COGNITIVE SCIENCE
A Multidisciplinary Journal

Cognitive Science 44 (2020) e12875

ISSN: 1551-6709 online
DOI: 10.1111/cogs.12875

Counting to Infinity: Does Learning the Syntax of the Count List Predict Knowledge That Numbers Are Infinite?

Junyi Chu,[a] Pierina Cheung,[b] Rose M. Schneider,[a] Jessica Sullivan,[c] David Barner[a,d]

[a]*Department of Psychology, University of California, San Diego*
[b]*Office of Education Research, National Institute of Education, Nanyang Technological University*
[c]*Department of Psychology, Skidmore College*
[d]*Department of Linguistics, University of California, San Diego*

Received 5 August 2019; received in revised form 12 May 2020; accepted 8 June 2020

Abstract

By around the age of 5½, many children in the United States judge that numbers never end, and that it is always possible to add 1 to a set. These same children also generally perform well when asked to label the quantity of a set after one object is added (e.g., judging that a set labeled "five" should now be "six"). These findings suggest that children have implicit knowledge of the "successor function": Every natural number, n, has a successor, $n + 1$. Here, we explored how children discover this recursive function, and whether it might be related to discovering productive morphological rules that govern language-specific counting routines (e.g., the rules in English that represent base-10 structure). We tested 4- and 5-year-old children's knowledge of counting with three tasks, which we then related to (a) children's belief that 1 can always be added to any number (the successor function) and (b) their belief that numbers never end (infinity). Children who exhibited knowledge of a productive counting rule were significantly more likely to believe that numbers are infinite (i.e., there is no largest number), though such counting knowledge was not directly linked to knowledge of the successor function, per se. Also, our findings suggest that children as young as 4 years of age are able to implement rules defined over their verbal count list to generate number words beyond their spontaneous counting range, an insight which may support reasoning over their acquired verbal count sequence to infer that numbers never end.

Keywords: Count list; Infinity; Conceptual change; Successor function; Highest count; Decade + unit rule

Correspondence should be sent to Junyi Chu, Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, MA 02138. E-mail: junyichu@mit.edu

1. Introduction

Human learners draw on a finite set of experiences to acquire information about the world, but nevertheless acquire systems of rules that permit the generation of unbounded representational outputs. For example, natural language is often touted as an example of how humans make "infinite use of finite means": A finite lexicon and system of combinatorial rules allows children to generate an unbounded number of possible utterances (Chomsky, 1965; von Humboldt, 1836/1999, p. 91). Similarly, numerate humans learn a set of symbols and combinatorial rules that permit an unbounded set of mathematical expressions. For example, the English base-10 numeral system expresses 80 numbers (from 20 to 99) by composing just 17 unique decade and unit words (20 through 90, and 1 through 9). The expressive power of this system is unbounded: To express larger numbers, one simply needs to learn the appropriate words representing powers of 10 (e.g., hundred, thousand, million, etc.), and recursively compose them following the syntactic rules of numerals (Hurford, 1975; Le Corre, Li, Huang, Jia, & Carey, 2016). As described below, although children initially believe that numbers are finite—and have limited knowledge of both the structure and meaning of number words—they ultimately come to believe that numbers never end. In the present study, we investigate how children learn that numbers are infinite, and whether the rule-governed structure of the verbal count list might play a role in children's inference that numbers form an infinite class.

When children first begin learning about numbers in early childhood, their knowledge is clearly item-based and finite. At around the age of 2 years, English-speaking children in the United States begin to recite a subset of the verbal count list (one, two, three, four, etc.), but often cannot count beyond 10 at this point (Fuson, 1988; Fuson & Hall, 1983; Gelman & Gallistel, 1978). Moreover, these number words appear to lack meanings at this early stage: When asked to give a number (e.g., to give one fish), children initially give a random amount (e.g., Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990). Some months later, children appear to acquire an exact meaning for the word *one*, and can give one object when asked, while failing to reliably give two when asked. These children are often called one-knowers. Another 6–9 months later, the children become two-knowers (and can reliably give two objects), then three-knowers several months after that. One by one, children add meanings to their number words in a way that suggests the lack of a productive logic governing these meanings (Sarnecka & Lee, 2009).

While children initially lack a productive rule for understanding number words, a breakthrough appears to happen at around the age of 3 1/2 or 4 (in U.S. middle-class English-speaking groups), when children appear to realize that they can correctly give any requested number by counting and giving all objects that are implicated in their count. These children are often called "Cardinal Principle Knowers" or CP-knowers. While the specific time-course may vary, the basic developmental sequence—that is, of progressing through discrete number-knower stages and finally using the counting routine to productively generate any quantity within the child's count list—has been reported for children learning a variety of languages around the world (Almoammer et al., 2013; Barner, Libenson, Cheung, & Takasaki, 2009; Le Corre et al., 2016; Piantadosi, Jara-

Ettinger, & Gibson, 2014; Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007). Also, in bilingual preschoolers, children progress through discrete number-knower stages independently in each language, but generally become CP-knowers at the same time in both languages (Wagner, Kimura, Cheung, & Barner, 2015). What remains unclear, however, is what, exactly, children learn when they become CP-knowers.

By some accounts, becoming a CP-knower not only involves mastery of an enumeration procedure for set sizes within one's familiar count list but also an inductive leap in understanding the meaning and structure of natural numbers. Multiple researchers have argued that, to learn how counting works, children construct a type of analogical mapping between the verbal count list and the ordered set of cardinalities that the list represents, beginning with the numbers one, two, and three (Carey, 2004; Gentner, 2010; Schaeffer et al., 1974; Wynn, 1992; for review, see Marchand & Barner, 2018). For example, Wynn (1992) argues that "in order to learn the counting system, children must implicitly make the analogy between the magnitudinal relationships of their own representations of numerosities, and the positional relationships of the number words" (p. 250). Similarly, according to Carey (2004), "Children may here make a wild analogy—that between the order of a particular quantity within an ordered list, and that between this quantity's order in a series of sets related by additional individuals. If the child recognizes this analogy, they are in a position to make the crucial induction: For any word on the list whose quantificational meaning is known, the next word on the list refers to a set with another individual added" (p. 67). Following this logic, Sarnecka and Carey (2008) created a measure that they called the Unit Task, in which children were told, for example, "OK. I'm putting FOUR frogs in the box," then saw one or two items added, and were then asked, "Now is it FIVE or SIX?" (trials included 4 + 1, 4 + 2, 5 + 1, 5 + 2). They found that children identified as CP-knowers by Wynn's Give-a-Number task succeeded on 67% of trials overall, whereas subset knowers performed at chance. Based on this, they concluded that becoming a CP-knower involves more than acquiring a procedural rule, and instead marks the moment at which children acquire the successor function.

On this account, the ability to accurately count objects therefore reflects the ability to link cardinal representations with implicit knowledge of a type of logical rule, called the "successor function," described by logicians and philosophers of mathematics including von Leibniz (1704/1996), Peirce (1881), Dedekind (1888/1963), and Peano (1889) in efforts to construct axioms to define the natural numbers. One such system, commonly referred to as the Peano–Dedekind axioms, includes principles akin to those in (1), inter alia (though various notational variants exist):

(1) i. 1 is a natural number.
ii. All natural numbers exhibit logical equality (e.g., $x = x$; if $x = y$, then $y = x$, etc.).
iii. For every natural number n, $S(n)$ (the successor of n) is a natural number.
iv. Every natural number has a successor.

Critically, because the Peano axioms state that every natural number has a successor, they generate an infinite number of numbers. Consequently, a child who has implicit knowledge of such rules would be expected to believe that it is always possible to add 1 to a number, and also that numbers never end. Thus, this account predicts that becoming a CP-knower represents a shift from representing numbers as a finite sequence of individual words to understanding them as products of a rule—the successor function—that generates an infinite set of positive integers (for discussion, see Sarnecka & Carey, 2008).

Does becoming a CP-knower involve learning to reason about cardinalities in terms of a recursive successor function? The results of Sarnecka and Carey (2008) leave open this question, since they do not test whether children who succeed on the Unit task generalize this knowledge to all numbers in their count list, let alone to all possible numbers. Also, while they showed that CP-knowers outperformed subset knowers on the Unit task, they did not show that success on the Unit task, and thus the ability to reason about cardinalities in terms of successor relations, was a prerequisite to becoming a CP-knower. Instead, they tested only very small numbers—four and five—leaving open the possibility that children's knowledge was item-based, rather than governed by an abstract logical rule. Also, the study left open the precise timeline according to which successor function knowledge is acquired.

Several subsequent studies have suggested that, if children acquire a rule akin to the successor function, this likely occurs several years later than predicted by Sarnecka and Carey (2008). First, Davidson, Eng, and Barner (2012) tested a large group of CP-knowers with the Unit Task, but included a slightly wider range of numbers, extending from 4 to 25. As a proxy for experience with number words, Davidson et al. asked children to count as high as they could, and binned them into low, medium, and high counters, analyzing Unit Task performance only for numbers within each child's count list. They found that almost all low counters (who could count up to 19) performed at chance on the Unit Task for numbers within their count range despite being CP-knowers, and that only the highest counters (who could count beyond 30) performed systematically well on small numbers. For larger numbers, all groups performed relatively poorly, even when those large numbers were well within their counting range. Similar results were found in a study of bilingual learners (Wagner et al., 2015), and in a training study which found that many CP-knowers lacked successor function knowledge (though CP-knowers were more likely than subset knowers to show improvement over 2–3 weeks of training; Spaepen, Gunderson, Gibson, Goldin-Meadow, & Levine, 2018). Also, children perform still poorer when asked to reason about predecessors in the Unit task (though this may reflect the working memory challenge associated with counting backwards, rather than children's understanding of how moving up and down the count list relates to number; see Kaminski, 2015; Sella & Lucangeli, 2020). Finally, in a study of slightly older children, Cheung, Rubenson, and Barner (2017) found that CP-knowers did not perform reliably above chance on the Unit Task for all numbers in their count list until around the age of 5 1/2—up to 2 years after children from comparable populations initially become CP-knowers. In sum, these studies suggest that children learn to reason productively about successor relations only after a protracted period of item-based learning. This raises the possibility that learning about successor relations depends on some gradual inductive inference or additional knowledge about count words that are not captured in these studies.

Critically, as noted above, the successor function does not merely state that for a particular number, n, there is a successor. Instead, it states that every number has a successor such that numbers are infinite (for similar treatments of the successor function, see e.g., Decock, 2008; Wright, 1983). Given that knowledge of particular numbers could plausibly reflect memorized knowledge rather than the application of a productive successor function, Cheung et al. (2017) paired the Unit Task (which tested how children implement the successor function in particular numbers) with an infinity interview first reported by Gelman and colleagues (e.g., Evans, 1983; Hartnett & Gelman, 1998), which tested children's beliefs about numbers as a class. In this battery, children were asked about the largest number they could name and whether it was the largest possible number, or whether it might be possible to repeatedly add 1 to it. This "successor question" tested whether children believe that numbers can be generated via a +1 rule. Children also answered an "endless question" about whether counting would get them to the end of numbers, or if numbers went on forever. Like earlier studies on this topic (Evans, 1983; Hartnett & Gelman, 1998), Cheung et al. (2017) found that children initially believe that numbers are finite, and that it is not always possible to add 1, but that by around the age of 6 many undergo a transition and begin to claim that numbers never end. Furthermore, they found that this knowledge emerged shortly after children were able to reason about successor relations and pass the Unit task for large numbers in their count list.

How might children learn, from knowledge of a finite count list, that numbers exhibit a successor function, and are infinite? Previous studies have found that children's ability to identify successors for known numbers, as measured by the Unit task, is related to how high they can count (Cheung et al., 2017; Davidson et al., 2012). For example, Cheung et al. (2017) found that children who could count up to at least 80 (many of whom could count even higher) were able to identify the cardinal value of successors for a wide range of known numbers within their count list, whereas children with lower highest counts could only do so reliably for the smallest numbers. This observation suggests at least two mutually compatible explanations for the relationship between counting experience and successor function knowledge. The first possibility is that there is no direct link between how high a child can count and successor knowledge, and that these two outcomes are correlated because they both result from general exposure to number. The second possibility is that there is a more direct causal link between the two: that children's understanding of how count words are syntactically structured might inform their intuitions regarding successor relations and infinity, and that this structure is only apparent after children have learned to count to relatively large numbers. Specifically, Cheung et al. (2017) noted that when children learn to count in English, they are required to learn a recursive base-10 structure wherein they first count from 1 to 9, then repeat this 1 through 9 structure with varying degrees of regularity for higher decades, which themselves are generated by multiplying 1–9 by 10 (for related proposals, see Barner, 2017; Hurford, 1987; Rule, Dechter, & Tenenbaum, 2015; Yang, 2016). Compatible with other cases of morphological learning (e.g., the past tense or plural; Pinker & Ullman, 2002), children may begin by simply memorizing items in their count list, but transition to a system of rules when the number of regular items is sufficiently large, resulting in a rule that can generate an unbounded number of exemplars (and in tandem, the intuition that numbers may never end). For example, after having learned to count to 40 or 50, a child might notice that after each decade term (20, 30, 40, etc.), the next number can be generated using an additive decade + unit rule, by adding the words one through nine in order as in Table 1.

Consistent with this hypothesis, previous studies have found that children who can count beyond 100 are better able to decompose numbers into decades and ones, whereas children who cannot count as high appear to store the count list as a memorized string (Fuson, Richards, & Briars, 1982; Siegler & Robinson, 1982). Also relevant is that children who cannot yet count all the way to 100 nevertheless make errors which suggest some knowledge of rules that structure counting. For example, when asked to count as high as they can, many children stop at decade transitions (Fuson et al., 1982; Siegler & Robinson, 1982; Wright, 1994), with the most frequent being 29 and 39 (Gould, 2017). If children were merely reciting a memorized and unstructured list as they do the alphabet, we might expect the distribution of their errors to be random rather than at decade transitions. Instead, their errors suggest that children have memorized an initial list, for example, up to 20 or 30, and use some form of morphological rule, like the one described above, to generate numbers up to the next decade transition (which requires memorized knowledge, since in English decade labels are irregular, and cannot be generated from a rule alone). Consistent with this, children exposed to languages with relatively transparent base-10 counting systems, like Mandarin or Cantonese Chinese, appear to count higher and make fewer errors than children learning less transparent counting systems, like English or Welsh (Miller, Smith, Zhu, & Zhang, 1995; Miller & Stigler, 1987; for related work, see Dowker, Bala, & Lloyd, 2008). Such evidence suggests that children make use of the linguistic structure of their count list to learn rules governing counting. An open question—and the main focus of the present study—is whether learning that number words are compositionally structured might facilitate insights into the conceptual structure of numbers, such as learning the successor function and infinity.

Critically, although the successor function logically entails that numbers are infinite, young children may not automatically compute the entailments of their beliefs, and may

Table 1
Example of decade + unit rule for "thirty" and "xty" and corresponding cardinalities

Decade Label	Unit Label	Decade + Unit Label	Cardinal Value (Defined by Successor Function)
thirty	one	thirty-one	31 = 30 + 1
	two	thirty-two	32 = 31 + 1
	three	thirty-three	33 = 32 + 1
	four	thirty-four	34 = 33 + 1
	n	thirty-n	3n = 3m + 1
	...	...	...
xty	n	xty-n	xtyn = xtym + 1

not infer from their successor knowledge that numbers never end. As noted by Cheung et al. (2017), many children in their study believed that it is always possible to add 1 to a set but nevertheless believed that numbers must ultimately end (children they called Successor Only Knowers), whereas only a handful of children held the opposite pattern of beliefs—that you cannot always add 1, but that numbers are nevertheless infinite (what they called Endless Only Knowers). Previous studies find the same pattern but report no Endless Only knowers at all (Evans, 1983; Hartnett & Gelman, 1998). Cheung et al. (2017) interpreted this pattern as evidence for a developmental sequence, whereby children learn some kind of bounded (item-based) successor rule that applies to a finite list, and only later learn that numbers never end. For example, children may first learn that known numbers exhibit a successor relation by empirically noticing this relation between familiar numbers, but may make the induction that this function is recursive by learning that *number words* can be productively generated via the additive decade + unit rule that governs counting. On this hypothesis, experience with counting may be related to both learning the successor function and that numbers are infinite, but for different reasons—for example, in the first instance providing opportunities for item-based observation regarding relations between known numbers, and in the second instance allowing children to discover rules that generate an unbounded set of numerical symbols.

While previous studies have shown that how high a child can count is related to their ability to identify successor relations of specific numbers (Cheung et al., 2017; see also Davidson et al., 2012), they have not tested whether knowledge of productive counting rules is related to the belief that numbers are endless, independently of the belief that every number has a successor. This is important, because learning rules that govern the structure of number words may allow children to make a broad inference about numbers—for example, by learning that number words can be decomposed into decades and ones, children may realize that such rules can generate an infinite set of number words, which may, in turn, form the basis for the belief that numbers are infinite. Notably, although this relation between counting and infinity might hinge on first acquiring knowledge of successor relations, this need not necessarily be the case. For example, children might learn that it is possible to generate indefinitely many numbers based purely on the syntax of the count list, and thereby infer that numbers must be infinite, even without yet understanding how this recursive rule relates to the possibility of adding 1. Given this possibility, it is important to test how children's knowledge of the structure of counting relates both to learning that it is always possible to add 1 to a number, and separately to learning that numbers are infinite.

In the present study, we had two goals. First, we sought to measure children's acquisition of the "decade + unit" rule to determine when such knowledge emerges, and how it is related to general counting experience (e.g., highest count). Second, we asked whether knowledge of such a rule was related to both acquisition of successor function knowledge and the belief that numbers are infinite. To do this, we presented 4- and 5-year-old children with three tasks.

We first assessed children's counting productivity using the "Highest Count" task. A critical difference between this study and previous work is how we evaluated children's highest count. Specifically, in previous studies of successor function knowledge, a child's highest count was defined as the highest number to which they could count before their first error. While this is likely a good first-pass measure of how much training children have received, it leaves open whether a particular child has acquired a productive rule or has simply memorized their count sequence. For this reason, we explored not only how high children could count without error but also which number they stopped on, and what they did when provided with the next number. As already noted, previous studies find that children's highest counts are not randomly distributed, and that instead their most common first error occurs at decade transitions, compatible with having learned a decade + unit rule. To test whether children count up to decades (e.g., up to 29, 39, or 49) by exploiting such a rule, we provided children who stopped at a decade transition with the next decade term (e.g., 30, 40, or 50) and asked whether they could then continue counting. We reasoned that children who have acquired a productive decade + unit rule should be able to count higher once decade terms are provided—that is, they should know that for any decade label N-ty, the next number in the counting sequence should be N-ty-one, followed by N-ty-two, etc.

Next, as an alternative test of whether children have acquired a productive rule for generating numbers, we used the Next Number task, in which children were told a number (e.g., "fifty-seven") and asked to generate the next number in the count sequence (i.e., "What comes next?"). We reasoned that children who understand the decade structure of counting should not merely represent the count list as a single memorized string but should be able to generate the next number for any decade—that is, they should exhibit knowledge of how the verbal count list implements the successor function. Based on this, we reasoned that productive knowledge of counting as measured by the Highest Count task should be correlated with performance on the Next Number task.

Finally, we tested children on a qualitative Infinity Interview and asked whether either of the two tests of counting productivity predicted children's intuitions about infinity. It is important to ask children questions that are not tied to specific numbers, because a child might be able to label the next number for a finite list of words, but not know the successor function—that is, that *every* number has a successor. In particular, we tested how our counting measures were related to children's belief that it is always possible to add 1 (i.e., successor function knowledge), and that numbers never end. Multiple past studies have reported ceiling performance on this task by age 6–7 years (Cheung et al., 2017; Evans, 1983; Hartnett & Gelman, 1998); thus, by studying younger children (ages 4 and 5), we investigated what factors contribute to variability in the acquisition of infinity knowledge.

To our knowledge, this is the first investigation linking children's productive counting ability to their beliefs about the successor function and infinity—beliefs which develop slowly but are integral components of number understanding. Consistent with previous studies of counting, we expected that a significant portion of our sample would have productive knowledge of the decade + unit rule. Finding that counting productivity predicts children's beliefs about the successor function and infinity would support the hypothesis that learning to count helps children to learn the successor function and suggest a specific connection between learning morphological rules of counting and discovering that numbers are infinite. Alternatively, if counting productivity does not predict either set of beliefs about infinity, we might interpret previous reports linking children's counting ability and successor function understanding as resulting from other shared factors, such as more generalized exposure to numbers, rather than a specific inference about the structure of the count list.

2. Method

2.1. Participants

We tested 122 4- and 5-year-old children (M = 5;0, SD = 7 months, range = 4;0 to 5;11, 58 males) recruited from preschools and museums in the San Diego metropolitan area. Although we did not collect demographic information from each participant, our sample was drawn from a population with the following statistics: White (75.5%), Black (5.5%), Asian (12.6%), American Indian or Alaska Native (1.3%), Pacific Islander (0.6%), and Multiracial (4.5)%.[1] All participants spoke English as a primary language. A stopping rule was defined such that 30 children were tested in each 6-month age bin within this range of ages. An additional 25 children participated but were excluded for the following reasons: their primary language was not English (n = 4), they did not complete the tasks (n = 10), parental interference (n = 1), experimenter error (n = 3), or failing the practice trials on the Next Number Task (n = 9).

2.2. Stimuli and procedure

2.2.1. Highest count task

We used the Highest Count task to measure children's spontaneous counting ability, and to classify children as either Productive or Non-Productive Counters based on their ability to recover from errors on decade transitions. Below, we describe procedures in detail. Fig. 1 shows the simplified decision tree describing the protocol and classification rules for determining productive counting knowledge.

To test children's knowledge of the counting sequence up to 99, we first asked each child to count as high as they could. If the child failed to respond, the experimenter said, "Let's count together! One..." with rising intonation to encourage the child to continue counting alone. We allowed children to count until they stopped naturally, and recorded errors made along the way. Each skipped number (e.g., "12, 13, 15"), skipped sequence of numbers (e.g., "18, 19, 30, 31"), or substitution error (e.g., "4, 9, 6") was counted as one error. Children were also allowed to self-correct or restart counting with no penalty (e.g., if a child counted "1, 2, 4, no, 1, 2, 3, 4, 5," 3 would not count as an error). To avoid underestimating children's counting ability due to lapses in attention, we always reminded children of the last number they had said whenever they stopped counting. For instance, a child who stopped at 25 was prompted with, "So what's after 25?" Children

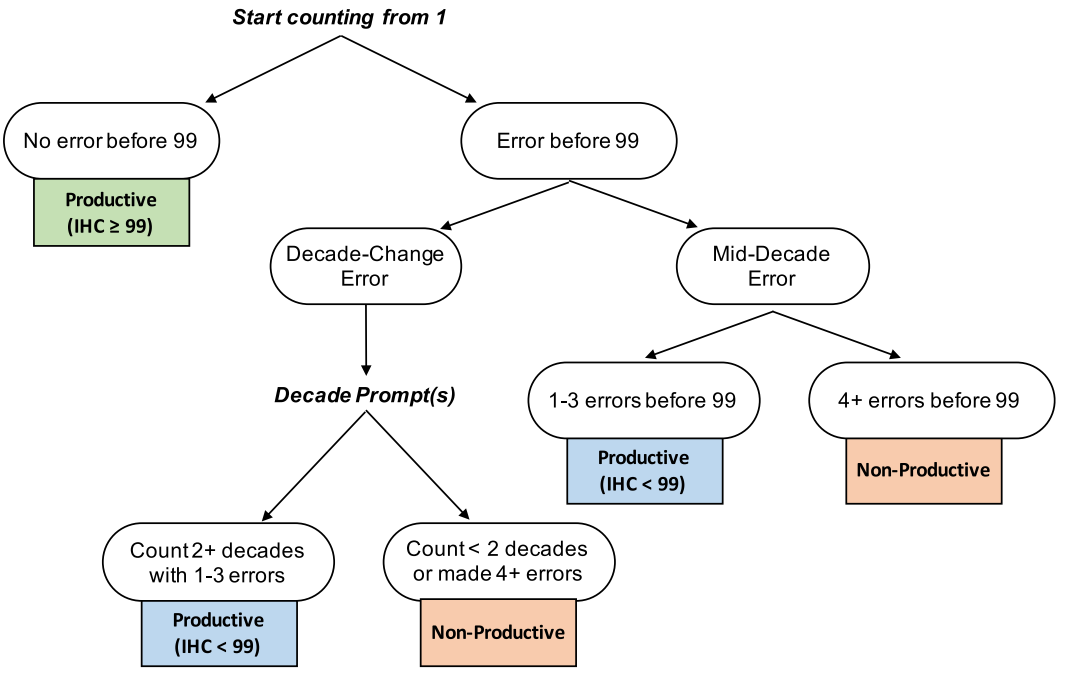

Fig. 1. Decision tree for productivity classification based on highest count task. Participants started counting from 1 and were classified as Non-Productive, Productive (IHC < 99), or Productive (IHC ≥ 99) counters.

were allowed to continue counting and receive as many reminders as necessary; these pauses and reminders were not considered errors. Using this method, we obtained children's initial highest count (IHC), which was the highest number children counted to before making any errors, either on their own or with reminders. We also obtained children's final highest count (FHC), which was the highest number children ever counted to that was part of a three-number consecutive sequence. We allowed for up to 10 errors, with at most three errors in a single decade. For some children, their FHC also included experimenter-provided decade prompts, as described below.

In this study, we were especially interested in testing whether children have a productive decade + unit rule for counting, as measured by their ability to either (a) count to 99 on their own with minimal errors or (b) extend their count sequence when provided with decade labels beyond their IHC. To measure the latter ability, during testing we first identified instances where children's first or second error occurred at a decade transition (e.g., stopping at 29, substituting "30" with "twenty-ten," or skipping 30 altogether). We called these Decade-Change Errors and provided children with a corrective decade prompt. For instance, a child who made an error after 29 was told, "After 29 is 30. Can you keep counting? 29, 30..." Children who successfully counted-up were provided with decade prompts for any subsequent Decade-Change errors they made, with decade prompts ranging from 20 to 90. Children who made no Decade-Change Errors did not receive any decade prompts and were simply allowed to continue counting until they stopped naturally.[2] The task ended whenever a child reached 99 or if they said they could not continue counting, whichever was earlier.

We also used the Highest Count task to classify participants as Productive or Non-Productive Counters, following criteria developed in a previously published, pre-registered report (Schneider et al., 2020). We reasoned that if a child has a productive decade + unit rule, then they should be able to count up to decade labels, but may not know what those decade labels are. These children should be expected to make errors on decade transitions (e.g., 39–40) but should also be able to continue counting once decade labels are provided. We therefore classified children as Productive if (a) they counted to 99 with three or fewer errors or (b) they could count at least two decades beyond their initial Decade-Change Error without making more than three errors in those two decades, including any decade-transition errors that elicited a decade prompt. This three-error criterion was developed because it allowed for two decade-transition errors and at most one mid-decade error. For example, a child whose initial error was at 29 but continued counting with decade prompts to 49 or higher was classified as a Productive Counter, but if they continued to only 39 or made too many errors before getting to 49, they were classified as a Non-Productive Counter.[3] Note that although alternative criteria are possible, we resisted exploring the full space of results that might result from different cut-offs. Still, in response to a reviewer request, in the Supplemental Materials, we provide one example of how a more conservative threshold (allowing just 1 error to Productive Counters) affects analyses. We show that this criterion generates a similar pattern of results, with only 5% of children classified differently (see Supplemental Material D for details).

2.2.2. *Next number task*

In this task, children were provided with a number and asked, "What comes next?" Children received two practice items (*one*, *five*) to ensure they understood the task, and corrective feedback was provided on these practice items if needed. For example, when asked, "*Five*. What comes next?," children who answered *four* were invited to count and figure out the correct answer (e.g., "No, *four* comes before *five*. What comes after *five*? Can you count and find out?"). All children in the final dataset successfully answered these practice questions before continuing.

Since we were interested in individual differences between children, all children received the same test items in a fixed order, ranging from 20 to 90: 23, 40, 62, 70, 37, 29, 86, and 59. These test items were designed to cover the range of counting abilities up to 99 and to utilize the decade rule for forming number words. No feedback was provided during the test trials.

2.2.3. *Infinity interview*

Following the protocol of previous studies of infinity knowledge (Cheung et al., 2017; Evans, 1983; Evans & Gelman, 1982; Hartnett & Gelman, 1998), we also assessed children's understanding of infinity by probing two types of belief: (a) that there is no biggest number ("Endless Knowledge") and (b) that it is always possible to add 1 to any number ("Successor Knowledge"). To probe this, we asked six questions as follows:

1. *"What is the biggest number you can think about?"* If the child did not answer, the experimenter probed them by asking how high they could count.
2. *"Is that the biggest number there could ever be?"*
    a. If yes, move on.
    b. If no, "*Can you think of a bigger number? Is that the biggest number there could ever be?"* The experimenter repeated this exchange up to four times or until the child affirmed that they had produced the biggest number.
3. "*If I keep counting, will I ever get to the end of numbers, or do numbers go on forever? Why?*"
4. "*If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more? Why/Why not?*"
5. "*You said the biggest number you know is X. Tell me, is it possible to add one to X, or is X the biggest number possible? Why?*" For this question, X was the largest number the child had stated in the entire testing session.
6. "*Could I keep adding one? Why/Why not?*"
    a. If yes, "*What would happen if I kept adding one?*"

We assigned each child a binary classification for both aspects of infinity understanding. Classifications were assigned by the first author and a coder blind to the hypotheses. The coding scheme was consistent with previous studies (Cheung et al., 2017; Evans, 1983; Hartnett & Gelman, 1998) and is provided in Supplemental Material (A), along with example transcripts. Initial agreement was 84.0% for Endless Knowledge coding (Cohen's Kappa = 0.63, $p < .001$) and 80.9% for Successor Knowledge coding (Cohen's Kappa = 0.62, $p < .001$). Disagreements were resolved through consulting a third coder.

To assess Endless knowledge of infinity, we coded whether children believed that there was a highest number such that numbers must end, or whether they believed that numbers go on forever. This coding was based on responses to questions 1–3 and 5. For example, in response to question 3, one child coded as lacking Endless knowledge claimed that 100 was the biggest number possible, and if we kept counting, it would get "long and get to one hundred." In contrast, another child with Endless knowledge of infinity responded, "Forever, [because] there's lots of numbers in the world." To assess Successor knowledge of infinity, we coded whether the child believed that it was always possible to add 1 to any number according to their responses to questions 4 and 6. Children who claimed it was impossible to keep adding 1, for instance because "it would get too big" or "it already has one," were coded as lacking Successor knowledge of Infinity. Finally, we identified children as "Full Infinity knowers" if they endorsed both aspects of infinity understanding.

3. Results

Compatible with the first goal of this study, our first set of analyses examined children's acquisition of the productive decade + unit rule using data from the Highest Count and Next Number tasks. Second, compatible with our second goal, we analyzed children's performance on the Infinity interview, and asked how their responses to the Successor Knowledge and Endless Knowledge items were related to counting experience and productive knowledge of the decade + unit rule.

All analyses were conducted in R (version 3.6.0, R Core Team, 2019). Regression models were constructed using either the R base stats package or, for models containing mixed effects, using lme4 (Bates, Mächler, Bolker, & Walker, 2015). For ease of interpretation, predictor variables were mean-centered for analyses.[4] To test for the significance of specific independent variables, we conducted likelihood ratio tests comparing models with and without particular effects of interest. For analyses containing within-subject measures, we constructed mixed-effects models containing random intercepts for all relevant grouping units (e.g., subject, item).

3.1. Characterizing decade + unit productivity

3.1.1. Productivity and counting ability

Fig. 2 presents the distribution of children's IHC by productivity classification. Overall, 73 children were classified as Productive Counters (M_{age} = 5;3, SD = 6 months) and 49 were classified as Non-Productive Counters (M_{age} = 4;7, SD = 5 months). In all, 32 children reached 99 on their IHC and were therefore classified as Productive Counters (IHC $\geq$ 99). Of the remaining 90 participants, 49 children (54%) made a Decade-Change error on either their first error (n = 37) or second error (n = 12) and thus received Decade Prompts. Of these, 32 (65%) continued counting two decades further with fewer than two additional errors and were classified as Productive, while 17 did not and were classified as Non-Productive. Another nine children were classified as Productive for counting past their IHC to reach 99 with up to three errors, without receiving any Decade Prompts. Note that this pattern of results is similar if a more conservative criterion of just one error is adopted. Using this criterion, six participants (5% of our sample) are reclassified from Productive to Non-Productive (for details, see Supplemental Material D).

Productive Counters had, on average, a higher IHC (M = 69.2, SD = 28.8, *Median* = 65) than Non-Productive Counters (M = 22.6, SD = 14.9, *Median* = 15). This difference remained if we considered only Productive Counters with IHC below 99 (M = 46.0, SD = 15.3, *Median* = 49). However, some Productive Counters (n = 9) had an IHC of 29 or lower.

Next, we analyzed precisely how far children could count past their initial error. Fig. 3 shows the participants' IHC, FHC, any errors made, and provided decade prompts, if any. While our classification criteria for productive counting only required children to count past their Decade-Change errors by two decades, most of the Productive Counters were

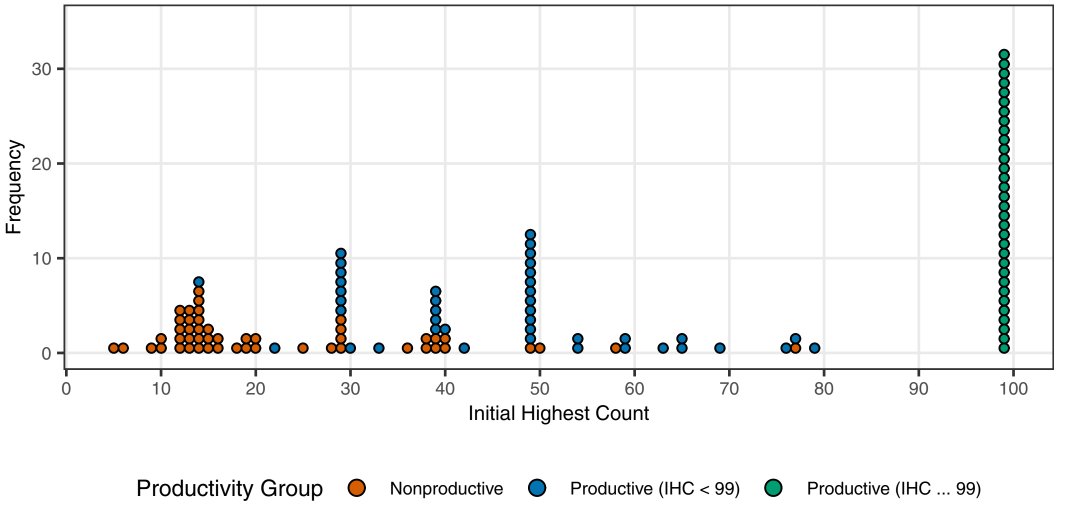

Fig. 2. Distribution of children's initial highest count by decade productivity.

able to count several decades further, and many of them reached 99 (*Mean number of prompts* = 3.5, *SD* = 1.6, *Range* = 1–7). Almost half of the Productive Counters (*n* = 32) received decade support on the Highest Count Task, and the remaining children (*n* = 41) counted to at least 99 without assistance and without making more than three errors. Productive Counters had a median FHC of 99 (*M* = 96.4, *SD* = 9.5, *Range* = 49–99), which was 34 numbers higher than their median IHC of 65. About a third of Non-Productive Counters (*n* = 15) received decade support (*Mean number of prompts* = 1.2, *SD* = 0.4, *Range* = 1–2), and the remaining children (*n* = 34) initially made mid-decade errors and therefore did not receive decade prompts. Non-Productive Counters had a median FHC of 29 (*M* = 32.0, *SD* = 17.6, *Range* = 5–99), which was only 14 numbers past their median IHC of 15. This difference is of course not surprising, since the ability to count at least two decades past the experimenter's first prompt was what defined the difference between Productive and Non-Productive children.

In summary, we found that between the ages of 4 1/2 and 5 1/2, many children in our study exhibited evidence of having learned a productive decade + unit rule. Overall, the average age of Productive Counters was 5 years, 3 months. Though some of these children (*n* = 41) were classified as Productive on the basis of having counted up to 99 with minimal errors, many received this classification because, upon stalling on a decade transition, they were able to recover once provided with a decade label, compatible with the use of a rule. These data not only provide evidence that children who stop on decade transitions do likely count using productive rules but also suggest that a child's IHC may not provide the best measure of their mastery of counting. Even many children who had quite low initial counts (e.g., below 30) were able to recover when provided a decade prompt, suggesting that some children may memorize only a small subset of the count list (e.g., less than 30) before extracting a rule.

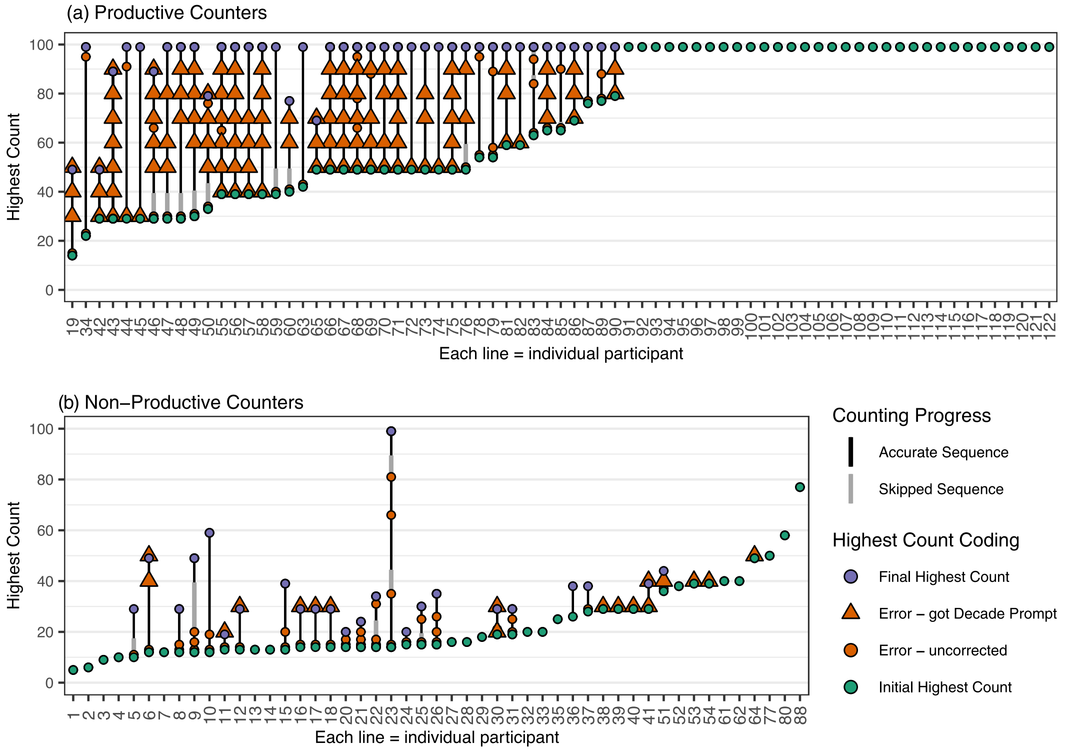

Fig. 3. Highest count profiles for (a) Productive Counters and (b) Non-Productive Counters. Each line represents highest counts achieved by an individual child. Initial highest count (green dots) shows highest number counted to before any error. Final highest count (purple dots) shows highest number counted to as part of a consecutive three-number sequence, allowing for up to 10 total errors and not more than three errors in a single decade. Errors include counting mistakes made on single numbers (orange dots and triangles) or over continuous intervals (orange lines). Decade Prompts (orange triangle) show decade terms provided by the experimenter after an error at a decade transition (e.g., stopping at 39). Black lines represent absolute gain from initial to final highest count.

3.1.2. Productivity and next number performance

The Productivity classification, above, was one of two measures of decade + unit rule knowledge that we explored in this study. We also tested this using the Next Number task. Here, we asked how our two candidate measures of decade + unit rule knowledge were related to one another.

First, we found that Productive Counters (71% correct; *SD* = 27%) significantly outperformed Non-Productive Counters (28%; *SD* = 26%) on the Next Number task ($t(120) = -8.76$, $p < .001$). However, recall that Productive Counters included children who could count to 99 on their own (IHC ≥ 99) and those who could not (IHC < 99). As shown in Fig. 4, Productive Counters with IHC ≥ 99 performed close to ceiling on the Next Number task (*M* = 91%, *SD* = 15%), likely because all the items tested were within their familiar count sequence. To obtain a stronger, more conservative test of our hypothesis that Productive Counters could utilize a decade rule to succeed on this task, we excluded Productive Counters with IHC ≥ 99 from the following analyses. Additionally, because Productive Counters have higher IHC and age than Non-Productive Counters, we deployed a regression strategy to include those two covariates.

Our second analysis thus used a mixed-effects logistic regression predicting trial-level accuracy from Productivity, IHC, age, and a Productivity by IHC interaction (with random intercepts for subject and item magnitude). Although accuracy was twice as high among Productive Counters (IHC < 99) (M = 56%, SD = 26%) compared to Non-Productive Counters (M = 28%, SD = 26%), this difference did not meet the threshold of statistical significance (β = 0.84, OR = 2.31, 95% CI = [1.01, 5.30], $\chi^2(1) = 3.00$, $p = .08$), though there was a significant effect of IHC (β = 0.90, OR = 2.47, 95% CI = [1.61, 3.78], $\chi^2(1) = 14.9$, $p < .001$). Age was also not a significant predictor ($\chi^2(1) = .0001$, $p = .99$). However, these effects are qualified by a significant Productivity by IHC interaction (β = −1.07, OR = 0.34, 95% CI = [0.16, 0.76], $\chi^2(1) = 7.01$, $p = .008$). Inspection of simple slopes within Productivity Group found that while IHC positively predicted accuracy among Non-Productive Counters (β = 1.39, OR = 4.01, 95% CI = [2.22, 7.24]), there was no effect for Productive Counters (β = 0.32, OR = 1.38, 95% CI = [0.78, 2.43]).[5]

This second analysis suggests the Next Number task and children's ability to countup from counting errors (i.e., our Productivity classification) capture different aspects of counting knowledge. Whereas IHC was a strong predictor of Next Number performance for Non-Productive Counters, it was not related to accuracy for Productive Counters. This was not because Productive Counters had a limited range of IHCs (indeed, some were as low as 29). Instead, one likely reason is that the Next Number and Productivity measures reflect performance on different ranges of the count sequence. Specifically, the Next Number task tests knowledge on both small and large numbers (e.g., from 23 to 86),

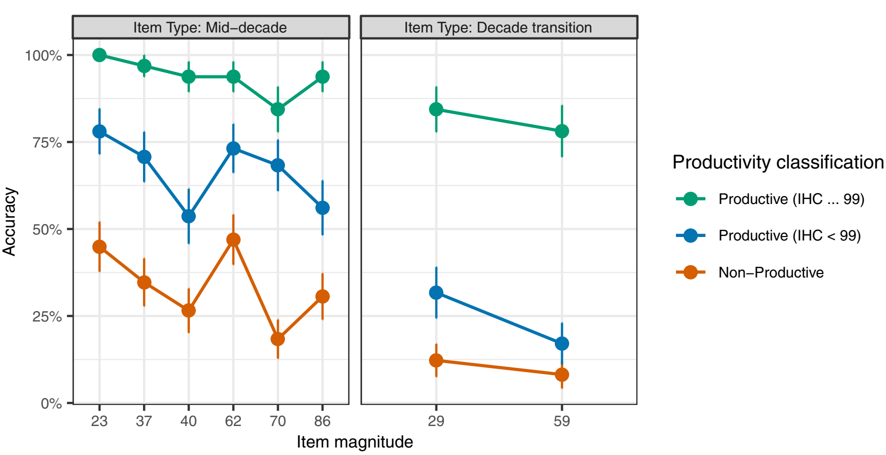

Fig. 4. Average proportion of correct response on Next Number task by item and decade productivity status. Error bars indicate standard error about the mean.

whereas to be classified as Productive, a child might be tested with only relatively small numbers (e.g., counting up to 49 from an initial error on 29, see Fig. 3). Also, the Next Number task may place greater working memory demands on children—since it requires counting up from arbitrary points in the count list—potentially making a more strongly routinized count list more valuable, requiring less working memory resources for the retrieval of numbers.

We further explored how the two tasks are related by conducting a *post-hoc* analysis testing whether Productive Counters (IHC < 99) show a selective advantage on the Next Number task for Mid-Decade items, where a productive decade rule ("*N*-ty-one, *N*-ty-two, ...") would be most beneficial, compared to decade transitions, where this rule would be less beneficial. Including this Item Type variable (Mid-Decade or Decade Transition) significantly improved model fit ($\chi^2(1) = 12.4$, $p < .001$). This model found a significant main effect of Item Type ($\beta = 2.27$, OR = 9.67, 95% CI = [4.03, 23.20], $\chi^2(1) = 25.79$, $p < .001$), with greater accuracy on Mid-Decade items ($M = 49\%$, $SD = 50\%$) than on Decade Transition items ($M = 17\%$, $SD = 37\%$). Adding a Productivity by Item type (Decade transition/Mid-decade) interaction did not significantly improve model fit ($\chi^2(1) = 0.29$, $p = .59$), indicating that both Productive Counters and Non-Productive Counters found mid-decade items easier than decade transition items.[6]

Next, we reasoned that, if a memorized count list is what allows children to generate successors on the Highest Count and Next Number tasks, then children should perform better on items within their IHC than beyond it. If, instead, children have acquired a rule that generates the decade structure, then we might find an interaction between Productivity Group and Item Range such that Productive Counters perform well both within and beyond their IHC, while Non-Productive Counters are only able to generate successors within their IHC. To test this prediction, we conducted a logistic mixed-effects regression predicting trial-level accuracy from Productivity, Item Range (Within/Beyond IHC), IHC, age, and the interaction of Productivity and Item Range, with random intercepts for subject and item magnitude. For this analysis, we excluded children with IHC $\geq$ 99 because all numbers tested on this task would be within their counting range.[7] Model comparison by likelihood ratio test found no significant main effect of either Productivity ($p = .07$) or Item Range ($p = .89$). Critically, there was a significant interaction effect of Productivity and Item Range ($\beta = 1.16$, OR = 3.20, 95% CI = [1.11, 9.21], $\chi^2(1) = 4.67$, $p = .031$). Planned contrasts indicate that performance on numbers within children's IHC was similar for Productive Counters (IHC < 99) ($M = 61\%$) and Non-Productive Counters ($M = 61\%$; $p = .99$ by t test), whereas accuracy for numbers beyond their IHC was significantly greater among Productive Counters ($M = 56\%$) than among Non-Productive Counters ($M = 27\%$; $t(88) = 4.98$, $p < .001$; see Fig. 5).

Collectively, analyses contrasting children's highest count behaviors and their performance on the Next Number task suggest that these measures capture slightly different phenomena. Both require children to draw on knowledge of the numeral sequence, but only the Next Number task requires children to generate number sequences from arbitrary points in the count list, without the benefit of the "momentum" afforded by the count routine for small and large numbers. Previous work finds that children's ability to count-up

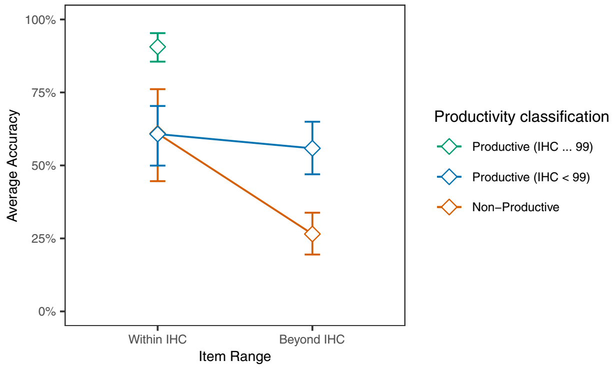

Fig. 5. Average proportion correct for each participant on the Next Number task by decade productivity status (between subjects) and Item Range (within subjects). Within IHC: correct answer is within a particular participant's initial highest count; Beyond IHC: correct answer is greater than that participant's initial highest count. Error bars show bootstrapped 95% confidence intervals about the mean. The largest number tested on this task was 86; thus, all trials were within IHC for productive counters with IHC $\geq$ 99.

from an arbitrary position in the count list is significantly affected by an experimenter verbally rehearsing preceding numbers, and thereby providing momentum (e.g., Fuson et al., 1982; Siegler & Robinson, 1982). In this sense, the Next Number task is a more difficult task, and perhaps a more stringent test of children's knowledge of the decade + unit rule.

3.2. *Characterizing infinity knowledge*

In the next set of analyses, we report children's successor function knowledge and beliefs about infinity. In the following section, we then probe how these are related to children's performance on the Highest Count and Next Number tasks.

We coded whether children thought every number has a successor ("Successor" knowledge) and separately, whether they thought numbers never end ("Endless" knowledge). Following previous work (Cheung et al., 2017), we also identified children who endorsed both beliefs as having "Full Infinity" knowledge (though we should note that, naturally, there is much more to fully grasping the notion of infinity than these two pieces of knowledge).

Table 2 reports the frequency of these outcome classifications. About half of our sample (48%, n = 59) demonstrated no knowledge of infinity, claiming that there was a biggest number and that it was not possible to keep adding 1. Non-Productive Counters were more likely to fall in this category than Productive Counters (67% of Non-Productive vs. 36% of Productive Counters, $\chi^2(1) = 10.58$, $p = .001$). A smaller fraction of our sample (19%, n = 23) exhibited full knowledge of infinity, claiming that there was no biggest number and that we could always keep adding 1. Productive Counters were more likely to exhibit full infinity knowledge than Non-Productive Counters (27% vs. 6%, $\chi^2(1) = 7.34$, $p = .007$). The remaining children (33%, n = 40) had partial knowledge of infinity—claiming either that you could always add 1 (n = 29) or that there was no biggest number (n = 11), but not both. The distribution of infinity knowledge categories is comparable to previously reported results (e.g., Cheung et al., 2017; Evans, 1983; Hartnett & Gelman, 1998), although our sample consists of slightly younger participants.

3.3. Predictors of infinity knowledge

In this section, we address how our measures of counting and productivity knowledge (IHC, Productivity Group, and Next Number task) were related to beliefs about infinity. Because the two components of infinity knowledge might develop independently, as indicated by the partial knowers who have one component knowledge but not the other, we conducted separate analyses to predict children's belief that every number has a successor (Successor Knowledge) and belief that numbers never end (Endless Knowledge). This allowed us to identify possible similarities or differences in relevant factors for acquiring each component of infinity knowledge. We also predicted children's status as Full Infinity Knowers, following Cheung et al. (2017).

Once again, our analyses excluded Productive Counters with IHC ≥ 99, so as to obtain a stronger, more conservative test of our hypothesis that Productive knowledge of counting might relate to beliefs about infinity.[8] Thus, we compared Productive Counters with IHC < 99 against Non-Productive Counters (total N = 90).

For our initial models, predictors included IHC, Productivity Group (Productive vs. Non-Productive Counters), and Next Number accuracy (proportion correct). Any predictor in these initial models that significantly predicted the outcome variable relative to a base model (with age as the only predictor) was added to a final model, to allow comparison among the predictors. Additionally, because IHC was significantly correlated with both Productivity Group ($\chi^2(1) = 40.27$, $p < .001$) and Next Number accuracy ($r(88) = .59$, $p < .001$), whenever these latter two variables were entered in the final model we also included IHC and its interaction to test for the role of decade + unit rule knowledge above and beyond rote counting ability. Models were constructed hierarchically, with model comparisons performed at every step using a likelihood ratio test, and with models selected on the basis of a significant chi-squared statistic and reduced AIC value. For details about model fits and model comparison results, see Tables 3–5.

Table 2
Frequency of infinity knowledge in Productive and Non-Productive Counters

Classification	Non-Productive Counters (N = 49)	Productive Counters, IHC < 99 (N = 41)	Productive Counters, IHC ≥ 99 (N = 32)	Total (N = 122)
No infinity knowledge	33 (67%)	18 (44%)	8 (25%)	59 (48%)
Only successor knowledge	12 (24%)	10 (24%)	7 (22%)	29 (24%)
Only endless knowledge	1 (2%)	4 (10%)	6 (19%)	11 (9%)
Full infinity knowledge	3 (6%)	9 (22%)	11 (34%)	23 (19%)

Interestingly, in predicting children's Successor Knowledge (Table 3), none of the three predictors explained a significant proportion of additional variance compared to the base model. In contrast, for models predicting children's possession of Endless Knowledge (Table 4), Productivity Group explained significant additional variance relative to the base model ($\chi^2(1) = 5.52$, $p = .019$; AIC = 84.9), though other measures of counting ability (IHC, Next Number accuracy) did not explain additional variance when controlling for age. This final model thus included only Productivity Group and age as predictors, and it estimated that Productive Counters were more likely than Non-Productive Counters to have Endless Knowledge ($\beta = 1.63$, $p = .03$, OR = 5.08, 95% CI = [1.30,23.50]), while Age was not a significant predictor ($\beta = 0.02$, $p = .94$). The effect of Productivity Group remained significant when controlling for IHC, although these more complex models did not explain any additional variance. Finally, we constructed models predicting children's status as Full Infinity Knowers (Table 5). None of our counting measures (IHC, Productivity Group, or Next Number accuracy) improved model fit compared to the base model with only age as a predictor.

In summary, we found that children who were classified as Productive, based on the Highest Count task, were significantly more likely than Non-Productive children to believe that numbers never end. Other measures, such as IHC and Next Number performance, were not as strongly related to infinity knowledge. In addition, none of our measures of counting knowledge were related to children's Successor Knowledge as measured by the infinity interview. Thus, in this study, we find that one measure of children's productive counting rules predicts a belief that numbers are endless, but none of these measures predict the belief that it is always possible to add +1 to a number. This suggests that children may learn successor relations between numbers independent of learning productive counting rules—for example, that this +1 rule describes the finite set

Table 3
Regression models for predicting successor knowledge on the infinity interview (participants IHC < 99, $N = 90$)

	Coefficient Estimates (β)				Summary Statistics		
Models	Age	IHC	Next Number Accuracy	Productivity Group	Log-Likelihood[a]	AIC	$R^2_{\text{Nagelkerke}}$
Base model							
Age	0.296				−58.75	121.51	0.027
Initial models							
Age + IHC	0.598*	−0.520			−57.05	120.10	0.077
Age + Next Number accuracy	0.358		−0.155		−58.55	123.09	0.034
Age + Productivity Group	0.157			0.496	−58.32	122.64	0.040

Notes. Coefficients were compared against 0 using *t* tests. Model comparisons done using likelihood ratio tests.
[a]Each initial model was compared against the base model.; $^*p < .05$.

Table 4
Regression models for predicting endless knowledge on the infinity interview (participants IHC < 99, $N = 90$)

Models	Coefficient Estimates (β)				Summary Statistics		
	Age	IHC	Next Number Accuracy	Productivity Group	Log-Likelihood[a]	AIC	$R^2_{\text{Nagelkerke}}$
Base model							
Age	0.451				−42.22	88.44	0.049
Initial models							
Age + IHC	0.261	0.349			−41.63	89.26	0.070
Age + Next Number accuracy	0.287		0.466		−41.04	88.07	0.090
Age + Productivity Group	0.025			1.625**	−39.46*	84.92	0.142
Final models							
Age + Productivity Group + IHC	0.012	0.044		1.586**	−39.45	86.91	0.142
Age + Productivity Group * IHC	0.009	0.105		1.555**	−39.40	88.79	0.144

Notes. Coefficients were compared against 0 using *t* tests. Model comparisons done using likelihood ratio tests.
[a]Each initial model was compared against the base model.; $^{*}p < .05$,; $^{**}p < .01$.

Table 5
Regression models for predicting full infinity knowledge on the infinity interview (participants IHC < 99, $N = 90$)

Models	Coefficient Estimates (β)				Summary Statistics		
	Age	IHC	Next Number Accuracy	Productivity Group	Log-Likelihood[a]	AIC	$R^2_{\text{Nagelkerke}}$
Base model							
Age	0.527				−33.91	71.81	0.058
Initial models							
Age + IHC	0.481	0.084			−33.88	73.76	0.059
Age + Next Number accuracy	0.417		0.307		−33.51	73.03	0.073
Age + Productivity Group	0.207			1.229	−32.71	71.43	0.104

Notes. Coefficients were compared against 0 using *t* tests. Model comparisons done using likelihood ratio tests.
[a]Each initial model was compared against the base model.

of that they know, but that a morphological decade rule may explain why children believe that numbers never end—that is, because number *words* can be productively generated.

4. Discussion

Given only finite experience with discrete quantities, number words, and counting, how do children learn that *every* natural number has a successor, and that numbers are endless? In this paper, we had two goals. First, we sought to characterize children's acquisition of productive morphological rules, and when this knowledge emerges in development. Second, we asked how such knowledge might be related to (a) their knowledge of the successor function (i.e., that it is possible to add 1 to any number) and (b) their beliefs regarding infinity (i.e., that numbers never end). Prior research suggests that how high children can count is related to their ability to identify successor relations for known numbers (e.g., Cheung et al., 2017), leading to the suggestion that counting experience causes children to notice the recursive base-10 structure of the count list, which, in turn, provides a basis for learning about successor relations and for generating unbounded number words (Barner, 2017; Cheung et al., 2017; Rule et al., 2015; Yang, 2016). Thus, learning a rule that generates successive number words might lead children to the belief that all numbers have successors, and that numbers never end. Our study found multiple pieces of evidence that some 4- and 5-year-old children, but not others, use a productive rule when counting. Also, we found that Productive Counters differed from Non-Productive Counters with respect to their understanding of numerical infinity, though, interestingly, not their successor function knowledge, per se.

Several results provide evidence that some, but not all, 4- and 5-year-old children use a productive decade rule when counting. First, when asked to count as high as they can, many children stopped at decade transitions, with about half making a decade-transition error as their first or second error. Whereas a memorized sequence predicts that errors should be randomly distributed over the count list, a decade rule predicts that they should occur disproportionately for irregular words that are not generated by a rule, such as decade labels (e.g., *twenty, thirty, fifty*). Furthermore, we found that when provided with decade prompts, many (65%) of the children who made decade transition errors could continue counting, often counting two or more decades further. Second, when tested on the Next Number task, some children correctly generated successors for numbers beyond their IHC, whereas some children did not, and could only name next numbers for items within their initial count. This, too, suggests that while some children's count list was purely memorized, unabetted by a decade rule, other children benefited from a rule that allowed them to identify next numbers on trials outside their familiar count routine. Third, we found some evidence that these two measures of productivity were related to one another, though imperfectly so. Although Productive Counters had higher average accuracy on the Next Number task than Non-Productive Counters, the strongest predictor of Next Number performance was children's IHC, not their Productivity classification. Also, whereas Productive and Non-Productive children performed similarly on the Next Number task for numbers within their IHC, Productive children performed significantly better than Non-Productive children for numbers outside their IHC, compatible with the use of a productive rule. The fact that IHC was a stronger predictor of Next Number performance than Productivity suggests that, though in their own ways compelling measures of children's decade rule knowledge, these tasks draw on different constructs, perhaps because only the Next Number task requires children to countup from arbitrary numbers without the benefit of momentum afforded by the count routine, and requires knowledge of both small and large numbers (Fuson et al., 1982; Siegler & Robinson, 1982).

In addition to characterizing several measures of counting productivity, our second goal was to explore how such measures might be related to children's intuitions about infinity. Overall, we found that children's belief that numbers never end was predicted by Productivity classification, but not other measures of counting proficiency, suggesting that children's ability to countup from a decade label provided by the experimenter is the best predictor of whether they think numbers never end. Interestingly, we also found that no measures of counting proficiency were related to children's successor function knowledge, as measured by the infinity interview. Specifically, children's judgment that it is always possible to add 1 to a number did not appear to be related to how high they could count, whether they could readily countup from a decade label, or whether they could identify the next number in an arbitrary position in the count list. While previous studies have found that children with full infinity knowledge often have a highest count around 100 (e.g., Cheung et al., 2017; Hartnett & Gelman, 1998), the present study is the first to investigate how counting proficiency relates to successor and endless knowledge of infinity separately. This combination of results suggests that children's knowledge of successor relations may be acquired separately from their intuition that numbers never end, either because the successor function is not necessarily the basis by which children learn about infinity, or because successor function knowledge is initially defined over a finite set of numbers, and only later rendered fully recursive—for example, by learning that recursive morphological rules of generate an infinite set of number words, and thus that the successor function can also be infinitely applied.

A starting point for this work was the observation in previous studies (Cheung et al., 2017) that children's acquisition of generalized successor function knowledge appears to be related to how high children can count. Critically, however, Cheung et al. only tested how counting abilities are related to children's reasoning about specific numbers, as measured by the Unit task (Sarnecka & Carey, 2008), but not whether counting was related to the two beliefs about how numbers behave in general. Our work tested whether counting—specifically learning a productive decade rule—might explain more general intuitions regarding the successor function and infinity. We reasoned that counting might be related to such intuitions in two broad ways. First, it might be the case that, as children are increasingly exposed to numbers, they acquire more knowledge about how those numbers operate (including but not restricted to successor relations), which they may generalize to all numbers, without making a specific connection between learning morphological rules of counting and discovering that numbers are infinite. An alternative, however, is that counting abilities might relate to knowledge of infinity specifically because the morphological rules that govern counting provide rules for generating ever larger numbers. Such rules might provide the basis for the belief that numbers never end. That is, learning the morphological rules may allow children to reason that number words can be productively generated, and thus conclude that numbers are endless. On this view, counting may be separately related to the belief that every number has a successor and to the belief that numbers are endless.

Our data are compatible with this distinction between successor function and infinity knowledge. Children may learn item-based successor relations early on, and may even believe that all numbers have a successor, despite believing that only a finite number of numbers exist—that is, they think the +1 function is bounded to a finite count list. About a quarter of our sample held such beliefs. Alternatively, children may have beliefs about numbers that support certain logical inferences, but not be aware of the entailments of these beliefs—for example, that a fully recursive function necessitates that numbers never end. As evidence for this, Harnett and Gelman (1998; see also Hartnett, 1991) note that children's beliefs about whether or not numbers are infinite can change over the course of a single testing session, simply by asking them what the highest number is, and whether it is possible to add 1 to it repeatedly. For instance, some children in their study were able to recognize the inconsistency between their belief that there was a largest number and their belief that it is always possible to add 1, and by the end of the testing session, had given up on either of these beliefs.

Our finding that children may acquire a productive counting rule by the time they have memorized a count list as short as 29 (see Figs. 2 and 3) presents a challenge to current computational models of number word learning. For example, Yang (2016) developed a "Tolerance Principle" which states that children invoke rules for explaining regularities in linguistic input when the number of exceptions or irregularities is below some threshold. According to his model, if there are N different linguistic tokens in the input, then a regular rule will be preferred only if the number of exceptions is below $N/\ln(N)$. In English, the early number words from 1 to 20 are exceptions to the decade rule, so the Tolerance Principle predicts that children would have to acquire a count list of at least length 72 before inducing a regular rule. Similar estimates in the 60–70 range were obtained by Rule et al. (2015) using a Bayesian architecture for inferring word to quantity mappings. However, our data suggest that some children can acquire a productive counting rule with much less data: The median IHC of Productive Counters was only 49 (ignoring those who reached 99 on their own). Future work should reconcile these empirical findings within computational models of number word learning.

Future work might also explore how intuitions regarding infinity may come from sources beyond the count list. One possible source, for example, is exposure to mathematical talk about infinity and explicit teaching about related ideas. Learning about formal mathematical operations like addition, or the recursive addition of zeros in Arabic notation, could help children appreciate ways of generating endlessly more quantities. For example, Singer and Voica (2008) describe how a group of fifth and sixth graders explained that rational numbers are infinite, because infinitely many digits could be added after the decimal point. Another possible source of intuitions is by analogy to quantities other than number, such as space, time, or geometry. One early study by Evans (1983) found that knowledge about infinity in number, time, and space developed similarly between kindergarten and third grade (e.g., from bounded to unbounded), but this research did not test if such beliefs were correlated within individual children (see

Hartnett & Gelman, 1998, for similar findings). Current studies in our labs are exploring this possibility. More recently, Smith, Solomon, and Carey (2005) found that first and second graders' intuitions about matter being infinitely divisible preceded their intuitions about the infinite divisibility of rational numbers. Numbers can be infinite in many ways (Monaghan, 2001); correspondingly, there is much room for future research into children's intuitions about infinity in different conceptual domains.

Much remains to be discovered about how children acquire rule-like knowledge of the count sequence and such knowledge helps shape beliefs about the logical structure of natural numbers. One limitation of the present work is that all tasks used real, attested numbers. Although our productivity analyses were meant to use errors as an indication of rule use, it remains possible that children's errors on decade terms were nevertheless affected by differences in rote memory for decade terms versus other count words (e.g., that somehow children's memory trace of "forty" is weaker than their memory trace of "forty-one" or "forty-two"). Any study that tests children's knowledge of attested number words faces this type of challenge. A stricter—and possibly more onerous—test of understanding the decade + unit rule might involve unfamiliar or made-up numbers. For instance, a child with productive knowledge might be able to judge that the successor of "a billion one" is "a billion two" and that the successor of "daxy-five" is "daxy-six." Recent work testing this prediction has found that children's ability to count up from these less familiar or novel numbers is correlated with the measures of productivity used here, and it is a strong predictor of their performance on a task assessing successor function knowledge (Schneider, Sullivan, Guo, & Barner, under review).

In addition to the factors we tested, it is likely that other domain-general capacities play a role in children's ability to succeed at our experimental tasks, including the ability to hold numbers in working memory and the more general ability to extract patterns—or rules—from linguistic data. Such differences may help explain why some children with low IHCs were able to count higher using provided decade labels, whereas other children with higher IHCs were not. However, previous studies suggest that domain-general limits are unlikely to fully explain individual differences in count list proficiency. In one cross-cultural study of successor function knowledge, English-speaking U.S. children outperformed children in India learning morphologically complex count lists in Hindi and Gujarati, even when controlling for working memory, age, and IHC (Schneider et al., 2020). Thus, while domain-general cognitive abilities are likely recruited in these tasks, they are importantly deployed over language-specific inputs that a child receives.

In conclusion, we suggest that sometime around 5 years of age (although surely variable by socioeconomic status and language), children learn to generate number words beyond their spontaneous counting range by implementing a recursive base-10 rule defined over their verbal count list. This insight may support an inductive inference over their acquired verbal count sequence, which facilitates conceptual insights into the infinite nature of the natural numbers. Consistent with recommendations from early childhood educators (Frye et al., 2013), the present results demonstrate another way that forward counting ability (i.e., counting up from one) may provide children with a useful foundation for learning more abstract natural number concepts such as cardinality, the successor function, and arithmetic. Our results suggest that understanding the logic of natural numbers is closely related to understanding the syntactic logic of the verbal numerals, suggesting one route by which learning language can impact learning about number concepts.

Acknowledgments

We thank Kaiqi Guo, Natasha Nowicka, Dawn Au, Eren Fukuda, and Sara Lee for their help with data collection and annotation. We are also grateful to the Fleet Science Center and participating families and preschools. This material is based upon work supported by the National Science Foundation under Grant #1749518 and by the James S. McDonnell Foundation.

Open Research badges

This article has earned Open Data and Open Materials badges. Data and materials are available at https://osf.io/z6ky3/.

Notes

1. For detailed demographic information regarding San Diego residents, see https://www.census.gov/quickfacts/fact/table/sandiegocountycalifornia,CA/PST045218
2. Upon coding the data, we noticed that some counting errors were not detected during the experiment such that some children (n = 10) received a decade prompt despite making mid-decade errors before their first Decade-Change Error. We classified these children as Non-Productive Counters irrespective of their subsequent counting progress, since they should not have received a decade prompt. Any counting past the erroneously provided decade prompt was excluded from our analyses.
3. Because we consider skipped sequences as single errors, this classification allows for the possibility that children improve by two decades with a single error that skips most of that sequence (e.g., “29, [prompt 30], 48, 49”). Post-hoc checks confirmed that no participants were classified as Productive Counters due to this loophole; all Productive Counters counted-up correctly from decade prompts (i.e., “[prompt 30], 31”).
4. Continuous variables were mean-centered and scaled by 1 standard deviation. Categorical variables (all binary in this paper) were also mean-centered and weighted by their group counts (i.e., weighted effect coding, see Grotenhuis et al., 2017).

This allows regression coefficients to be interpreted as standardized main effects and to be compared across models.

5. We obtain similar results when all Productive counters are included: there was a significant effect of Initial Highest Count, but not Productivity or Age. However, the interaction of Productivity and Initial Highest Count was no longer significant, likely because Productive Counters (IHC $\geq$ 99) performed at ceiling. Full analysis results are reported in Supplemental Material (B).
6. Again, we obtain similar results when all participants are included in the analysis.
7. We do not perform this analysis including all participants because Productive counters with Initial Highest Count $\geq$ 99 did not receive any trials outside their Initial Highest Count.
8. Analyses incorporating the full sample are included in Supplemental Material (C). Those analyses qualitatively mirror the results presented here, though with some differences: None of the counting productivity measures were predictive of Successor knowledge. All three measures, however, were predictive of children's Endless knowledge, with Productivity Group showing the largest effect. No single counting measure remained significant when entered into a full model predicting Endless knowledge, suggesting that the three measures explain overlapping variance. Finally, Next Number performance explained significant additional variance in predicting Full Infinity Knowledge, but the coefficient was not significantly different from 0 (p = .059), suggesting overlapping variance with age.

References

Almoammer, A., Sullivan, J., Donlan, C., Marušič, F., Žaucer, R., O'Donnell, T., & Barner, D. (2013). Grammatical morphology as a source of early number word meanings. *Proceedings of the National Academy of Sciences of the United States of America*, *110*(46), 18448–18453.

Barner, D. (2017). Language, procedures, and the non-perceptual origin of number word meanings. *Journal of Child Language*, *44*(3), 553–590. https://doi.org/10.1017/S0305000917000058

Barner, D., Libenson, A., Cheung, P., & Takasaki, M. (2009). Cross-linguistic relations between quantifiers and numerals in language acquisition: Evidence from Japanese. *Journal of Experimental Child Psychology*, *103*(4), 421–440. https://doi.org/10.1016/j.jecp.2008.12.001

Bates, D., Mächler M., Bolker B., Walker S. (2015). Fitting Linear Mixed-Effects Models Using lme4. *Journal of Statistical Software*, *67*, (1). https://doi.org/10.18637/jss.v067.i01

Carey, S. (2004). Bootstrapping & the origin of concepts. *Daedalus*, *133*(1), 59–68. https://doi.org/10.1162/001152604772746701

Cheung, P., Rubenson, M., & Barner, D. (2017). To infinity and beyond: Children generalize the successor function to all possible numbers years after learning to count. *Cognitive Psychology*, *92*, 22–36. https://doi.org/10.1016/j.cogpsych.2016.11.002

Chomsky, N. (1965). *Aspects of the theory of syntax*. Cambridge: MIT Press. Retrieved from https://mitpress.mit.edu/books/aspects-theory-syntax

Davidson, K., Eng, K., & Barner, D. (2012). Does learning to count involve a semantic induction? *Cognition*, *123*(1), 162–173. https://doi.org/10.1016/j.cognition.2011.12.013

Decock, L. (2008). The conceptual basis of numerical abilities: One-to-one correspondence versus the successor relation. *Philosophical Psychology*, *21*, 459–473. https://doi.org/10.1080/09515080802285255

Dedekind, R. (1888/1963). *Essays on the theory of numbers*. (W.W. Beman, Trans.). New York, NY Dover Publications.

Dowker, A., Bala, S., & Lloyd, D. (2008). Linguistic influences on mathematical development: How important is the transparency of the counting system? *Philosophical Psychology*, *21*(4), 523–538. https://doi.org/10.1080/09515080802285511

Evans, D. W. (1983). Understanding zero and infinity in the early school years (Unpublished doctoral dissertation). University of Pennsylvania. Retrieved from https://repository.upenn.edu/dissertations/AAI8326285/

Evans, D. W., & Gelman, R. (1982). Understanding infinity: A beginning inquiry (Unpublished manuscript).Philadelphia, PA: University of Pennsylvania.

Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). Teaching math to young children: A practice guide (NCEE 2014–4005). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Available at the NCEE website: http://whatworks.ed.gov

Fuson, K. C. (1988). *Springer series in cognitive development. Children's counting and concepts of number*. Heidelberg, Germany: Springer-Verlag.

Fuson, K., & Hall, J. W. (1983). The acquisition of early number word meanings: A conceptual analysis and review. In H. Ginsburg (Ed.), *The development of children's mathematical thinking* (pp. 49–107). New York: Academic Press.

Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition* (pp. 33–92). Springer Series in Cognitive Development. New York, NY: Springer.

Gelman, R., & Gallistel, C. R. (1978). *The child's concept of number*. Cambridge: Harvard University Press. Retrieved from https://www.hup.harvard.edu/catalog.php?isbn=9780674116375

Gentner, D. (2010). Bootstrapping the mind: Analogical processes and symbol systems. *Cognitive Science*, *34* (5), 752–775. https://doi.org/10.1111/j.1551-6709.2010.01114.x

Gould, P. (2017). Mapping the acquisition of the number word sequence in the first year of school. *Mathematics Education Research Journal*, *29*(1), 93–112. https://doi.org/10.1007/s13394-017-0192-8

Grotenhuis, M., Pelzer, B., Eisinga, R., Nieuwenhuis, R., Schmidt-Catran, A., & Konig, R. (2017). When size matters: advantages of weighted effect coding in observational studies. *International Journal of Public Health*, *62*(1), 163–167. https://doi.org/10.1007/s00038-016-0901-1

Hartnett, P. M. (1991). The development of mathematical insight: From one, two, three to infinity (Unpublished doctoral dissertation).University of Pennsylvania.

Hartnett, P. M., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction*, *8*(4), 341–374. https://doi.org/10.1016/S0959-4752(97)00026-1

Hurford, J. R. (1975). *The linguistic theory of numerals. Cambridge studies in linguistics* (Vol. 16). Cambridge: Cambridge University Press.

Hurford, J. R. (1987). *Language and number: The emergence of a cognitive system*. Oxford, UK: Blackwell.

Kaminski, J. (2015). Young children's understanding of the successor function. Proceedings of the 37th Annual Conference of the Cognitive Science Society Meeting (pp. 1045–1050). Pasadena, CA: Cognitive Science Society.

Le Corre, M., Li, P., Huang, B. H., Jia, G., & Carey, S. (2016). Numerical morphology supports early number word learning: Evidence from a comparison of young Mandarin and English learners. *Cognitive Psychology*, *88*, 162–186. https://doi.org/10.1016/j.cogpsych.2016.06.003

Marchand, E., & Barner, D. (2018). Analogical mapping in numerical development. In D.B. Berch, D.C. Geary, & K.M. Koepke (Eds.), *Language and culture in mathematical cognition (Vol. 4)*. Cambridge, MA: Academic Press.

Miller, K. F., Smith, C. M., Zhu, J., & Zhang, H. (1995). Preschool origins of cross-national differences in mathematical competence. *Psychological Science*, *6*(1), 56–60. https://doi.org/10.1111/j.1467-9280.1995.tb00305.x

Miller, K. F., & Stigler, J. W. (1987). Counting in Chinese: Cultural variation in a basic cognitive skill. *Cognitive Development*, *2*(3), 279–305. https://doi.org/10.1016/S0885-2014(87)90091-8

Monaghan, J. (2001). Young peoples' ideas of infinity. *Educational Studies in Mathematics*, *48*, 239–257.

Peano, G. (1889). The principles of arithmetic presented by a new method. In J. van Heijenoort (Ed), *From Frege to Gödel. A source book in mathematical logic. 1879–1931* (pp. 83–97). Cambridge, UK: Harvard University Press.

Peirce, C. S. (1881). On the logic of number. *American Journal of Mathematics*, *4*(1/4), 85.

Piantadosi, S. T., Jara-Ettinger, J., & Gibson, E. (2014). Children's learning of number words in an indigenous farming-foraging group. *Developmental Science*, *17*(4), 553–563. https://doi.org/10.1111/desc.12078

Pinker, S., & Ullman, M. T. (2002). The past and future of the past tense. *Trends in Cognitive Sciences*, *6* (11), 456–463. https://doi.org/10.1016/S1364-6613(02)01990-3

R Core Team. (2019). *R: A language and environment for statistical computing*. Vienna, Austria. Available at: https://www.r-project.org

Rule, J., Dechter, E., & Tenenbaum, J. (2015). Representing and learning a large system of number concepts with latent predicate networks. In D.C. Noelle, R. Dale, A.S. Warlaumont, J. Yoshimi, T. Matlock, C.D. Jennings, & P.P. Maglio (Eds.), *Proceedings of the 37th annual meeting of the Cognitive Science Society*. Austin, TX: Cognitive Science Society.

Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition*, *108*(3), 662–674. https://doi.org/10.1016/j.cognition.2008.05.007

Sarnecka, B. W., Kamenskaya, V. G., Yamana, Y., Ogura, T., & Yudovina, Y. B. (2007). From grammatical number to exact numbers: Early meanings of 'one', 'two', and 'three' in English, Russian, and Japanese. *Cognitive Psychology*, *55*(2), 136–168. https://doi.org/10.1016/j.cogpsych.2006.09.001

Sarnecka, B. W., & Lee, M. D. (2009). Levels of number knowledge during early childhood. *Journal of Experimental Child Psychology*, *103*(3), 325–337. https://doi.org/10.1016/j.jecp.2009.02.007

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974). Number development in young children. *Cognitive Psychology*, *6*(3), 357–379. https://doi.org/10.1016/0010-0285(74)90017-6

Schneider, R. M., Sullivan, J., Guo, K., & Barner, D. (under review) What counts? Sources of knowledge in children's acquisition of the successor function.

Schneider, R. M., Sullivan, J., Marušič, F., Žaucer, R., Biswas, P., Mišmaš, P., Plesničar, V., & Barner, D. (2020). Do children use language structure to discover the recursive rules of counting? *Cognitive Psychology*, *117*, 101263. https://doi.org/10.1016/j.cogpsych.2019.101263

Sella, F., & Lucangeli, D. (2020). The knowledge of the preceding number reveals a mature understanding of the number sequence. *Cognition*, *194*, 104104. https://doi.org/10.1016/j.cognition.2019.104104

Siegler, R. S., & Robinson, M. (1982). The development of numerical understandings. *Advances in Child Development and Behavior*, *16*, 279–312. https://doi.org/10.1016/S0065-2407(08)60072-5

Singer, F. M., & Voica, C. (2008). Between perception and intuition: Learning about infinity. *Journal of Mathematical Behavior*, *27*(3), 188–205. https://doi.org/10.1016/j.jmathb.2008.06.001

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, *51*(2), 101–140. https://doi.org/10.1016/j.cogpsych.2005.03.001

Spaepen, E., Gunderson, E. A., Gibson, D., Goldin-Meadow, S., & Levine, S. C. (2018). Meaning before order: Cardinal principle knowledge predicts improvement in understanding the successor principle and exact ordering. *Cognition*, *180*, 59–81. https://doi.org/10.1016/j.cognition.2018.06.012

von Humboldt, W. (1836/1999). On language (P. Heath, Trans.). Cambridge: Cambridge University Press.

von Leibniz, G. W. (1996). New essays on human understanding (P. Remnant & J. Bennett, Trans.). Cambridge: Cambridge University Press.

Wagner, K., Kimura, K., Cheung, P., & Barner, D. (2015). Why is number word learning hard? Evidence from bilingual learners. *Cognitive Psychology*, *83*, 1–21. https://doi.org/10.1016/j.cogpsych.2015.08.006

Wright, C. (1983). *Frege's conception of numbers as objects*. Aberdeen, UK: Aberdeen University Press.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics*, *26*(1), 25–44.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*(2), 155–193. https://doi.org/10.1016/0010-0277(90)90003-3

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251. https://doi.org/10.1016/0010-0285(92)90008-P

Yang, C. (2016). The linguistic origin of the next number. Available at: https://ling.auf.net/lingbuzz/003824. Accessed November 1, 2019.

Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article:

Supplementary Material. The Supplementary Material contains details about the Infinity interview and supplementary analyses using different productivity classification criteria.