Question 1:

a)

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

$$\left.\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta \\ df &= d\vec{l}\cdot\nabla f \end{aligned}\right\} \longrightarrow \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta = d\vec{l}\cdot\vec{\nabla f}$$

$\longrightarrow$ Let $\vec{\nabla f} = a\frac{\partial f}{\partial r}\hat{r} + b\frac{\partial f}{\partial \theta}\hat{\theta}$

$$\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta = (dr\,\hat{r} + r\,d\theta\,\hat{\theta})\cdot\left(a\frac{\partial f}{\partial r}\hat{r} + b\frac{\partial f}{\partial \theta}\hat{\theta}\right)$$

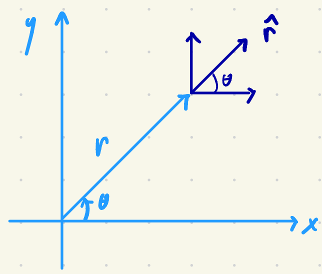
$$= a\frac{\partial f}{\partial r}dr + br\frac{\partial f}{\partial \theta}d\theta$$

by observation, $a = 1$, $b = \frac{1}{r}$

$$\boxed{\therefore\ \vec{\nabla f} = 1\frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}}$$

b) show $\nabla\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$

$\longrightarrow$ first, establish basis for polar (r,θ) i.t.o cartesian (x,y)



$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$

$\hat{\theta} = ?$

$\longrightarrow$ note that $\hat{r}$ and $\hat{\theta}$ must be orthogonal so $\hat{\theta}\cdot\hat{r} = 0$

$\longrightarrow$ let $\hat{\theta} = \hat{x}a + \hat{y}b$, $a,b \neq 0$ $\because \|\hat{\theta}\| = 1$

$$\hat{\theta}\cdot\hat{r} = a\cos\theta + b\sin\theta$$
$$0 = a\cos\theta + b\sin\theta$$

two options: $a_1 = -\sin\theta$, $b_1 = \cos\theta$

or

$a_2 = \sin\theta$, $b_2 = -\cos\theta$

$\because$ we define $\theta > 0$ in ccw direction, we choose

$a = a_1$, $b = b_1$

$$\boxed{\hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta} \longrightarrow \frac{d}{d\theta}\hat{r} = -\sin\theta\hat{x} + \cos\theta\hat{y} = \hat{\theta}$$

$\longrightarrow \quad \vec{F}(r,\theta) = F_r\,\hat{r} + F_\theta\,\hat{\theta}$

$\longrightarrow$ gradient <u>OPERATOR</u> $\quad \nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right) \cdot \left(F_r\,\hat{r} + F_\theta\,\hat{\theta}\right)$$

$$= \hat{r} \cdot \frac{\partial}{\partial r}\left(F_r\,\hat{r}\right) + \hat{r} \cdot \frac{\partial}{\partial r}\left(F_\theta\,\hat{\theta}\right) + \frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}\left(F_r\hat{r}\right) + \frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}\left(F_\theta\,\hat{\theta}\right)$$

$$= \frac{\partial}{\partial r}F_r + \underbrace{\cancel{\hat{r} \cdot \frac{\partial}{\partial r}(F_\theta)\hat{\theta}}}_{0} + \underbrace{\cancel{\hat{r} \cdot \frac{\partial}{\partial r}(\hat{\theta})F_\theta}}_{0} + \underbrace{\cancel{\frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}(F_r)\hat{r}}}_{0} + \frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}(\hat{r})F_r + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta$$

$$= \frac{\partial}{\partial r}F_r + \frac{1}{r}\hat{\theta} \cdot \hat{\theta}\,F_r + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta$$

$$= \underbrace{\frac{\partial}{\partial r}F_r + \frac{1}{r}F_r}_{\downarrow} + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta$$

$$\boxed{= \frac{1}{r}\frac{\partial}{\partial r}(r\,F_r) + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta} \quad \blacksquare$$

Question 2

a) Unit Circle CCW.

parametrize in t

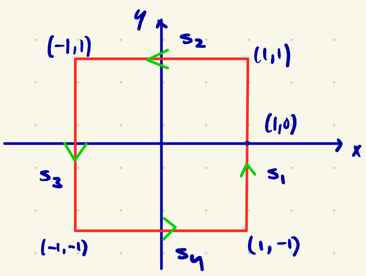
$$\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \xrightarrow{\frac{d}{dt}} \vec{r}\,'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \qquad 0 \le t < 2\pi$$

$$\int_\gamma F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi} \left(\frac{\cos t - \sin t}{2(\cancel{\cos^2 t + \sin^2 t})} \hat{x} + \frac{\cos t + \sin t}{2(\cancel{\cos^2 t + \sin^2 t})} \hat{y} \right) \cdot \left(-\sin t\, \hat{x} + \cos t\, \hat{y} \right) dt$$

$$= \int_0^{2\pi} \frac{-\sin t \cos t + \sin^2 t}{2} + \frac{\cos^2 t + \sin t \cos t}{2}\, dt$$

$$\boxed{= \frac{1}{2} \int_0^{2\pi} 1\, dt = \pi}$$

b) parametrize $s_1 - s_4$: all defined on $-1 \le t \le 1$



s_1 $\quad r_1(t) = \langle 1, t \rangle \qquad r_1'(t) = \langle 0, 1 \rangle$

s_2 $\quad r_2(t) = \langle -t, 1 \rangle \qquad r_2'(t) = \langle -1, 0 \rangle$

s_3 $\quad r_3(t) = \langle -1, -t \rangle \qquad r_3'(t) = \langle 0, -1 \rangle$

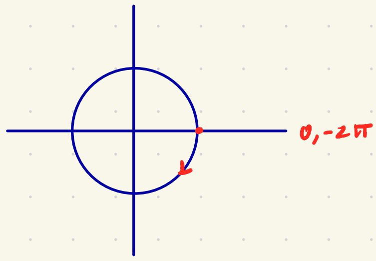
s_4 $\quad r_4(t) = \langle t, -1 \rangle \qquad r_4'(t) = \langle 1, 0 \rangle$

$$\oint \vec{F} \cdot d\vec{l} = \int_{s_1} F(r_1(t)) \cdot r_1'(t)\, dt + \int_{s_2} F(r_2(t)) \cdot r_2'(t)\, dt + \int_{s_3} F(r_3(t)) \cdot r_3'(t)\, dt + \int_{s_4} F(r_4(t)) \cdot r_4'(t)\, dt$$

$$= \int_{-1}^{1} \left(\frac{1+t}{2+2t^2} \right) - \left(\frac{-t-1}{2t^2+2} \right) - \left(\frac{-1-t}{2+2t^2} \right) + \left(\frac{t+1}{2t^2+2} \right) dt$$

$$\boxed{= \int_{-1}^{1} 4 \left(\frac{1+t}{2+2t^2} \right) dt = \pi}$$

c) unit circle CW



$$\vec{r}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \xRightarrow{\frac{d}{dt}} \vec{r}\,'(t) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} \quad -2\pi < t \le 0$$

$$= \int_0^{-2\pi} \left(\frac{\cos t + \sin t}{2} \hat{x} + \frac{\cos t - \sin t}{2} \hat{y} \right) \cdot \left(-\sin t \, \hat{x} - \cos t \, \hat{y} \right) dt$$

$$= \int_0^{-2\pi} \frac{-\sin t \cos t - \sin^2 t \quad - \cos^2 t + \sin t \cos t}{2} \, dt$$

$$\boxed{= \int_0^{2\pi} -\frac{1}{2} \, dt = -\pi}$$

d) $\vec{F} = \nabla \phi$

$$\frac{\partial \phi}{\partial x} = \frac{x - y}{2x^2 + 2y^2} \hat{x} \qquad \frac{\partial \phi}{\partial y} = \frac{x + y}{2x^2 + 2y^2} \hat{y}$$

$$\int \frac{\partial \phi}{\partial x} dx = \phi + g(y)$$ $\Rightarrow$ not a nice integral.

$\longrightarrow$ convert $\vec{F}$ to polar

$\longrightarrow$ $x = r\cos\theta$ $\quad \hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$

$y = r\sin\theta$ $\quad \hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta$

$\Big\} \longrightarrow$ express $\underline{\hat{x}, \hat{y} \text{ i.t.o } \hat{r}, \hat{\theta}}$

$$\underbrace{\hat{y} = \frac{\hat{r} - \hat{x}\cos\theta}{\sin\theta} \quad \text{and} \quad \hat{y} = \frac{\hat{\theta} + \hat{x}\sin\theta}{\cos\theta}}$$

$$\hookrightarrow \frac{\hat{r} - \hat{x}\cos\theta}{\sin\theta} = \frac{\hat{\theta} + \hat{x}\sin\theta}{\cos\theta}$$

$$\hat{r}\cos\theta - \hat{x}\cos^2\theta = \hat{\theta}\sin\theta + \hat{x}\sin^2\theta$$

$$\hat{x} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\hat{y} = \frac{\hat{\theta} + (\hat{r}\cos\theta - \hat{\theta}\sin\theta)(\sin\theta)}{\cos\theta}$$

$$\hat{y} = \frac{\hat{\theta}(1 - \sin^2\theta)}{\cos\theta} + \hat{r}\sin\theta$$

$$\hat{y} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

$\longrightarrow$ now express F_x, F_y i.t.o polar coords

$$\vec{F}(r,\theta) = \frac{\cos\theta - \sin\theta}{2}(\hat{r}\cos\theta - \hat{\theta}\sin\theta) + \frac{\cos\theta + \sin\theta}{2}(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$$

$$F_r = \frac{1}{2r}\left(\cos^2\theta - \cancel{\sin\theta\cos\theta} + \cancel{\cos\theta\sin\theta} + \sin^2\theta\right)$$
$$= \frac{1}{2r}$$

$$F_\theta = \frac{1}{2r}\left(-\cancel{\cos\theta\sin\theta} + \sin^2\theta + \cos^2\theta + \cancel{\cos\theta\sin\theta}\right)$$
$$= \frac{1}{2r}$$

$$\vec{F}(r,\theta) = \frac{1}{2r}\hat{r} + \frac{1}{2r}\hat{\theta}$$

$\longrightarrow$ recall that $$F = \nabla\phi(r,\theta) = \underbrace{\frac{\partial\phi}{\partial r}}\hat{r} + \underbrace{\frac{1}{r}\frac{\partial\phi}{\partial\theta}}\hat{\theta}$$
$$= \frac{1}{2r}\hat{r} + \frac{1}{2r}\hat{\theta}$$

by observation $\frac{\partial\phi}{\partial r} = \frac{1}{2r}$, $\frac{\partial\phi}{\partial\theta} = \frac{1}{2}$

$$\phi = \int\frac{\partial\phi}{\partial r}\,dr = \int\frac{1}{2r}\,dr = \frac{1}{2}\ln|r| + g(\theta) = \frac{1}{2}\ln(r) + g(\theta) \quad (r \geq 0)$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{2}\ln(r) + g(\theta)\right) = \frac{1}{2}$$

$$g'(\theta) = \frac{1}{2} \Rightarrow \quad g(\theta) = \frac{1}{2}\theta + 0$$

$$\phi = \frac{1}{2}\ln(r) + \frac{1}{2}\theta$$

c) $$(\vec{\nabla}\times\vec{F}) = \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{z}$$

$$\vec{F} = \frac{1}{2r}\hat{r} + \frac{1}{2r}\hat{\theta}$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{1}{2}\right) - 0\right)\hat{z}$$

$$\boxed{= 0\,\hat{z}}$$

f)

THM: if F is C(1) on a simply connected region omega which is a subset of Rn, and the partials commute then F is conservative.

We are given that (x,y) are elements in R2 \ {(0,0)}. This hole at the origin is a discontinuity and therefore, the region omega is NOT simply connected: in questions a, b and c we travel around the hole at the origin.

Thus F cannot be a conservative force.

Moreover, from questions an and c, we have path dependence on two different closed loops, which equate to +- pi.

Question 3

a) from spherical coordinates:

$$\left.\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}\right\} f(r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi,\ \omega) = g(r,\theta,\phi,\omega)$$

$$S(g,P) = \sum_i \sum_j \sum_k \sum_l g(r_{ijkl}, \theta_{ijkl}, \phi_{ijkl}, \omega_{ijkl})\,\Delta A$$

$\longrightarrow$ where ΔA is the elemental 4D "volume" of the spherinders we integrate over

$$\int_\Omega g(r,\theta,\phi,\omega)\,d\Omega = \lim_{\|P\|\to 0} \sum g(r^*,\theta^*,\phi^*,\omega^*)\underbrace{r^2\sin\phi\ \Delta r\,\Delta\theta\,\Delta\phi\,\Delta\omega}_{\Delta A \to d\Omega}$$

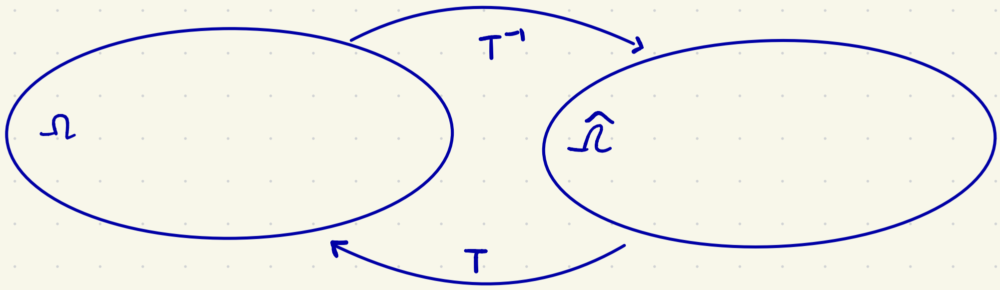
$\longrightarrow$ where $\|P\|$ is the norm of the partition (the longest 4D "length" of the spherinders, that we integrate over)

b) Fubini's Theorem depends on the specificity of the function g. If each intermediate iterated integral is Riemann integrable, then Fubini's theorem applies. Riemann integrability means that each iterated integral converges to a finite value as the norm of the partition approaches 0.

c) $$\int_\Omega f(\vec{x})\,d\Omega = \int_{\hat{\Omega} = T^{-1}(\Omega)} f(T(u))\,\left|\det(D_u T)\right|\,d\hat{\Omega}$$

$$T^{-1}(\vec{x}) = \vec{u}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix}$$



$$\vec{u} = \begin{bmatrix} r \\ \theta \\ \phi \\ \omega \end{bmatrix}$$

$$\left|\frac{\partial(x,y,z,\omega)}{\partial(r,\theta,\phi,\omega)}\right| = \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \omega} \\ \frac{\partial \omega}{\partial r} & \frac{\partial \omega}{\partial \theta} & \frac{\partial \omega}{\partial \phi} & \frac{\partial \omega}{\partial \omega} \end{vmatrix}$$

$x = r\cos\theta \sin\phi$
$y = r \sin\theta \sin\phi$
$z = r\cos\phi$
$\omega = \omega$

$$= \det \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi & 0 \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ \cos\phi & 0 & -r\sin\phi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

expand across bottom now:

$$= \left| 0\cdot(-1)^5 + 0\cdot(-1)^6 + 0\cdot(-1)^7 + 1\cdot(-1)^8 \det \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \right|$$

$$= \left| (-1)^{3+1}\cos\phi\left(-r^2\sin^2\theta\cos\phi\sin\phi - r^2\cos^2\theta\sin\phi\cos\phi\right) + 0 - r\sin\phi\left(r\cos^2\theta\sin^2\phi + r\sin^2\theta\sin^2\phi\right) \right|$$

$$= \left| -r^2\cos^2\phi\sin\phi\left(\cancelto{1}{\sin^2\theta + \cos^2\theta}\right) - r^2\sin^3\phi\left(\cancelto{1}{\cos^2\theta + \sin^2\theta}\right) \right|$$

$$= \left| -r^2\sin\phi\left(\cancelto{1}{\cos^2\phi + \sin^2\phi}\right) \right|$$

$$= \left| -r^2\sin\phi \right| = r^2\sin\phi \quad \because\ 0 \le \phi \le \pi,\ r \ge 0$$

$$T = \begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi & 0 \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ \cos\phi & 0 & -r\sin\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Omega : \{x, y, z, w \in \mathbb{R}\}$

$\tilde{\Omega} : \{r, \theta, \phi, \omega \mid r \ge 0,\ 0 \le \theta < 2\pi,\ 0 \le \phi < \pi,\ \omega \in \mathbb{R}\}$ $\quad \boxed{d\tilde{\Omega} = r^2\sin\phi \, dr\, d\theta\, d\phi\, d\omega}$

The transformation T is a diffeomorphism because of the way we have defined our variables. Note r, theta and phi are bounded, which ensures that every point in omega has a one to one mapping. As an example, if theta was left unbounded on omega tilde, then the points (0, 0, 0, 0) and (0, 2pi, 0, 0) would map to the same point in omega, (x=0,y=0,z=0,w=0).

Moreover, all the transformations that were made in relating variables to each other, are differentiable and continuous on the domains we have specified.

We also have that the transformation T is invertible since det(T) != 0. Thus our transformation can be considered a diffeomorphism.

d) Radius R, height h

$$\int_{\hat{\Omega}} g(r,\theta,\phi,\omega)\, d\hat{\Omega} = \int_{\hat{\Omega}} g(r,\theta,\phi,\omega)\, r^2 \sin\phi \, dr\, d\theta\, d\phi\, d\omega .$$

let $g(r,\theta,\phi,\omega) = 1$ $\because$ 1 is a Riemann integrable function for all intermediate iterated integrals, the Fubini applies: $d\hat{\Omega} = r^2 \sin\phi \, dr\, d\theta\, d\phi\, d\omega$

$$\int_{\hat{\Omega}} g(r,\theta,\phi,\omega)\, d\hat{\Omega} = \int_{\hat{\Omega}} 1 \, d\hat{\Omega} = \int_0^h \int_0^{\pi} \int_0^{2\pi} \int_0^R 1\, r^2 \sin\phi \, dr\, d\theta\, d\phi\, d\omega$$

$$= \int_0^h \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \cdot \int_0^R r^2 \, dr \quad d\phi\, d\omega$$

$$= \int_0^h \int_0^{2\pi} [2] \cdot \frac{1}{3} R^3 \, d\phi\, d\omega$$

$$= 2 \cdot \frac{1}{3} R^3 \int_0^h \int_0^{2\pi} 1 \quad d\phi\, d\omega$$

$$= 2 \cdot \frac{1}{3} R^3 \cdot h \cdot 2\pi$$

$$\boxed{= \frac{4}{3}\pi R^3 h}$$

Question 4

a) $$T_3(x,y) = f(x_0,y_0) + h_x \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + h_y \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} + \frac{1}{2!}\left[h_x^2 \frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + 2h_x h_y \frac{\partial^2 f}{\partial x \partial y}\bigg|_{(x_0,y_0)} + h_y^2 \frac{\partial^2 f}{\partial y^2}\bigg|_{(x_0,y_0)}\right]$$
$$+ \frac{1}{3!}\left[h_x^3 \frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + 3h_x^2 h_y \frac{\partial^3 f}{\partial x^2 \partial y}\bigg|_{(x_0,y_0)} + 3h_x h_y^2 \frac{\partial^3 f}{\partial x \partial y^2}\bigg|_{(x_0,y_0)} + h_y^3 \frac{\partial^3 f}{\partial y^3}\bigg|_{(x_0,y_0)}\right]$$

b)

$$\frac{f_1 - f_0}{h} = \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$
$$= \frac{1}{h}\left[\cancel{f(x_0,y_0)} + h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + \frac{1}{2}h^2\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + \frac{1}{6}h^3\frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + \dots\right] - \cancel{\frac{f(x_0,y_0)}{h}}$$
$$= \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + \frac{h}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + \frac{h^2}{6}\frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + \dots$$

$\blacksquare$

$$\frac{f_1 - f_{-1}}{2h} = \frac{f(x_0+h, y_0) - f(x_0-h, y_0)}{2h}$$
$$= \frac{1}{2h}\left[f(x_0,y_0) + h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + \frac{1}{2}h^2\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} + \frac{1}{6}h^3\frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + \dots\right]$$
$$- \frac{1}{2h}\left[f(x_0,y_0) - h\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + \frac{1}{2}h^2\frac{\partial^2 f}{\partial x^2}\bigg|_{(x_0,y_0)} - \frac{1}{6}h^3\frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + \dots\right]$$
$$= \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} + \frac{1}{6}h^2\frac{\partial^3 f}{\partial x^3}\bigg|_{(x_0,y_0)} + \dots$$

$\blacksquare$

c)

a) take h small enough
In theory, taking h -> 0 would work, and the finite difference approaches zero:

$$\lim_{h \to 0} \frac{f_n - f_{-n}}{nh} \longrightarrow \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)} + \underbrace{\cancel{\frac{h^n}{(n+1)!} \frac{\partial^{n+1}}{\partial x^{n+1}}}}_{\to 0}$$

However, in applications to the real world, we might not be able to achieve these infinitesimally small distances because the initial conditions we “observe” are measured to a finite degree of accuracy. Consider the camera used in the Flow Viz lab, which had a finite resolution. Thus, by measuring smaller and smaller h, f_n and f_(-n) reach a certain level of accuracy which is bounded by the current ability of the measurement equipment.
Moreover, we must also consider the computational cost of calculating these finite differences, while we could, for example, increase the mesh refinement, which would allow CFD to solve for increasingly small areas/volumes in space, but in doing so, we add a large computational cost. Thus, we cannot simply “take h small enough” to approximate the derivatives of f “as well as we want”.

b) take enough sample points
As stated in the question, “we could keep going using more point evaluations” to get better approximations. However, it’s not just about the quantity of sampled points, rather we need to take sample point that are as close as possible to (xo, yo). Once more, this goes back to how accurately we can measure the points that surround the initial conditions.

In all, the statements assume that the data we collect is continuous, which isn’t the case. Nonetheless, by taking more and more sample points around (xo, yo), at smaller and smaller DISTINGUISHABLE distances, we can in fact increase the accuracy of the partial derivatives.

d) $\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = \frac{(f_{1,0}-f_{-1,0})(f_{0,1}-f_{0,-1})}{4f_{0,0}\, h_x h_y} + er$

(remove " $\Big|_{(x_0,y_0)}$ " for ease of reading)

RHS:

$$= \frac{(f_{1,0}-f_{-1,0})(f_{0,1}-f_{0,-1})}{4f_{0,0}\, h_x h_y} + er$$

$$= \frac{(f_{1,0}-f_{-1,0})}{2h_x} \cdot \frac{(f_{0,1}-f_{0,-1})}{2h_y} \cdot \frac{1}{f_{0,0}} + er$$

$$= \left(\frac{\partial f}{\partial x} + \frac{h_x^2}{6}\frac{\partial^3 f}{\partial x^3} + \dots\right)\left(\frac{\partial f}{\partial y} + \frac{h_y^2}{6}\frac{\partial^3 f}{\partial y^3} + \dots\right) \cdot \frac{1}{f_{0,0}} + er \quad \text{(from 4b)}$$

$$= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{1}{f_{0,0}} + \underbrace{\frac{1}{f_{0,0} \cdot 6}\left(\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial y^3}h_y^2 + \frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial x^3}h_x^2\right) + \dots}_{er \text{ (anything with } h_x, h_y \text{ term)}}$$

$\therefore$ we must show

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot \left.\frac{1}{f_{0,0}}\right|_{(x_0,y_0)}$$

note, we assume $er \to 0$, as $\|h\| \to 0$, so we omit it from this equality.

$\longrightarrow$ before proving the above, lets look @ $f(x,y) = \phi(x)\,\psi(y)$ (2nd equality)

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = \frac{(\phi_{1,0}-\phi_{-1,0})(\psi_{0,1}-\psi_{0,-1})}{4\, h_x h_y} + \underbrace{\cancel{er}}_{0}$$

RHS:

$$\frac{(\phi_{1,0}-\phi_{-1,0})(\psi_{0,1}-\psi_{0,-1})}{4\, h_x h_y} + er$$

$$= \frac{(\phi_{1,0}-\phi_{-1,0})}{2h_x} \cdot \frac{(\psi_{0,1}-\psi_{0,-1})}{2h_y} + er$$

$$= \left(\frac{\partial \phi}{\partial x} + \frac{h_x^2}{6}\frac{\partial^3 \phi}{\partial x^3} + \dots\right)\left(\frac{\partial \psi}{\partial y} + \frac{h_y^2}{6}\frac{\partial^3 \psi}{\partial y^3} + \dots\right) + er$$

$$= \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} + \underbrace{\frac{1}{6}\left(\frac{\partial \phi}{\partial x}\frac{\partial^3 \psi}{\partial y^3}h_y^2 + \frac{\partial \psi}{\partial y}\frac{\partial^3 \phi}{\partial x^3}h_x^2\right) + \dots}_{er}$$

$$\boxed{= \frac{d\phi}{dx}\Big|_{(x_0,y_0)} \cdot \frac{d\psi}{dy}\Big|_{(x_0,y_0)} + \cancel{\epsilon_h}}$$

note, we assume $\epsilon_h \to 0$, as $\|h\| \to 0$, so we omit it from this equality.

going to LHS: $\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x_0,y_0)} \longrightarrow$ plug in $f = \phi(x)\psi(y)$

$$\frac{\partial^2}{\partial x \partial y}\left[\phi(x)\psi(y)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\phi(x)\psi(y)\right)\right]$$

$$= \frac{\partial}{\partial x}\phi(x) \cdot \frac{\partial}{\partial y}\psi(y)$$

$$\boxed{= \frac{d}{dx}\phi(x)\Big|_{(x_0,y_0)} \cdot \frac{d}{dy}\psi(y)\Big|_{(x_0,y_0)}}$$

LHS = RHS ■

now back to the first equality:

$$\frac{\partial^2 f}{\partial x \partial y}_{(x_0,y_0)} = \frac{\partial f}{\partial x}_{(x_0,y_0)} \frac{\partial f}{\partial y}_{(x_0,y_0)} \cdot \frac{1}{f_{0,0}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial \phi(x)}{\partial x}\psi(y) \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial \psi(y)}{\partial y}\phi(x)$$

$$\frac{\partial f}{\partial x} = \frac{d\phi}{dx}\psi(y) \qquad \frac{\partial f}{\partial y} = \frac{d\psi}{dy}\phi(x)$$

plug

plug

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}\Big|_{x_0,y_0} = \frac{d\phi}{dx}\Big|_{(x_0,y_0)} \cdot \frac{d\psi}{dy}\Big|_{(x_0,y_0)}}$$

note. this equality was proven above

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{x_0,y_0} = \frac{\partial f}{\partial x}\frac{1}{\psi(y)}\Big|_{(x_0,y_0)} \cdot \frac{\partial f}{\partial y}\frac{1}{\phi(x)}\Big|_{(x_0,y_0)}$$

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{x_0,y_0} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \cdot \frac{1}{f_{0,0}}$$

which proves the equality ■

$f(x,y)$:
$$\underbrace{f_{0,0}\frac{1}{6}\left(\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial y^3}h_y^2 + \frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial x^3}h_x^2\right)+\dots}_{cr}$$

$\phi(x)\,\psi(y)$:
$$\underbrace{\frac{1}{6}\left(\frac{\partial \phi}{\partial x}\frac{\partial^3 \psi}{\partial y^3}h_y^3 + \frac{\partial \psi}{\partial y}\frac{\partial^3 \phi}{\partial x^3}h_x^3\right)+\dots}_{cr}$$

The error terms are the same, just presented in a different form.