

# Thermal Diffusivity

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## 1 Abstract

This report covers the Thermal Diffusivity experiment using a rubber cylinder thermometer setup placed in alternating hot and cold baths. In this experiment, the thermal diffusivity was calculated as  $m = 6.141e - 7 \pm 4.415 * 10^{-7} \frac{\text{m}^2}{\text{s}}$ , which compared to typical values on the order of magnitude of  $10^{-7} \text{m}^2/\text{s}$  for insulators, falls within that order of magnitude [3].

## 2 Introduction

Thermal diffusivity, denoted  $m$  (or  $\alpha$ ), is a material property dependent on its thermal conductivity  $k$ , density  $\rho$ , and specific heat capacity  $c_p$ . It is a measure of how quickly heat can diffuse through a material [1]. If a material has a high thermal diffusivity,  $m$ , it will transfer heat quickly. Rubber, the material used in this experiment, generally has a low thermal diffusivity, so it tends to transfer heat more slowly. Due to its low thermal diffusivity, rubber is known as a good insulator. It is widely used in industrial applications for seals, gaskets, shock absorbers and electrical insulation around copper wires [2]. We will wrap a cylindrical piece of rubber around a thermometer and measure the temperature change over time as the rubber is heated and cooled multiple times.

### 2.1 Mathematical Background

The temperature of a material as it is subject to heat transfer can be described in time and space by the equation of thermal diffusion:

$$\frac{\partial T}{\partial t} = m \nabla^2 T \quad (1)$$

The first term,  $\frac{\partial T}{\partial t}$ , is the rate of change of temperature with time, which represents how quickly the temperature is changing.  $m$  is called the thermal diffusivity and is a property of the material. From the (1), we can see that the rate of temperature change is proportional to  $m$ . The final term,  $\nabla^2 T$ , is called the Laplacian of the temperature—it is a measure of how the temperature changes with distance. We can find the Laplacian in cylindrical coordinates:

$$\nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Where  $r$  and  $\phi$  are the polar coordinates and  $z$  is the projection onto the  $z$  axis. However, since the rubber piece and thermometer are assumed to be cylinders, we can assume that the temperature only changes in the radial direction (i.e. it only gets hotter or colder as we move in the direction of the cylinder's radius). Thus, the partials w.r.t  $\phi$  and  $z$  are effectively 0. As a result, we can simplify the Laplacian:

$$\nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \quad (2)$$

Subbing in our new expression for the Laplacian into (1):

$$\frac{\partial T}{\partial t} = m \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)$$

These simplifications allows the use of the method of separation of variables, where we write the temperature as a function of the radial distance and time:  $T(r, t) = R(r) \exp(i\omega t)$ ,  $\omega$  is the angular frequency of the temperature oscillations. Substituting our "guess" solution into the PDE, we get:

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \frac{i\omega}{m} R = \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \lambda^2 R = 0$$

where  $\lambda^2 = i\omega/m$ . The differential equation is called Bessel's Equation of order zero. A solution to this equation is called  $J_0$ , a Bessel function of order zero. Now if we set  $z = \lambda r$  the Bessel Equation becomes:

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \dots$$

However, by solving for  $\phi$ , the phase difference of the inner and outer radii, we can solve for the thermal diffusivity. The phase of the oscillating temperature can be measured by  $\Delta t$ , which is the time delay it takes for the inner temperature to reach its maximum/minimum after the outer temperature has reached its maximum/minimum. Thus, the Bessel function can be used to find a function for the phase difference  $\phi$  by taking the arc tangent of the ratio of two Bessel functions:

$$\tan(\phi(x)) = \frac{\frac{x^2}{2} - \frac{x^6}{2^2 4^2 6^2}}{1 - \frac{x^4}{2^2 4^2}}$$

Where  $x^2 = \frac{r^2 \omega}{m}$ . Note that the only free variable in  $x$  is  $m$ , we know both  $r$  and  $\omega$ , which are the radii and angular frequency respectively. Thus, we can rewrite the phase difference as a function of  $r$ ,  $\omega$ ,  $\phi$ :

$$\tan(\phi(x)) = \tan(\phi(r, \omega, m))$$

In taking the arc tangent of the ratio, we have an expression for the phase difference:

$$\phi(r, \omega, m) = \arctan \left( \frac{\frac{x^2}{2} - \frac{x^6}{2^2 4^2 6^2}}{1 - \frac{x^4}{2^2 4^2}} \right)$$

Finally, applying the fact that the angular frequency is equal to  $\omega = 2\pi/T$ , we can express the time delay  $\Delta t$  relationship as

$$\Delta t \omega = \Delta \phi = \phi_1(r_1, \omega, m) - \phi_2(r_2, \omega, m) \quad (3)$$

where  $\phi_2$  and  $\phi_1$  are the phase differences for the two radii, and  $r_2$  and  $r_1$  are the outer and inner radii of the rubber cylinder. Note that WLOG,  $r_2$  and  $r_1$  do not have an order as the calculated difference would result in a simple sign flip, which is mitigated by the even nature of the function with respect to  $m$ . In solving for equation 3, we can thus solve for our free variable,  $m$ .

### 3 Materials:

- |                                                                                    |                |
|------------------------------------------------------------------------------------|----------------|
| • 2 Thermometers                                                                   | L capacity     |
| • Rubber tube with measured $r_1 = 3.14 \text{ mm}$<br>and $r_2 = 7.92 \text{ mm}$ | • Hot plate    |
| • Calipers                                                                         | • Scissor lift |
| • 2 cylindrical, glass containers with around 1                                    | • Ice          |
|                                                                                    | • Water        |

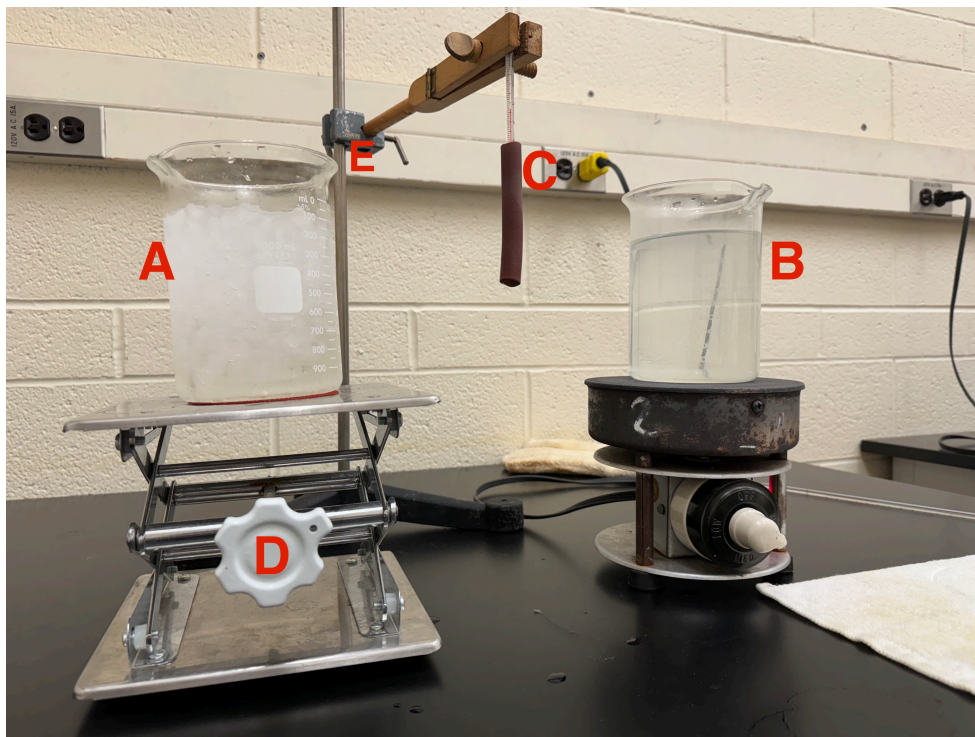


Figure 1: Experimental Setup. A) cold ice bath with ice chips. B) hot bath on the hotplate. C) thermometer with rubber cylinder covering the end. D) scissor stand. E) retort stand. Notice how A and B are at the same height and contain around the same volume. Instead of rotating the clamp, we found it easier to rotate the entire stand E by picking it up from its base—this limited the amount of time, C was exposed to the air.

- Clamp and retort stand
- Stopwatch

## 4 Procedure

**Setup:** We filled up one of the glass containers with water to around  $3/4$  of its capacity, placed it on the hot plate and set it to the "high" setting. It took around 10 minutes for the water to boil, so we measured the inner and outer radii of the rubber tube using callipers in that time. We then filled up the second container with  $2/3$  of its capacity with ice; this may seem like a lot; however, a lot of the ice will melt during the experiment. We added the water to the ice and stirred it, ensuring it floated. We adjusted the height of the ice water beaker using the knob on the scissor to match the height of the boiling water, as shown in Figure 1. We adjusted the clamp on the retort stand to hold the thermometer in place and ensured that it was submerged in the water, but did not touch the glass. Using the second thermometer, we measured the temperatures of the hot and cold water; once the temperatures were around  $95^{\circ}\text{C}$  and  $0^{\circ}\text{C}$ , we started our trials.

**Trials:** We took 6 trials with varying periods  $T = 120, 180, 240, 300, 225, 300$  seconds. For each distinct trial, we exchanged between the hot and cold baths five times. For  $T = 60, 90, 120, 150$  seconds, we took temperature readings every 5 seconds, whereas for  $T = 225, 300$ , we took readings every 10 seconds. To log the readings, one person read out the value of the thermometer while the other logged the data into a spreadsheet. Every  $T$  seconds, we would transfer the rubber piece into the opposite bath; this transfer was done by lifting the entire retort stand instead of adjusting the clamp. Lifting the entire stand minimized the time to transfer the rubber from one bath to the other.

**Note on safety:** With time, the hot bath will lose a significant amount of water due to evaporation and/or boiling (if temperature reaches  $100^{\circ}\text{C}$ ). To refill the hot bath, transport it with two gloves to the sink and fill it with tap water.

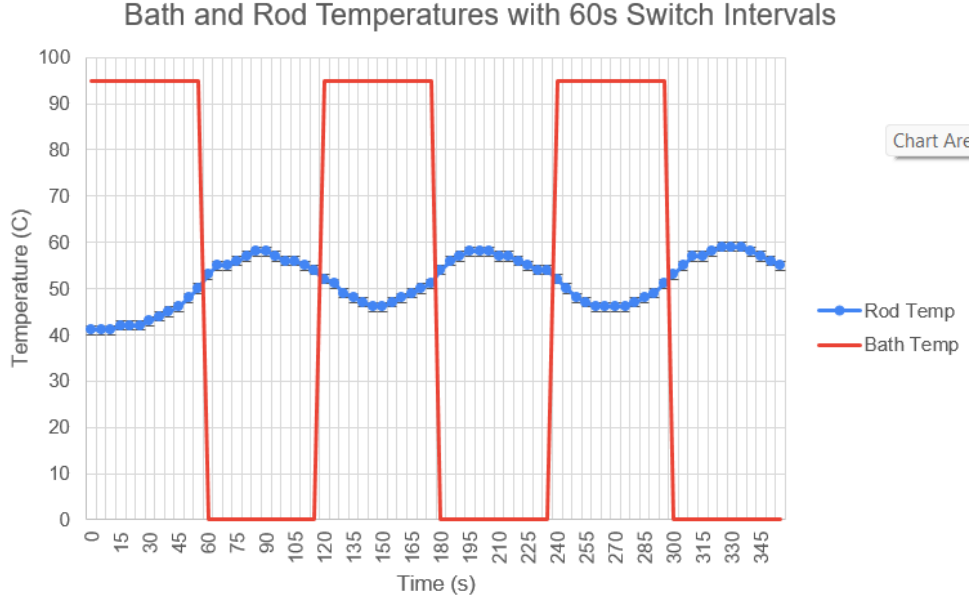


Figure 2: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 60s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 5s, and may be too small to be seen.

## 5 Data Processing and Analysis

### 5.1 Uncertainty Analysis

The uncertainty of any time measurements was taken to be the interval in which we took measurements in, either 5 or 10s ( $\pm 5 - 10$  s). The uncertainties of the measured radii were ( $\pm 0.000005$  m), half of the caliper's smallest unit. Any uncertainty propagation formulae used are consistent with the provided document:

$$\Delta f = \sqrt{\sum \left( \frac{\partial f}{\partial x_i} \Delta x_i \right)^2} \quad (4)$$

where  $\Delta f$  is the calculated uncertainty,  $\Delta x_i$  the uncertainty of the  $i$ -th input variable and  $\frac{\partial f}{\partial x_i}$  is the partial derivative of  $f$  w.r.t  $x_i$ . For more complex calculations where this formula is very difficult to apply, such as equation 3, the uncertainty of our final  $m$  value was calculated as the standard error  $s/\sqrt{n}$  using the  $m$  values found, where  $s$  is the sample standard deviation and  $n$  is the number of values.

### 5.2 Data Extraction and Calculations

From Figures 2-7, we can see that the desired trend, shown in the lab manual [3], is accurately represented by our data, especially for longer switch intervals. For shorter switch intervals, the rod does not seem to have enough time to reach its full peak, thus potentially interfering with the  $\Delta t$  (time of peak difference from switch time) values that we measure.

Measuring the  $\Delta t_i$  values over the five peaks and taking an average, for the  $i$ -th trial (60s = 1st trial...etc), and corresponding  $w = 2\pi/T$  for each trial, we get:

$$\Delta t_1 = 25 \pm 5s, \omega_1 = 0.0524 \pm 0.0467[1/s]$$

$$\Delta t_2 = 25 \pm 5s, \omega_2 = 0.0349 \pm 0.0311[1/s]$$

$$\Delta t_3 = 27 \pm 5s, \omega_3 = 0.0262 \pm 0.0234[1/s]$$

$$\Delta t_4 = 25 \pm 5s, \omega_4 = 0.0209 \pm 0.0187[1/s]$$

$$\Delta t_5 = 20 \pm 10s, \omega_5 = 0.0140 \pm 0.0176[1/s]$$

$$\Delta t_6 = 20 \pm 10s, \omega_6 = 0.0105 \pm 0.0132[1/s]$$

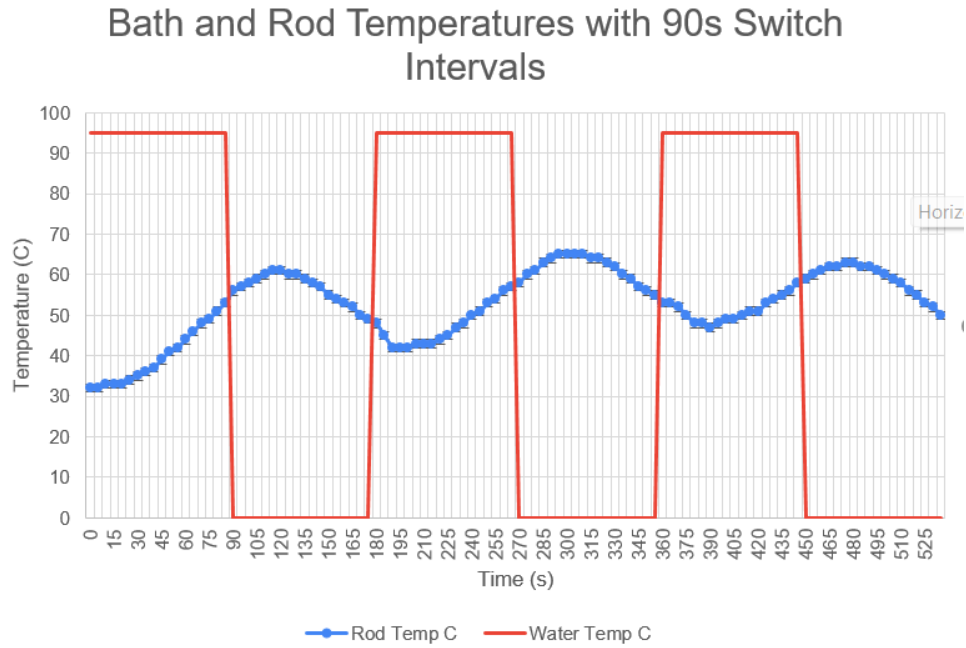


Figure 3: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 90s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 5s, and may be too small to be seen.

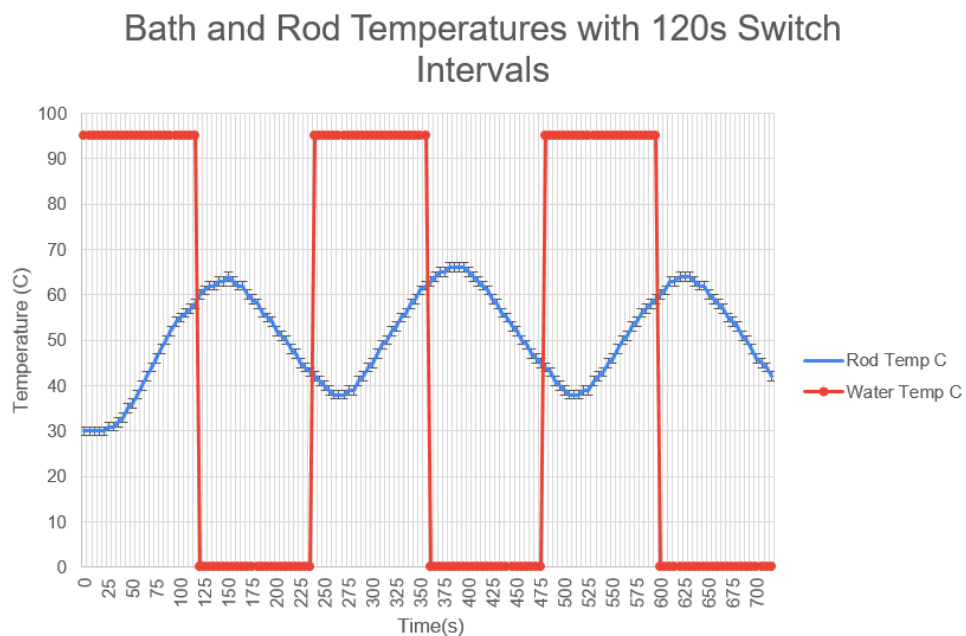


Figure 4: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 120s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 5s, and may be too small to be seen.

### Bath and Rod Temperatures with 150s Switch Intervals

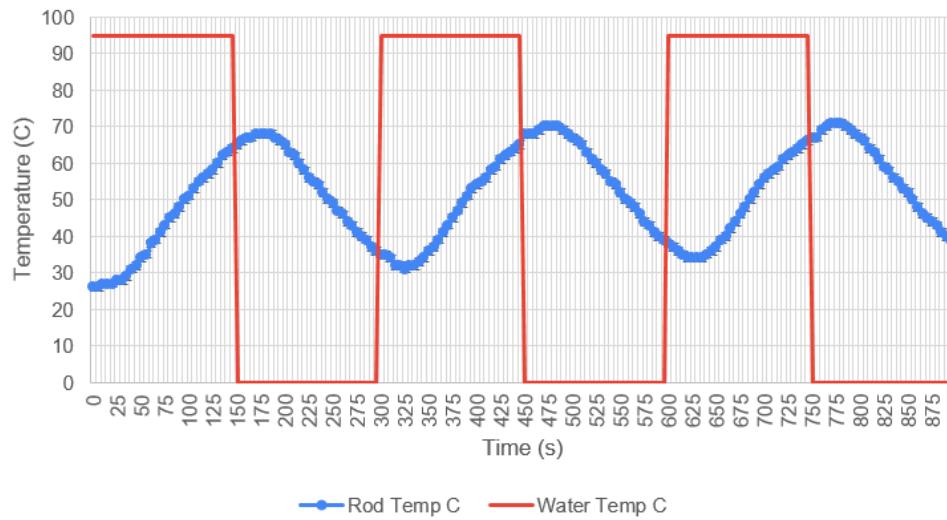


Figure 5: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 150s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 5s, and may be too small to be seen.

### Bath and Rod Temperatures with 225s Switch Intervals

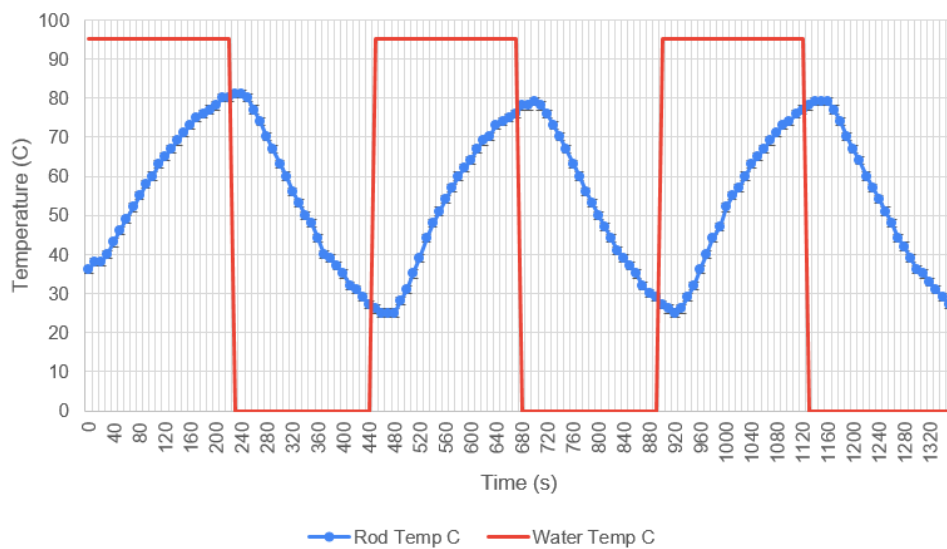


Figure 6: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 225s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 10s, and may be too small to be seen.

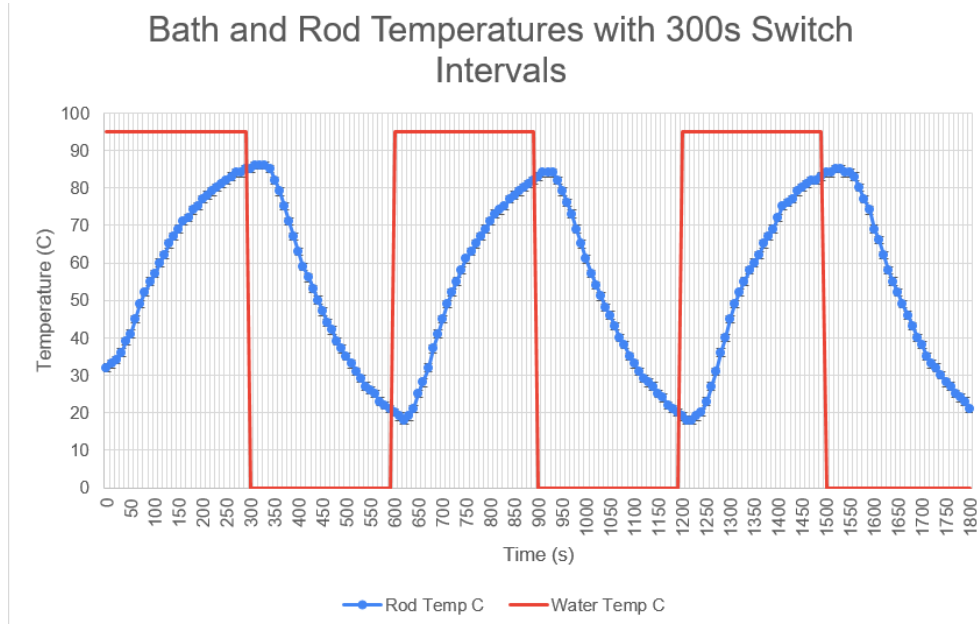


Figure 7: Measured Rod Temperatures plotted with Bath Temperature in degrees Celsius and Time in seconds for the 300s Interval Trials. The uncertainties on the y-axis are 1 degree and on the x-axis are 10s, and may be too small to be seen.

Using equation (3), we get, for  $\Delta\phi$  :

$$\Delta\phi_1 = 1.309 \pm 0.262$$

$$\Delta\phi_2 = 0.873 \pm 0.175$$

$$\Delta\phi_3 = 0.707 \pm 0.131$$

$$\Delta\phi_4 = 0.524 \pm 0.105$$

$$\Delta\phi_5 = 0.2793 \pm 0.0698$$

$$\Delta\phi_6 = 0.2094 \pm 0.0524$$

Applying equation (3) and using the code snippet in the Appendix, we calculated our  $m$  parameter to be, for each trial:

$$m_1 = 1.014e - 6$$

$$m_2 = 5.738e - 7$$

$$m_3 = 4.064e - 7$$

$$m_4 = 3.056e - 7$$

$$m_5 = 1.874e - 7$$

$$m_6 = 1.371e - 7$$

We can calculate the average  $m$ , which we take as the result of this experiment to be  $\bar{m} = 6.141e-7[m^2/s]$  and the sample standard deviation to be  $s = 1.082 * 10^{-6}$  . The standard error of this experiment,  $s/\sqrt{n}$ , where  $n = 6$ , is thus  $4.415 * 10^{-7}[m^2/s]$ .

Therefore, we have found the thermal diffusivity  $m$  in this experiment to be  $m = 6.141e - 7 \pm 4.415 * 10^{-7}[m^2/s]$ .

## 6 Discussion

### 6.1 Sources of Error

1. Temperature is not uniform throughout the water bath, so the temperature of the outer layer of rubber may not have been constant throughout trials. While convection currents in the water bath help distribute the heat, heat loss or gain to/from the environment occurs most along the glass or air boundaries. In the case of the hot bath, the bottom of the beaker was the hottest, while the top and the sides were cooler. If the rubber were placed at a different location each time

it was transferred, there would be consistency in the temperature readings. To mitigate this, we could have drawn a mark at the bottom of each beaker, ensuring that the rubber was placed at the exact location each time.

2. The ice water was not perfectly at  $0^{\circ}C$ . As opposed to the hot water that is constantly being supplied heat, the ice would melt away as the experiment took place. This was exacerbated from the heat that was emitted from the hot bath. Although we measured the initial temperature to be  $0^{\circ}C$ , the temperature would have increased as the ice melted. To mitigate this, we could have better insulated the two baths from each other, perhaps by surrounding the cold beaker with Styrofoam, or using a better insulator than the glass beaker for the ice bath.
3. The initial temperature of the rubber was not consistent between trials. The range of our initial temperatures was from  $41^{\circ}C$  to  $26^{\circ}C$ . This would have caused the temperature peaks to reach different values, as more or less heat would have been required to reach the maximum temperature.
4. The rubber tube was not perfectly cylindrical, in fact it's geometry was skewed to one side, as seen in figure 1. In equation 2, we simplify the Laplacian by assuming that the temperature only changes in the radial direction; however, due to the non-cylindrical shape of the rubber, the temperature could have changed in the  $z$ -direction or  $\phi$  direction. In addition, the inner and outer radii  $r_1$  and  $r_2$ . To mitigate this, we could have measured the radii at multiple points along the rubber tube and taken the average, or we could have used a rubber tube with a more uniform geometry (like the other lab groups).
5. The rubber tube did not perfectly fit the thermometer. There was a small gap between the bulb of the thermometer and the rubber tube. The measured temperature of the inner rubber may have reacted to the changes in water baths much more quickly than if the thermometer was perfectly wrapped around the rubber. This would have caused the phase difference between the inner and outer radii to be smaller than it actually was, i.e. the thermal diffusivity would have been overestimated.
6. There was human error in measuring thermometer readings at 5 and 10 second intervals. In this case, there are two possible human errors. One stems from the reading of the analog thermometer, and the other comes from the reading of the stopwatch. To avoid human error, in both cases, a video recording of the process could be taken. By analyzing the video, we would have the exact time at which the temperature was read and a better reading of the thermometer.
7. We can also consider the instrumental error associated with the thermometer. The thermometer was only accurate to the nearest whole number, so it had an uncertainty of  $\pm 0.5^{\circ}C$ . As such, the lack of accuracy could have caused temperature peaks to differ. To remedy this, we can implement a digital thermometer with a higher degree of precision. Also referring to error 6, we could have had a digital thermometer that took values for us at shorter intervals, assuring that we had a more accurate reading of the temperature and how it behaves with time.
8. The timing of the trials was not perfect, as the time delay between the transfer of water baths was not exactly  $T$  seconds. There is no way to mitigate this error, as there will always be some time where the rubber and thermometer spends time outside of the water, in the air. Additionally, reading error for stopwatch units could be considered in measurements of time, which contributes to error in our  $\Delta t$  and  $\omega$  terms.
9. We only used a two-term approximation for  $\tan\phi$ , which leads to some error terms being dropped. An approximation with more terms would offset this error. Additionally, MATLAB's `fzero` function error should be noted with such small values in this experiment, as the values returned are numerically approximated.



## 7 Conclusion

The thermal diffusivity  $m$  in this experiment was found to be  $m = 6.141e - 7 \pm 4.415 * 10^{-7}[m^2/s]$ . From [3], this generally agrees and falls within the order of magnitude of  $10^{-7}m^2/s$  as described for insulators, as the rubber covering used within the experiment should be a thermal insulator. However, the large discrepancy between trials for measured diffusivities ( $1.371 * 10^{-7}$  compared to  $1.014 * 10^{-6}$ ) can be explained by a variety of errors unaccounted for. Notably, consistent temperature of the bath and rod between trials, instrumental errors for the stopwatch, thermometer, and times when the baths were changed would all account for non-negligible uncertainties for the intermediate variables within this experiment. Future research could expand on refining the techniques used in this experiment, or by finding a more accurate numerical method to compute the parameter  $m$ .

## References

- [1] R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, John Wiley and Sons, Inc., 1960. Eq. 8.1-7. ISBN 978-0-471-07392-5.
- [2] "Industrial Uses of Rubber," Industrial Rubber. [Online]. Available: <https://www.industrial-rubber.com/news/industrial-uses-of-rubber/>. [Accessed: Apr. 8, 2024].
- [3] B. Wilson, "Thermal Diffusivity," University of Toronto, Toronto, ON, Canada, Dec. 05, 2023. [Accessed: Apr 1, 2024].

## 8 Appendix

```

dt = [25, 25, 27, 25, 20, 20];
ddt = [5, 5, 5, 5, 10, 10];
periods = [120, 180, 240, 300, 450, 600];
warray = [2*pi/120, 2*pi/180, 2*pi/240, 2*pi/300, 2*pi/450, 2*pi/600]; % PERIODS
wdt = warray.*dt;
wdterrors = [0.262, 0.175, 0.131, 0.105, 0.0698, 0.0524];

% C = wdt
% phi1 - phi2 - wdt = 0 i.e. phi = 0
ans1 = zeros(6,1);
% w = 0;
for i = 1:6
    w = warray(i);
    C = wdt(i);
    phi = @(m) atan((576*m^4*0.00314^4*w^2 - 0.00314^12*w^6)/(2304*m^6 - 36*m^2*0.00314^8*w^4)) - atan((576*m^4*0.00792^4*w^2 - 0.00792^12*w^6)/(2304*m^6 - 36*m^2*0.00792^8*w^4)) - C;
    ans1(i) = fzero(phi, 10^-7);
end

syms m
syms w
% syms C
syms r
syms r2
phi = atan((576*m^4*r^4*w^2 - r^12*w^6)/(2304*m^6 - 36*m^2*r^8*w^4)) - atan((576*m^4*r2^4*w^2 - r2^12*w^6)/(2304*m^6 - 36*m^2*r2^8*w^4));
% partial_C = diff(phi, C);
partial_r = diff(phi, r);
partial_r2 = diff(phi, r2);
partial_w = diff(phi, w);
partial_m = diff(phi, m);

dw = zeros(6,1);
for i = 1:6
    dr = 0.000005;
    dr2 = 0.000005;
    dw(i) = sqrt(2*pi/(periods(i)^2)*ddt(i));
    a = vpa(subs(partial_r, {m, w, r, r2}, {ans1(i), warray(i), 0.00314, 0.00792}));
    b = vpa(subs(partial_r2, {m, w, r, r2}, {ans1(i), warray(i), 0.00314, 0.00792}));
    c = vpa(subs(partial_w, {m, w, r, r2}, {ans1(i), warray(i), 0.00314, 0.00792}));
    d = vpa(subs(partial_m, {m, w, r, r2}, {ans1(i), warray(i), 0.00314, 0.00792}));

    newfunc = @(dm) ((a*dr)^2 + (b*dr2)^2 + (c*dw)^2 + (d*dm))^0.5 - wdterrors(i);
    % fzero(newfunc, 10^-2)
end

```

Figure 8: Source code in MATLAB to solve for  $m$  values using equation (3). Note that after the first for loop, the code was not used but note the attempt at computing error propagation using the standard formulae, which MATLAB's built in `fzero` failed to find accurate results for,