Assignment 6

https://github.com/jchryssanthacopoulos/quantum_information/tree/main/assignment_6

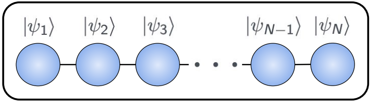
Quantum Information and Computing
AA 2022–23

James Chryssanthacopoulos
13 December 2022

Università degli Studi di Padova

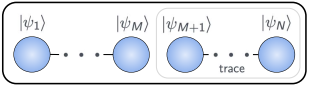
Many-body Quantum System

- Quantum system composed of N subsystems, each with wavefunction $|\psi_i\rangle = \sum_{\alpha_j} c_{ij} |\alpha_j\rangle$ in D-dimensional Hilbert space



- Overall wavefunction is $|\Psi\rangle = \sum_{\alpha_1 \cdots \alpha_N} C_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\rangle$, where $C_{\alpha_1 \cdots \alpha_N}$ are D^N complex coefficients
  - With normalization and phase constraints, there are $2D^N - 2$ real DOFs
- If state is **separable**, $|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$, which has $(2D-2)N$ real DOFs
- **Density matrix** of a pure state is $\rho \equiv |\Psi\rangle\langle\Psi|$. To describe state of part of system, **reduced density matrix** is computed, tracing over rest of system. For example:

$$\rho_M = \mathrm{Tr}_{M+1} \cdots \mathrm{Tr}_N |\Psi\rangle\langle\Psi| =$$



Separable versus Generic States

- **Separable**. $|\Psi\rangle$ needs only $\mathcal{O}(DN)$ parameters (linear in N), but it represents non-interacting systems and is less flexible (i.e., mean-field approximation)
- **Generic**. $|\Psi\rangle$ needs $\mathcal{O}(D^N)$ parameters (exponential in N), but it can capture arbitrary interactions

```
! generate separable state containing N * d parameters,
! making sure to correctly normalize each wavefunction
do ii = 1, N
    norm = 0
    do jj = 1, D
        separable_state(ii, jj) = cmplx(rand(0) * 2 - 1, rand(0) * 2 - 1)
        norm = norm + separable_state(ii, jj) * conjg(separable_state(ii, jj))
    end do
    separable_state(ii, :) = separable_state(ii, :) / sqrt(norm)
end do

if (debug_level .ge. DEBUG_LEVEL_1) then
    print *, "Wavefunction factorization = "
    call print_complex_matrix(separable_state)
end if

! populate tensor
state = 1

do ii = 1, dim
    mat_indices = tensor2mat(ii, N, D)

    if (debug_level .ge. DEBUG_LEVEL_2) then
        print *, "Matrix indices for tensor index = ", ii, " are ", mat_indices
    end if

    do jj = 1, N
        state(ii) = state(ii) * separable_state(jj, mat_indices(jj))
    end do
end do
```

Each wavefunction separately normalized, leading to overall normalization

Since $|\Psi\rangle$ is still D^N-dimensional, function used to convert between generic and separable indices

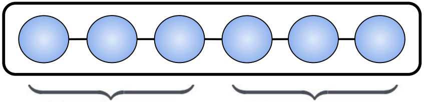
```
function prepare_generic_state(N, D) result(state)
    implicit none

    integer N, D, dim
    real*8 norm
    integer ii

    complex*16, dimension(:), allocatable :: state

    dim = D ** N

    allocate(state(dim))

    norm = 0

    do ii = 1, dim
        state(ii) = cmplx(rand(0) * 2 - 1, rand(0) * 2 - 1)
        norm = norm + state(ii) * conjg(state(ii))
    end do

    state = state / sqrt(norm)

end function
```

```
function tensor2mat(tt, N, D) result(mat_idx)
    implicit none

    integer tt, N, D
    integer ii
    integer mat_idx(N)

    do ii = 1, N
        mat_idx(ii) = modulo((tt - 1) / D ** (ii - 1), D) + 1
    end do

end function
```

Computing Density Matrices


```
compiled/density_matrix --N 6 --D 4 --M 3 --type generic --output_filename data/density_mat_N6_D4_M3.txt
```

- `--N 6`: number of subsystems
- `--D 4`: number of dimensions
- `--M 3`: number of subsystems in right bipartition
- `--type generic`: type of state to prepare (e.g., separable, generic, bell)
- `--output_filename data/density_mat_N6_D4_M3.txt`: matrices saved to file



$$\rho = \begin{pmatrix} 5.64 \times 10^{-4} & \cdots & -2.27 \times 10^{-4} - 1.08 \times 10^{-4} i \\ \vdots & \ddots & \vdots \\ -2.27 \times 10^{-4} + 1.08 \times 10^{-4} i & \cdots & 1.11 \times 10^{-4} \end{pmatrix}$$

$D^N = 4^6 = 4096$ rows and columns

Computing left reduced matrix has $\mathcal{O}(D^{2N-M})$ complexity

```
do ii = 1, dim
    do jj = 1, dim
        do kk = 1, D ** M
            idx1 = kk + (ii - 1) * D ** M
            idx2 = kk + (jj - 1) * D ** M
            rho_reduced_L(ii, jj) = rho_reduced_L(ii, jj) + rho(idx1, idx2)
            if (debug_level .ge. DEBUG_LEVEL_2) then
                print "('Added rho index ', (i4), (i4), ' to rho reduced L index ',
                    idx1, idx2, ii, jj
            end if
        end do
    end do
end do
```

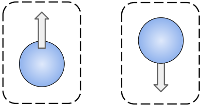
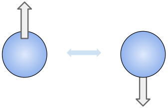
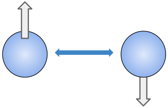
$D^M = 4^3 = 64$ rows and columns

$$\rho_L = \begin{pmatrix} 1.62 \times 10^{-2} & \cdots & -1.14 \times 10^{-3} - 3.08 \times 10^{-4} i \\ \vdots & \ddots & \vdots \\ -1.14 \times 10^{-3} + 3.08 \times 10^{-4} i & \cdots & 1.59 \times 10^{-2} \end{pmatrix}$$

$$\rho_R = \begin{pmatrix} 1.63 \times 10^{-2} & \cdots & -1.02 \times 10^{-3} - 1.36 \times 10^{-3} i \\ \vdots & \ddots & \vdots \\ -1.02 \times 10^{-3} + 1.36 \times 10^{-3} i & \cdots & 1.74 \times 10^{-2} \end{pmatrix}$$

Qubit Systems

- Density matrices are Hermitian with trace one, as expected
- $\rho_{L,R}$ of maximally entangled state exhibits total loss of coherence
- Von Neumann entropy $-\mathrm{Tr}(\rho \log \rho)$ computed by diagonalizing ρ

	Separable	**Generic (Partially Entangled)**	**Maximally Entangled**
			
$\lvert\Psi\rangle$	$\left[(-0.743-0.548i)\lvert 0\rangle+(0.380-0.061i)\lvert 1\rangle\right]\times \left[(0.058-0.499i)\lvert 0\rangle+(-0.804+0.318i)\lvert 1\rangle\right]$	$(-0.57-0.42i)\lvert 00\rangle+(0.29-0.05i)\lvert 01\rangle+ (0.04-0.32i)\lvert 10\rangle+(-0.52+0.20i)\lvert 11\rangle$	$0.71\lvert 01\rangle-0.71\lvert 10\rangle$
ρ	$\begin{pmatrix} 0.21 & -0.06-0.06i & -0.18+0.33i & 0.15-0.04i \\ -0.06+0.06i & 0.04 & -0.05-0.15i & -0.03+0.06i \\ -0.18-0.33i & -0.05+0.15i & 0.64 & -0.19-0.19i \\ 0.15+0.04i & -0.03-0.06i & -0.19+0.19i & 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.50 & -0.15-0.15i & 0.11-0.20i & 0.21+0.33i \\ -0.15+0.15i & 0.09 & 0.03+0.09i & -0.16-0.04i \\ 0.11+0.20i & 0.03-0.09i & 0.10 & -0.08+0.16i \\ 0.21-0.33i & -0.16+0.04i & -0.08-0.16i & 0.31 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & -0.50 & 0 \\ 0 & -0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
ρ_L	$\begin{pmatrix} 0.25 & -0.21+0.38i \\ -0.21-0.38i & 0.75 \end{pmatrix}$	$\begin{pmatrix} 0.59 & -0.05-0.23i \\ -0.05+0.23i & 0.41 \end{pmatrix}$	$\begin{pmatrix} 0.50 & 0 \\ 0 & 0.50 \end{pmatrix}$
ρ_R	$\begin{pmatrix} 0.85 & -0.25-0.25i \\ -0.25+0.25i & 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.60 & -0.23+0.01i \\ -0.23-0.01i & 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.50 & 0 \\ 0 & 0.50 \end{pmatrix}$
$S_{L,R}$	0	0.56	$\log 2 = 0.69$

Entropy of Qubit Systems

- Entropy computed for different N and sizes of right bipartition M
- When $M = 0$, entropy is computed over entire system, so it is equal to zero since state is pure
- As M increases, entropy increases since more of system is traced over, but when $M > N/2$, entropy decreases as more of system is included