General

- midnight: $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
- y-axis is axis 1, x-axis is axis 2
- $a^1 = a$
- $a^0 = 1$
- $a^{-n} = 1/a^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^m a^n = a^{m+n}$
- $a^m / a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a a^n = a^{n+1}$
- $y = \lg_b x \Leftrightarrow b^y = x$
- $\lg(uv) = \lg u + \lg v$
- $\lg(u/v) = \lg u - \lg v$
- $\lg(1/v) = -\lg v$
- $\lg(u^r) = r \lg u$
- $\lg_a x = \lg_b x / \lg_b a$

- $\dim(P_n) = n+1$
- $\dim(\mathbb{R}^n) = n$

Complex Numbers

- imaginary number i: $i^2 = -1$
- complex number z: $z = x + iy$ $x, y \in \mathbb{R}$
  - Real part: $\mathrm{Re}\, z = x \in \mathbb{R}$
  - Imaginary part: $\mathrm{Im}\, z = y \in \mathbb{R}$
- set of complex numbers $\mathbb{C}$: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$
  - $\mathbb{R} \subset \mathbb{C}$: $\mathbb{R} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z = 0\}$
  - rein-imaginär: $0 + iy$, $y \in \mathbb{R}$
- $a + ib = c + id \Leftrightarrow a = c, b = d$
- konjugate of $z = a + ib$: $\bar{z} = a - ib$
  - $z \cdot \bar{z} = x^2 \cdot y^2 = |z|^2$
  - $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \quad \forall z \in \mathbb{C}$
  - $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \quad \forall z \in \mathbb{C}$
- Betrag $|z|$ is distance between z and origin
  - $|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$
  - $|z| = |\bar{z}|$
  - $|z_1 + z_2| \leq |z_1| + |z_2|$
  - $|z_1 z_2| = |z_1||z_2|$
  - $|z_1 / z_2| = |z_1| / |z_2|$
- arithmetic:

  $+$ $z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$

  $*$ $z_1 * z_2 = (a+ib) * (c+id) = (ac - bd) + i(ad + bc)$
  - Polar: $z_1 z_2 = r s e^{i(\varphi + \psi)}$

  $:$ $z_1 / z_2 = (z_1 \cdot \bar{z}_2) / (z_2 \cdot \bar{z}_2) = (z_1 \cdot \bar{z}_2) / |z|^2$
  - Polar: $z_1 / z_2 = r/s\, e^{i(\varphi - \psi)}$

  $\sqrt[n]{z_1} = z_2 \Rightarrow z_1 = z_2^n = s^n e^{in\psi} \overset{!}{=} r e^{i\varphi}$ $\left\{ \begin{array}{l} s = \sqrt[n]{r} \\ n\psi = \varphi + 2\pi k \quad k = 0 \ldots n-1 \end{array} \right.$
- Polar Coordinates:
  - $z = x + iy \Leftrightarrow z = r(\cos\varphi + i \sin\varphi) \overset{\text{Euler}}{\Leftrightarrow} z = r e^{i\varphi}$
    - $x = r \cos\varphi$, $y = r \sin\varphi$
    - $r = |z|$, $\varphi = \cos^{-1}(x/r) = \sin^{-1}(y/r)$

    $\hookrightarrow \arg(z) = |z| \in [0, 2\pi) \Rightarrow r$ is uniquely defined
- Euler Formula: $e^{i\varphi} = \cos\varphi + i \sin\varphi$
  - $|e^{i\varphi}| = 1$
-

deg	rad	sin	cos	deg	rad	sin	cos	deg	rad	sin	cos	deg	rad	sin	cos
0	0	0	1	90	$\pi/2$	1	0	180	π	0	-1	270	$3\pi/2$	-1	0
30	$\pi/6$	$1/2$	$\sqrt{3}/2$	120	$2\pi/3$	$\sqrt{3}/2$	$-\frac{1}{2}$	210	$7\pi/6$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	300	$5\pi/3$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	135	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	225	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	315	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$
60	$\pi/3$	$\sqrt{3}/2$	$1/2$	150	$5\pi/6$	$1/2$	$-\sqrt{3}/2$	240	$4\pi/3$	$-\sqrt{3}/2$	$-\frac{1}{2}$	330	$11\pi/6$	$-1/2$	$\sqrt{3}/2$

LGS and Gauss Elimination

- m equations, n unknowns
  - $m > n$: 0, 1, ∞ solutions
  - $m < n$: 0, ∞ solutions
  - $m = n$: 1, 0, ∞ solutions
- $Ax = b$
  - A: Coefficient Matrix, x: Unknown Vector, b: RHS
  - A: $m \times n$, x: $n \times 1$, b: $m \times 1$
- Eliminationsschema: $(A|b)$
- Upper Triangular Form: $a_{ij} = 0 \quad j > i$
- Lower Triangular Form: $a_{ij} = 0 \quad i > j$
- Row Echelon Form: Special Triangular Form (RREF)
- Back Substitution: recursive solve of LGS in e.g. RREF
  - define free Variables, then solve for others
- Pivot: Element $\tilde{a}_{ii}$ of reduced system ($\neq 0$)
- Reduced System: System w/o pivot rows/columns
- Elementary row operations: Solutionset is equivalent
  - swap two rows
  - add multiple of pivotrow to another row
  - (multiply row by a constant $\neq 0$)
- equation $(0 \ldots 0 | 0)$ can be removed
- degenerated equation: $(0 \ldots 0 | c_i \neq 0) \Rightarrow$ no solution
- Rank $A = r$: # Pivots in RREF
  - $r \leq \min(m, n)$
- Verträglichkeitsbedingung: $\exists$ if $m > r$
  - $c_{r+1} = c_{r+2} = \cdots = c_m \overset{!}{=} 0$
  - exists but not fulfilled $\Rightarrow$ no solutions
- Free Variables: columns w/o pivot
  - # free Variables $n - r$
- Solution Set: $\mathrm{Sol}(A, b) = \{x \in \mathbb{K}^n \mid Ax = b\}$
- Equivalent equations: Have the same solution set
- $n > m$ System
  - has (at least) one solution if one of:
    - $r = m$ (no Verträglichkeitsbedingungen)
    - $r < m$ and $c_{r+1} = \cdots = c_m = 0$ (Vtgk b fulfilled)
  - If there exists a solution then:
    - if $r = n$: unique solution
    - $r < n$: $(n-r)$ parameter solution set

Gauss II

- exactly one solution $\Leftrightarrow \begin{cases} r = n = m & \text{or} \\ r = n < m & \text{and} \\ c_{r+1} = \dots c_m = 0 \end{cases}$
- for every b at least one solution $\Leftrightarrow$ $r = m$
- for every b exactly one solution $\Leftrightarrow$ $r = m = n$
- Square LGS has unique solution $\Leftrightarrow$ LGS is solveable for every b $\Leftarrow$ LGS has only trivial Solution for the homogenious system
- homogenious system: $b = 0$
  - has always trivial solution
- inhomogenious system: $b \neq 0$
  - has only trivial solution if $r = n$
  - has $(n-r)$ parameter solution set if $r < n$
- trivial Solution: $x_1 = \dots x_n = 0$
- non-trivial solution: $x_1 \neq \dots x_n \neq 0$
- regulär / non-singular: LGS has unique solution
- singular: LGS has 0 solutions or a solution set
- Square system: either
  - $r = n \Leftrightarrow A$ is non-singular
  - for every b there is at least one solution $\Leftrightarrow$ Verträglichkeitsbedingungen
  - for every b there is maximum one solution $\Leftrightarrow$ no free Variables
  - for every b there is exactly one solution
  - the homogenous sistem has the trivial solution only
  - $r < n \Leftrightarrow A$ is singular
  - for some b there is no solution
  - for no b exists a unique solution
  - for some b there are ∞ solutions
  - the homogenious system has non-trivial solutions

Matrices and Vectors over $\mathbb{K}$

- $\mathbb{K}$ is Körper: normally $\mathbb{R}$ or $\mathbb{C}$
- $m \times n$ Matrix: m rows, n columns, mn Elements
- $a_{ij} = (A)_{ij}$ = Element at index i,j of A
- a_{jj}: Diagonal Elements
- $n \times n$: Square matrix, has order n
- $m \times 1$: column vector
- $1 \times n$: row vector
- Zero Matrix: $(0)_{ij} = 0 \ \forall i,j \in \{1..n\}$
- Diagonal Matrix: $(D)_{ij} = 0 \ j \neq i$, $D = \mathrm{diag}(a_{11}, \dots a_{nn})$
- Unit Matrix: $I_n = \mathrm{diag}(1 \dots 1)$
  - $AI = A = IA$
- Upper Triangular Matrix: $(R)_{ij} = 0 \ i > j$
- Lower Triangular Matrix: $(L)_{ij} = 0 \ j > i$
- A, B are equivalent: same Shape and $a_{ij} = b_{ij}$
- Arithmetics
  - Scalar Multiplication αA: $(\alpha A)_{ij} := \alpha (A)_{ij}$
  - Addition of $m \times n$ Matrices A and B: $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$
  - Multiplication of $m \times n$ Matrix A by $n \times p$ B: $(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik} (B)_{kj}$, (AB) has shape $m \times p$
- Rules
  - $(\alpha\beta)A = \alpha(\beta A)$
  - $(\alpha A)B = \alpha(AB) = A(\alpha B)$
  - $(\alpha + \beta)A = \alpha A + \beta A$
  - $\alpha(A+B) = \alpha A + \alpha B$
  - $A + B = B + A$
  - $(A+B)+C = A+(B+C)$
  - $(AB)C = A(BC)$
  - $(A+B)C = (AC)+(BC)$
  - $A(B+C) = (AB)+(AC)$
  - $AB \neq BA$ normally
    - if $AB = BA \Rightarrow A$ and B commute
  - $A + 0 = A = 0 + A$
  - $\exists (-A)_{ij} = -(A)_{ij} \ \forall i,j \ \forall A \in \mathrm{Mat}_{m \times n}$
    - $A + (-A) = 0 = (-A) + A$
- Set of all matrices $\mathbb{K}^{m \times n} = \mathrm{Mat}_{m,n}(\mathbb{K})$
- Nullteiler: $A \neq 0, B \neq 0 \Rightarrow AB = 0$

Matrices and Vectors over $\mathbb{K}$

- $A, B = \mathrm{Mat}_{m,n} \ \exists X, (X)_{ij} = (B)_{ij} - (A)_{ij}$ s.d. $A + X = B$
- Linear Combination: $\alpha_1 a_1 + \dots + \alpha_n a_n$, $\alpha_i \in \mathbb{E}$, $a_i \in \mathbb{E}^m$
  - $Ax = b$, $A = (a_1 | \dots | a_n)$
    - Has a solution $\Leftrightarrow$ b is a linear combination of a_i
    - $A e_i = a_i$
- $A: m \times n$, B $n \times p$ $\Rightarrow$
  - $AB = (Ab_1 | \dots | Ab_p) = (a_1 B | \dots a_m B)^T$
- Transposed A^T of A: $(A^T)_{ij} = (A)_{ji}$
- Complex Conjugate / Hermitian Transposed A^H of A $(A^H)_{ij} = (\bar{A})_{ij}^T = \overline{(A)_{ij}^T} = \overline{(A)_{ji}}$
- Symmetric: $A = A^T \Leftrightarrow A_{ij} = A_{ji}$
- Hermitian: $A = A^H \Leftrightarrow A_{ij} = \overline{A_{ji}}$
- Schiefsymmetrisch: $A^T = -A$
  - $A \in \mathbb{R} \Rightarrow \forall$ EV strictly imaginary or $= 0$
- Rules
  - $(A^T)^T = A$ $\quad (A^H)^H = A$
  - $(\alpha A)^T = \alpha A^T$ $\quad (\alpha A)^H = \bar{\alpha} A^H$
  - $(A+B)^T = A^T + B^T$ $\quad (A+B)^H = A^H + B^H$
  - $(AB)^T = B^T A^T$ $\quad (AB)^H = B^H A^H$
- A, B Symmetric: $AB = BA \Leftrightarrow AB$ Symmetric
- $A^T A$ and AA^T are symmetric (same for hermitian)
- outer Produkt: Product of column and row vector
  - $n \times 1 \cdot 1 \times m \to n \times m$
  - Rang $A = 1 \Leftrightarrow$ A is a outer product
  - useful for orthogonal Projections
- $A \in \mathbb{E}^{m \times n}$ describes a linear mapping
  - $A: \mathbb{E}^n \to \mathbb{E}^m$, $x \to Ax$
  - linear if $A(\alpha a + b) = \alpha(Aa) + (Ab)$
  - Image: $\mathrm{Im}\, A = \{Ax \in \mathbb{E}^m \mid x \in \mathbb{E}^n\}$
- Inverse A^{-1} of A:
  - $AA^{-1} = I = A^{-1}A$
  - A^{-1} is unique
  - A non-singular $\Leftrightarrow$ A is invertable
  - A, B are invertable:
    - $(A^{-1})^{-1} = A$
    - $(AB)^{-1} = B^{-1}A^{-1}$
    - $(A^T)^{-1} = (A^{-1})^T$ (same for hermitian)

Matrices and Vectors over $\mathbb{K}$

- Inverse Calculation:
  - $Ax = I \Rightarrow A^{-1}Ax = A^{-1} \Rightarrow Ix = A^{-1}$
    - Gauss: $(A \mid I) \rightsquigarrow (I \mid A^{-1})$
  - 2×2: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, if $ad-cb \neq 0$
- Unitary Matrix A: $AA^H = I_n$
- Orthogonal Matrix A: $AA^T = I_n$
- Rules: A, B are unitary (same for orthogonal)
  - A is regular and $A^{-1} = A^H$
  - $AA^H = I_n$
  - A^{-1} is unitary
  - AB is unitary
- Permutation Matrices: every row and column contains only one 1. All other elements are 0
  - $PA \Rightarrow$ rows get permutated
  - $AP \Rightarrow$ columns get permutated
  - $\begin{pmatrix} 1 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & & & \\ & \alpha & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & & & \\ & 1 & \alpha & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$

  $\begin{cases} \text{swap } i^{th} \text{ and } j^{th} \\ \text{column/row} \end{cases}$ $\begin{cases} \text{scale } i^{th} \text{ row/} \\ \text{column by } \alpha \end{cases}$ $\begin{cases} \text{replace } i^{th} \text{ row/column} \\ \text{by } (i + \alpha j) \text{ row/column} \end{cases}$
- Unitary / Orthogonal transformations preserve length, and angle
  - $\|Ax\| = \|x\|$ and $\langle Ax, Ay \rangle = \langle x, y \rangle$
- Lower/Upper Bidiagonal Matrix:
  $(B)_{ij} = 0 \quad i < j$ and $j > j+1$ / $j < i$ and $j > i+1$
- Tridiagonal Matrix: $(T)_{ij} = 0 \quad i > j+1$ and $j > i+1$
  - is squared
- Banded Matrix with lower bandwith p and upper bandwith q: $(B)_{ij} = 0 \quad i > j+p$ or $j > i+q$
  - total bandwith: $p+q+1$
- Hesseberg Matrices: have lower bandwith 1
- Toeplitz Matrix: a_{ij} depends on difference $i-j$
- Hankel Matrix: a_{ij} depends on sum $i+j$
- Block Matrix: Every matrix may be split into a block matrix
  - Blocksize may be different

LR-Zerlegung

- $A = LR$ w/o row swaps, $A: n\times n$ & non-singular
  - L: Lower Triangular Form: Stores the Operations
  - R: Upper Triangular Form
- $Lc = b$: Forward Substitution
- $Rx = c$: Back Substitution
- $PA = LR$ with row swaps, $A: n\times n$ & non-singular
  - $Lc = Pb$
  - $Rx = c$
- $Ax = P^{-1}PAx = P^{-1}LRx = P^{-1}Lc = P^{-1}Pb = b$
- $PA = LR$ $A: m\times n$ & singular
  - $P: m\times m$
  - $R: m\times n$
  - $c: m\times 1$
  - $L: m\times m$, non-singular
  - $PA = LR$
  - $Lc = Pb$
  - $Rx = c$: free Variables are freely choosable
- Block LR Zerlegung
  - Pivot Blocks must be invertable
  - $\tilde{A} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \Rightarrow \left(\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline 0 & D-CA^{-1}B \end{array}\right)$

    $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline CA^{-1} & I \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline 0 & D-CA^{-1}B \end{array}\right)$
- Pivot Strategy
  - Correctness and accuracy of decomposit. depends on chosen pivot
  - Select largest element as pivot (abs)
    - Prevents rounding errors

LR-Zerlegung

- Analysis:
  - Memory: n^2+n total
  - Runtime:
    - j^{th} Step of Gauss:
      - 1 division
      - $n-j$ multiplications
      - $(n-j)^2$ add.+mult.
      - $\ell(n-j)$ add.+mult.
    - k^{th} Step of ℓ-times backsubstitution
      - $\ell(n-k)$ add.
      - $\ell(n-k+1)$ mult.
    - total for step j
      - 1 division
      - $(n-j)^2 + (2\ell+1)(n-j) + \ell$ mult.
      - $(n-j)^2 + 2\ell(n-j)$ add.
    - Total LR-Zerlegung:
      - n division
      - $\frac{1}{3}n^3 + O(n)$ mult.
      - $\frac{1}{3}n^3 - \frac{1}{2}n^2$ add.
    - Total For-/Backward Substitution
      - $n^2 - n$ add.
      - n^2 mult.
    - Solve $AX = B$:
      - n div.
      - $\frac{4}{3}n^3 + O(n)$ mult.
      - $\frac{4}{3}n^3 - \frac{3}{2}n^2 + O(n)$ add
    - Solve $AX = I$ (find A^{-1})
      - n div, $n^3 + O(n)$ mult, $n^3 + O(n)$ add.

LR-Zerlegung

- Solve $Ax = b$ using LR w/o Pivoting
  - $A = LR$
    - $(A \mid I) \rightsquigarrow (R \mid L)$
      - gauss only on LHS, on RHS record the operations: $a_{21} = a_{21} - l_{21} a_{11} \to a_{21} = 0$
      - Do not swap lines
      - Do not multiply line by a constant
  - $Lc = b$: Forward substitution (normal gauss)
  - $Rx = c$: Backsubstitution (normal gauss)
- Solve $Ax = b$ using LR w Pivoting
  - $PA = LR$
    - $(I \mid A \mid I) \rightsquigarrow (P \mid R \mid L)$
      - gauss only on middle
      - RHS records the operations
      - LHS the swaps
        - Swap elements of middle as well as on the RHS, but not the diag. elements
  - $Lc = Pb$: Mat. Mul, then Forward substitution
  - $Rx = c$: Backsubstitution

Vectorspace

- Körper / Field: non-empty set $\mathbb{K}$ with add. and mult.
  $+ : \mathbb{K} + \mathbb{K} \to \mathbb{K}, \quad a, b \mapsto a + b$
  $* : \mathbb{K} \cdot \mathbb{K} \to \mathbb{K}, \quad a, b \mapsto a \cdot b$
  - axioms:
    - K1: $a + b = b + a$
    - K2: $(a+b)+c = a+(b+c)$
    - K3: $a + 0 = a$
    - K4: $a + (-a) = 0$
    - K5: $a \cdot b = b \cdot a$
    - K6: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
    - K7: $a \cdot 1 = a$
    - K8: $a \cdot a^{-1} = 1$
    - K9: $a(b+c) = a \cdot b + a \cdot c$
    - K10: $(a+b) \cdot c = a \cdot c + b \cdot c$
  - normally $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$
  - Important for Comp. Science is $\mathbb{Z}_2 = \{0, 1\}$
- Vector Space: non-empty set V with addition and scalar multiplication over $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{C}$
  $+ : V + V \to V, \quad a, b \mapsto a + b$
  $* : \mathbb{E} * V \to V, \quad \alpha, a \mapsto \alpha \cdot a$
  - axioms:
    - V1: $a + b = b + a$
    - V2: $(a+b)+c = a+(b+c)$
    - V3: $a + 0 = a$
    - V4: $a + (-a) = 0$
    - V5: $\alpha(a+b) = \alpha a + \alpha b$
    - V6: $(\alpha + \beta) a = \alpha a + \beta a$
    - V7: $(\alpha \beta) a = \alpha(\beta a)$
    - V8: $1 a = a$
  - Vector: Element of V
  - Nullvector: $0 \in V$
  - Scalar: $\alpha \in \mathbb{E}$
  - Show set is V: Prove all axioms
  - $\mathbb{K}^n$
  - $Mat_{m,n}(\mathbb{K}) = \mathbb{K}^{m \times n}$
  - P_m: Field of all polynoms of deg $\leq m$
  - $C^m[a,b]$: Field of all m times diff. functions in $[a,b]$

Vectorspace

- Korrolar:
  - $0a = 0$
  - $\alpha 0 = 0$
  - $\alpha a = 0 \Rightarrow \alpha = 0$ or $a = 0$
  - $(-\alpha)a = \alpha(-a) = -(\alpha a)$
  - $\forall ab \; \exists c$ s.d. $a + c = b$, c is unique
- Subspace: non-empty subset U of V
  - axioms: $\forall a, b \in U, \forall \alpha \in \mathbb{E}$
    - U1: $0 \in U$
    - U2: $a + b \in U$
    - U3: $\alpha a \in U$
  - Subspace is a Vectorspace
  - Every V is a U of itself
  - Every V has the $U = \{0\}$
  - $A = Mat_{mn}(\mathbb{K}) \Rightarrow U = \{x \in \mathbb{K}^n \mid Ax = 0\} \subseteq \mathbb{K}^n$
- Lineare Hülle: $span\{a_1 .. a_n\} = \{\sum_{k=1}^{l} \alpha_k a_k\}$
  - Subspace spanned by the set of all linear combinations
  - Erzeugungssystem: the set of all a_i $i: 1..l$
- Linearely Dependant: $\sum_{k=1}^{l} \alpha_k a_k = 0$ and not all $\alpha_k = 0$
- Linearely Independant: $\sum_{k=1}^{l} \alpha_k a_k = 0 \Rightarrow \alpha_k = 0 \quad \forall i = 1 \ldots l$
  - Finite set of vectors is linearely dependant if one vector can be expressed as a linear combination of the other vectors
  - Infinite set of vectors is linearely independant if every finite subset is linearely independant
- Basis: Linearely Independant Erzeugungssystem
  - $v_1 .. v_n$ is a basis of V
  - $v_1 .. v_n$ is a minimal Erzeugungssystem
  - $v_1 .. v_n$ is a maximal lin. ind. Subset of V
  - Every $v \in V$ is a lin. comb of $v_1 .. v_n$
  - $B \in \mathbb{E}^{n \times n}$ is non-singular $\Rightarrow$ n column vectors form a basis
  - Koordinates: unique $\xi_1 .. \xi_n$ of lin. comb. for a $v \in V$ for a given basis $v_1 .. v_n$

Vector space

- Basis:
  - if $v_1 \dots v_n \in V$ are spanning and $w_1 \dots w_m \in V$ lin. ind. $\Rightarrow m \leq n$
  - V has a finite span $\Rightarrow$ every basis of V has the same number of vectors
  - Dimension: number of Basis vectors
  - Every set of lin. indep. vectors may be extended to a basis
  - complementary: two subspaces $U^1 U^2 \subset V$ if every $v \in V$ may be expressed as $v = u^1 + u^2$ $u^1 \in U^1$ $u^2 \in U^2$
  - Direct sum: Complementary subspaces $V = U^1 \oplus U^2$
- Is A a basis, lin. indep. or lin dep.?
  - Bring A in RREF using Gauss
  - Rang $A = n \Rightarrow$ column vectors are lin. ind.
  - Rang $A < n \Rightarrow$ column vectors are lin. dep.
  - Rang A = Rang $V \Rightarrow$ column vectors span V
  - Rang A = Rang $V = n \Rightarrow$ column vectors are basis of V

Change of Basis

- $\{b_1, b_2, \dots b_n\}$ old basis of V
- $\{b_1', b_2', \dots b_n'\}$ new basis of V
- express b_k' in terms of b_k: $b_k' = \sum_{i=1}^{n} T_{ik} b_i$, $k = 1 \dots n$
- Transformationsmatrix: $T = (T_{ik}) \in Mat_{n,n}(K)$
  - T: new $\to$ old $\Leftrightarrow T^{-1}$: old $\to$ new
  - k^{th} column of T is the coordinate vector b_k' in terms of the old basis
- $\xi = (\xi_1, \xi_2 \dots \xi_n)^T$ coordinate vector $v \in V$ in old basis
- $\xi' = (\xi_1', \xi_2' \dots \xi_n')^T$ coordinate vector $v \in V$ in new basis
- $\sum_{i=1}^{n} \xi_i b_i = v = \sum_{i=1}^{n} \xi_i' b_i'$
- $\xi_i = \sum_{i=1}^{n} T_{ik} \xi_i'$, $k = 1 \dots n$
- $\xi = T\xi' \Leftrightarrow \xi' = T^{-1}\xi$

$$\begin{tikzcd} \xi \in \mathbb{K}^n \arrow[dr, "X"'] & & \xi' \in \mathbb{K}^n \arrow[ll, "T"'] \arrow[dl, "X'"] \\ & v \in V & \end{tikzcd}$$

- Special case: $V = \mathbb{E}^n$:
  - $B := \{b_1, b_2, \dots b_n\}$ old basis of V
  - $B' := \{b_1', b_2', \dots b_n'\}$ new basis of V
  - $B' = BT \Leftrightarrow T = B^{-1}B'$ (follows from $b_k' = \sum_{i=1}^{n} T_{ik} b_i$)
  - $B = B'T^{-1} \Leftrightarrow T^{-1} = B'^{-1}B$
  - $B\xi = v = B'\xi'$
  - $\xi = T\xi'$
  - Find T: (old $\to$ new)
    - $(B' | B) \xrightarrow{\text{Gauss ohne Zeilenvertausch.}} (I | T)$
    - Inverse of T^{-1} if T^{-1} is known
    - $T = B^{-1}B'$
  - Find T^{-1}: (new $\to$ old)
    - $(B | B') \xrightarrow{\text{Gauss ohne Zeilenvertausch.}} (I | T^{-1})$
    - inverse of T if T is known
    - $T^{-1} = B'^{-1}B$

Linear Mapping

- Linear Mapping: $F: X \to Y$, $x \mapsto Fx$ if $F(a + \alpha b) = F(a) + \alpha F(b)$
  - Convention: $F(X) \Rightarrow FX$, $F(x) \Rightarrow Fx$
- Domain: X, Image Space: Y
  - Lineares Funktional: $Y = \mathbb{E}$
  - Linearer Operator: X and Y are Funktionenräume
  - Selbstabbildung: $X = Y$
- Surjective: $F(X) = Y$
- Injective: $F(x) = F(x') \Rightarrow x = x'$
  - F Injective $\Leftrightarrow \ker F = \{0\}$
- Bijective: Injective and surjective
  - Inverse mapping F^{-1} is defined
- Matrices define Lin. Mappings: $A \in \mathbb{K}^{m,n}$
  $A: x \in \mathbb{K}^n \mapsto Ax \in \mathbb{K}^m$
  - ableitungsoperator: $D: f \in P(\mathbb{R}) \mapsto f' \in P(\mathbb{R})$
  - Differentialoperator: $L: f \in P(\mathbb{R}) \mapsto \int_0^x f(x)\,dx$
  - Evaluationsabbildung: $E_t: f \in P(\mathbb{R}) \mapsto f(t)$
- X has basis $b_1 \dots b_n$, Y has basis $c_1 \dots c_m$
- Abbildungsmatrix A: $m \times n$
  - l^{th} column: coordinates (in basis of Y) of the image of the lth basis vector of X
- $Fb_l = \sum_{k=1}^{m} a_{kl} c_k$ $l = 1 \dots n$: Images $Fb_l \in Y$ in the basis of X as a linear combination of c_k
- Commutative Diagramm:

$$\begin{tikzcd} x \in X \arrow[r, "F"] \arrow[d, shift right, "\kappa_x"'] & Y \ni y \arrow[d, shift right, "\kappa_y"'] \\ \xi \in \mathbb{E}^n \arrow[r, "A"] \arrow[u, shift right, "\kappa_x^{-1}"'] & \mathbb{E}^m \ni \eta \arrow[u, shift right, "\kappa_y^{-1}"'] \end{tikzcd}$$

$F = \kappa_y^{-1} A \kappa_x$ | $\eta = A\xi$

$A = \kappa_y F \kappa_x^{-1}$

- Coordinate Mapping:
  $\kappa_x: X \to \mathbb{E}^n$ $x \mapsto \xi$
  - Bijection, linear $\Rightarrow$ Isomorphismus
- Get Abbildungsmatrix: of $F: X \to Y$
  - Insert basisvectors of X into F
  - Construct A of the transformed basisvectors
  - May have to apply coordinate transformation
    - e.g. when $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$
  - $F: \mathbb{R}^n \to \mathbb{R}^m \Rightarrow A \in \mathbb{R}^{m \times n}$

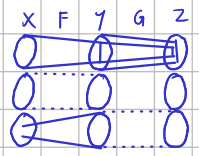
Linear Mapping

- **Isomorphismus:** unique linear mapping $F: X \to Y$
  - F^{-1} is also isomorphismus, A is regular
- **Automorphismus:** Isomorpismus where $X = Y$
- **Concatination of Mappings:** $F: X \to Y$, $G: Y \to Z$

$$G \circ F = GF : X \to Z$$

$$\begin{tikzcd} x \in X \arrow[r, "F"'] \arrow[d, shift right, "x_1"'] & y \in Y \arrow[r, "G"'] \arrow[d, shift right, "x_2"'] & z \in Z \arrow[d, shift right, "x_2"'] \\ \xi \in \mathbb{E}^n \arrow[r, "A"] \arrow[rr, bend right, "BA"'] & \eta \in \mathbb{E}^m \arrow[r, "B"] & \zeta \in \mathbb{E}^p \end{tikzcd}$$

- **Image:** $F(X) = \operatorname{im} F = \{F(x) \in Y \mid x \in X\} \subseteq Y$
  - $\operatorname{Im} F =$ set of b, for which $Ax = b$
- **Preimage:** $F^{-1}(y) = x$ (F^{-1} is not inverse mapping)
- **Kernel:** $\ker F = \{x \in X \mid Fx = 0\} \subseteq X$
  - $\ker F = \{0\} \Leftrightarrow F$ is injective
  - $\ker A =$ solution set of the system $Ax = 0$
- **Dimensionsformel:** $\dim X - \dim \ker F = \dim \operatorname{im} F = \operatorname{Rang} F$
- **Rang** $F := \operatorname{Rang} \operatorname{im} F$
  - $F: X \to Y$ is injective $\Leftrightarrow \operatorname{Rang} F = \dim X$
  - $F: X \to Y$ is surjective $\Leftrightarrow \operatorname{Rang} F = \dim \operatorname{im} F$
  - $F: X \to Y$ is bijective/isomorphism $\Leftrightarrow \operatorname{Rang} F = \dim X = \dim Y$
  - $F: X \to X$ is bijective/automorphism $\Leftrightarrow \operatorname{Rang} F = \dim X$
    $\Leftrightarrow \ker F = \{0\} \Leftrightarrow F$ is surjective $\Leftrightarrow F$ is injective
    - $\det(F) \neq 0$: bijective
- **Isomorph:** $\exists$ isomorphism $F: X \to Y$
  - Two vector spaces have same dimension $\Leftrightarrow$ they are isomorphe
- For $F: X \to Y$ and $G: Y \to Z$:
  - $\operatorname{Rang} FG \leq \min(\operatorname{Rang} F, \operatorname{Rang} G)$
  - G injective $\Rightarrow \operatorname{Rang} GF = \operatorname{Rang} F$
  - F is surjective $\Rightarrow \operatorname{Rang} GF = \operatorname{Rang} G$



- **Column Space** $R(A) = \operatorname{span}\{a_1 .. a_n\} = \operatorname{im} A$
- **Row Space:** Subspace spanned by the row vectors
- **Null Space** $\mathcal{N}(A) = \{x \mid Ax = 0\} = \ker A$
- $Ax = b$
  - $Ax = b$ has a solution $\Leftrightarrow b \in R(A)$
  - $Ax = b$ has a unique solution $\Leftrightarrow b \in R(A)$ and $\mathcal{N}(A) = \{0\}$
  - $\dim \mathcal{N}(A) = \dim \ker A = n - \operatorname{rank} A$

Linear Mapping

- **Find Basis of** $\ker A$:
  - Get RREF using Gauss
  - Take all colums with free variables and set one variable per column to 1, all other to 0
  - Fill the non-free variables by creating a system of equations: $Ax = 0$ and solve the system for the non-free variables
- **Find Basis of** $\operatorname{Img} A$:
  - Get RREF using Gauss
  - Get Pivot Columns
  - Take the Pivot Colums, but from the unreduced A
- **Rank of** $A \in \mathbb{E}^{m \times n}$
  - # Pivots in RREF
  - $\dim \operatorname{im} A$
  - Dimension of the column space / # lin. indep. colums
  - Dimension of the row space / # lin indep. rows
- $\operatorname{Rang} A = \operatorname{Rang} A^T$
- **For** $A \in \mathbb{E}^{m \times n}$, $B \in \mathbb{E}^{p \times m}$
  - $\operatorname{Rang} BA \leq \min(\operatorname{Rang} B, \operatorname{Rang} A)$
  - $\operatorname{Rang} B = m \ (\leq p) \Rightarrow \operatorname{Rang} BA = \operatorname{Rang} A$
  - $\operatorname{Rang} A = m \ (\leq n) \Rightarrow \operatorname{Rang} BA = \operatorname{Rang} B$
- **For** $A \in \mathbb{E}^{n \times n}$, $B \in \mathbb{E}^{m \times m}$
  - $\operatorname{Rang} BA \leq \min(\operatorname{Rang} B, \operatorname{Rang} A)$
  - $\operatorname{Rang} B = m \Rightarrow \operatorname{Rang} BA = \operatorname{Rang} A$
  - $\operatorname{Rang} A = m \Rightarrow \operatorname{Rang} BA = \operatorname{Rang} B$
- **For** $A \in \mathbb{E}^{n \times n}$
  - A is invertable
  - A is non-singular
  - $\operatorname{Rang} A = n$
  - n column vectors are lin. indep.
  - n row vectors are lin. indep.
  - $\operatorname{im} A = R(A) = \mathbb{E}^n$
  - $\ker A = \mathcal{N}(A) = \{0\}$
  - Lin. mapping $A: \mathbb{E}^n \to \mathbb{E}^n$ is automorphism
  - A is a transformation matrix of a coordinate-transformation

Linear Mapping

- **affine (Sub)space:** $u_0 + U = \{u_0 + u \mid u \in U\}$, $u_0 \in V$, $U \subset V$
- **Affine Mapping:** $H: X \to y_0 + Y$, $x \mapsto y_0 + Fx$
  where $F: X \to Y$ is a lin. mapping and $y_0 \in Y$
- $u_0 \notin U \Rightarrow u_0 + U$ is not a subspace
- $y_0 \neq 0 \Rightarrow H$ is not linear
- x_0 is any solution to $Ax = b$ and $\mathcal{L}_0$ is the solution set of $Ax = 0$, then is the solution set of $Ax = b$ $\mathcal{L}_b = x_0 + \mathcal{L}_0$
-

$$\begin{tikzcd} x \in X \arrow[r, "F"] \arrow[d, shift right, "x_1"'] & y \in Y \arrow[d, shift right, "x_2"'] \\ \xi \in \mathbb{E}^n \arrow[r, "A"] \arrow[d, shift right, "T^{-1}"'] & \eta \in \mathbb{E}^m \arrow[d, shift right, "S^{-1}"'] \\ \xi' \in \mathbb{E}^n \arrow[r, "B"] & \eta' \in \mathbb{E}^m \end{tikzcd}$$

  - $y = Fx$
  - $\eta = A\xi$
  - $\xi = T\xi'$
  - $\eta = S\eta'$
  - $\eta' = B\xi'$
  - $B = S^{-1}AT$, $A = SBT^{-1}$
  - $\operatorname{Rang} F = \operatorname{Rang} A = \operatorname{Rang} B$
  - If $Y = X$: $B = T^{-1}AT$, $A = TBT^{-1}$
- $A, B \in \mathbb{E}^{n \times n}$ are similar if non-singular T exists s.d. $B = T^{-1}AT$ and $A = TBT^{-1}$
- **ähnlichkeitstransformation:** $A \mapsto B = T^{-1}AT$
- $F: X \to Y$ is a lin. mapping where $\dim X = n$ and $\dim Y = m$ and $\operatorname{Rang} F = n$. Then F has the abbildungsmatrix $A = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$

Vectorspace with Scalar product

- **Norm**: $\|\cdot\| : V \to \mathbb{R},\ x \mapsto \|x\|$
  - Properties $\forall x, y \in V, \forall \alpha \in \mathbb{E}$
    - N1: Positive defined: $\|x\| \geq 0,\ \|x\| = 0 \Rightarrow x = 0$
    - N2: Homogenious: $\|\alpha x\| = \alpha \|x\|$
    - N3: Triangular Inequality: $\|x+y\| \leq \|x\| + \|y\|$
  - Normed Vector Space: V.S. with a norm
  - Euclidean Norm / 2-Norm: (inuzierte Norm) $\|x\|_2 = \sqrt{x_1^2 + \dots x_n^2} = \sqrt{x^T x} = \sqrt{\langle x, x\rangle}$
  - Max Norm: $V = P[a,b]$ $\|p\|_\infty = \max_{x \in [a,b]} |p(x)|$
- **Scalar Product**: $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{E},\ x, y \mapsto \langle x, y\rangle$
  - Properties: $\forall x, y, z \in V, \forall \alpha \in \mathbb{E}$
    - S1: Linear in the second factor:
      $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
      $\langle x, \alpha y\rangle = \alpha \langle x, y\rangle$
      $\Rightarrow \langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle$
    - S2: Symmetric if $V = \mathbb{E}$: $\langle x, y\rangle = \langle y, x\rangle$
      Hermitian if $V = \mathbb{C}$: $\langle x, y\rangle = \overline{\langle y, x\rangle}$
    - S3: Positiv definite: $\langle x, x\rangle \geq 0,\ \langle x, x\rangle = 0 \Rightarrow x = 0$
  - Euclidian V.S. / orthogonal V.S.: $V = \mathbb{E}$ with S.P.
  - Unitary V.S.: $V = \mathbb{C}$ with scalar product
  - Euclidian S.P: $\langle x, y\rangle = x^T y = \sum_{j=1}^{n} x_j y_j$ $(x^H y)$
  - Weighted S.P: $\langle x, y\rangle_W = x^H W y$, W is diag. matrix
  - For all V.S. can a S.P. be defined
- **Angel**: $\varphi = \sphericalangle(x, y)$, $0 \leq \varphi \leq \pi$
  - $\mathbb{E} = \mathbb{R}$: $\varphi = \arccos \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}$
  - $\mathbb{E} = \mathbb{C}$: $\varphi = \arccos \frac{\mathrm{Re}(\langle x, y\rangle)}{\|x\| \cdot \|y\|}$
  - x, y are orthogonal, if $\langle x, y\rangle = 0$: $x \perp y$
  - $M, N \subseteq V$ are orthogonal, if $\langle x, y\rangle = 0\ \forall x \in M, y \in N$: $M \perp N$
  - Zero vector is orthogonal to all vectors
  - Zerospace is orthogonal to all Subspaces
  - Pythagoras: $x \perp y \Rightarrow \|x \pm y\|^2 = \|x\|^2 + \|y\|^2$
- **Cauchy-Schwarz Inequality**:
  $|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \Leftrightarrow |\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$
  $|\langle x, y\rangle| = \|x\| \cdot \|y\| \Rightarrow x, y$ are lin. dep.

Vectorspace with Scalar product

- Set M of pairwise orthogonal Vectors is lin. indep if $0 \notin M$
- $\dim V = n \Rightarrow n$ paiswise orthogonal Vectors $(\neq 0)$ form a basis for V
- **Orthogonal Basis**: Basisvectors are pairwise orthogonal $\langle b_k, b_l\rangle = 0$ if $k \neq l$
- **Orthonormal Basis**: Orthogonal Basis where all basisvectors have length 1: $\langle b_k, b_k\rangle = 1 \Rightarrow \langle b_k, b_l\rangle = \delta_{kl}$
- **Kronecker-Delta**: $\delta_{kl} = 0$ if $k \neq l$, 1 if $k = l$
- V is V.S., $\dim V = n$, $\{b_1 \dots b_n\}$ is orthonormal basis, then $\forall x \in V$ $x = \sum_{k=1}^{n} \langle b_k, x\rangle b_k$
  $\Rightarrow$ for coordinates of an orthogonal basis: $\xi_k = \langle b_k, x\rangle$

- **Parsevalsche Formel**: $\langle x, y\rangle_V = \langle \xi, \eta\rangle_{\mathbb{E}^n}$: Given two vectors $\in V$, V has a orthonormal basis, the S.P is equals the S.P of their coordinate vectors in $\mathbb{E}^n$
  $\Rightarrow \|x\| = \|\xi\|$, $\sphericalangle(x, y) = \sphericalangle(\xi, \eta)$, $x \perp y \Leftrightarrow \xi \perp \eta$
- **Triangle Inequality**: $\|x+y\| \leq \|x\| + \|y\|\ \forall x, y \in V$
- **Gram-Schmidt-Orthogonalisierungsverfahren**
  - $\{a_1 \dots a_n\} \subseteq V$ lin. indep., find $\{b_1 \dots b_n\}$
    - $b_1 := \frac{a_1}{\|a_1\|}$
    - $\tilde{b}_k := a_k - \sum_{j=1}^{k-1} \langle b_j, a_k\rangle b_j$ $\Big\}\ k = 2 \dots n$
    - $b_k := \tilde{b}_k / \|\tilde{b}_k\|$
  - $\{b_1 \dots b_n\}$ are paiswise orthogonal and normed
  - after k steps: $\mathrm{span}\{a_1 \dots a_n\} = \mathrm{span}\{b_1 \dots b_k\}$
  - $\{a_1 \dots a_n\}$ is basis of $V \Rightarrow \{b_1 \dots b_n\}$ is orthonormal basis of V
- $\forall$ V.S. with SP $\exists$ orthonormal basis
- **Orthogonal Complement**: $U \subseteq V$, $U' \subseteq V$, s.d. $U \oplus U' = V$
  - $U := \mathrm{span}\{b_1 \dots b_k\}$ of orthogonal vectors
  - $U' := \mathrm{span}\{b_{k+1} \dots b_n\}$ of orthogonal vectors $\Big\}\ U \perp U'$
  - $x \in V = \underbrace{\xi_1 b_1 + \dots + \xi_k b_k}_{=: u \in U,\ u \perp U'} + \underbrace{\xi_{k+1} b_{k+1} + \dots + \xi_n b_n}_{=: u' \in U',\ u' \perp U} = u + u'$
  - $U' \Leftrightarrow U^\perp$
  - $(U^\perp)^\perp = U$
  - $\dim U + \dim U^\perp = \dim V$

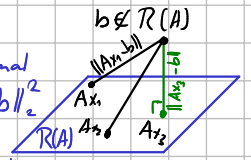
Vectorspace with Scalar product

- Orthogonal Complement:
  - $A \in \mathbb{E}^{m \times n}$, $x \in \mathbb{E}^n$, $Ax = 0 \Leftrightarrow x \perp$ all rows of A
  - $x \in \mathcal{N}(A) \Leftrightarrow Ax = 0 \Leftrightarrow x \perp \mathcal{R}(A^T) \Leftrightarrow x \in (\mathcal{R}(A^T))^\perp$ (same A^H)
  - $A \in \mathbb{C}^{m \times n}$ Rank $A = r$ (same for A^T):
    - $\mathcal{N}(A) = (\mathcal{R}(A^H))^\perp \subseteq \mathbb{E}^n$ | $\mathcal{N}(A^H) = (\mathcal{R}(A))^\perp \subseteq \mathbb{E}^m$
    - $\mathcal{N}(A) \oplus \mathcal{R}(A^H) = \mathbb{E}^n$ | $\mathcal{N}(A^H) \oplus \mathcal{R}(A) = \mathbb{E}^m$
    - $\dim \mathcal{R}(A) = r$ | $\dim \mathcal{N}(A) = n - r$
      $\dim \mathcal{R}(A^H) = r$ | $\dim \mathcal{N}(A^H) = m - r$
  - Fundamental Subspaces: $\mathcal{N}(A), \mathcal{R}(A^H) \in \mathcal{N}(A^H), \mathcal{R}(A)$
- Change of basis:
  - Transformationmatrix is unitary / orthogonal for orthonormal bases: $T^{-1} = T^H$
  - $\xi = T \xi'$ $\quad \xi' = T^H \xi$
  - $B = B' T^H$ $\quad B' = B T$
  - $\eta' = T^H \eta$
    - $\langle \xi', \eta'\rangle = \langle \xi, \eta\rangle$
      - $\|\xi'\| = \|\xi\|$ $\sphericalangle(\xi', \eta') = \sphericalangle(\xi, \eta)$, $\xi' \perp \eta \Leftrightarrow \xi \perp \eta$
- **Lin. Mapping**: $F: X \to Y$, B, B' orth. basis of X, D, D' ortho. basis of Y
  - $B = S^H A T$, $A = S B T^H$
  - $X = Y$: $B = T^H A T$ $\quad A = T B T^H$
  - A Hermitian $\Rightarrow B$ Hermitian (symetric)
  - A unitary $\Rightarrow B$ unitary (orthogonal)
- **Unitary Mapping**: $\langle Fx, Fy\rangle_Y = \langle x, y\rangle_X$, X, Y unitary / orthogonal V.S., $F: X \to Y$ Lin. Mapping $x, y \in X$
  - $\|Fx\|_Y = \|x\|_X$: F is length preserving or isometric
  - $x \perp y \Rightarrow Fx \perp Fy$: angle preserving
  - $\ker F = \{0\}$: injective
  - If $\dim X = \dim Y < \infty$
    - F is isomorphismus
    - $\{b_1 \dots b_n\}$ Orthogonal Basis of $X \Rightarrow \{Fb_1 \dots Fb_n\}$ orthogonal basis of Y
    - F^{-1} is unitary (orthogonal)
    - Abbildungsmatrix A of orthogonal bases of X and Y is unitary (orthogonal)

Method of the Least Square

- V.S. $\mathbb{E}^n$ with S.P
- Linear Regression: minimize vertical distance between datapoints and a line

- LGS with more unknowns than equations $m<n$ has generally no solution



- Find x s.d. $\|Ax-b\|_2^2$ is minimal
- Minimal x: $x^* = \arg\min \|Ax-b\|_2^2$
- $\|Ax-b\|^2$ is minimal $\Rightarrow (Ax-b) \perp R(A)$
  $\Rightarrow (Ax-b) \in R(A)^\perp = N(A^H) \Rightarrow A^H Ax - b = 0$
  $\Rightarrow A^H A x = A^H b$
- Normalgleichungen: $A \in \mathbb{E}^{m\times n}$, $m \geq n$, $b \in \mathbb{E}^m$, $x \in \mathbb{E}^n$
  $\min \|Ax-b\|^2 \Leftrightarrow \underline{A^H A} x = A^H b$
- $A^H A$ non-singular $\Leftrightarrow$ Rang $A = n$
  - $x^* = (A^H A)^{-1} A^H b$
- Moore-Penrose-Pseudoinverse: $A^+ := (A^H A)^{-1} A^H$
  $\in \mathbb{E}^{n\times m}$
  - $A^+ A = I$
- Rang$(A) < n$: Pseudoinverse with Singulärwert-zerlegung. $\Rightarrow$ Normalgleichung has no unique solution
- Approximation of Data using analytic functions
  - Measured: $(t_1, f_1) \dots (t_N, f_N)$
  - Approximate using $\mathbb{P}^n$
  - $\sum_{j=1}^{N} |f(t_j) - f_j|^2 \to \min$, find $a_1 \dots a_n$ of f
  - Vandermonde Matrix U: $\underbrace{\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^n \end{pmatrix}}_{N\times(n+1)} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}}_{(n+1)\times 1} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}}_{N\times 1}$
    $\Rightarrow Ua = f$
    $a^* \overset{?}{=} \arg\min \|Ua - f\|^2$
    $\Rightarrow U^T U a^* = U^T f$

Determinant

- Permutation: $p: \{1 \dots n\} \to \{1 \dots n\}$
  $(1, \dots n) \mapsto (p(1), \dots p(n))$
  - unique mapping (bijection)
  - $n!$ Permutations of $(1, \dots n)$
  - Symmetric group S_n: set of all $n!$ permut.
  - Transposition t_k: Swap of neighbouring elements
    - Every permutation is a product of transposit.
      - $p = t_v \circ \dots \circ t_1$
    - Not unique
    - $\# t_k$ is either always even or odd
- Signum of permutation: sign $p = +1$ if v is even, -1 if v is odd
- Determinant: $\det: \mathbb{E}^{n\times n} \to \mathbb{E}$, $A \mapsto \det A$, $|A|$
- $\det A \in \mathbb{E}^{n\times n} = \sum_{p\in S_n} \text{sign}(p) \cdot a_{1p(1)} \cdot \dots \cdot a_{np(n)}$
  - Runtime: $O(n!)$ $\to$ practically unusable
- $|a_{11}| = a_{11}$
- $\begin{vmatrix} a_{11} & a_{21} \\ a_{11} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- $|A \in \mathbb{E}^{3\times 3}| = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$
  (Sarrus) $- a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}$
- Triangular & Diagonal Matrix: $\prod_{i=1}^{n} a_{ii}$
- Entwicklung nach k-ter
  - Zeile: $\det A = \sum_{j=1}^{N} (-1)^{k+j} a_{kj} \cdot \det A_{[k,j]}$
  - Spalte: $\det A = \sum_{j=1}^{N} (-1)^{j+l} a_{jl} \cdot \det A_{[j,l]}$
  - $A_{[k,j]}$: $(n-1)(n-1)$ is the matrix A without row k and column j
  - Kofaktor $K_{kl} := (-1)^{k+l} \det A_{[k,l]}$
- Geometric interpretation: Transformation of the hyperplane under mapping described by A
  - $\det A = 1$: area/volume stays unchanged
  - $\det A = 0$: loss of dimension
  - $\det A = -1$: Flip of the coordinate axis
  - $\det A = a \neq 0$: Scaling of area/volume

Determinant

- Properties: (also for columns)

Characteristic:
- D1: linear in each row: $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \alpha a_{i1} + \beta b_{i1} & \cdots & \alpha a_{in} + \beta b_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \beta \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$
- D2: swap of two rows $\Rightarrow$ swap of sign
- D3: $\det I = 1$

  - A has row of only 0 (singular) $\Rightarrow |A| = 0$
  - $\det(\alpha A) = \alpha^n \det A$
  - A has two equal rows (singular) $\Rightarrow |A| = 0$
  - adding a multiple of another row to the current row does not change det
  - A is diagonal matrix $\Rightarrow |A| = \prod_{i=1}^{n} a_{ii}$
  - A is triangle matrix $\Rightarrow |A| = \prod_{i=1}^{n} a_{ii}$
  - $\det A \neq 0 \Leftrightarrow$ Rang $A = n \Leftrightarrow A$ is non-singular
- A reduced to RREF using gauss:
  $\det A = (-1)^v \prod_{k=1}^{r} r_{kk}$, v is # row swaps and r_{kk} are the diagonal elements of RREF
  - Runtime: $O(n^3)$ instead of $O(n!)$
- $A, B \in \mathbb{E}^{n\times n} \Rightarrow \det(AB) = \det A \cdot \det B$
- A is non-singular: $\det(A^{-1}) = \det(A)^{-1}$
- $\det A^T = \det A$, $\det A^H = \overline{\det A}$
- Block Matrixes: $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = \det A \det B$
- $\det A = \prod_{j=1}^{n} \lambda_j$, λ_i is EW of A
- $\det A \pm 1$ if A is unitary/orthogonal
- $\det A \in \mathbb{R}$ if A is hermitean

Eigenwert and Eigenvalue

- The Eigenvalue $\lambda \in \mathbb{E}$ is the stretching factor of the Eigenvector $v \in V$, $v \neq 0$ under the lin. transformation $F: V \to V$, $x \mapsto Fx$
  $\Rightarrow Fv = \lambda v$
  - "Eigenvector is invariant to F"
- Eigenraum $E_\lambda := \{v \in V \mid Fv = \lambda v\}$, set of all Eigenvectors and the Zerovector form a subspace of V
  - Each Eigenraum has at least a 1D. subspace
  - $E_\lambda = \ker(F - \lambda I)$
- Geometrische Vielfachheit: $\dim E_\lambda = \dim \ker(A - \lambda I) = n - \operatorname{Rang}(A - \lambda I)$
- Algebraic Vielfachheit: Vielfachheit of λ as a root of $\chi_A(\lambda) = (\lambda - \lambda)^k (\ldots)$
- $\forall \lambda$ geom. Vielf. $\leq$ algeb. Vielf.
- Spectrum $\sigma(F)$: Set of all Eigenvalues
  - $0 \in \sigma(A) \Leftrightarrow A$ is singular
- Same for square matrices: $A\xi = \xi\lambda$, $A \in \mathbb{E}^{n \times n}$, $\xi \in \mathbb{E}^n \neq 0$
- When a Abbildungsmatrix A describes F, A and F have the same Eigenvalues and Eigenvectors
- Scalar multiple of a Eigenvector is a Eigenvector to: $Fv = \lambda v \Rightarrow F(\alpha v) = \lambda(\alpha v)$
- x is E.V. to E.W. $\lambda \Leftrightarrow Fx = \lambda x$ and $x \neq 0 \Leftrightarrow x \in \ker(F - \lambda I)$ and $x \neq 0 \Leftrightarrow (F - \lambda I)$ is singular $\Leftrightarrow \det(F - \lambda I) = 0$
- Characteristic Polynomial $\chi_A(\lambda) := \det(A - \lambda I)$
  - $A \in \mathbb{E}^{n \times n} \Rightarrow \chi_A \in \mathcal{P}^n$
  - $\chi_A(\lambda) = (-\lambda)^n + \operatorname{spur} A\,(-\lambda)^{n-1} + \ldots + \det A$
    - $A \in \mathbb{R}^{2\times 2}$: $\chi_A(\lambda) = \lambda^2 - \operatorname{spur}(A)\lambda + \det(A)$
- Characteristic Equation $\chi_A(\lambda) = 0$
  - $\chi_A(\lambda) = 0 \Leftrightarrow \lambda$ is E.V.
- Spur: Sum of the diagonal elements: $\operatorname{spur} A = \sum_{j=1}^{n} a_{jj}$
- Ähnliche matrices ($B = T^{-1}AT$) have the same characteristic polynomial, det, spur and λ. The geom. Viel. and algeb. Viel are equal too.
- A and A^T have the same EW.
- A is full rank: $\lambda_i \neq 0 \; \forall i$

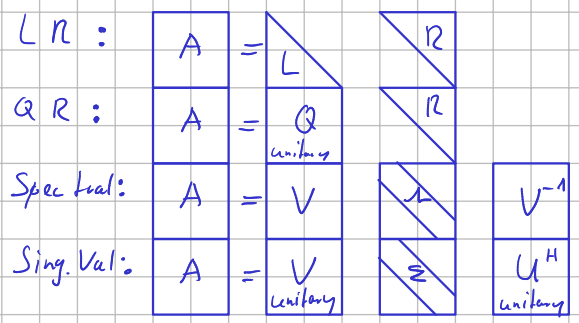
Eigenwert and Eigenvalue

- E.V. to different E.W. are lin. indep.
  - $\Rightarrow A$ can have at most dim
- Eigenbasis: Basis consisting of Eigenvectors of A
  - Does not always exist
  - n different E.V $\Rightarrow \exists$ Eigenbasis
  - $\exists$ Eigenbasis $\Rightarrow A$ diagonalisierbar
- Find E.V. and E.W. of A:
  - Calculate $\chi_A(\lambda) = \det(A - \lambda I) = 0$
  - Calculate root of $\chi_A(\lambda)$ and the algebraic Vielfachheit
  - Calculate v_k for every λ_k of $(A - \lambda_k I)v = 0$ using Gauss
- Spectral Decomposition / Eigenwertszerlegung
  - does not always exist
  - $\exists$ Eigenbasis $\Rightarrow \exists$ to A similar Diag. Matrix Λ
  - V is matrix with Eigenbasis as columns
  - $AV = V\Lambda \Leftrightarrow A = V\Lambda V^{-1}$
  - A is diagonalisierbar by V
  - $\exists V \in \mathbb{E}^{n \times n}$ and $\exists \Lambda$ diag. s.d. $A = V\Lambda V^{-1} \Rightarrow$ columns of V are E.V. and diag. elem. Λ are E.W.
  - A diagonalisierbar $\Leftrightarrow \forall$ E.W. the geom. Viel. = algeb. Viel.
  - $A \in \mathbb{C}^{n \times n}$ Hermitesch ($\mathbb{R}^{n \times n}$ symmetric):
    - $\lambda_i \in \mathbb{R} \; \forall i \in \{1 \ldots n\}$
    - E.V. to different E.W. are pairwise orthogonal in $\mathbb{C}$ (in $\mathbb{R}$)
    - $\exists$ Eigenbasis of $\mathbb{C}^n$ ($\mathbb{R}^n$) of E.V. of A
    - For unitary (orthogonal) $U = (u_1 \ldots u_n)$
      $U^T A U = \Lambda := \operatorname{diag}(\lambda_1 \ldots \lambda_n)$
- get Λ of A:
  - Calculate E.W. and n lin. indep. E.V. of A
  - def. $\Lambda = \operatorname{diag}(\lambda_1 \ldots \lambda_n)$, $\lambda_i \geq \lambda_{i+1}$
  - def. $V = (v_1 | \ldots | v_n)$
  - $A = V\Lambda V^{-1}$
- Spur: $\sum \lambda$ (including algebraic multiplicity)
- det: $\prod \lambda$ (including algebraic multiplicity)
- Diag. elem. of lower triag. Mat. are λ_i

Eigenwert and Eigenvalue

- $A \in \mathbb{E}^{n \times n}$ is normal if $A^H A = A A^H$
  - Hermitic and symmetric A are normal
  - $A \in \mathbb{E}^{n \times n}$ is normal $\Leftrightarrow A$ is Diagonalisierbar by an unitary matric.
- $A \in \mathbb{E}^{n \times n}$ hermitesch is positive definit if $\forall x \in \mathbb{E}^n, x \neq 0 \quad x^H A x > 0$
- $A \in \mathbb{E}^{n \times n}$ hermitesch is positive semidefinit if $\forall x \in \mathbb{E}^n, x \neq 0 \quad x^H A x \geq 0$
- A positive definit $\Rightarrow \lambda_i > 0 \; \forall i \in 1 \ldots n$
- A positive semidefinit $\Rightarrow \lambda_i \geq 0 \; \forall i \in 1 \ldots n$
- Singulärwertzerlegung / Singular Value Decomposition:
  - does always exist. Even if $\mathbb{E}^{n \times m}$
  - Orthonormalbasis $\{v_1 \ldots v_n\}$ and $\{u_1 \ldots u_n\}$ and $\tau_1 \ldots \tau_n \geq 0$ s.d.: $A \cdot \underbrace{(v_1 \cdots v_n)}_{V} = \underbrace{(u_1 \ldots u_n)}_{U} \underbrace{\operatorname{diag}(\tau_1 \ldots \tau_n)}_{\Sigma}$
  - $AV = U\Sigma \Leftrightarrow A = U\Sigma V^H$
    - V, U unitary, Σ diag ≥ 0
  - $\tau_j := \sqrt{\lambda_j}$
  - $\forall A \in \mathbb{E}^{m \times n}$, $\exists U \in \mathbb{E}^{m \times m}$ unitary, $V \in \mathbb{E}^{n \times n}$ unitary and $\Sigma \in \mathbb{E}^{m \times n}$ diagonal, s.d. $A = U\Sigma V^H$
    - diag elements of Σ: $\tau_1 \geq \tau_2 \ldots \geq \tau_r > \tau_{r+1} = \tau_n = 0$ are called Singulärwerte
    - U diagonalisiert $AA^H = U \left(\begin{array}{c|c} \begin{matrix} \tau_1^2 & & \\ & \ddots & \\ & & \tau_r^2 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) U^H$
    - V diagonalisiert $A^H A = V \left(\begin{array}{c|c} \begin{matrix} \tau_1^2 & & \\ & \ddots & \\ & & \tau_r^2 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) V^H$
    - $\exists A^{-1} \Rightarrow A^{-1} = V \underbrace{\begin{pmatrix} 1/\tau_1 & & \\ & \ddots & \\ & & 1/\tau_n \end{pmatrix}}_{\Sigma^{-1}} U^H$
- Runtime: worse than $O(n^3)$
- $m = n$: $A = U \cdot \Sigma \cdot V^H$
- $m > n$: $A = U \cdot \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} \cdot V^H$
- $m < n$: $A = U \cdot (\Sigma \;\; 0) \cdot V^H$

Decompositions



- given square matrix D with one unknown α. What are the possible values for α s.d. D is diagonalisierbar by an orthogonal Matrix over $\mathbb{R}$.
  - A must be normal: $DD^T = D^TD$

- given $B \in \mathbb{R}^{n\times n}$, with $\lambda \in \{\dots\}$. Is it possible that B has $b_{ij} = x \ \forall j$
  - B is similar to diag Mat. Λ $\Rightarrow$ must have same Span, Ew.
  - $\sum \lambda_i = \sum b_{ii} \Rightarrow$ possible

- given $A \in \mathbb{R}^{3\times 3}$, $\lambda = \{a,b,c\}$. What is $\det(1+A^{50})$
  - $A^{50} = \operatorname{diag}(|a|,|b|,|c|)$
  - $1+A^{50} + \operatorname{diag}(1+|a|, 1+|b|, 1+|c|)$
  - $\det(\operatorname{diag}(1+|a|, 1+|b|, 1+|c|)) = \prod 1+|\lambda|$

- $V := \{p(x) \in P_3 \mid p(0) = p(1) = 0\}$. Find a basis of V
  - $p(x) = ax^3 + bx^2 + cx + d$
  - Conditions: • $p(0) = 0 \Rightarrow d = 0$
    • $p(1) = 0 \Rightarrow a+b+c = 0$

$$\Rightarrow p(x) = ax^3 + bx^2 + (-a-b)x$$
$$= a\underbrace{(x^3 - x)} + b\underbrace{(x^3 - x)}$$

- Mapping $F: P_3 \to \mathbb{R}$ is given by $F(p) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$ and Basis $B = \{1, t, t^2, t^3\}$, $C = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. Find the transformation matrix:
  - Let B be transformed by $F \Rightarrow$ get Transformation-matrix

- $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$. Prove $\|Ax\|_2 \geq \delta_{min}\|x\|_2$
  δ_{min} is the smallest singular value
  $\Rightarrow \|Ax\| = \|U\Sigma V^T x\|$: Sing. decomp. of A
  $\Rightarrow \|U\Sigma V^T x\| = \|\Sigma V^T x\|$: orthonorm. Mat. does not change the Norm
  $\Rightarrow \tilde{\Sigma} = \operatorname{diag}(\delta_{min})$ to get a lower bound
  $\Rightarrow \|\tilde{\Sigma} x\| = \sqrt{\sum_i \delta_{min}^2 x_i^2} = \delta_{min}\|x\| \leq \|\Sigma V^T x\|$
  $\Rightarrow \|Ax\| \geq \delta_{min}\|x\|$

- Show that $U = \{A \in \mathbb{R}^{2\times 2} \mid \operatorname{Span}(A) = 0\}$ is a linear subspace of $\mathbb{R}^{2\times 2}$
  - $A, B \in U$
    - $\operatorname{span}(\alpha A + B) = \alpha \operatorname{span}(A) + \operatorname{span}(B)$
    $\Rightarrow \alpha A + \beta B \in U \Rightarrow U$ is a linear subspace

- Find basis of $U = \{A \in \mathbb{R}^{2\times 2} \mid \operatorname{Span}(A) = 0\}$
  - $\operatorname{Span}(A) = 0 \Leftrightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = A$

$$\Rightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\Rightarrow \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$$

- For which values a,b,c is D diagonalisierbar over $\mathbb{R}$:
$$D = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$
  - D is schief symetric and all elements are real
  - All Eigenvalues are complex or equal 0
  - Matrix is only diagonalisierbar over $\mathbb{R}$ if all Eigenvalues are real
  - D diagonalisierbar $\Leftrightarrow a = b = c = 0$

- $a, b, c \in \mathbb{R}^3$ are pairwise orthonormal $\Rightarrow \det(2a, 3b, 4c) = \pm 24$

- Show $F: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $A \to AM$ is linear
  - $F(\lambda x + y) = (\lambda x + y)M = \lambda xM + yM$
  $= \lambda F(x) + F(y)$

- Given $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, create orthogonal basis:
  - $b_1 = \frac{v_1}{\|v_1\|}$
  - $\tilde{b}_2 = v_2 - b_1 \langle b_1, v_2 \rangle$
  - $b_2 = \frac{\tilde{b}_2}{\|\tilde{b}_2\|}$ (not necessary for orthogonal basis)

- Get kern of a mapping: Use the mapping itself and not the transformation-matrix to get it.

- $$\begin{pmatrix} 1 & 5 & 2 & 3 & 7 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0 \quad \left| \begin{array}{l} a + 5b + 2c + 3d + 7e = 0 \\ b + \quad e = 0 \\ c + d + e = 0 \end{array} \right.$$

$$d \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + e \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Solve for non free (a, b, c)

$c = -d - e$

$b = -e$

$a = -5b - 2c - 3d - 7e$

$= 5e + 2d + 2e - 3d - 7e$

$= -d$

- given $g(t) = a + tb$, $h(s) = c + sd$, $a, b, c \in \mathbb{R}^3$
  - Pts with minimal distance: $\arg\min_{t,s \in \mathbb{R}} \|g(t) - h(s)\|^2$
    $= \arg\min \|a + tb - c - sd\|^2$
  - $M \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} m & n \\ o & p \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}$, find a, b, c, d, k, l:
    $\arg\min \| a - c + (b, -d) \begin{pmatrix} t \\ s \end{pmatrix} \|^2$
    $\Rightarrow$ LSS: $(b, -d)^T (b, -d) \begin{pmatrix} t \\ s \end{pmatrix} = (b, -d)^T (c - a)$
    $$\begin{pmatrix} b^T \\ -d^T \end{pmatrix} (b, -d) \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} b^T \\ -d^T \end{pmatrix} (c - a)$$
    $m = b^T b$, $n = -b^T d$, $o = -d^T b$, $p = d^T d$
    $k = b^T (c - a)$, $l = -d^T (c - a)$
  - Rank M tells: when not full rank $\Rightarrow \infty$ many sol.
    $\Rightarrow$ lines are parallell
  - Same lines in $\mathbb{R}^4$: nothing changes except that we add another dimension to the vectors.

- $F(A) = \frac{1}{2}(A - A^T)$. find $\ker F$:
  $F(A) = 0 \Rightarrow A = A^T \Rightarrow$ a symetrisch
  $\Rightarrow \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

- Characterice Matrices of img F, $F(A) = \frac{1}{2}(A - A^T)$
  $\Rightarrow B \in \operatorname{im} F \Rightarrow B = \frac{1}{2}(A - A^T)$
  $B^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -B$
  $\Rightarrow B$ is schiefsymmetric
  $\Rightarrow \operatorname{im} F \subset \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$
  $\Rightarrow B$ is schiefsymmetric: $B = \frac{1}{2}B + \frac{1}{2}B = \frac{1}{2}B - \frac{1}{2}B^T =$
  $\frac{1}{2}(B - B^T) = F(B)$
  $\Rightarrow \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\} \subset \operatorname{im} F$

- MC:
  - Vectors a, b, c are lin. indep when pairwise lin. indep: ✗
  - $B \in \mathbb{E}^{3\times 1}$, $C \in \mathbb{E}^{3\times 1}$. It is possible that $\operatorname{rang}(BC) = 3$: ✗
  - $D \in \mathbb{E}^{2\times 2}$, $\dim(\ker(D)) = 2$ genau dann, wenn $D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: ✓
  - $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0 \Rightarrow$ algebraische Vielfachheit for all λ is always 1: ✗
  - $AB = I \in \mathbb{R}^{n\times n} \Rightarrow A, B$ inversable: ✗
  - $A, B \in \mathbb{R}^{n\times n}$, $AB = -BA$
    - $\det(AB) = \det(-BA)$: ✓
    - $\det(A)\det(B) = -\det(A)\det(B)$: ✗
      - $d(AB) = d(-BA) \Rightarrow d(A)d(B) = d(-I)d(B)d(A)$
        $\Rightarrow d(A)d(B) = (-1)^n d(B)d(A) \Rightarrow n$ must be even
    - $d(A)$ or $d(B)$ must be zero: ✗
      - $d(A)d(B) \overset{!}{=} 0 \Leftrightarrow n$ odd
    - A and B must be singular: ✗
      - only when n is odd
    - $ABx = 0$ has solution schar: ✗
      - only if $d(AB) = 0$
    - $ABx = c$ may have no, one, multiple solutions for $c \neq 0 \in \mathbb{R}^n$: ✓
      - 1 sol: $d(AB) \neq 0$
      - ∞ sol: $d(AB) = 0$ and $c \in \operatorname{img} AB$
      - 0 sol: $d(AB) = 0$ and $c \notin \operatorname{img} AB$
    - $A = 0$ or $B = 0$: ✗
  - rows lin. indep $\Leftrightarrow$ columns indep.: ✗
    - only for square matrices
  - $R(A) = R(A^T)$, $A \in \mathbb{R}^{3\times 3}$: ✗
  - $\dim R(A) = \dim R(A^T)$, $A \in \mathbb{R}^{3\times 3}$: ✓
  - $A \in \mathbb{R}^{n\times n}$
    - $|d(A)| = 1 \Rightarrow A^{-1} = A^T$: ✗
    - $A^{-1} = A^T \Rightarrow |d(A)| = 1$: ✓
    - $A^{-1} = A^T \Rightarrow \lambda_i = 1$: ✗
    - $A = A^T$ and $\lambda_i = 1 \Rightarrow A = Id$: ✓
      - $A = U\Lambda U^T \Rightarrow \Lambda = I \Rightarrow A = UU^T \Rightarrow UU^T = I$
        $\Rightarrow A = I$
  - $\langle x, y \rangle \leq \min(\|x\|, \|y\|)$, $x, y \in \mathbb{R}^n$: ✗