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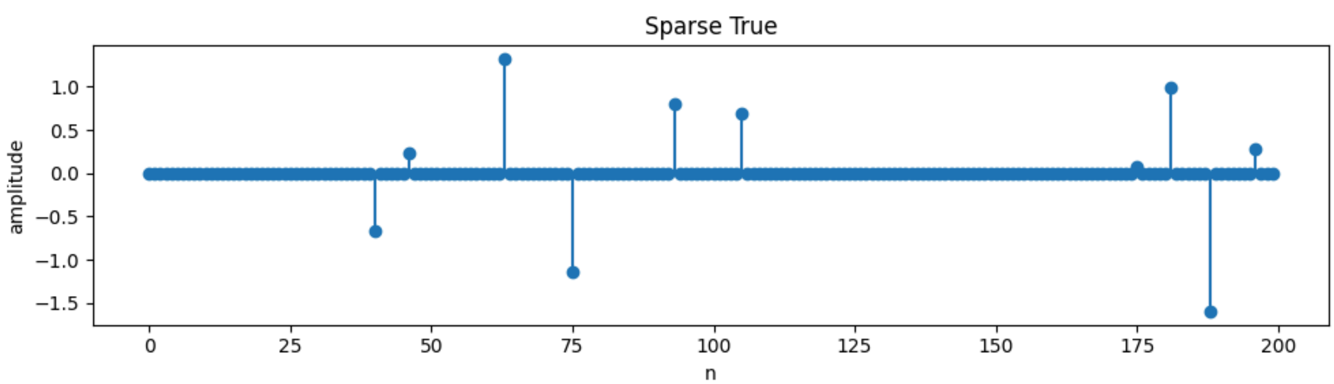
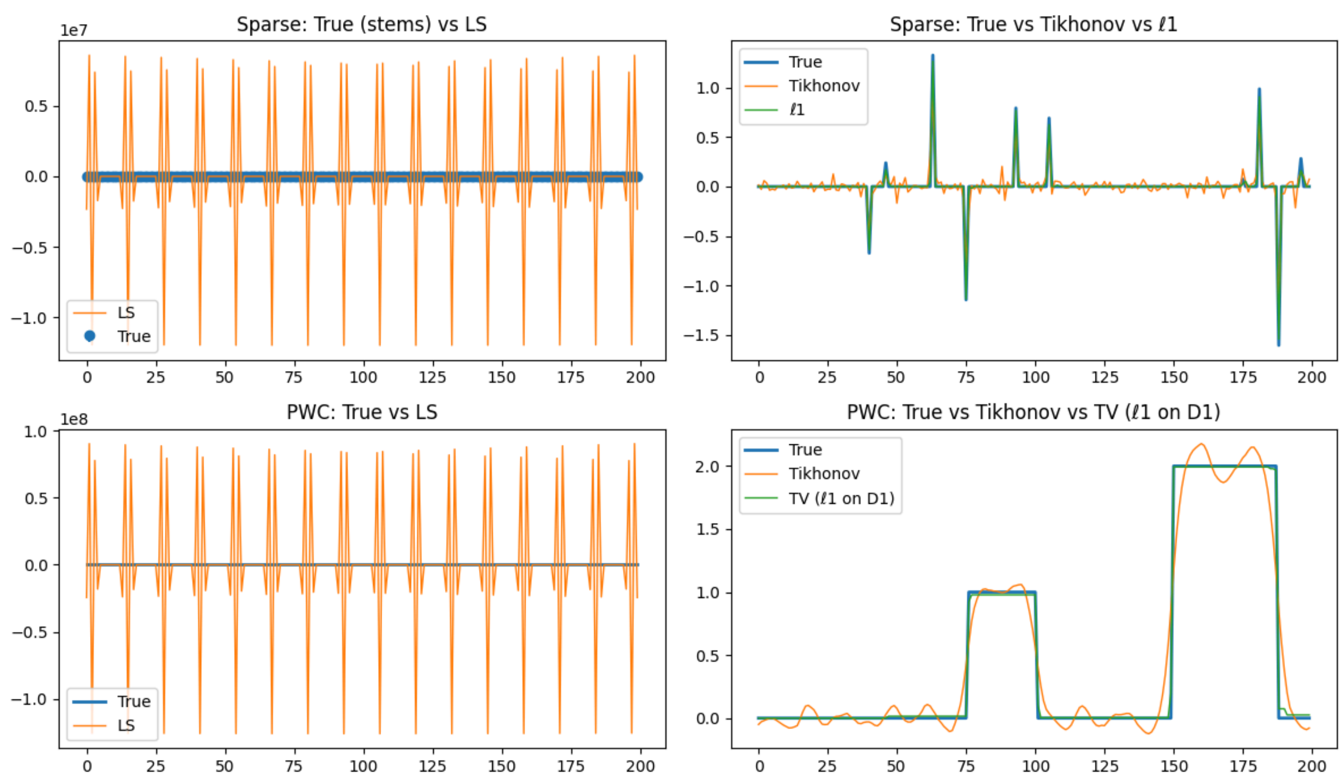
Problem 1: Function Recovery: LS, Modified Tikhonov Regularization, ℓ1 Regularization

```
n = 200
m_kernel = 6
sigma = 3.0

λ sparse = 0.03292896837493047
μ sparse = 0.001985416206020707

λ TV= 0.49580682416846555
μ TV = 0.40315193628604395
```

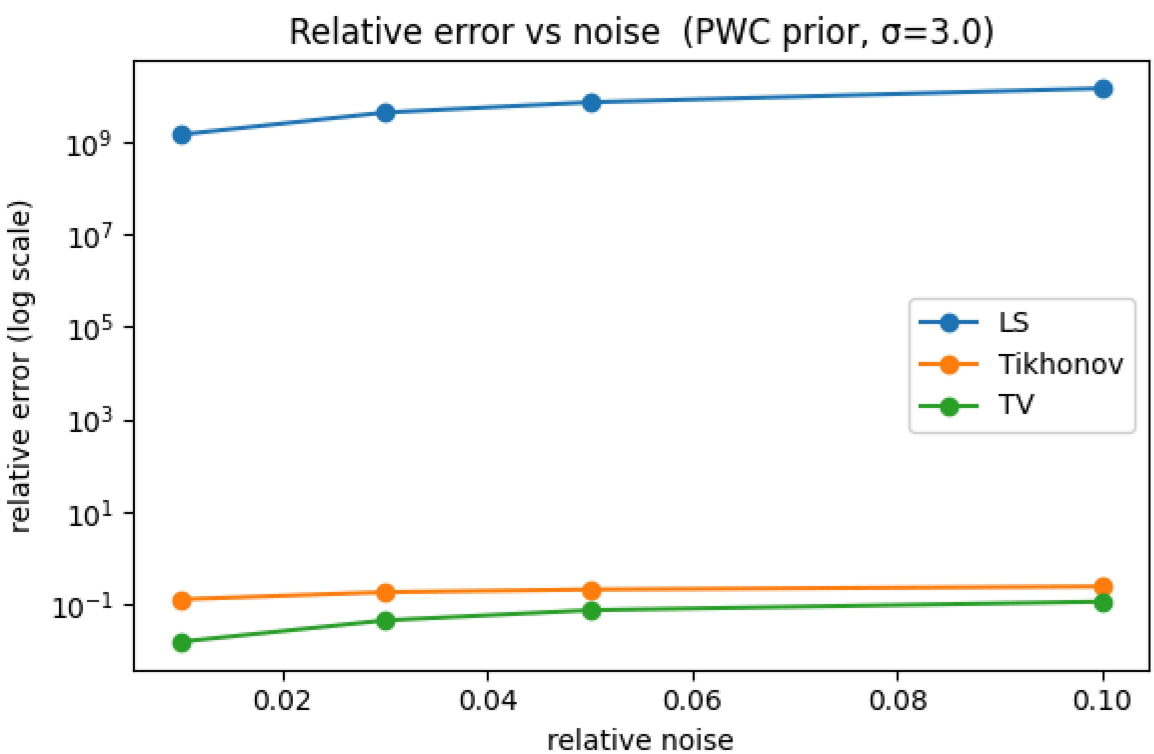


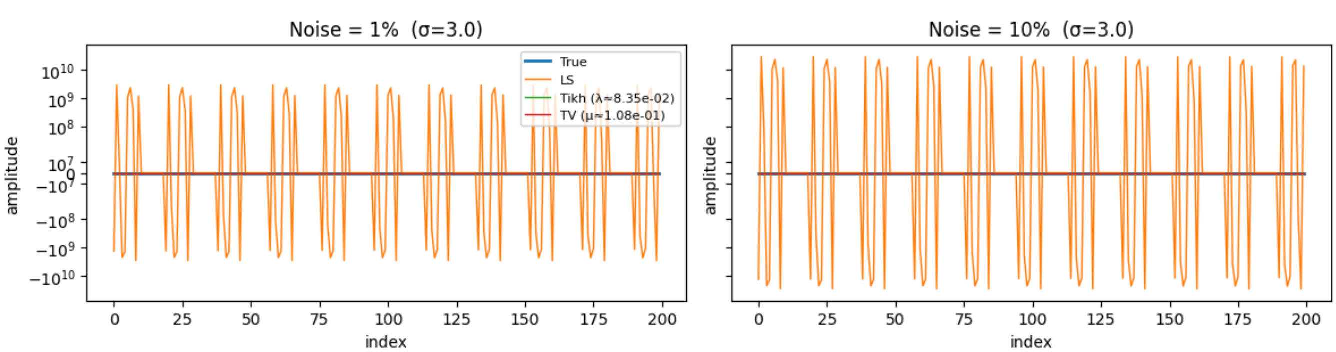


```
Double Check
Sparse:
  LS rel err: 23197115.70162896
  Tik rel err: 0.3886608267526403
  L1 rel err: 0.0833700658019523

PWC:
  LS rel err: 52776950.370476924
  Tik rel err: 0.19331640032310368
  TV rel err: 0.023960038501548934

blur cond(A): 59498148715.25219 , smallest σ ≈ 1.6779794820581446e-11
```

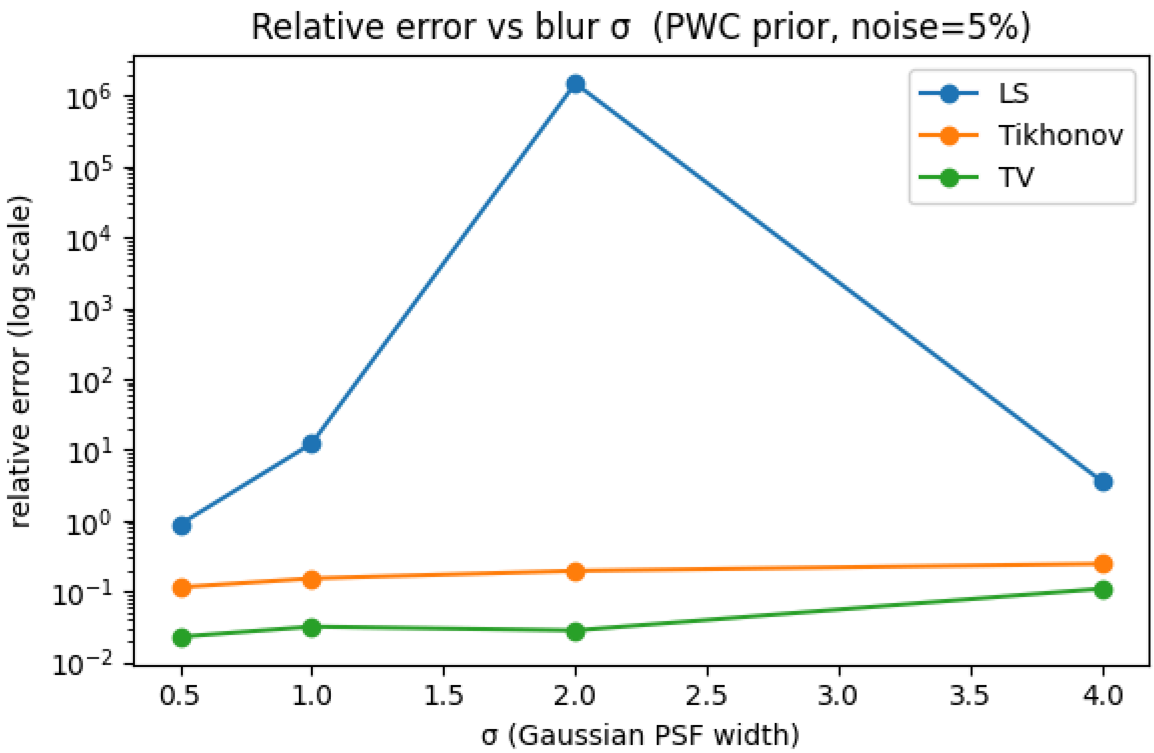
Changes in Noise and Blur

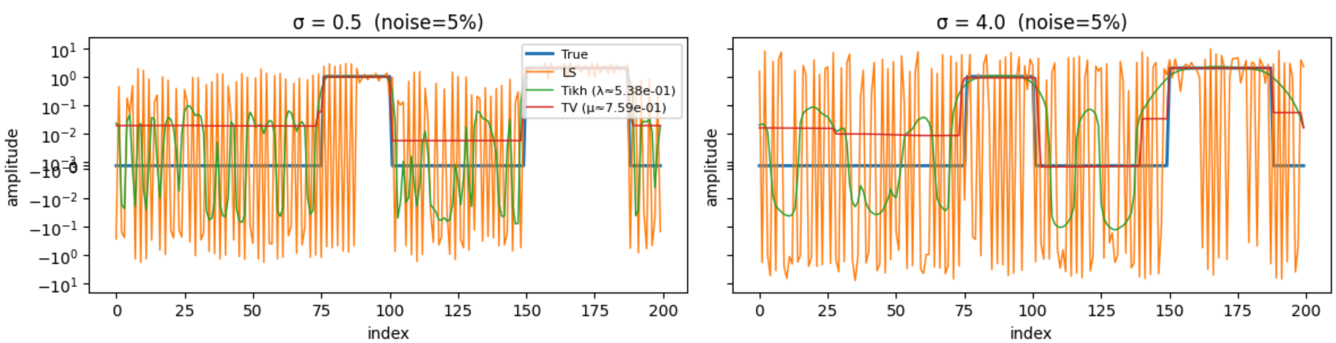
```
n=200, tau=1.05
sigma = 3.0
noise_levels =  [0.01, 0.03, 0.05, 0.1]
```






```
sigmas = [0.5, 1.0, 2.0, 4.0]
noise_fixed =  0.05
```





Sources for regularization inspiration:

- L. Ding, L. Li, W. Han, and W. Wang, On existence of a variational regularization parameter under Morozov's discrepancy principle, arXiv:2506.11397 (math.NA), 2025, version 1.
- P. C. Hansen, Discrete Inverse Problems: Insight and Algorithms, SIAM, Philadelphia, 2010, Chap. 5, pp. 85–105.
- Y.-W. Wen and R. H. Chan, "Parameter selection for total-variation-based image restoration using discrepancy principle," IEEE Trans. Image Process. 21 (2012), no. 4, 1770–1781. DOI: 10.1109/TIP.2011.2181401.

Response:

- **Regularization Parameters**: I tried to challange myself and used Morozov's discrepancy principle rather than the suggested L-curve or trial-and-error method. Because I generated the noise and know the noise level, so I set a target residual and chose the regularization parameter α to satisfy $\|Af_\lambda - g_{\text{noisy}}\|_2^2 \approx \|e\|_2^2$, with a small safety factor $\tau \in [1.0, 1.2]$, hence $\|Af_\lambda - g_{\text{noisy}}\|_2^2 \approx \tau\,\|e\|_2^2$. I computed λ (and μ for TV/ℓ_1) via a log-bisection search. In comparison, the L-curve requires a grid

of solves (computationally expensive) and its corner can be ambiguous, while trial-and-error is subjective and can over or under-regularize.
- The Hansen et al. textbook also brings GCV and NCP as potential methods, but they would also not be the most efficient methods. GCV assumes a linear estimator and undersmooths for nonlinear TV/ℓ1. NCP assumes a white residual and low-frequency noise can be mistaken as signal and undersmooths.
- I did some further digging and found a paper by L. Ding et al. (2025) goes even one step further and proves that one can select α to keep the residual within a modified Morozov band, establishing existence and covergance for nonlinear inverse problems.

Observations:

- With Gaussian blur, the forward operator A is extremely ill-conditioned (as seen in the double check and graphs), so unregularized least squares amplifies noise and fails.
- Tikhonov regularization stabilizes the inversion but smooths edges and leaves small ripples. The bias grows as noise or blur increase, so the optimal λ must increase accordingly.
- For the piecewise-constant signal, TV (ℓ1 on 1st differences) matches the prior and gives the best recoveries (as seen in above).
- In conclusion, when noise increase, LS error explodes, Tikhonov's error grows gradually, and TV stays robust up to a higher threshold. On the other hand, if blur increases, the problem becomes more ill-posed. LS is unusable, Tikhonov oversmooths more, and TV still preserves edges.

Problem 2: Fourier Data

Problem 2(a): Calculate Fourier Coefficients $\hat{f}\,(k)$ as a function of k

$$f(t) = \begin{cases} 1, & -\frac{1}{4} < t \le 0, \\ 2, & \frac{1}{2} \le t \le \frac{7}{8}, \\ 0, & \text{else.} \end{cases}$$

Definition (Fourier coefficients with basis $e^{ik\pi t}$):

$$\hat{f}\,(k) \;=\; \frac{1}{2}\int_{-1}^{1} f(t)\, e^{-ik\pi t}\, dt.$$

Only integrate where $f \neq 0$, for $I_1 = (-\frac{1}{4}, 0], \quad I_2 = [\frac{1}{2}, \frac{7}{8}]$:

$$\hat{f}\,(k) = \frac{1}{2}\left(\int_{-1/4}^{0} e^{-ik\pi t}\, dt \;+\; 2\int_{1/2}^{7/8} e^{-ik\pi t}\, dt\right)$$

DC term (k=0)

$$\hat{f}(0)=\frac{1}{2}\int_{-1}^{1} f(t)\,dt=\frac{1}{2}\Big(\underbrace{\tfrac{1}{4}}_{I_1}\cdot 1+\underbrace{\tfrac{3}{8}}_{I_2}\cdot 2\Big)=\frac{1}{2}\Big(\tfrac{1}{4}+\tfrac{3}{4}\Big)=\boxed{\tfrac{1}{2}}$$

Oscillatory terms $k \neq 0$

Use the antiderivative

$$\int e^{-ik\pi t}\,dt=\frac{e^{-ik\pi t}}{-ik\pi}$$

First interval $I_1=(-\frac{1}{4},0]$:

$$\int_{-1/4}^{0} e^{-ik\pi t}\,dt=\frac{e^{-ik\pi t}}{-ik\pi}\bigg|_{-1/4}^{0}=\frac{1-e^{ik\pi/4}}{-ik\pi}$$

Second interval $I_2=[\frac{1}{2},\frac{7}{8}]$:

$$2\int_{1/2}^{7/8} e^{-ik\pi t}\,dt=2\,\frac{e^{-ik\pi t}}{-ik\pi}\bigg|_{1/2}^{7/8}=\frac{2\left(e^{-ik\,7\pi/8}-e^{-ik\,\pi/2}\right)}{-ik\pi}$$

Combine and include the leading factor $\frac{1}{2}$:

$$\hat{f}(k)=\frac{1}{2}\left[\frac{1-e^{ik\pi/4}}{-ik\pi}+\frac{2\left(e^{-ik\,7\pi/8}-e^{-ik\,\pi/2}\right)}{-ik\pi}\right]$$

Clean the sign by factoring $\frac{1}{-ik\pi}$:

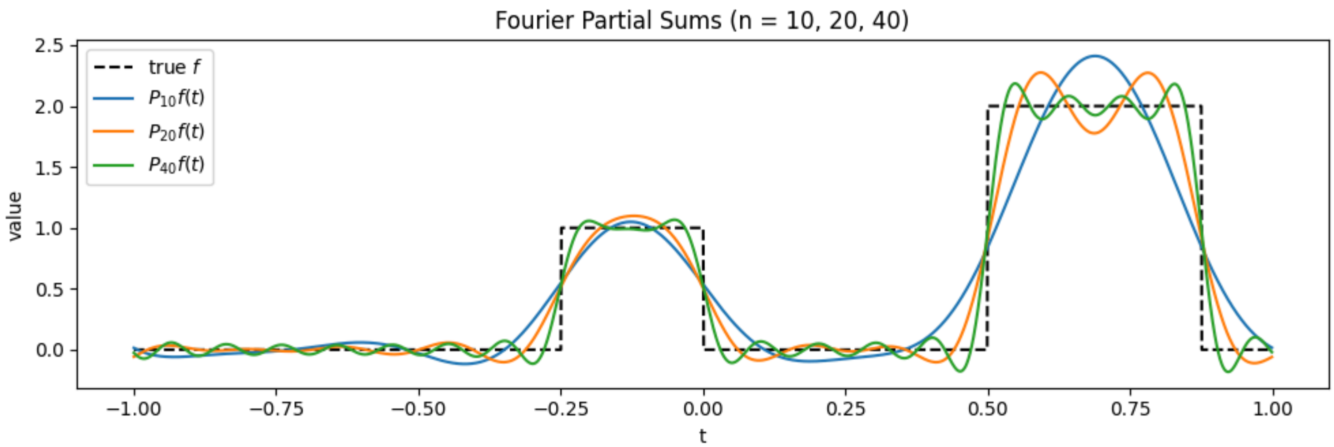
$$\hat{f}(k)=\frac{1}{2ik\pi}\Big[e^{ik\pi/4}-1\;+\;2\Big(e^{-ik\pi/2}-e^{-ik\,7\pi/8}\Big)\Big],\quad (k\neq 0);$$

Final result

$$\boxed{\hat{f}(0)=\tfrac{1}{2},\qquad \hat{f}(k)=\frac{1}{2ik\pi}\Big[e^{ik\pi/4}-1+2\Big(e^{-ik\pi/2}-e^{-ik\,7\pi/8}\Big)\Big]\ ,\quad (k\neq 0)}$$

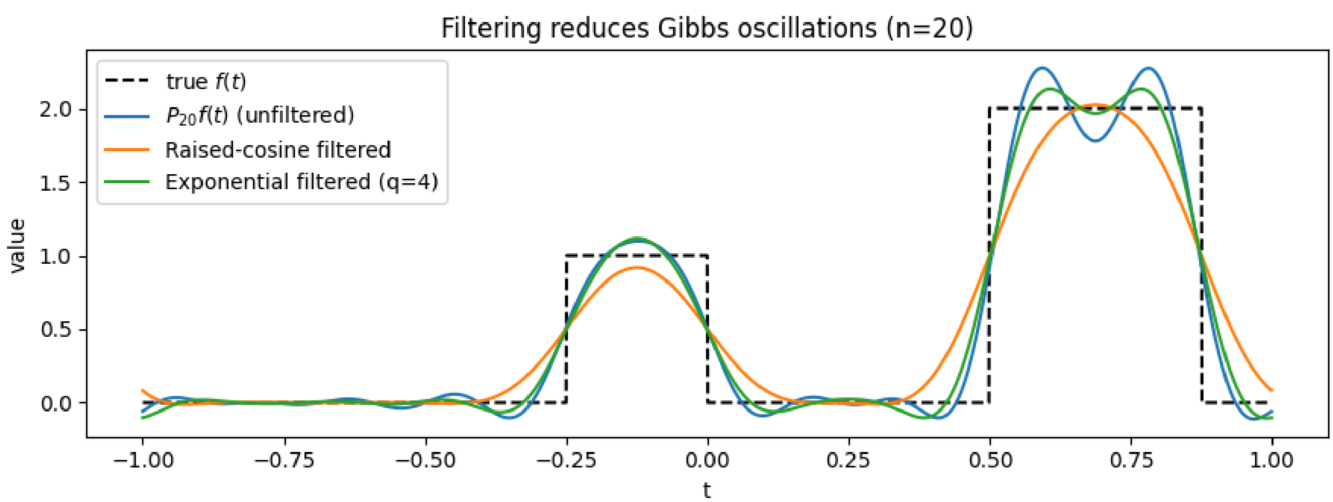
```
k: [-6 -5 -4 -3 -2 -1  0  1  2  3  4  5  6]
hat f(k): [-0.064 -0.0421j -0.0274-0.0787j -0.0796+0.j      0.2416-0.05j
 -0.033 -0.3513j -0.084 +0.2475j  0.5   +0.j      -0.084 -0.2475j
 -0.033 +0.3513j  0.2416+0.05j   -0.0796-0.j      -0.0274+0.0787j
 -0.064 +0.0421j]
```

Problem 2(b)



Response: As n increases from 10 to 40, the Fourier partial sums track the flat plateaus of the true piecewise-constant f(t) more accurately and the transition zones become steeper, while the oscillations localize closer to the jump points.

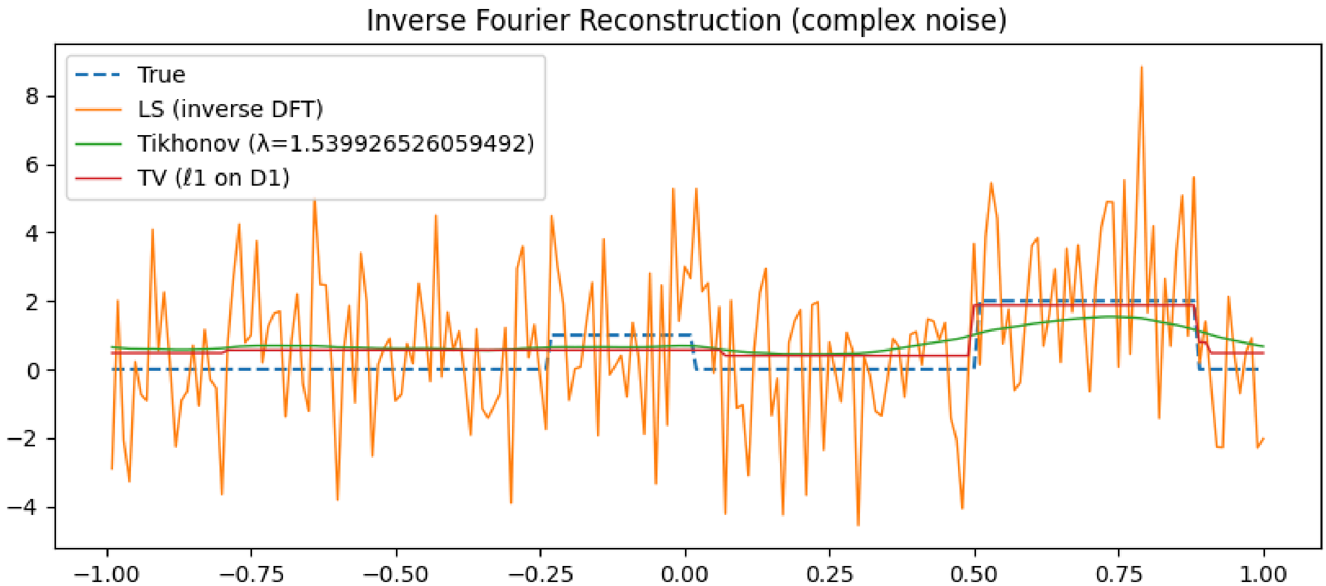
Problem 2(c)



Response: Raised-cosine gives the least ringing but more smoothing bias. Exponential offers a balanced compromise, less oscillations than unfiltered with better approximations on flat regions than raised-cosine.

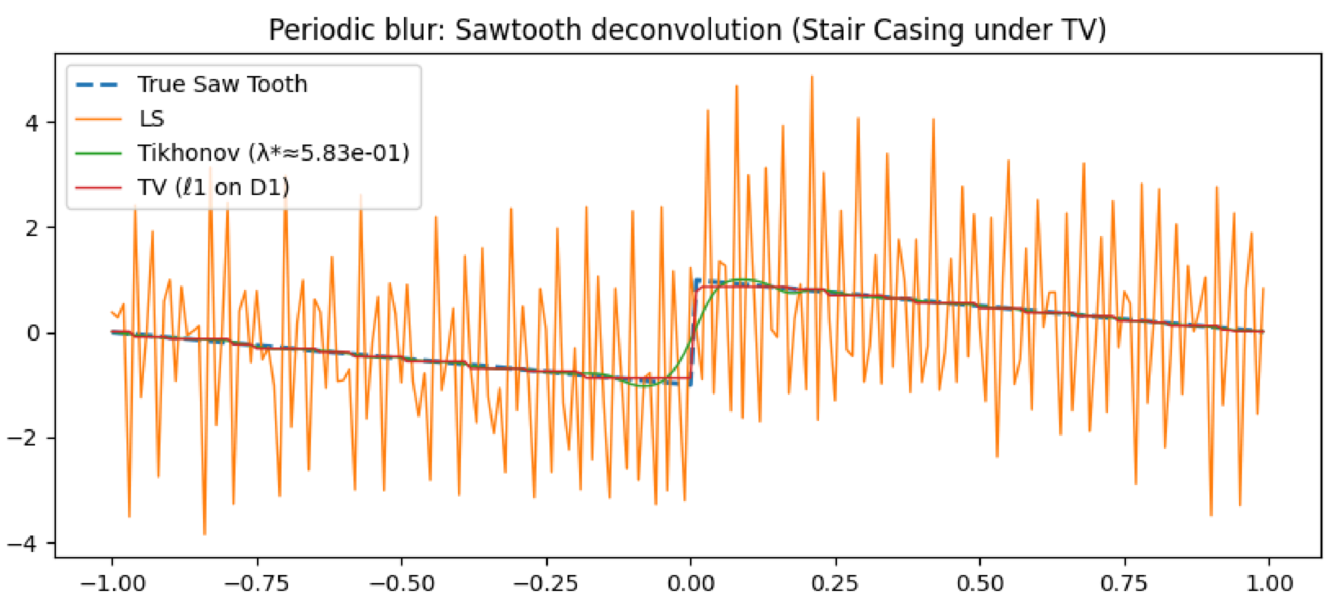
Problem 3: Inverse Method Approach for Fourier Data

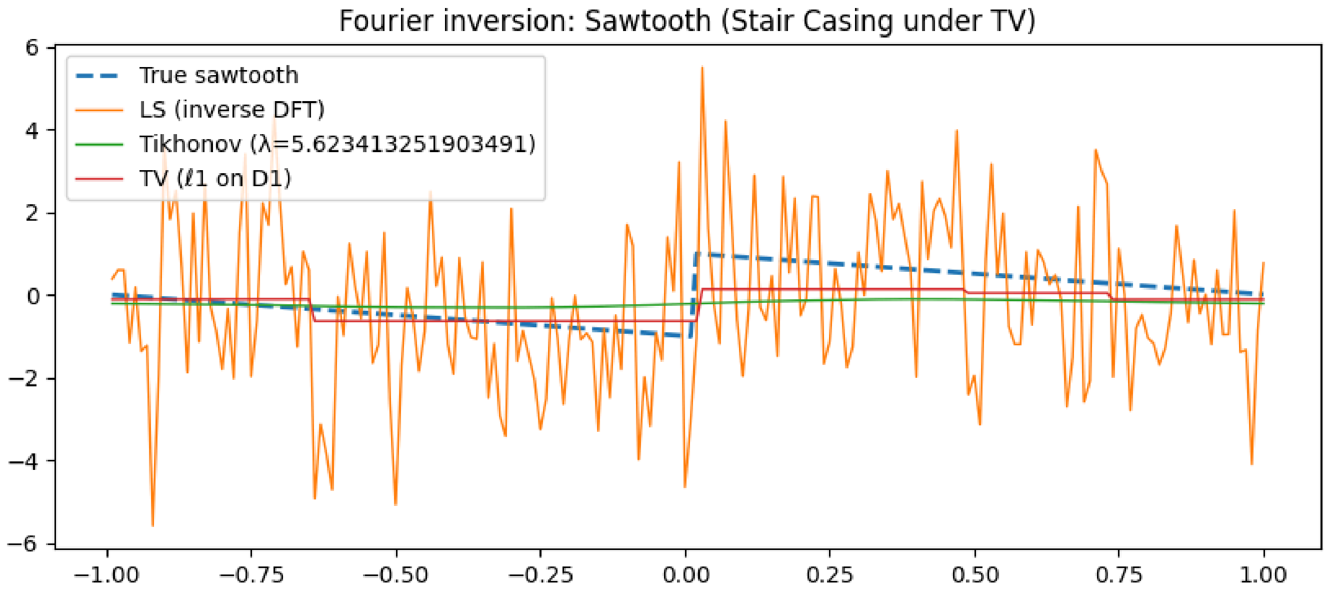
```
noise std = 0.2
TV μ* (discrepancy) = 0.143
```



Response: The plot show a bias-variance trade off for the Inverse-Fourier problem with complex noise. LS is too noisy. Tikhonov suppresses most noise but rounds the jump and depresses the pleteaus. Therefore, TV is the best match to the true piecewise constant function.

Problem 4: Saw Tooth Function with Periodic Blur and Fourier Inversion





Response: We can observe the stair casing effect in the TV (ℓ1) regularization case at $t \approx 0$ in the periodic blur case more mildly in the Fourier inversion.