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Diagrams benefit symbolic problem-solving

Junyi Chu[1]*, Bethany Rittle-Johnson[2] and Emily R. Fyfe[3]

[1]Department of Psychology, University of California San Diego, La Jolla, California, USA

[2]Department of Psychology and Human Development, Vanderbilt University, Nashville, Tennessee, USA

[3]Department of Psychological and Brain Sciences, Indiana University, Bloomington, Indiana, USA

Background. The format of a mathematics problem often influences students' problem-solving performance. For example, providing diagrams in conjunction with story problems can benefit students' understanding, choice of strategy, and accuracy on story problems. However, it remains unclear whether providing diagrams in conjunction with symbolic equations can benefit problem-solving performance as well.

Aims. We tested the impact of diagram presence on students' performance on algebra equation problems to determine whether diagrams increase problem-solving success. We also examined the influence of item- and student-level factors to test the robustness of the diagram effect.

Sample. We worked with 61 seventh-grade students who had received 2 months of pre-algebra instruction.

Method. Students participated in an experimenter-led classroom session. Using a within-subjects design, students solved algebra problems in two matched formats (equation and equation-with-diagram).

Results. The presence of diagrams increased equation-solving accuracy and the use of informal strategies. This diagram benefit was independent of student ability and item complexity.

Conclusions. The benefits of diagrams found previously for story problems generalized to symbolic problems. The findings are consistent with cognitive models of problem-solving and suggest that diagrams may be a useful additional representation of symbolic problems.

Concrete external representations, such as diagrams and physical models, are commonly used tools in mathematics that can influence students' learning of concepts and problem-solving performance (Belenky & Schalk, 2014). One way that these external representations may influence student outcomes is by changing how students internally represent problems, which in turn influences how they solve the problems (Koedinger, Alibali, & Nathan, 2008).

Diagrams are one type of concrete representation thought to aid learning and problem-solving. Indeed, research in cognitive science provides abundant evidence for the benefits

*Correspondence should be addressed to Junyi Chu, Department of Psychology, University of California San Diego, 9500 Gilman Drive, MC 0109, La Jolla, CA 92093, USA (email: junyichu@ucsd.edu).

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of diagrams in conjunction with relevant text (Mayer, 2005). Important features distinguish diagrams from other types of illustrations. In this article, we define diagrams as schematic visual representations that express information via spatial relationships. Specifically, we were interested in diagrams which reflect quantitative information via spatial units that obey mathematical part–whole relationships. Consider a diagrammatic versus a pictorial representation of a story problem about using a measuring stick to estimate the depth of the sea (Figure 1). The drawing on the left is a diagram that spatially represents quantitative information about length and depth as physical distances, but the drawing on the right is a picture that includes boats, fish, and other information irrelevant to the problem solution. Previous research has found that diagrams, but not pictures, are beneficial to problem-solving (e.g., Hegarty & Kozhevnikov, 1999). Thus, in this study, we focused on diagrams as our concrete external representation. Specifically, we compared students' performance on symbolic algebra equations in the presence or absence of a relevant diagram.

We selected symbolic algebra equations as our target task because students have difficulty reasoning about abstract, unknown quantities in algebra. According to Koedinger *et al.* (2008), algebra is the 'first abstract symbolic language' (p. 367) that most people encounter in school after learning natural language. Students often have difficulty comprehending and producing algebraic equations (Knuth, Stephens, McNeil, & Alibali, 2006; Payne & Squibb, 1990). Further, algebra is an important topic for secondary students as it acts as a gatekeeper subject that influences later academic and career success (Atanda, 1999; Horn, Kojaku, & Carroll, 2001; U.S. Department of Education, 1997). Thus, we explored whether diagrams could help beginning algebra students to better understand and solve symbolic algebraic equations.

There are at least three reasons why diagrams might be helpful for solving symbolic equations. First, diagrams may highlight key relations between quantities and operations in the problem and help students extract this relevant information efficiently (Larkin & Simon, 1987). In algebraic problems involving variables with unknown values, diagrams may describe mathematical relationships between known and unknown quantities, highlighting relevant units for subsequent operations and allowing students to use part-part-whole concepts to solve the problem (Ng, 2015). Second, diagrams may decrease cognitive demands (Munez, Orrantia, & Rosales, 2013; Murata, 2008). For example, a neuroimaging study with adults revealed that solving a story problem by

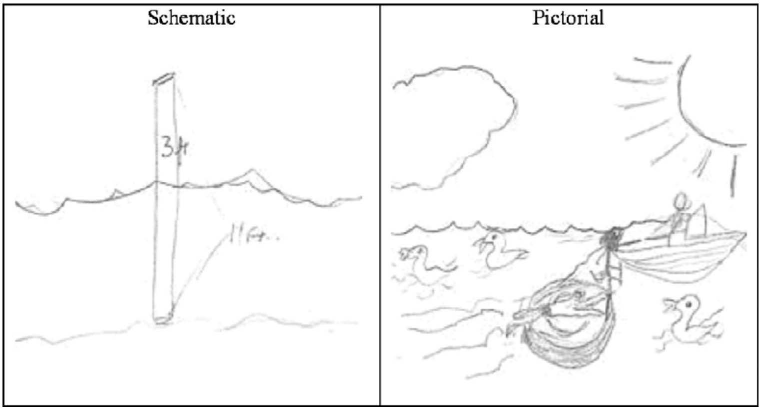


Figure 1. Example of schematic and pictorial drawing. *Note.* Adapted from Edens and Potter (2008), p. 186.

constructing a mental diagram required fewer attentional resources than by constructing a mental equation (Lee *et al.*, 2007). Third, diagrams may facilitate connections between concrete and abstract representations (Koedinger & Terao, 2002; Lee, Khng, Ng, & Ng, 2013). Specifically, diagrams may elicit students' intuitive, informal knowledge, and strategies, which may allow students to connect this knowledge to formal, symbolic problem formats.

Although some research has found benefits of using diagrams during problem-solving, the vast majority of it has focused on story problems. For example, one line of research focuses on individual differences in the spontaneous use of diagrams during story problem-solving. More accurate story problem solvers were more likely to construct diagrams, either by drawing on paper or by mental visualization (Edens & Potter, 2008; Hegarty & Kozhevnikov, 1999; Van Garderen & Montague, 2003).

A second line of research on diagram benefits comes from cross-cultural comparisons of instructional practice. Countries such as Japan and Singapore have long incorporated diagrams into math instruction, and these countries typically are the top performing countries in international mathematics assessments (Mayer, Sims, & Tajika, 1995; Murata, 2008; Ng & Lee, 2009). For example, in Singapore, first- and second-grade students are introduced to horizontal bar diagrams to solve story problems (Ng & Lee, 2009). Students who construct a diagram are able to use informal arithmetic strategies to solve algebraic problems, thus making algebraic problems accessible earlier – beginning in third grade in Singapore, as opposed to seventh grade in most U.S. classrooms (Lee *et al.*, 2013). Promisingly, teaching sixth-grade pre-algebra students in the United States to use this diagram enabled them to solve algebra story problems successfully (Koedinger & Terao, 2002). Further, interventions that include practice with diagrams have helped U.S. elementary school children improve their story problem-solving success (Jitendra *et al.*, 2007).

A third line of research experimentally manipulated whether diagrams were presented alongside story problems. Munez *et al.* (2013) used novel diagrams consisting of vertical rectangular bars. These diagrams were clearly labelled with relevant variable names and numerical values from the story problem statements. High-school students had greater accuracy and lower response times on arithmetic story problems when diagrams were present than not. Further, the diagram benefit was greatest on more difficult problems. These findings are consistent with similar research with undergraduates (Lewis, 1989).

However, the diagram benefit may be less robust for middle-school students. Booth and Koedinger (2012) assessed sixth- to eighth-grade students on three algebraic problems presented in one of three formats: equation, story, or story-with-diagram. While high-ability students of all grades performed equally with or without diagrams, low-ability students in seventh- and eighth-grade (but not sixth-grade) were more accurate and made fewer conceptual errors on story-with-diagram problems than the other two problem formats. Further, these students primarily benefited from diagrams on more complex problems where the unknown variable appeared twice (e.g., $N - 1/5 * N = 30$), in line with other research that the diagram benefit is strongest on more complex problems (Lewis, 1989; Munez *et al.*, 2013). This suggests that age, ability, and problem complexity may be key moderators of any potential diagram benefit.

In sum, diagrams have been shown to help students interpret and solve story problems, including those with an algebraic structure, although results are less consistent with middle-school students than with high-school and college students. However, little is known about the use of diagrams with more symbolic tasks such as algebraic equations, which are inherently different from story problems. For example, equations are more abstract than story problems and require students to decode the symbolic language of algebra (Payne & Squibb, 1990). Further, it may be unnecessary to construct diagrams for equations, as equations already highlight problem structure and contain less irrelevant information than story problems. In fact, beginning algebra students in Singapore report preferring algebraic over diagram solutions to equation problems, despite years of using diagrams to solve similar problems in story problem format (Ng, 2003). This suggests that a diagram benefit may not easily transfer to algebra equation-solving. Given the potential benefits of diagram representations, and the strong relationship between early algebra competence and future career and life success, it is important to study to what extent a diagram benefit generalizes to algebra equation problems.

In this study, we adopted the general approach from Booth and Koedinger (2012) to test for a diagram benefit, but paired diagrams alongside algebraic equations instead of story problems. Further, we made several changes to overcome design limitations. For example, of the three test problems used in Booth and Koedinger (2012), two included spatial diagrams and the third used a pictorial representation. As a result, a diagram benefit was found for the first two problems, but not for the third problem. However, there was a potential confound between problem difficulty and diagram type as the problem with the pictorial representation was also the most structurally complex problem. In this study, we used consistent, spatial diagrams and systematically manipulated problem complexity. Booth and Koedinger (2012) also found that seventh- and eighth-grade students, but not sixth-grade students, benefited from diagrams. However, this may be explained by two confounded factors which both improve over time: (1) diagram-specific skills due to increased exposure and practice with diagrams or (2) general mathematical problem-solving ability. In this study, we used novel diagrams and implemented a diagram introduction phase to control for students' prior diagram familiarity. Further, we tested students with different levels of general mathematical ability.

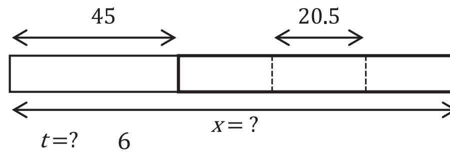
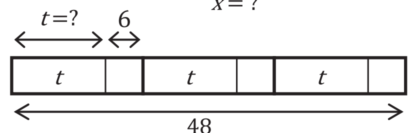
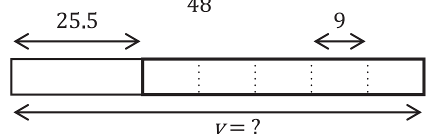
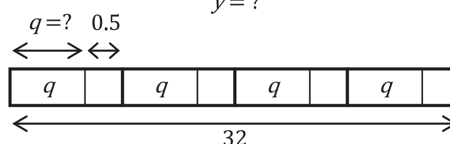
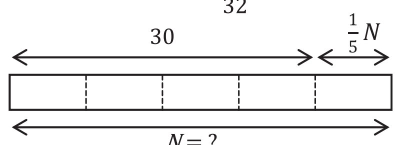
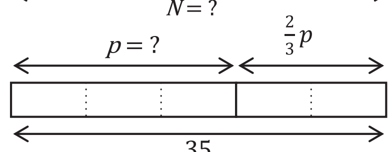
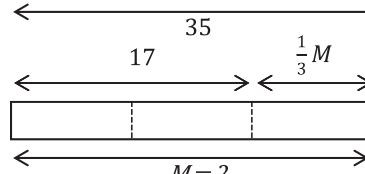
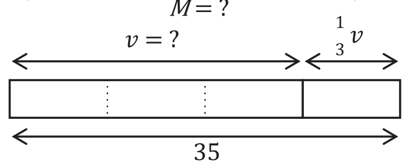
In line with findings from story problem experiments (e.g., Booth & Koedinger, 2012; Koedinger & Nathan, 2004), we hypothesized that providing diagrams alongside equations would improve problem-solving accuracy by eliciting students' intuitive knowledge of quantitative relations in the problem. This should increase both the usage of non-algebraic strategies and the accuracy of algebraic strategies, while reducing the frequency of conceptual errors. We also explored whether the effect of diagrams would depend on item or student characteristics, including problem complexity, general math ability, and students' understanding of the relationship between a diagram and equation representation. We varied problem complexity by including both single- and double-reference problems (Table 1), and we explored general math ability by working with students from advanced and regular mathematics classes. Finally, we measured students' understanding of the diagram by asking them to convert diagrams into equations, and vice versa.

Method

Participants

Participants were 62 seventh-grade students from four classrooms attending an independent private school in the south-eastern United States serving a middle-class population, all taught by the same math teacher. Students were tested approximately 2 months into their first pre-algebra course. Two of the four classrooms (34 students) were in an advanced math class, which covered the same breadth of content, using the same textbook, but in greater depth. Students had experience with reading algebraic expressions and solving simple one step equations, but had not studied the equation forms

Table 1. Equations and diagrams from the equation-solving assessment

Problem complexity	Equation	Diagram
Single-reference	$\frac{(x-45)}{3} = 20.5$	
	$3(t+6) = 48$	
	$\frac{(y-25.5)}{5} = 9$	
	$4(q+0.5) = 32$	
Double-reference	$N - \frac{1}{5}N = 30$	
	$p + \frac{2}{3}p = 35$	
	$M - \frac{1}{3}M = 17$	
	$v + \frac{1}{3}v = 32$	

or the type of diagrams used in the experiment. We excluded data from one student who showed visible frustration throughout the experiment, had difficulty understanding the task, and did not attempt any of the assessment items. The final sample contained 61 students (33 males) aged 12–13 years (M = 12.7, SD = 0.3).

Design

Students participated in an experimenter-led classroom session during their regular 50-min math period. All students completed three tasks in the same order: (1) introduction and practice with diagrams, (2) diagram/equation translation task, and (3) equation-solving. During equation-solving, we manipulated the presence of diagrams using a within-subjects design. Specifically, students solved algebra problems in two matched formats (equation and equation-with-diagram). Calculators were not permitted and there were no time limits on any task.

Materials and procedure

Throughout the experiment, two types of algebra equations were used as follows: (1) single-reference problems, which involved only one instance of the variable, and (2) double-reference problems, in which the unknown variable appeared twice, as in previous research (Booth & Koedinger, 2012; Koedinger *et al.*, 2008). Double-reference problems are more difficult because simpler strategies such as guess and test and substitution are less likely to be effective. The diagrams were adapted from the Singapore curriculum and constructed according to these guidelines: (1) rectangular bars with solid borders represented quantities, (2) dotted vertical lines were used to divide the rectangular bars into equal portions, (3) rectangular bars were labelled internally with variables, and (4) horizontal arrows over the length of a bar indicated the quantity's magnitude and were labelled with known values or with a question mark. Diagrams were drawn to approximate the relative quantities in each problem, but were not of the same scale across problems. Table 1 shows examples of equations and corresponding diagrams for each type of problem complexity.

Diagram introduction and practice

At the beginning of the session, the experimenter (the first author) provided a general introduction describing diagrams as a special kind of picture that represents information about numbers and quantities. She used four examples to explain the guidelines used to construct diagrams, to highlight important features (e.g., arrows, labels, dotted lines), and to describe how diagrams could represent each operation (i.e., addition, subtraction, multiplication, and division). For each example, students were asked to copy the diagram onto a worksheet. These example diagrams were simpler than the diagrams in the rest of the experiment. The intent was to develop basic knowledge of interpreting the diagrams as they were unfamiliar to the students. No references to equations were made. This took approximately 8 min. Then, students were asked to work individually on four diagram problems (refer to Appendix S1 for exact problems). Students had to solve for an unknown variable from a given diagram. The corresponding equation was not included. The first three problems were single-reference problems, and the fourth was a double-reference problem. After 10 min of individual work, the experimenter provided the correct answer for each problem. The intent was to increase familiarity and use with the diagrams.

Representation translation task

Students then completed a four-item task intended to measure how well they could translate between diagrams and equations (refer to Appendix S1 for exact problems). The first two items required students to select which of two diagrams corresponded to a given equation. Students received one point for circling the correct diagram. The next two items required students to write an equation that corresponded to a given diagram. Students received one point for writing a valid equation; expressions (e.g., $2x$) were not valid. For both pairs of problems, a single-reference problem was presented before a double-reference problem. Students read directions and a completed example before attempting each pair of problems. No feedback was provided.

Equation-solving assessment

Finally, students solved eight algebra equations (see Table 1), which were adapted from Booth and Koedinger (2012)'s choice of single- and double-reference problems. The participating math teacher checked the final set of eight problems and verified that they were of suitable difficulty to provide us with a range of responses.

We designed eight algebra problems contrasting two factors: presentation format and problem complexity. Each student solved four problems in an equation-only format and an isomorphic set of four problems in an equation-with-diagram format. For each presentation format, students first solved two single-reference problems followed by two double-reference problems. We used four counterbalanced forms of the assessment to control for order of presentation (equation-only first or equation-with-diagrams first) and version (which problems had diagrams). Problems were blocked by presentation format; the first four problems were in one format followed by four in the other format. On the four equation-with-diagram problems, students indicated whether they had used the diagram to solve each problem by circling 'yes' or 'no' inside a box below the diagram. This provided a measure of diagram use. Students reported using the diagram on 59% of equation-with-diagram items (with 16 non-responses excluded). Students reported using the diagram more often on double-reference problems (72% of problems) than single-reference problems (49%), $t(49) = 3.34$, $p < .01$. One version of the equation-solving assessment is included in the Appendix S1.

Coding

Each problem on the equation-solving assessment was scored as correct (1) or incorrect (0) based on exact accuracy of their solution. We also coded students' strategy use and types of errors based on their written work, using a coding scheme adapted from previous research (Koedinger *et al.*, 2008). These schemes are outlined in Tables 2 and 3. Strategies were coded based on the general approach taken, even if the implementation was incomplete or inaccurate. Apart from problems with no written work, strategies generally fell into one of three categories: algebra, unwind, or guess and check. We were primarily interested in the algebra and unwind strategies, which constitute formal and

Table 2. Children's strategies on equation-solving problems

Strategy	Definition	Example responses $(x - 45)/3 = 20.5$
Algebra	Uses algebraic manipulations to derive solution	$x - 45 = 20.5 * 3$ $x = 61.5 + 45$
Unwind	Works backward using arithmetic strategies	$20.5 * 3 = 61.5$ $61.5 + 45 = 106.5$
Guess and check	Substitutes different values into the provided equation	$(90 - 45)/3 = 15$ $(108 - 45)/3 = 21$
Other	Uses other non-algebraic strategies or strategy is ambiguous	$45 * 3 = 135$
Answer only	Answer provided with no work	$x = 106.5$
No attempt	Student leaves problem blank or writes 'I don't know'	

Table 3. Children's error types on the equation-solving assessment

Error	Definition	Example problem $(x - 45)/3 = 20.5$
Arithmetic	Student makes a computational error, but solution is otherwise correct	$3 * 20.5 = 60.15$ $x = 45 + 60.15$ ...
Conceptual	Student employs invalid or incomplete solution step	$x - 45 = 3 + 20.5$ ...
Copy slip	Student miscopies a value from the problem or from own work, but solution is otherwise correct	$(x - 45)/3 = 205$ $x - 45 = 615$ ...
No Work	Student writes an incorrect answer without any work shown.	$x = 50$
No Attempt	Student leaves problem blank.	

informal versions of logical problem-solving, respectively. Similarly, inaccurate or incomplete attempts received one of three main error codes: arithmetic, conceptual, or copy slip. A second rater coded the responses of 25% of the children, demonstrating high inter-rater agreement as indicated by Cohen's kappa ($\kappa = .85$ for strategies, $\kappa = .92$ for errors).

Results

Below, we analyse student performance separately for each task. We begin with descriptive information on the diagram practice task, followed by an analysis of accuracy, errors, and strategies on the equation-solving assessment. Finally, we examine accuracy on the representation translation task and test whether it relates to equation-solving performance.

Diagram practice task

A diagram introduction phase was included to familiarize all students with the novel diagram format. As a group, students gained some basic familiarity with the diagrams, solving 65% of the diagram practice problems correctly (SD = 27%, Range = 0–100%), with only 13 students (21%) scoring below 50%.

Equation-solving assessment

To evaluate the effect of counterbalanced forms for the equation-solving assessment, a 2 (order) x 2 (version) ANOVA on accuracy scores was carried out. This revealed no significant effects of either factor or their interaction, Fs < 1. Thus, the subsequent analyses treated the four forms as equivalent. Recall, we hypothesized that diagrams would improve problem-solving accuracy by eliciting students' intuitive knowledge of quantitative relations. This should potentially reduce the rate of conceptual errors, increase the use of non-algebraic strategies, and increase the accuracy of algebraic strategies. To test our hypotheses, we analysed children's accuracy, error rates, and strategy use separately using repeated-measures ANOVAs. Each analysis included format (equation or equation-with-diagram) and complexity (single- or double-reference) as within-subject factors, and ability (regular or advanced class) as a between-subjects factor.

Accuracy

For problem accuracy, we found significant main effects of format, $F(1, 59) = 11.5$, $p = .001$, $\eta_p^2 = .16$, complexity, $F(1, 59) = 17.5$, $p < .001$, $\eta_p^2 = .23$, and ability, $F(1, 59) = 24.3$, $p < .001$, $\eta_p^2 = .29$. As shown in Figure 2, students were more accurate on equations-with-diagrams (M = 47%, SD = 36%) than on equations-without-diagrams (M = 35%, SD = 32%, Cohen's $d = 0.44$), and on single-reference problems (M = 50%, SD = 36%) than on double-reference problems (M = 32%, SD = 34%, Cohen's $d = 0.55$). Also, students in advanced classes solved more problems correctly (M = 56%, SD = 29%) than students in regular classes (M = 22%, SD = 23%, Cohen's $d = 1.27$). None of the two-way or three-way interactions were significant, Fs < 1.

Errors

Figure 3 shows the frequencies of each error type. For conceptual errors, we found a main effect of problem format, $F(1, 59) = 12.39$, $p < .001$, $\eta_p^2 = .17$, but not complexity, $F(1, 59 = 1.49$, n.s., or ability, $F(1, 59) = 2.51$, n.s. Students made fewer conceptual errors on equation-with-diagram problems (M = 21%, SD = 24%) compared to equation-only problems (M = 34%, SD = 32%, Cohen's $d = 0.45$). Figure 4 shows example student work illustrating how diagrams may reduce conceptual errors. No other main effects or interactions were found for conceptual errors or other error types.

Strategies

Table 4 shows the frequencies and success rates of each strategy type. Overall, students used an unwind strategy twice as often as an algebra strategy (36% vs. 16% of all problems), $t(60) = 4.31$, $p < .001$), and 12% of all items were not attempted. This reflects their status as beginning algebra learners.

We analysed the frequencies of algebra and unwind strategy use separately. For algebra strategy use, we found a main effect of ability, $F(1, 59) = 13.11$, $p < .001$, $\eta_p^2 = .18$, but not format or complexity, Fs < 1. Students in advanced classes used the algebra strategy more frequently (M = 25%, SD = 27%) than students in regular classes (M = 5%, SD = 7%, Cohen's $d = 0.99$). For unwind strategy use, we found a main effect of problem complexity, $F(1, 59) = 19.15$, $p < .001$, $\eta_p^2 = .25$, but not format, $F(1, 59) = 2.05$, n.s., or ability, $F(1, 59) = 1.54$, n.s. Students used the unwind strategy more frequently on single-reference problems (M = 46%, SD = 37%) than on double-reference problems (M = 24%,

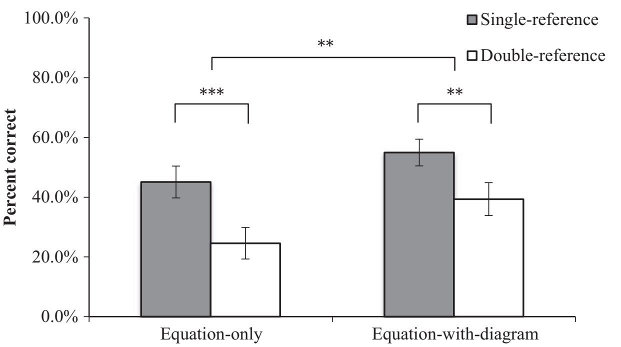


Figure 2. Accuracy on equation-solving assessment by problem complexity and presentation format. *Note*. Error bars represent standard errors. $^{**}p < .01$; $^{***}p < .001$.

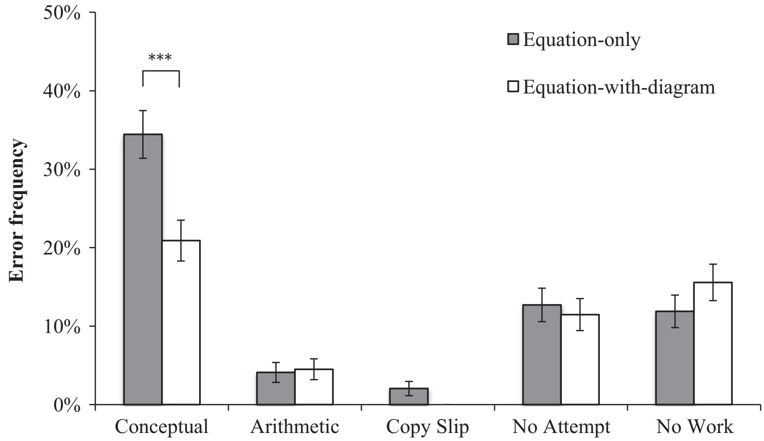


Figure 3. Percentage of problems with different errors by presentation format. *Note.* Error bars represent standard errors. ***$p < .001$.

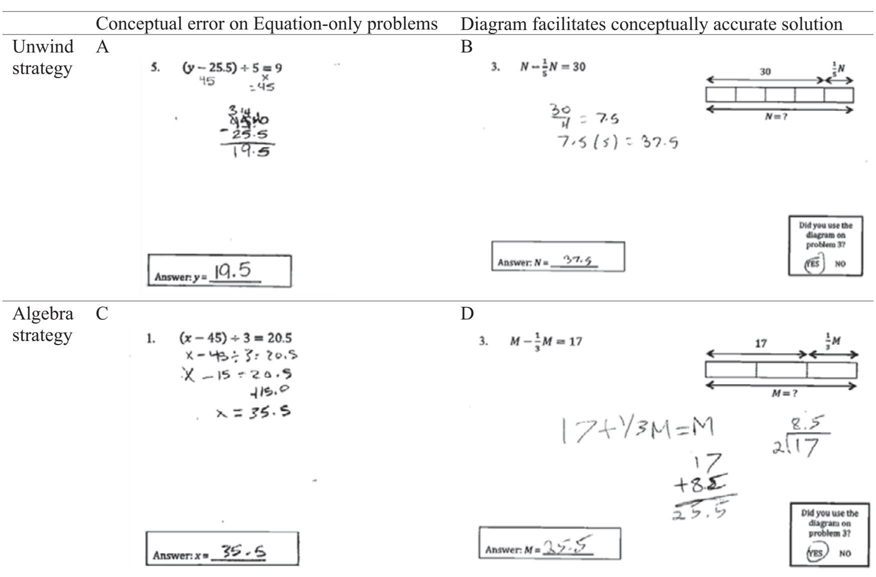


Figure 4. Examples of algebraic and unwind strategies. *Note*. Using the unwind strategy (A, B), students reverse arithmetic operations in the equation to obtain the unknown value. Using the algebra strategy (C, D), students solve for the unknown variable by re-writing equations following algebraic rules. On equation-only problems, students demonstrate common errors such as subtracting items that should be added (A) or ignoring parentheses (B). Diagrams can improve accuracy by helping students to use appropriate arithmetic operations (C) or to construct valid alternative equations (D).

SD = 30%, Cohen's *d* = 0.58). There were no two- or three-way interactions for either strategy. Other strategies were used infrequently, making statistical analyses inappropriate. Visual inspection of mean use of these strategies suggested no effect of problem format (see Table 4). Overall, providing diagrams did not impact which strategy students selected.

Table 4. Frequency and success rate of strategy types on equation-only and equation-with-diagram problems, by problem complexity

	Single-reference % Used	Double-reference % Used	Total	
			% Used	% Correct
Equation-only				
Algebra	17	16	17	51
Unwind	44	20	32	51
Guess and check	7	3	5	46
Other	10	20	15	19
Answer only	16	21	19	24
No attempt	6	20	13	0
Equation-with-Diagram				
Algebra	16	14	15	62
Unwind	48	29	38	69
Guess and check	3	4	4	67
Other	7	9	8	32
Answer only	20	28	24	27
No attempt	7	16	12	0

Note. Percentages are computed out of all problems in that presentation format and complexity; values may not add up to 100 because of rounding error.

We also examined the accuracy of the algebra and unwind strategies. Students were 56% accurate with the algebra strategy and 61% accurate with the unwind strategy. Given the high number of students who never used a particular strategy or used it only once, overall frequencies were too small to conduct meaningful statistical analyses. Nonetheless, inspection of means suggests that students were more accurate when using each strategy on equations-with-diagram problems than equation-only problems (M = 62% vs. 51% correct for the algebra strategy and 69% vs. 51% for the unwind strategy).

Representation translation task

Finally, recall that we were interested in whether the effect of diagrams would depend on students' understanding of the diagram representation. Students generally had some success on the representation translation task, but with large individual differences (M = 61%, SD = 34%, Range = 0–100%). Indeed, students in the advanced classes had higher translation scores (M = 76%, SD = 25%) than students in the regular classes (M = 43%, SD = 35%), $t(45.86) = 4.19$, $p < .0001$, Cohen's $d = 1.12$. Importantly, student's representation translation scores significantly predicted accuracy on equation-with-diagram problems, even after controlling for accuracy on equation-only problems, $\beta = .307$, $t(58) = 2.89$, $p = .005$), suggesting that diagram understanding played a role in success.

Discussion

We introduced middle-school students to a novel diagram during their regular math class and tested them on algebraic equation problems with or without accompanying diagrams. Students were significantly more accurate when diagrams were provided. Analyses of problem-solving errors and strategies, and individual differences in diagram understanding, suggest potential explanatory mechanisms for the diagram benefit. We integrate our findings with past research and reflect on potential explanations and implications for theories of problem-solving.

The current study extends earlier correlational and experimental research on the benefit of diagrams in solving story problems (e.g., Booth & Koedinger, 2012; Koedinger & Terao, 2002) to the more abstract domain of equation-solving. Moreover, we replicate the specific finding that diagrams improved problem-solving accuracy by reducing the rate of conceptual errors. This diagram benefit was robust across different levels of problem complexity and student math ability, in contrast to some previous work (e.g., Booth & Koedinger, 2012). Several features of our design may have promoted a more robust diagram benefit, including using a consistent diagram type across all problems, providing an introduction to understanding the target diagram type, and using problems that were free of floor and ceiling effects. Diagrams seem to have the potential to promote better problem-solving success as long as problems are within students' zone of proximal development (hard to solve without scaffolding but accessible with scaffolding; Vygotsky, 1978).

What potential explanatory mechanisms might account for the diagram benefit? Our findings support arguments that constructing appropriate mental models of a problem, and thus having access to relevant problem information, is key to problem-solving success (e.g., Johnson-Laird, 1983; Koedinger & Nathan, 2004). Providing students with an alternative diagram representation might help them to construct an accurate and usable internal problem representation in at least three ways.

First, diagrams may have enhanced *information access* by representing quantitative relationships among abstract algebra symbols in a more concrete or transparent manner. Indeed, diagram use increased on more complex problems which involve less transparent quantitative relationships. According to Ng (2015), diagrams such as the ones used in this study are useful because they efficiently represent the right mathematical unit. A diagram representing the problem $N - \frac{1}{5}N = 30$ (see Table 1 for the question and Figure 4B for sample student work) makes clear that we can treat $\frac{1}{5}N$ as a common unit, allowing students to simplify the relationship between N and 30.

Second, diagrams may enhance *knowledge access* by facilitating informal reasoning. Consistent with previous research on story problems (Koedinger & Nathan, 2004), students in our study made fewer conceptual errors when diagrams were present (see Figure 3). Diagrams might help activate knowledge of operations and quantitative relations, reducing errors that violate the structure of the problem. Further, the presence of diagrams tended to increase the general effectiveness of each strategy (see Table 4), perhaps by allowing students to check their conceptual understanding of the problem and the accuracy of their procedures.

There are likely additional processes at work that might explain the robust diagram benefit found in this study. For example, the *cognitive load* hypothesis suggests that diagrams reduce demands on students' working memory and attention by offloading some cognitive processing onto perceptual processing (e.g., Larkin & Simon, 1987; Lee *et al.*, 2007).

While we found an overall positive effect of diagrams on algebraic problem-solving, our results also suggest a possible individual difference in students' ability to benefit from a diagram representation. Specifically, students' representation translation scores (the ability to construct a diagram from an equation, or vice versa) were positively related to accuracy on problems with diagrams, even after controlling for accuracy on problems without diagrams. We also found higher representation translation scores for students in the advanced math classes. It is thus unclear whether the diagram benefit can be attributed to diagram understanding specifically or to more general abilities. Future research utilizing more targeted measures of representation translation or diagram comprehension abilities could help clarify the mechanisms underlying the diagram benefit.

Implications

This study adds to an emerging line of research suggesting that combining concrete and symbolic representations can integrate their advantages and mitigate their disadvantages (e.g., Fyfe, McNeil, Son, & Goldstone, 2014; Goldstone & Son, 2005). Concrete representations, including diagrams, tend to be more visuospatial and grounded in familiar experiences or contexts, but they also contain extraneous perceptual details that can distract learners or inhibit transfer of knowledge to novel situations (Harp & Mayer, 1997; Kaminsky, Sloutsky, & Heckler, 2008). In contrast, abstract, symbolic representations eliminate extraneous surface details and are more arbitrarily linked to their referents, thus allowing for greater generalizability, but possibly appearing as meaningless symbols to learners with limited familiarity (Nathan, 2012). One technique, known as 'concreteness fading', uses concrete materials to introduce learners to new concepts, before gradually removing perceptual details so as to encourage learners to generalize their understandings. Using diagrams in combination with equations, and gradually fading the inclusion of diagrams may facilitate learning and performance in algebra. Future research can test this possibility using intervention designs.

Finally, integrating diagrams with symbolic representations is in line with the view that developing facility with multiple representations is an important outcome of elementary math education (NCTM, 2000). Learners' ability to integrate multiple representations is a crucial component of learning and benefiting from multiple representations (Ainsworth, 2006). While we found some individual difference in representation translation ability across students, domain novices (pre-algebra students) in our study were nonetheless able to benefit from diagrams alongside equations with minimal instruction, suggesting that diagrams may be a valuable tool to include in one's repertoire of representations for symbolic equations.

Conclusion

The current study successfully extends previous findings of a diagram benefit in problem-solving to a symbolic domain. Providing novel diagrams enhanced students' accuracy on difficult algebra equation problems independent of problem complexity and student ability. Integrating with prior work on problem-solving helps us to better understand how diagrams aid students in interpreting, representing, and using appropriate strategies when solving mathematical problems. Diagrams may be more powerful than previously leveraged, providing a useful additional representation of symbolic problems.

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Supporting Information

The following supporting information may be found in the online edition of the article:

Appendix S1. Materials for equation-solving assessment.