hitting two trees A and B at the same instant of time.
- We draw light cones for both observers in their respective reference frames, by listing all the events:
  - 1: light strikes A
  - 2: light strikes B
  - 3: A is observed by Mark
  - 4: B is observed by Mark
  - 5: A is observed by Nancy
  - 6: B is observed by Nancy
- Note that in both cases reference frames, Mark observes the strike at the same time, and Nancy observes strike B before A.

Muon Decay

- Muons have a half life of $2.2\mu s$
- Muons are created at the top of the atmosphere (about 60 km above the surface), when cosmic rays collide with air
- Their speed is around $v = 0.999c \rightarrow \gamma = 22$. The time taken for these particles to reach the surface is

$$\Delta t = \frac{60\ km}{c} \approx 2 \times 10^{-4}\ s \approx 100 \times t_{0.5}$$

- So, in the time it takes for a particle to reach the surface, about 100 half lives occur, so given 1 particle, we would expect to see around $(\frac{1}{2})^2 \approx 10^{-30}$ particles.

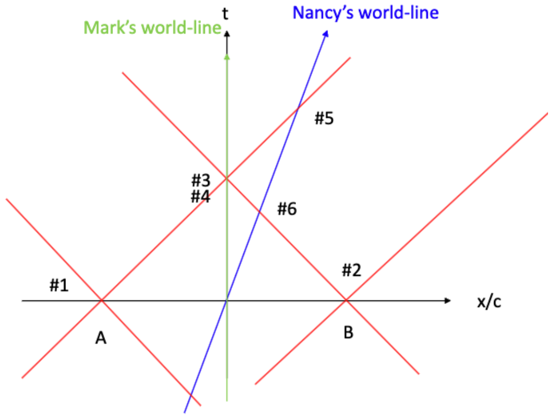

Figure 10: Relative to Mark, the equidistant trees will meet his eyes at the same time. Nancy is moving at $+0.8c$, thus her world-line has a slope of $+0.8$. She sees tree B first, and later, tree A. Both events 1 and 2 occur at the same time.

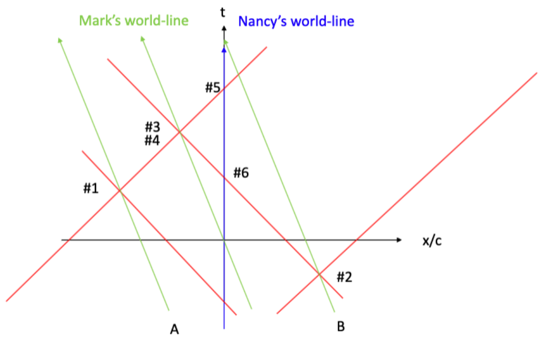

Figure 11: To Nancy, she is not moving. The light travelling from tree B has to travel less distance that that travelled by A. Since the speed of light is the same in every inertial reference frame, the light from B, must reach her faster than A. Thus, the time of event 1 must occur after the time of event 2. Thus, 2 happens when time is negative, because t = 0, corresponds to when Nancy is directly in between A and B, thus from her point of view, the lightning has already struck B. N

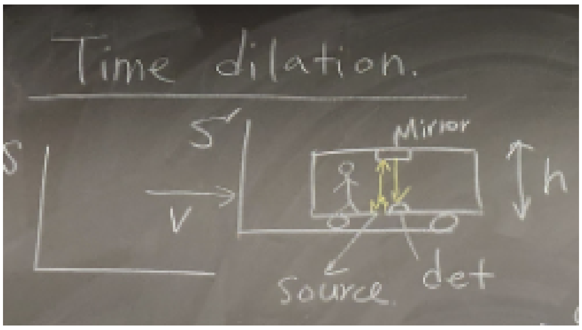

Figure 12:

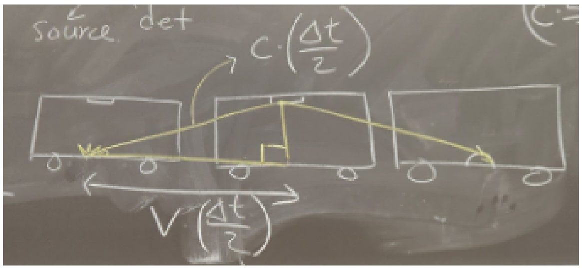

Figure 13:

- However, almost 10 % detected (I don't know the number of initial particles)
- This is due to time dilation. In earth's reference frame: the half life is time dilated

$$t_{\frac{1}{2}}^{earth} = \gamma t_{\frac{1}{2}} \approx 5 \times 10^{-5}$$

- Thus while it takes the same time to arrive to the surface, it only takes around 4 earth half lives.
- This is explained by two observers with their own clocks, one moving near c and the other stationary, while the clocks would appear to be moving normally in their own reference frames, the stationary observer will observe the moving clock to be operating slower and vice versa.
- This is the reason, for the half life of muon's appearance of being slower according to the earth.
-

Time Dilation

- Consider the following thought experiment.
- According to reference frame S', the time taken for a beam of light to reflect back to the detector is $\Delta t' = \frac{2h}{c}$ in frame S' .
- Frame S views S' in motion, thus the beam of light not only has to travel in the y-direction, but also in the x-direction
- If Δt is the time taken for the beam to get shot out and then reflected back, then the distance travelled by the light in the first half is equal to $c\frac{\Delta t}{2}$.
- We also know that the distance travelled in the x-direction when the beam hits the mirror is $v\frac{\Delta t}{2}$
- By applying Pythagorean:

$$(c\frac{\Delta t}{2})^2 = (v\frac{\Delta t}{2})^2 + h^2 \rightarrow \Delta t = \frac{2h}{\sqrt{c^2 - v^2}} = \frac{2h}{c\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\frac{2h}{c} = \gamma(\Delta t')$$

- **$\Delta t'$ is called proper time in this case, or Δt_0 because it measures time where the experiment takes place.**
- proper time is always the shortest period of time. However, in the case of muons, the experiment takes place in the earth reference frame, does that not make it proper? Even though it is the longer period of time. We see that what happens in reality is that far fewer half lives elapse in the earth reference frame (signifying a longer time period), than in the muon's reference frame. Is the muon's time frame proper time even though the number of particles we observe in reality is far greater than expected.
- Muons moving at relativistic speeds measure time experienced by themselves, which is shorter than the time measured by an observer at rest on earth.
- At high speeds, muons are allowed to experience less time during their journey, allowing more of them to reach the earth's surface before decaying.
- Again this is as a result of $\Delta t = \gamma(\Delta t')$, since $\gamma > 1$, $\Delta t > \Delta t'$
  - Recall that the muons have a half-life measured relative to their own clock, and now it can fit in less halftimes.

Lecture 15

Recap

- Time dilation $\Delta t = \gamma \Delta t_o$, where the proper time is measured by the stationary clock relative to the experiment

Length Contraction

- Length contraction is $l = \frac{l_o}{\gamma}$, where l_o is the proper length (measured in the rest frame)
- Take the example of a reference frame that is moving. If there is a car, with a length l' is s', the reference frame that is moving, then $v\Delta t' = l'$. For the stationary observer $v\Delta t = l$, where l is the length of the car relative to the stationary reference frame. Now we can make the following deductions:
$$l' = v\ \Delta t' = v\ \gamma\ \Delta t = \gamma\ l$$
- Thus the length is
$$l = \frac{l'}{\gamma} = \frac{l_o}{\gamma}$$
where $l' = l_o$ is the proper length
  - Time dilation:
$$\Delta t = \gamma \Delta t_o$$
  - Length contraction:
$$l = \frac{l_o}{\gamma}$$
basically, the faster an object is moving relative to a stationary observer, the more it seems as though it's length is contracted.

The Twin Paradox:

- Alice has a twin brother named Bob
- At age 20, Alice flies to a space station that is 3.25 ly away and then returns. The speed is 0.65 c ($\gamma = 1.32$)

Bob says the following:

- The total distance travelled is $6.5/\gamma = 4.92\ ly$
- The journey takes $4.92\ ly/0.65c = 7.57$ years, so she will be 27.6 years old after the trip

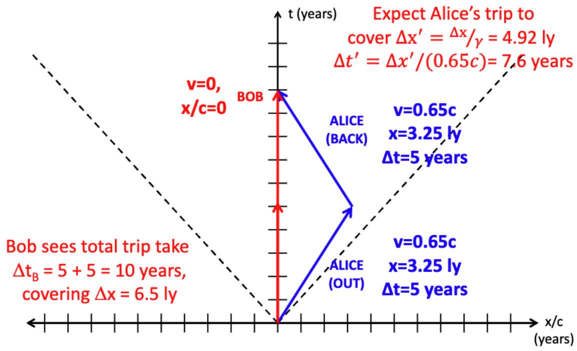

Figure 14: According to Bob's reference frame, he remains stationary with time. Alice is moving at relativistic speeds, so according to Bob, Alice will travel a shorter distance than the actual distance travelled in Alice's reference frame which is 6.5 ly. However, Bob, will perceive this round trip distance as 4.92 ly (through gamma). Thus, the time taken by Alice according to Bob is 7.6 years. According to bob, the time dilation will set him back 10 years.

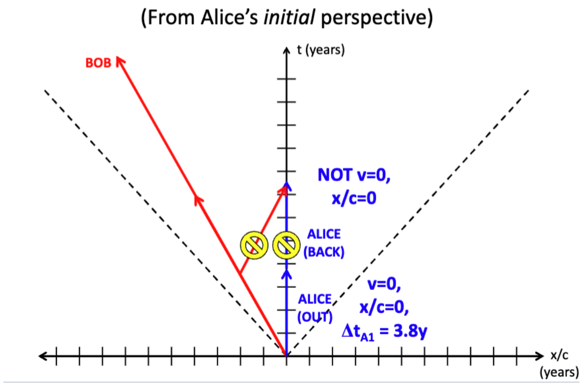

Figure 15: Note that we must consider the fact that Alice's reference frame changes as she returns back from the space station. So she does not keep going along the t-axis of the light cone, nor does Bob meet her along the t-axis.

- Bob's time will be time-dilated to 10 years $\Delta t = \gamma \times \Delta t = 1.32 \times 7.57'$, and he will be 30 years old.
- This number can also be found by considering the non length contracted distance of Alice's journey: $6.5\ ly/0.65c = 10$

Alice says the following:

- Alice's time that should be dilated since Bob is technically moving ($v = -0.65c$).
- So bob's time should be considered as the proper time, so Alice's time in this case is Δt, so $\Delta t' = \Delta t/\gamma = \frac{7.57}{1.32}$

Who is right?

Lecture 15

Doppler Shift

- Recall for classical waves

$$f = \frac{f_O}{1 - \frac{v_{source}}{v_{wave}}}$$

- Relativistic frequency shift needs to take time dilation into account:

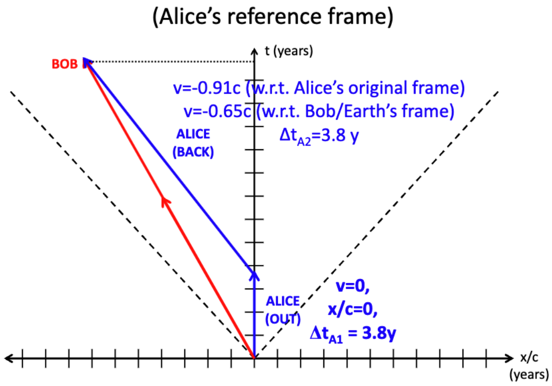

Figure 16: Instead, Bob, remains at the same v = -0.65 c, And as soon, as she turns back, she goes at v = -0.91 c according to her original frame. Thus, in the end, Alice is 27.6 and Bob is 30 in both reference frames. So, the trick was accounting for the change of reference frame.

- Time dilation: time is getting longer, $\gamma > 1$, so proper time must be multiplied: $\gamma \Delta t_o$ (some logic)
- $$f = f_o \sqrt{\frac{c+v}{c-v}}$$
- Formula changes with time dilation
- Considering both moving away and moving towards:
- $$f = f_o \sqrt{\frac{c \pm v}{c \mp v}}$$
- So if a observer and source are moving towards each other, frequency should always increase, so denominator, decreases, Numerator increases. Vice versa

Application: Redshift

- We can measure the absorption spectra from distant stars and they appear at lower frequency, meaning that these starts must be moving away from us (expanding universe)
- All absorption liens are red shifted, (frequency decreases)

Relativistic Momentum

Consider two particles moving in a stationary reference frame S Particles are the same mass with same speed but moving directly at each other Particles collide with each other, and their velocity mov

- Classical definition of momentum does not apply to moving reference frames at relativistic speeds, so we define relativistic momentum:

$$\vec{p} = \gamma_p m \vec{u}$$

- Where $\gamma_p = \frac{1}{\sqrt{1-\frac{u^2}{v^2}}}$
- The γ_p is NOT the same as γ

Lecture 16

Relativistic Energy

- Define work kinetic energy theorem: then using Lorentz transformation to derive relativistic energy
-