1 Allgemein

1.1 Potenzen und Wurzeln

- – $a^1 = a$ – $a^0 = 1$
- $a^{-n} = \frac{1}{a^n}$
- – $a^{\frac{1}{n}} = \sqrt[n]{a}$ – $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- $a^m a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^n b^n = (ab)^n$
- $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$
- $a^b = e^{b \ln a}$
- $e^{a \ln b} = b^a$
- $e^{a+b} = e^a \cdot e^b$
- $e^0 = 1$

1.2 Logarithmen

$y = \log_a x \iff a^x = x$

- $a^{\log_a x} = x$
- $\log_a a^x = x$
- – $\log_a a = 1$ – $\log_a 1 = 0$
- $\log(uv) = \log(u) + \log(v)$
- $\log(\frac{u}{v}) = \log(u) - \log(v)$
- $\log(u^r) = r \log(u)$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\ln(a \cdot b) = \ln(a) + \ln(b)$
- $\ln(1) = 0$

1.3 Trigonometrische Funktionen

Sinussatz: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r$

Cosinussatz: $a^2 = b^2 + c^2 - 2ab\cos\alpha$

Tangens: $\tan(z) := \frac{\sin(z)}{\cos(z)}, z \notin \{\frac{\pi}{2} + \pi k\}$

Cotangens: $\cot(z) := \frac{\cos(z)}{\sin(z)}, z \notin \{\pi k\}$

- $\exp(iz) = \cos(x) + i\sin(z)$
- – $\cos(z) = \cos(-z)$ – $\sin(-z) = -\sin(z)$
- – $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ – $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
- – $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
  – $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
- $\cos(z)^2 + \sin(z)^2 = 1$
- – $\sin(2z) = 2\sin(z)\cos(z)$
  – $\cos(2z) = \cos(z)^2 - \sin(z)^2 = 1 - 2\sin^2(z) = 2\cos^2(z) - 1$
- $\tan a = \frac{\sin a}{\cos a}$
- $1 + \tan^2 a = \frac{1}{\cos^2 a}$

1.3.1 Komposition

$\sin(\arccos(x)) =$	$\sqrt{1 - x^2}$	$\sin(\arctan(x)) =$	$\frac{x}{\sqrt{1+x^2}}$
$\cos(\arctan(x)) =$	$\frac{1}{\sqrt{1+x^2}}$	$\cos(\arcsin(x)) =$	$\sqrt{1 - x^2}$
$\tan(\arcsin(x)) =$	$\frac{x}{\sqrt{1-x^2}}$	$\tan(\arccos(x))$	$\frac{\sqrt{1-x^2}}{x}$

1.3.2 Hyperbolisch

- – $\sinh = \frac{e^x - e^{-x}}{2}$ – $\cosh = \frac{e^x + e^{-x}}{2}$ – $\tanh = \frac{\sinh(x)}{\cosh(x)}$
- $\cosh^2(x) - \sinh^2(x) = 1$
- $\sinh(a + b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$
- $\cosh(a + b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$

1.3.3 Winkel

deg	rad	sin	cos	deg	rad	sin	cos
0	0	0	1	30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
90	$\frac{\pi}{2}$	1	0	120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{-1}{2}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$
180	π	0	-1	210	$\frac{7\pi}{6}$	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$
225	$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	240	$\frac{4\pi}{3}$	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$
270	$\frac{3\pi}{2}$	-1	0	300	$\frac{5\pi}{3}$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$
315	$\frac{7\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	330	$\frac{11\pi}{6}$	$\frac{-1}{2}$	$\frac{\sqrt{3}}{2}$

$\arcsin(0) = 0$	$\arccos(0) = \frac{\pi}{2}$	$\arctan(0) = 0$
$\arcsin(1) = \frac{\pi}{2}$	$\arccos(0) = 0$	$\arctan(1) = \frac{\pi}{4}$
$\arcsin(-1) = -\frac{\pi}{2}$	$\arccos(0) = \pi$	$\arctan(-1) = -\frac{\pi}{4}$
		$\arctan(\sqrt{3}) = \frac{\pi}{3}$

- $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$

1.3.4 Reduktion

$\sin\frac{\pi}{2} - a = \cos a$	$\cos\frac{\pi}{2} - a = \sin a$	$\tan\frac{\pi}{2} - a = \frac{1}{\tan a}$
$\sin\frac{\pi}{2} + a = \cos a$	$\cos\frac{\pi}{2} + a = -\sin a$	$\tan\frac{\pi}{2} + a = \frac{-1}{\tan a}$
$\sin\pi - a = \sin a$	$\cos\pi - a = -\cos a$	$\tan\pi - a = -\tan a$
$\sin\pi + a = -\sin a$	$\cos\pi + a = -\cos a$	$\tan\pi + a = \tan a$
$\sin 2\pi - a = -\sin a$	$\cos 2\pi - a = \cos a$	$\tan 2\pi - a = -\tan a$
$\sin -a = -\sin a$	$\cos -a = \cos a$	$\tan -a = -\tan a$

1.4 Sonstiges

Mitternacht: $x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- $\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Ellipse Gleichung: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ellipse Volumen: πab

1.5 Komplexe Zahlen

Imaginary Number: $i, i^2 = -1$

Complex Number: $z, z = x + iy,\ x, y \in \mathbb{R}$

Set: $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$

Konjugate: $\overline{z} = x - iy,\ z = x + iy$

- $z \cdot \overline{z} = x^2 \cdot y^2 = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Betrag: $|z|$ Distanz zwischen z und Origin

- $|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$
- $|z| = |\overline{z}|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 \cdot z_2| = |z_1||z_2|$
- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$

Euler: $e^{i\gamma} = \cos\gamma + i\sin\gamma$

- $|e^{i\gamma}| = 1$

1.5.1 Arithmetic

Für $z_1 = a + ib = re^{i\gamma}, z_2 = b + id = se^{i\delta}$:

- $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$
- $z_1 \cdot z_2 = (a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$
  – $z_1 z_2 = rse^{i(\gamma+\delta)}$
- $\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{z_1 \cdot \overline{z_2}}{|z_2|}$
  – $\frac{z_1}{z_2} = \frac{r}{s} e^{i(\gamma - \delta)}$
- $\sqrt[n]{z_1} = z_2 \implies z_1 = z_2^n = r^n e^{in\delta} \stackrel{!}{=} re^{i\gamma}$
  – $s = \sqrt[n]{r}$
  – $n\gamma = \gamma + 2\pi k, k = 0 \ldots n - 1$

1.5.2 Polar Coordinates

$z = x + iy \iff z = r(\cos\gamma + i\sin\gamma) \stackrel{Euler}{=} z = re^{i\gamma}$

- – $x = r\cos\gamma$ – $y = r\sin\gamma$
- $r = |z|$
- $\gamma = \arccos\frac{x}{r} = \arcsin\frac{y}{r}$

- $\arg z = |z| \in [0, 2\pi[\implies r$ ist eindeutig bestimmt

1.6 Rechenregeln Ableitung

$\mathbf{f+g:}$ $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

$\mathbf{f \cdot g:}$ $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

$\mathbf{\frac{f}{g}:}$ $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}, \ g(x_0) \neq 0$.

$\mathbf{g \circ f:}$ $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

1.7 Rechenregel Integral

Partiell: $\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\mathrm{d}x$

Substitution: $\int_{\phi(a)}^{\phi(b)} f(x)\mathrm{d}x = \int_a^b f(\phi(t))\phi'(t)\mathrm{d}t$

1.8 Bekannte Reihen

Reihe		Wert	konv.	div.
Geometrische Reihe		$q \in \mathbb{C}$		
$\sum_{k=0}^{\infty} aq^k$	$a + aq +$	$\frac{a}{1-q}$	$\lvert g\rvert < 1$	$\lvert q\rvert \geq 1$
$\sum_{k=0}^{\infty}(k+1)q^k$	$1 + 2q +$	$\frac{1}{(1-q)^2}$		
Harmonische Reihe				
$\sum_{k=1}^{\infty} \frac{1}{k}$		∞		
$\sum_{k=1}^{\infty} \frac{1}{k^2}$		$\frac{\pi^2}{6}$		
$\sum_{k=1}^{\infty} \frac{1}{k^4}$		$\frac{\pi^4}{90}$		
$\sum_{k=1}^{\infty} \frac{1}{k^a}$			$a > 1$	$a \leq 1$
Alternierende Harmon. Reihe				
$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$		$\ln 2$		
$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$		$\frac{\pi^2}{12}$		
$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$		$\frac{\pi^4}{720}$		
$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$	$1 - \frac{1}{3} + \frac{1}{5} -$	$\frac{\pi}{4}$		
Teleskopreihe				
$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$		1		
Exponentialfunktion $z \in \mathbb{C}$, konv. abs.				
$\sum_{k=0}^{\infty} \frac{z^k}{k!}$	$1 + z + \frac{z^2}{2!} +$	$\exp z$		
$\sum_{k=0}^{\infty} \frac{(-a)^k}{k!}$		$\frac{1}{e^a}$		
$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} -$	$\sin x$		
$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$	$1 - \frac{x^2}{2} + \frac{x^4}{4!} -$	$\cos x$		
$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} +$	$\sinh x$		
$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$	$1 + \frac{x^2}{2} + \frac{x^4}{4!} +$	$\cosh x$		

1.9 Spezielle Summen

$\sum_{k=1}^{n} k$	$1 + 2 + \cdots + n$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$1 + 4 + \cdots + n^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$1 + 8 + \ldots n^3$	$\left(\sum_{k=1}^{\infty} k\right)^2$
$\sum_{k=1}^{n}(2k-1)$	$1 + 3 + \cdots + (2n-1)$	n^2
$\sum_{k=1}^{n}(2k-1)^2$	$1 + \cdots + (2n-1)^2$	$\frac{n(2n-1)(2n+1)}{3}$
$\sum_{k=0}^{n-1} q^k$	$1 + q + \cdots = \frac{q^n - 1}{q - 1}$	$\frac{1-q^n}{1-q}, \ q \notin \{0, 1\}$

1.10 Funktionen und deren Grenzwert

Funktion	Grenzwert	Bedingung
$\lim_{n\to\infty} a^n$	0	$\lvert a\rvert < 1$
$\lim_{n\to\infty} \sqrt[n]{a}$	1	$a > 0$
$\lim_{n\to\infty} \sqrt[n]{n^a}$	1	$a > 0$
$\lim_{n\to\infty} \sqrt[n]{n}$	1	
$\lim_{n\to\infty} \frac{\log_a n}{n}$	0	$a > 1$
$\lim_{n\to\infty} \frac{n^k}{a^n}$	0	$a > 1$
$\lim_{n\to\infty} \frac{a^n}{n!}$	0	
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$	∞	
$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$	e	
$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$	e^a	
$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$	$\frac{1}{e}$	
$\lim_{n\to 0} \frac{\sin n}{n}$	1	
$\lim_{n\to 1} \frac{\ln n}{n-1}$	1	
$\lim_{n\to\infty} \frac{n^m}{\exp(an)}$	0	$m \in \mathbb{R}, a > 0$
$\lim_{n\to 0} \frac{\exp(n)-1}{n}$	1	
$\lim_{n\to 0} \frac{\ln(1+n)}{n}$	1	
$\lim_{n\to\infty} \frac{\ln n}{n^a}$	0	$a > 0$
$\lim_{n\to 0} \frac{a^n - 1}{n}$	$\ln a$	$a > 0$
$\lim_{n\to 0}(n^a \ln n)$	0	$a > 0$

1.11 Auf- und Ableitungen

$f'(x)$	$f(x)$	$F(x)$
sax^{s-1}	ax^s	$\frac{a}{s+1}x^{s+1}$
$\frac{1}{x}$	$\ln(ax)$	$x\ln(ax) - x$
$\frac{1}{x+a}$	$\ln(x+a)$	$(x+a)\ln(x+a) + x$
ae^{ax}	e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{-a}{x^2}$	$\frac{a}{x}$	$a\ln(\lvert x\rvert)$
$\ln(a)ba^{bx}$	a^{bx}	$\frac{a^{bx}}{\ln(a)b}$
$\frac{1}{2\sqrt{x}}$	$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
$\frac{1}{x\ln(a)}$	$\log_a(x)$	$\frac{x(\ln(x)-1)}{\ln(a)}$
$\cos(x)$	$\sin(x)$	$-\cos(x)$
$\frac{1}{x\ln(a)}$	$\log_a(x)$	$\frac{x(\ln(x)-1)}{\ln(a)}$
$\cos(x)$	$\sin(x)$	$-\cos(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$	$x\arcsin(x) + \sqrt{1-x^2}$
$2\sin(x)\cos(x)$	$\sin^2(x)$	$\frac{x - \sin(x)\cos(x)}{2}$
$-\sin(x)$	$\cos(x)$	$\sin(x)$
$\frac{-1}{\sqrt{1-x^2}}$	$\arccos(x)$	$x\arccos(x) - \sqrt{1-x^2}$
$-2\sin(x)\cos(x)$	$\cos^2(x)$	$\frac{x + \sin(x)\cos(x)}{2}$
$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\tan(x)$	$-\ln(\lvert\cos(x)\rvert)$
$\frac{1}{1+x^2}$	$\arctan(x)$	$x\arctan(x) - \frac{\ln(x^2+1)}{2}$
$2a(\cos(ax))^2 - a$	$\sin(ax)\cos(ax)$	$-\frac{1}{4a}\cos(2ax) = \frac{-(\cos^2(ax))}{2a}$
	$\sin(x)\sin^2(x)$	$\frac{-\sin^2(x)-2}{3}\cos(x)$
	$\cos(x)\cos^2(x)$	$\frac{\sin(x)\cos^2(x)+2\sin(x)}{3}$
	$\sin(x)\cos^2(x)$	$\frac{-\cos^3(x)}{3}$
	$\sin^2(x)\cos(x)$	$\frac{\sin^3(x)}{3}$
	$\sin(x)\sin(2x)$	$\frac{-\sin(3x)+3\sin(x)}{6}$
	$\cos(x)\cos(2x)$	$\frac{\sin(3x)+3\sin(x)}{6}$
$\frac{2x}{(x^2+1)^2}$	$\frac{x^2}{x^2+1}$	$x - \arctan(x)$
$\frac{-4x}{(x^2+1)^3}$	$\frac{1}{(1+x^2)^2}$	$\frac{x}{2x^2+2} + \frac{\arctan(x)}{2}$
	$\frac{u'(x)}{u(x)}$	$\ln\lvert u(x)\rvert$
	$u'(x) \cdot u(x)$	$\frac{1}{2}(u(x)^2)$
$\frac{-x}{\sqrt{a^2-x^2}}$	$\sqrt{a^2-x^2}$	$\frac{a^2\arcsin(\frac{x}{\lvert a\rvert})+x\sqrt{a^2-x^2}}{2}$
$\frac{x}{\sqrt{x^2+a^2}}$	$\sqrt{a^2+x^2}$	$\frac{a^2\ln(\lvert\sqrt{x^2+a^2}+x\rvert)+x\sqrt{a^2+x}}{2}$
$x^x(1+\ln x)$	x^x	
$e^{\ln\lvert x\rvert}(\ln(\lvert x\rvert)+1)$	$\lvert x\rvert^x = e^{x\ln x}$	
$\frac{-2}{(x+a)^3}$	$\frac{1}{(x+a)^2}$	$\frac{-1}{x+a}$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$\frac{1}{\sqrt{1+x^2}}$	$\operatorname{arcsinh}(x)$	
$\sinh(x)$	$\cosh(x)$	$\sinh(x)$
$\frac{1}{\sqrt{x-1}\sqrt{x+1}}$	$\operatorname{arccosh}(x)$	
$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$	$\tanh(x)$	$\ln(e^{2x}+1) - x$
$\frac{1}{2x+2} - \frac{1}{2x-2}$	$\operatorname{arctanh}(x)$	

2 Ordinary Differential Equations (ODE)

Equation where the unknown is a function $y = f(x)$ and the equation relates values of y and its derivatives $y', y'' \dots$ at a single point x.

2.1 Properties

Order: Highest derivative appearing in the DE.

2.2 Linear Differential Equations

A linear ODE of order k is a DE of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1 y' + a_0 y = b$ where a_i, b, y are functions on $I \subset \mathbb{K}$. Meaning, all terms $y, y' \dots$ appear linearly.

2.2.1 Types

Homogenous Linear ODE: RHS $b = 0$
Inhomogenous Linear ODE: RHS $b \neq 0$

2.2.2 Solutions

The general solution to a inhomogeneous linear ODE is $y = y_h + y_p$. The solutions set is $S_b = \{y_h + y_p\}$.

$\mathbf{y_h}$**:** General solution to the homogeneous linear DE. S_0 is the solution set.

- S_0 is a vector space with dimension equal to the order of the DE

$\mathbf{y_p}$**:** A particular solution the inhomogeneous linear DE.

Construction of Solutions The following follows from the linearity of the DE:

- If y_1 and y_2 are two different solutions to an inhomogeneous linear DE, then $y = c_1 \cdot y_1 + c_2 \cdot c_2$ is also a solution.
- Superposition: If y_1 solves the linear DE for RHS b_1 and y_2 solves the linear DE for RHS b_2, then $y_1 + y_2$ solves the linear DE for RHS $b_1 + b_2$.

2.2.3 Initial Value Problem

For any DE of order k the initial condition is a set of k equations $y^{(i)}(x_0) = y_i, 1 \leq i \leq k$ at some initial point x_0. For any k initial conditions there exists a unique solution y (valid for homogeneous and inhomogeneous).

R: Solve $y' + ay = b$

1. Compute $\int a dx = A(x)$
2. Formulate $y_h = ze^{-A(x)}$ for $z \in C$.
3. Calculate $y_p = (\int b(x)e^{A(x)} \mathrm{d}x)e^{-A(x)}$.
4. Form general solution $y = y_h + y_p$.
5. If initial condition is given, solve for z.

2.3 Linear ODE With Constant Coefficients

Equivalent to regular linear ODE with the difference that $a_i \in \mathbb{C}$.

2.3.1 Characteristic Polynomial

For lin. ODE with constant coefficients $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1 y' + a_0 y = 0$ the characteristic polynomial is $P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_0$.

The k zeros of $P(\lambda)$ are the eigenvalues.

2.3.2 Solution

Any general solution is again of the form $y = y_h + y_p$.

Homogeneous y_h

- For district eigenvalues $\lambda_1, \dots, \lambda_r$ with multiplicity $m_1, \dots, m_r$ the function $f_{j,l}, x \mapsto x^l e^{\lambda_j x}, \quad 1 \leq j \leq r, 0 \leq l < m_j$ gives a system of solutions. I.e. all terms are multiplied with a constant z_i and summed up.
- If $\lambda_i = \beta + i\gamma$ is a EV then so is $\bar{\lambda_i} = \beta - i\gamma$.
- If a root λ_i is complex, i.e. $\lambda_i = \beta_i \pm i\gamma_i$, with multiplicity m_i then they contribute the system $x^l e^{\beta_i x}(\cos(\gamma_i x) + i \sin(\gamma_i x))$ to the solution.
- For complex roots $\beta + i\gamma$, which are part of a solution system, the following transformation is useful: $z_1 e^{(\beta+i\gamma)x} + z_2 e^{(\beta-i\gamma)x} = \tilde{z_1} e^{\beta x}\cos(\gamma x) + \tilde{z_2} e^{\beta x}\sin(\gamma x)$.

Inhomogeneous y_p

Method of Undetermined Coefficients/Ansatz Solution will be similar to disturbance function $b(x)$.

$b(x)$	Ansatz
a	b
$P_n(x)$	$Q_n(x)$
$P_n e^{\alpha x}$	$Q_n e^{\alpha x}$
$P_n \sin(\beta x)$	$Q_n \sin(\beta x) + R_n \cos(\beta x)$
$P_n \cos(\beta x)$	$Q_n \sin(\beta x) + R_n \cos(\beta x)$
$P_n \cos(\beta x)$	$Q_n \sin(\beta x) + R_n \cos(\beta x)$
$P_n \sin(\beta x) + G_n \cos(\beta x)$	$Q_n \sin(\beta x) + R_n \cos(\beta x)$
$ae^{\alpha x} \sin(\beta x)$	$e^{\alpha x}(c \sin(\beta x) + d \cos(\beta x))$
$be^{\alpha x} \cos(\beta x)$	
$P_n(x)e^{\alpha x}$	$R_n(x)e^{\alpha x}$
$P_n(x)e^{\alpha x} \sin(\beta x)$	$e^{\alpha x}(R_n(x) \sin(\beta x) + S_n(x) \cos(\beta$
$Q_n(x)e^{\alpha x} \cos(\beta x)$	

- If $b(x)$ is a lin. combination of functions we try – Lin. comb. of Ansatz functions – Superposition
- If λ_i is a EW with multiplicity m_i then we multiply the Ansatz by x^{m_i}.
- If a Ansatz is equal to solution of the homogeneous equation we have to multiply the Ansatz by x.

R: Solve using Ansatz

1. Select a suitable Ansatz and set y_p = chosen Ansatz (check special cases listed above).
2. Calculate required derivatives $y_p^{(i)}$.
3. Insert y_p and derivatives into the ODE.
4. Solve for the constant in the Ansatz.
5. y_p is equal the Ansatz with replaces constant.

Variation of Constants Assume $k = 2$, i.e. $y'' + a_1 y' + a_0 y = b$ and a homogeneous solution $y_h = z_1 f_1 + z_2 + f_2$, where f_1, f_2 are independent solutions.

For y_p we try $y_p = z_1(x)f_1 + z_2(x)f_2$. To determine the unknown functions $z_1(x), z_2(x)$ we try:

$$
\begin{aligned}
z_1' f_1 + z_2'(x) f_2 &= 0 \\
z_1' f_1 + z_2'(x) f_2 &= b
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}}_{=A} \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \implies \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix}
$$

Which gives the particular solution $y_p = z_1(x)f_1 + z_2(x)f_2$.

2.4 Separation of Variables

A ODE is separable if it is of the form $y' = b(x)g(y)$.

We can separate the variables x, y.

$\frac{dy}{dx} = b(x)g(y) \implies \frac{dy}{g(y)} = b(x)dx \implies \int \frac{1}{g(y)}\mathrm{d}y = \int b(x)\mathrm{d}x$

3 Differential Calculus in $\mathbb{R}^n$

3.1 Basics

3.1.1 Norm

For $x \in \mathbb{R}^n$ we use the norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, which satisfies

- $|x| > 0 \quad \forall x \neq 0$ • $|tx| = |t||x|$ • $|x+y| \leq |x| + |y|$

3.1.2 Set Properties

Subset $X \subset \mathbb{R}^n$ is:

Bounded: if $\{|x||x \in X\}$ is bounded.
Closed: if every sequence (x_k) in X for which $\lim_{k \to a}(x_k) = y, y \in \mathbb{R}^n$ we have $x \in X$.
Compact: if closed and bounded.
Open: if $\forall x \in X, \exists r > 0$ s.t. $\{y \in \mathbb{R}^n||y - x| < r\} \subset X$.
- Equivalently, if its complements $\mathbb{R}^n \setminus X$ is closed.

- If $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ are bounded (or closed or compact or open), then $X \times Y = \{(x,y) \in \mathbb{R}^{n+m}|x \in X, y \in Y\}$ is bounded (or closed or compact or open).
- For $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous, $\forall Y \subset \mathbb{R}^m$ closed, the set $f^{-1}(y) = \{x \in \mathbb{R}^n | f(x) \in Y\} \subset \mathbb{R}^n$ is closed. I.e. the inverse image of a closed set under a continuous map is closed.

3.1.3 Sequences

For sequence $(x_k)_k, x_k \in \mathbb{R}^n = (x_{k,1}, \ldots x_{k,n})$ and $y \in \mathbb{R}^n = (y_1, \ldots y_n)$, x_k converges to y as $k \to \infty$ if $\forall \epsilon > 0, \exists N \geq 1$, s.t. $\forall n \geq N, |x_k - y| < \epsilon$.

We write $x_k \to y$ or $\lim_{k\to\infty} x_k = y$.

I.e. $\lim_{k\to\infty} x_k = y$, if one of the following holds:

- $\forall 1 \leq i \leq n$, the sequence $(a_{k,i})$ converges to y_i.
- $\lim_{k\to\infty} |x_k - y| = 0$.

3.1.4 Limit of f

Function $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ has limit $y \in \mathbb{R}^m$ if for $x \to x_0, x_0 \in X, \quad \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in X, x \neq x_0$, s.t. $|x - x_0| < \epsilon \implies |f(x) - y| < \delta$. In that case $\lim_{x \to x_0} f(x) = y$.

I.e. $\lim_{x\to x_0} f(x) = y \iff \forall$ sequences (x_k) in X, where $\lim_{k\to\infty} x_k = x_0$, the sequence $\lim_{k\to\infty} f(x_k) = y$.

3.1.5 Min-Max-Theorem

For $X \subset \mathbb{R}^n$ compact and non-empty and for $f : X \to \mathbb{R}$ continuous. Then f is bounded and achieves a maximum $x^+, f(x^+) = \sup_{x \in X} f(x)$ and minimum $x^-, f(x^-) = \inf_{x\in X} f(x)$.

3.2 Continuity

at $\mathbf{x_0}$: for $f : X \subset \mathbb{R}^n \to \mathbb{R}^m, x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. for $x \in X$ satisfies $|x - x_0| < \delta \implies |f(x) - f_0(x)| < \epsilon$.
on X: for $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ if x_0 continuous $\forall x_0 \in X$.

Function $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in X \iff \forall$ sequences (x_k) in X, where $\lim_{k\to\infty}(a_k) = x_0$, the sequence $(f(x_k))$ in $\mathbb{R}^m$ converges to $f(x_0)$. I.e. $\lim_{k\to\infty} f(x_k) = f(\lim_{k\to\infty} a_k)$.

3.2.1 Rules

If f, g are continuous then $f + g, f \cdot g, \frac{f}{g}, f \circ g$ are continuous.

3.2.2 Sandwich Lemma

If $f, g, h : \mathbb{R}^n \to \mathbb{R}$, where $f(x) < g(x) < h(x) \quad \forall x \in \mathbb{R}^n$ and for some $a \in \mathbb{R}^n, \lim_{y\to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

R: Polar Coordinates Trick

For $f : \mathbb{R}^2 \to \mathbb{R}$ check if continuous at 0. Let $f(x,y) = f(r\cos\delta, r\sin\delta)$. If $\lim_{(x,y)\to 0} f(x,y) = \lim_{r \to 0}$ is dependant on $\delta \implies$ limes does not exists $\implies$ not continuous at 0.

3.3 Partial Derivatives

For $X \subset \mathbb{R}^n$ open, the function $f : X \to \mathbb{R}^m$ has a partial derivative on X with respect to the $1 \leq i \leq n$-th variable if $\forall x_0 = (x_{0,1}, \ldots x_{0,n}) \in X$ the function $g_i(t) := f(x_{0,1}, \ldots x_{0,i-1}, t, x_{0,i+1}, \ldots x_{0,n})$ on the set $I = \{t \in \mathbb{R} | (x_{0,1}, \ldots x_{0,i-1}, t, x_{0,i+1}, \ldots x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. The derivative $g'(x_{0,i})$ is denoted as $\frac{\partial f}{\partial x_i}(x_0) = \partial x_i f(x_0) = \partial_i f(x_0)$.

- A partial derivative $\partial_i f$ can be derived again $\partial_{x_j}(\partial_{x_i} f) = \partial_{x_i,x_j} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Or for repeated variables $\partial_{x_i}(\partial_{x_i}(f)) = \partial_{x_i^2} f = \frac{\partial^2 f}{\partial x_i^2}$.

3.3.1 Properties

For $X \subset \mathbb{R}^n$ open and $f, g : X \to \mathbb{R}^m$ and $1 \leq i \leq n$:

- If $\partial_i(f)$ and $\partial_i(g)$ exist, then $\partial_i(f + g) = \partial_i(f) + \partial_i(g)$.
- If $m = 1$ and if $\partial_i(f)$ and $\partial_i(g)$ exist, then $\partial_i(fg) = \partial_i(f)g + f\partial_i(g)$.
- If $m = 1$, if $\partial_i(f)$ and $\partial_i(g)$ exist and if $g(x) \neq 0 \forall x \in X$, then $\partial_i(\frac{f}{g}) = \frac{\partial_i(f)g - f\partial_i(g)}{g^2}$.

3.3.2 Jacobi Matrix

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$ with partial derivatives on X, $f(x) = f_1(x), \ldots f_m(x))$, the matrix $J_f(x) = (\partial_j f_i(x))_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 & \ldots & \partial_n f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \ldots & \partial_n f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_m & \partial_2 f_m & \ldots & \partial_n f_m \end{pmatrix} \forall x \in X$ with m rows and n columns is the Jacobi matrix of f at x.

- $\det J_f(x)$ tells when the linear map f is invertible and when not.

3.3.3 Gradient

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}$ with partial derivatives on X, the column vector $\begin{pmatrix} \partial_1 f(x_0) \\ \vdots \\ \partial_n f(x_0) \end{pmatrix} = (J_f(x_0))^T$ is the gradient of f at x_0 denoted as $\bigtriangledown f(x_0)$.

- Slop is maximal in the direction of the gradient.

3.4 Differentiability

For $X \subset \mathbb{R}^n$ open, $x_0 \in X$ and $f : X \to \mathbb{R}^m$. f is differentiable at x_0 with differential $u : \mathbb{R}^n \to \mathbb{R}^m$ if $\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{1}{|x-x_0|}(f(x) - f(x_0) - u(x - x_0)) = \frac{R(x,x_0)}{|x-x_0|} = 0$.

- The linear map u is called **total differential** of f at x_0 and is denoted as $df(x_0)$ or $d_{x_0}f$.
- If f is differentiable $\forall x_0 \in X$ then f is differentiable on X.
- The map $df(x_0)$ can be represented as a $m \times n$ matrix $Df(x_0)$.
- $f(x) = f(x_0) + u(x - x_0) + E(f, x, x_0)$, where $\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{E(f,x,x_0)}{|x-x_0|} = 0$
  - $f(x_0) + u(x - x_0)$ is a linear affine function.

3.4.1 Properties

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$. f is differentiable at x_0:

- f is continuous on x_0
  - if f is differentiable on X, then f is continuous on X
- f has all partial derivatives at x_0
- $df(x_0)$ is given by $Df(x_0) = J_f(x_0)$
  - if $m = 1$, then $df(x_0) = J_f(x_0) = (\partial_i f(x_0), \dots \partial_n f(x_0))$

For $X \subset \mathbb{R}^n$ open and $f, g : X \to \mathbb{R}^m$ differentiable on X:

- $f + g$ is differentiable and $d(f + g)(x_0) = df(x_0) + dg(x_0)$.
- if $m = 1$, then $f \cdot g$ is differentiable
- if $m = 1$ and $f(x) \neq 0 \quad \forall x \in X$, then $\frac{f}{g}$ is differentiable

Chain Rule For $X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open and $f : X \to Y$ differentiable on X and $g : Y \to \mathbb{R}^p$ differentiable on Y. Then $g \circ f : X \to \mathbb{R}^p$ is differentiable and $d(g \circ f)(x_0) = dg(f(x_0)) \cdot df(x_0)$.

- The Jacobi satisfies $J_{g \circ f}(x_0) = J_g(f(x_0))J_f(x_0)$

3.4.2 Condition

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$. If f has all partial derivatives and they are continuous on X, then f is differentiable in X.

- $\substack{\text{partial derivatives exist} \\ \text{+partial derivatives are continuous}} \implies f$ is differentiable.
- $df(x_0) = J_f(x_0)$.
- This implies that most functions are differentiable.

3.4.3 Tangent Space

For $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ differentiable on X and $x_0 \in X$, the differential $u = df(x_0)$. The graph of the affine linear approximation $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = f(x_0) + u(x - x_0)$ is the tangent space of f.

- The pts of the tangent space are in $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y = f(x_0) + u(x - x_0)\}$.
- If $m = 1$ then $u(x - x_0) = df(x_0)(x - x_0) = \bigtriangledown f(x_0) \cdot (x - x_0)$.

R: Calculate Tangent Space

Given function $f : \mathbb{R}^2 \to \mathbb{R}$, calculate the tangent space at pt (x_0, y_0).

1. Calculate $\partial_x f$ and $\partial_y f$.
2. Calculate $f(x_0, y_0)$.
3. Form $z = f(x_0, y_0) + \partial_x f(x_0, y_0) \cdot (x - x_0) + \partial_y f(x_0, y_0) \cdot (y - y_0)$.

3.4.4 Directional Derivative

For $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$, $v \in \mathbb{R}^n$ a non-zero vector and $x_0 \in X$. f has a directional partial derivative $w \in \mathbb{R}^m$ in the direction v, if $g(t) = f(x_0 + tv)$, defined on $\{t \in \mathbb{R} | x_0 + tv \in X\}$, holds $g'(0) = w$.

- I.e. $w = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f(x_0+tv)-f(x_0)}{t} = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{g(t)-g(0)}{t} = g'(0)$.
- Different notations: $w = df(x_0)(v)$

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$ differentiable. Then $\forall x \in X$ and $\forall v \in \mathbb{R}^n$ non-zero, f has directional partial derivative at x_0 in the direction v and is equal to $df(x_0)(v) = J_f(x_0)(v)$.

- If $m = 1$ and $|v| = 1$, then $df(x_0)(v) = J_f(x_0)(v) = \langle \bigtriangledown f(x_0), v \rangle = | \bigtriangledown f(x_0)||v| \cos(\theta)$.
  - This is maximal if we maximize $\cos\theta$ which is the case for $\theta = 0$.

R: Calculate Directional Derivative in Direction v

Given $f : \mathbb{R}^n \to \mathbb{R}$ differentiable, vector v and pt (x_0, y_0). Calculate $df(x_0, y_0)(v)$.

1. Calculate all partial derivatives of order 1.
2. Form $J_f(x, y)$.
3. Normalize $\tilde{v} = \frac{v}{|v|}$
4. Calculate $J_f(x_0, y_0) \cdot \tilde{v}$

3.4.5 Special Coordinates

Polar Coordinates
- $f : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$
- $(r, \theta) \mapsto (x, y) = (r\cos\theta, r\sin\theta)$
- $J_f(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$
- $\det J_f(r, \theta) = r$

Cylindrical Coordinates

- $f : [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$
- $(0, \theta, z) \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- $J_f(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\det J_f = r$

Spherical Coordinates

- $f : [0, \infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$

- $(r,\theta,\varnothing) \mapsto \begin{pmatrix} r\cos\theta\sin\varnothing \\ r\sin\theta\sin\varnothing \\ r\cos\varnothing \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- $J_f(r,\theta,\varnothing) = \begin{pmatrix} \cos\theta\sin\varnothing & -r\sin\theta\sin\varnothing & r\cos\theta\cos\varnothing \\ \sin\theta\sin\varnothing & r\cos\theta\sin\varnothing & r\sin\theta\cos\varnothing \\ \cos\varnothing & 0 & -r\sin\varnothing \end{pmatrix}$
- $\det J_f = -r^2\sin\varnothing$

3.5 Change of Variable

For $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ differentiable and $x_0 \in X$. f is a change of variable around x_0 if $\exists$ radius $r > 0$ s.t. the restaction of f to the ball $B = \{x \in \mathbb{R}^n \| x - x_0| < r\}$ of radius r and center x_0 has the property that the image $Y = f(B)$ is open in $\mathbb{R}^n$ and $\exists$ differentiable $f : Y \to B$ s.t. $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_B$.

- I.e. $f|_{B_r(x_0)}$ is a bijection to the image with a inverse f is is also differentiable.

3.5.1 Inverse Function Theorem

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$ differentiable. If $\exists x_0 \in X$ s.t. $J_f(x_0) \neq 0$ the f is a change of variable around x_0.

- The Jacobi of g at x_0 is $J_g(f(x_0)) = J_f(x_0)^{-1}$.
- $\det J_f(x_0) \neq 0 \iff$ Jacobi matrix of f at x_0 is invertible.
- If $f \in C^k$ then $g \in C^k$.

3.6 Higher Derivatives

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$

- $f \in C^1$ if f is differentiable on X and all its partial derivatives are continuous.
- For $k \geq 2$, $f \in C^k$ is f is differentiable and each $\partial_i f : X \to \mathbb{R}^m \in C^{k-1}$.
- If $f \in C^k \quad \forall k \geq 1$ then $f \in C^\infty$ is smooth.
- The set of function $g : X \to \mathbb{R}^m \in C^k$ is $C^k(X; R^m)$.

3.6.1 Notation

For derivative $\frac{\partial^k}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}$ of order $k \leq n$ of $f : \mathbb{R}^n \to \mathbb{R}$, we write $\frac{\partial^{|m|}}{\partial x^m} f$, with: • $m = (m_1, \dots m_n)$ the vector of number of derivatives for each partial derivative. • $|m| := m_1 + \dots + m_n$ • $x^n := (x_1^{m_1}, \dots, x_n^{m_n})$ • $m! := m_1! \dots m_n!$

3.6.2 Mixed Derivates Commute

For $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m \in C^k$ and $k \geq 2$. The derivatives of order k are independent of the order in which the partial derivatives are taken.

- I.e. $\partial_{x,y} f = \partial_{y,x} f$

3.6.3 Hessian

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m \in C^2$. The Hessian matrix of x is $\mathrm{Hess}_f(x) = H_f(x) = (\partial_{x_i,x_j} f)_{1 \leq i,j \leq n} =$
$$\begin{pmatrix} \partial_{x_1^2} f & \partial_{x_1,x_2} f & \dots & \partial_{x_1,x_n} f \\ \partial_{x_2,x_1} f & \partial_{x_2^2} f & \dots & \partial_{x_2,x_n} f \\ \vdots & & \ddots & \vdots \\ \partial_{x_n,x_1} f & \partial x_n, x_2 f & \dots & \partial_{x_n^2} f \end{pmatrix}.$$

- If partial derivatives commute, then $H_f(x)$ is symmetric, square, $n \times n$.

3.7 Taylor Approximation

3.7.1 Order 1

For $f : \mathbb{R}^n \to \mathbb{R}$, the Taylor Polynom of f at x_0 of degree 1 is $T_1 f(y; x_0) := f(x_0) + \triangledown f(x_0) \cdot y$.

For a point x_0 close to x the approximation of x by a affine linear polynomial is $T_1(x - x_0; x_0)$.

- T_1 is equivalent to the affine linear approximation of f we have seen before.

3.7.2 Order k

For $X \subset \mathbb{R}^n, x_0 \in X, f : X \to \mathbb{R}, f \in C^k$, the k-th Taylor polynomial of f at x_0 is $T_k(y; x_0) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \dots + \sum_{m_1+\dots+m_n=k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}(x_0) y_1^{m_1} \dots y_n^{m_n} = \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$. Where $y^m := (y_1^{m_1}, \dots, y_n^{m_n})$.

3.7.3 Approximation

For $f \in C^k(X; \mathbb{R}), x_0 \in X$. T_k is approximation of f at x with x_0: $f(x) = T_k f(x - x_0; x_0) + E_k(f, x, x_0)$. For the error E_k it holds $\lim_{x \to x_0} \frac{E_k(f,x,x_0)}{|x-x_0|^k} = 0$.

R: Taylor Polynom for $f : \mathbb{R}^2 \to \mathbb{R}$

Give Taylor of order $k \in \{1,2,3\}$ of $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ for a pt (x_0, y_0).

- Computer all parietal derivatives of order up to k.
- Form Taylor according to:

- $T_1 f((\alpha,\beta);(x_0,y_0)) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x} \cdot \alpha + \frac{\partial f(x_0,y_0)}{\partial y} \cdot \beta$
- $T_2 f((\alpha,\beta);(x_0,y_0)) = T_1 f((\alpha,\beta);(x_0,y_0)) + \frac{1}{2} \frac{\partial^2 f(x_0,y_0)}{\partial x \partial x} \cdot \alpha^2 + \frac{1}{2} \frac{\partial^2 f(x_0,y_0)}{\partial y \partial y} \cdot \beta^2 + \frac{\partial^2 f(x_0,y_0)}{\partial x \partial y} \cdot \alpha\beta$
- $T_3 f((\alpha,\beta);(x_0,y_0)) = T_2 f((\alpha,\beta);(x_0,y_0)) + \frac{1}{6} \frac{\partial^3 f(x_0,y_0)}{\partial^3 x} \alpha^3 + \frac{1}{2} \frac{\partial^3 f(x_0,y_0)}{\partial^2 x \partial y} \alpha^2\beta + \frac{1}{2} \frac{\partial^3 f(x_0,y_0)}{\partial x \partial^2 y} \alpha\beta^2 + \frac{1}{6} \frac{\partial^3 f(x_0,y_0)}{\partial^3 y} \beta^3$

R: Approximate $f : \mathbb{R}^2 \to \mathbb{R}$

Given $f : \mathbb{R}^2 \to \mathbb{R}$ and pt (x,y). Approximate $f(x,y)$ using Taylor.

- Find pt $(x_0, y_0) \approx (x,y)$ which approximates (x,y) and is easy to evaluate.
- Form Taylor $T_k((x - x_0, y - y_0); (x_0, y_0))$ of desired order k.

3.8 Critical Points

3.8.1 Extrema

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}$ differentiable. We define a neighbourhood of x_0 as $B_r(x_0) = \{x \in \mathbb{R}^n \| x - x_0| < r\} \subset X$. If $x_0 \in X$:

Local Maximum: at x_0 is $f(x) \leq f(x_0) \quad \forall x \in B_r(x_0)$.

Local Minimum: at x_0 is $f(x) \geq f(x_0) \quad \forall x \in R_r(x_0)$.

In that case: • $df(x_0) = 0$ • $\triangledown f(x_0) = 0$ • $\frac{\partial f}{\partial x_i}(x_0) = 0, 1 \leq i \leq n$

3.8.2 Critical Point

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}$ differentiable. $x_0 \in X$, $\triangledown f(x_0) = 0$ is a critical point.

- Critical points are candidates for local extrema.

- Critical points which are not a local extrema are saddle points.

3.8.3 Global Extrema

For $\bar{X} = \text{innerPart}(X) \cup \text{boundary}(X) \subset \mathbb{R}^n$ compact (closed and bounded) and $f : \bar{X} \to \mathbb{R}$ differentiable. A global extrema exists and it is either at a critical point ($\in \text{innterPart}(X)$) or on the boundary ($\in \text{boundary}(X)$) of $\bar{X}$.

3.8.4 Non-Degenerate Critical Point

For $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R} \in C^2$. A critical point $x_0 \in X$ is called non-degenerate if $\det \text{Hess}_f(x) \neq 0$.

3.8.5 Definite/Indefinite

A symmetric matrix $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$, with $\det A \neq 0$ is:

Positive Definite ($\mathbf{A > 0}$): iff $xAx^t > 0 \forall x \in \mathbb{R}^n \setminus \{0\}$
- All eigenvalues are positive.

Neg. Definite ($\mathbf{A < 0}$): iff $xAx^t < 0 \forall x \in \mathbb{R}^n \setminus \{0\}$
- All eigenvalues are negative.

Indefinite: otherwise
- Mixed eigenvalues.

Criteria A is positive definite $\iff \det A_j > 0 \quad \forall 1 \leq j \leq n$ and $A_j = (a_{k,l})_{\substack{1\leq k\leq j\\ 1\leq l\leq j}}$

3.8.6 Local Min/Max

For $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R} \in C^2$ and $x_0 \in X$ a non-degenerate critical point of f ($\triangledown f(x_0) = 0, \det \text{Hess}_f(x_0) \neq 0$). Then:

- If $\text{Hess}_f(x_0) > 0 \implies x_0$ is a local minimum.
- If $\text{Hess}_f(x_0) < 0 \implies x_0$ is a local maximum.
- If $\text{Hess}_f(x_0)$ is indefinite $\implies x_0$ is saddle point.

In case that $\text{Hess}_f(x_0) = 0$ (i.e. x_0 is degenerate) we cannot use this criteria.

R: Extreme Pts using 2nd Derivative Method (open)

Given $f : \mathbb{R}^n \to \mathbb{R}$ find extreme points and determine if they are max/min/saddle.

1. Calculate all partial derivatives up to order two.
2. Form $df(x) = (\partial_{x_1} f, \dots \partial_{x_n} f)$
3. Find all zeros x_0 of $df(x_0)$. They are the critical pts.
4. If $X \subset \mathbb{R}^n$, check boundary for max/min pts.
5. Form $\text{Hess}_f(x)$
6. Calculate $\det \text{Hess}_f(x)$
7. Insert all x_0 into $\det \text{Hess}_f(x_0)$ and evaluate.
8. If $\det \text{Hess}_f(x_0) = 0 \implies$ this method does not work.
9. Use previous theorem to determine if x_0 is min/max/saddle.

R: Extreme Pts using 2nd Derivative Method (closed)

As before, but $f : X \to \mathbb{R}$.

1. Follow previous recipe
2. Evaluate all corners and record their value.
3. For site i:
   (a) Find parametrisation $\gamma_i(t)$.
   (b) Calculate $\frac{d\gamma_i}{dt}$.
   (c) Find t_0 for which $\frac{d\gamma_i}{dt} = 0$.
   (d) Calculate $f(\gamma_i(t_0))$ and record their values.
4. Compare all evaluated points and determine the min/max/saddle pts.

R: det A with Sarrus

Given matrix $A : 3 \times 3$ calculate $\det A$.

$\det A = A_{1,1}A_{2,2}A_{3,3} + A_{2,1}A_{3,2}A_{1,3} + A_{3,1}A_{1,2}A_{2,3} - A_{1,3}A_{2,2}A_{3,1} - A_{1,2}A_{2,1}A_{3,3} - A_{1,1}A_{2,3}A_{3,2}$

R: Eigenwerte

Given square matrix A. Find the eigenvalues of A.

1. Form characteristic polynomial $\det (A - \lambda I)$.
2. Find zeros of the characteristic polynomial. They are the eigenvalues.

4 Integration in $\mathbb{R}^n$

For $I = [a, b] \subset \mathbb{R}$ compact and $f : I \to \mathbb{R}^n, f(t) = (f_1(t), \dots f_n(t))$ continuous ($\implies f_i$ continuous $\forall 1 \leq i \leq n$). Then $\int_a^b f(t)\text{d}t = (\int_a^b f_1(t)\text{d}t, \cdots \int_a^b f_n(t)\text{d}t)$.

For $f, g : [a, b] \to \mathbb{R}^n$:

- $\int_a^b (f(t) + g(t))\text{d}t = \int_a^b f(t)\text{d}t + \int_a^b g(t)\text{d}t$
- $\int_a^b f(t)\text{d}t = -\int_b^a f(t)\text{d}t$

4.1 Vector Field

For $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$. f is a vector fields which sends each $x \in X$ to a vector $v \in \mathbb{R}^n$.

4.2 Parametrized Curve

For a curve (represented by a map) $\gamma : [a, b] \to \mathbb{R}^n$ continuous and piecewise $\in C^1$. γ is a parametrized curve between $\gamma(a)$ and $\gamma(b)$.

- $\gamma : [0, 2\pi] \to (a \cos t, b \sin t)$ is a parametrisation of a ellipse.
  - If $a = b$ is is a circle of radius a.
  - If t is replaced by $2\pi - t$, the ellipse turns in opposite direction.
- If $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ and $\gamma : [0, 1] \to \mathbb{R}^3, t \mapsto (a_1 + b_1t, a_2 + b_2t, a_3 + b_3t)$ is the parametrisation of the line segment between vector a and $a + b$.
- For $f : [a, b] \to \mathbb{R} \in C^1$, the normal graphs we are used to are a parametrisation of $\gamma : [0, 1] \to \mathbb{R}^2, t \mapsto (t, f(t))$.

4.3 Line Integral

For $\gamma : [a, b] \to \mathbb{R}^n$ parametrised curve, $X \subset \mathbb{R}^n$ containing the image of γ and $f : X \to \mathbb{R}^n$ continuous. The integral $\int_a^b f(\gamma(t)) \cdot \gamma'(t)\text{d}t \in \mathbb{R}$ of f along γ.

- Denoted as $\int_\gamma f(s) \cdot \text{d}s$

4.3.1 Properties

Independent of Oriented Reparametrisation For $\gamma : [a, b] \to \mathbb{R}^n$ parametrized curve and $\delta : [c, d] \to [a, b]$ with: • $\in C^1$ • differentiable on $]c, d[$ • strictly increasing • $\delta(a) = c, \delta(b) = d$ An orientation reparametrisation of γ is $\sigma : [c, d] \to \mathbb{R}^n, \sigma = \gamma \circ \delta$.

For $X \subset \mathbb{R}^n$ containing the image of γ and $f : X \to \mathbb{R}^n \in C^1$, $\int_\gamma f\text{d}s = \int_\sigma f\text{d}s$.

Connecting Paths For $\gamma_1 : [a, b] \to \mathbb{R}^n, \gamma_2 : [c, d] \to \mathbb{R}^n$ parametrized paths, with $\gamma_1(b) = \gamma_2(c)$. For the paths formed by connecting to two paths, it holds $\int_{\gamma_1+\gamma_2} f\mathrm{d}s = \int_{\gamma_1} f\mathrm{d}s + \int_{\gamma_2} f\mathrm{d}s$.

Reverse Path For $\gamma : [a,b] \to \mathbb{R}^n$ parametrized path and $-\gamma : [a,b] \to \mathbb{R}^n$ the same path traced in opposite direction, then $\int_{-\gamma} f\mathrm{d}s = -\int_\gamma f\mathrm{d}s$.

Independent of Path For $X \subset \mathbb{R}^n$, vector field $f : X \to \mathbb{R}^n, g : X \to \mathbb{R} \in C^1$ s.t. $\nabla g = f$ and parametrized curve $\gamma : [a,b] \to \mathbb{R}^n, \gamma([a,b]) \subset X$ then: $\int_\gamma f\mathrm{d}t = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t = \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t = \int_a^b \frac{d}{dt}(g \circ \gamma)\mathrm{d}t = (g\circ\gamma)(b) - (g \circ \gamma)(a)$. I.e. the integral only depends on the endpoints of the curve.

4.3.2 Potential

For $X \subset \mathbb{R}^n$, vector field $V : X \to \mathbb{R}^n$ and $f : X \to \mathbb{R}$, where $\nabla f = V$, then f is a potential for V.

- Does always exists for $n = 1$ and is equivalent to the primitive (Aufleitung) of f.
- Necessary condition for existence of g: $\partial_j f_i = \partial_i f_j \quad \forall 1 \le i \ne j \le n$.
  - If X is star-shapend, then this condition is sufficient.

R: Find Potential for Given Vector Field

Given vector field $V = (v_1, v_2, v_3) : \mathbb{R}^3 \to \mathbb{R}^3$ find $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ with $\nabla f = V$.

1. Check necessary condition $\partial_j f_i = \partial_i f_j \quad \forall 1 \le i \ne j \le n$.

f will be of the form $f(x,y,z) = a(x,y,z)+b(y,z)+c(z)$

1. Calculate $a(x,y,z) = \int v_1 \mathrm{d}x$.
2. Using $v_2 \stackrel{!}{=} \partial_y a(x,y,z) + \partial_y b(y,z) + \partial_y H(z)$ find $b(y,z)$.
3. Using $v_3 \stackrel{!}{=} \partial_z a(x,y,z)+\partial_z b(y,z)+\partial_z c(z)$ find $c(z)$.

4.3.3 Conservative

For $X \subset \mathbb{R}^n$, vector field $f : X \to \mathbb{R}^n \in C^1$. If for any $x_1, x_2 \in X$, the integral $\int_\gamma f\mathrm{d}s$ is independent of the curve from x_1, x_2 then the vector field f is conservative.

- f is conservative $\iff \int_\gamma f(s)\mathrm{d}s = 0$ for all closed ($\gamma(a) = \gamma(b)$) parametrized curves in X.
- if f is conservative, then $\partial_j f_i = \partial_i f_j \quad \forall 1 \le i \ne j \le n$.
  - if f is in addition star-shaped, this condition holds in both directions.
  - $\implies$ its Jacobian matrix is symmetric.

R: Lineintegral of Conservative Vector Field

Given conservative vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ and curve γ, $a \le t \le b$. Calculate $\int_\gamma f\mathrm{d}s$.

- Calculate potential g with $\nabla g = V$.
- Calculate boundaries $(a, \gamma(a))$ and $(b, \gamma(b))$
- Calculate $g(b,\gamma(b)) - g(a, \gamma(a))$

4.3.4 Path Connected

For $X \subset \mathbb{R}^n$ open. X is path connected if $\forall x_1, x_2 \in X \quad \exists \gamma : (0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$.

If X is path connected and $f : X \to \mathbb{R}^n \in C^1$ then:

- f is gradient of $g : X \to \mathbb{R}$. I.e. $f = \nabla g$, g is potential for f.
- The line integral of f is independent of the path between any two points. I.e. f is conservative.
- the line integral of any closed curve is 0.

4.3.5 Star Shaped

$X \subset \mathbb{R}^n$ is star shaped if $\exists x_0 \in X$ s.t. $\forall x \in X$ the line segment connecting x and x_0 is in X.

- $\mathbb{R}^n$ is star-shaped.

4.3.6 curl

For $X \in \mathbb{R}^3$ open, $f : X \to \mathbb{R}^3 \in C^1$ vector field. Then curl $f = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix} = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$ is a vector field.

- curl f is continuous
- curl $f = 0 \implies \partial_j f_i = \partial_i f_j \quad \forall 1 \le i \ne j \le n \implies$ f is conservative.
- $\mathrm{curl}(\nabla f) = 0$

4.4 Riemann Integral

4.4.1 Partition

For $Q = I_1 \times \cdots \times I_n, I_k = [a_k, b_k]$. A partition P if Q is a sub-collection of rectangular boxes $Q_1, \ldots, Q_k$, s.t.:
- $Q = \bigcup_{j=1}^k Q_j$ • int $Q_i \cap$ int $Q_j \ne 0, i \ne j$
- Q is compact.

Volume $\mathrm{vol}(Q) = \prod_{i=1}^n (b_i - a_i) = \mu(Q)$

Norm $\mathrm{Norm}(P) = \delta_p := \max{}_{j=1}^{k}(\mathrm{diameter}(Q_j))$

4.4.2 Riemann Sum

For each Q_j we choose a $\xi_i \in Q_j$. The Riemann sum of f for partition P and intermediate point $\{\xi\}$ is $R(f, P, \xi) := \sum_{j=1}^k f(\xi_i)\mathrm{vol}(Q_j)$.

Lower R. Sum: $L_f(P) = \sum_{j=1}^k \inf_{x \in Q_j}(f(x))\mathrm{vol}(Q_j)$

Upper R. Sum: $U_f(P) = \sum_{j=1}^k \sup_{x \in Q_j}(f(x))\mathrm{vol}(Q_j)$

4.4.3 Riemann Integral

Lower R. Integral: $\underline{I}(f) = \int_{\underline{Q}} f\mathrm{d}x = \sup\{L_f(P) | \forall \text{ partitions } P \text{ of } Q\}$

Upper R. Integral: $\bar{I}(f) = \int_Q^{-} f\mathrm{d}x = \inf\{U_f(P) | \forall \text{ partitions } P \text{ of } Q\}$

f is integrable: if $\underline{I}(f) = \bar{I}(f)$
- it is denoted as $\int_A f(x)\mathrm{d}x$

For $f : \mathbb{R}^n \to \mathbb{R}$ continuous on rectangular box $Q \in \mathbb{R}^n$ then f is integrable.

4.4.4 Properties

For $X \subset \mathbb{R}^n$ compact, $f, g : Q \to \mathbb{R}$ integrable (continuous) and $\alpha, \beta \in \mathbb{R}$. Then:

- $\int_X \alpha f + \beta g\mathrm{d}x = \alpha \int_X f(x)\mathrm{d}x + \beta \int_X g(x)\mathrm{d}x$
- If $f(x) \le g(x) \quad \forall x \in Q$ then $\int_X f(x)\mathrm{d}x \le \int_X g(x)\mathrm{d}x$.
- If $f(x) \ge 0$ then $\int_X f(x)\mathrm{d}x \ge 0$
- $\left|\int_X f(x)\mathrm{d}x\right| < \int_X |f|\mathrm{d}x \le (\sup_X |f|)\mathrm{vol}(X)$.
- $\left|\int_X f(x) + g(x)\mathrm{d}x\right| < \int_X |f|\mathrm{d}x + \int_X |g(x)|\mathrm{d}x$.
- Fubini: If $Q = I_1 \times \cdots \times I_n$ and f continuous on Q then $\int_Q f(x_1, \ldots x_n)\mathrm{d}x_1, \ldots, x_n = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_n)\mathrm{d}x_n \ldots \mathrm{d}x_2\mathrm{d}x_1$.

- If $f=1$ then $\int_X 1\mathrm{d}x = \mathrm{vol}(X)$
- If $f \geq 0$ then $\int_X f(x)\mathrm{d}x = \mathrm{vol}(\{(x,y)\in X\times\mathbb{R} \mid 0 \leq y \leq f(x)\})$.

4.4.5 Fubini

For $X \subset \mathbb{R}^n$ compact, $f: X \to \mathbb{R}, n = n_1+n_2, n_i \geq 1$. Let $x_1 \in \mathbb{R}^{n_1}$, then $X_{x_1} = \{x_2 \in \mathbb{R}^{n_2} \mid (x_1,x_2)\in X\}$. Let $X_1 = \{x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset\}$. The in general X_1 is compact in $\mathbb{R}^{n_1}$ and X_{x_1} is compact in $\mathbb{R}^{n_2} \quad \forall x_1 \in X_1$.

If $g(x_1) := \int_{X_1} f(x_1,x_2)\mathrm{d}x_2$ continuous on X_1 then $\int_X f(x_1,x_2)\mathrm{d}x = \int_{X_1} g(x_1)\mathrm{d}x_1 = \int_{X_1}\int_{X_{x_1}} f(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1$.

Switching x_1 and x_2 we have $\int_X f(x_1,x_2)\mathrm{d}x = \int_{X_2}\int_{X_{x_2}} f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2$.

n = 2: $n_1 = n_2 = 1$:
- $D_1 := \{(x,y) \mid a \leq x \leq b, g(x) < y < h(x)\}$
  - $\int_{D_1} f\mathrm{d}x\mathrm{d}y = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\mathrm{d}x$
- $D_2 := \{(x,y) \mid c \leq y \leq d, G(y) < x < H(y)\}$
  - $\int_{D_2} f\mathrm{d}x\mathrm{d}y = \int_c^d \int_{G(x)}^{H(x)} f(x,y)\mathrm{d}y\mathrm{d}x$

n = 3: • $n_1 = 1, n_2 = 2$ • $n_1 = 2, n_2 = 1$

n = 4: • $n_1 = 1, n_2 = 3$ • $n_1 = 2, n_2 = 2$ • $n_1 = 3, n_2 = 1$

- If the lines cross, we have to integrate two regions separately.
- Sometimes we are required to change the order or integration to be able to determine a certain integral.
- If $g(x_1)$ is not continuous on X_1 we have to split in into continuous parts and apply Frobini on them.

4.4.6 Domain Additivity

For $X = A_1 \cup A_2$ where $A_1, A_2 \subset \mathbb{R}^n$ are compact, and $f: X \to \mathbb{R}$ continuous on X then $\int_{X=A_1\cup A_2} f(x)\mathrm{d}x + \int_{A_1\cap A_2} f(x)\mathrm{d}x = \int_{A_1} f(x)\mathrm{d}x + \int_{A_2} f(x)\mathrm{d}x$.

- If $A_1 \cap A_2 = \emptyset$ then $\int_{A_1\cup A_2} f(x)\mathrm{d}x = \int_{A_1} f(x)\mathrm{d}x + \int_{A_2} f(x)\mathrm{d}x$.
- If $\mathrm{vol}_n(A_1\cap A_2) = 0$ then $\int_{A_1\cap A_2} f(x)\mathrm{d}x = 0 \quad \forall f$.

4.4.7 Parametrized m-Set

For $1 \leq m \leq n$. The function $\gamma: [a_1,b_1]\times\cdots\times[a_m,b_m] \to \mathbb{R}^n$ is a parametrized m-set.

- γ is continuous.
- $\gamma \in C^1$ for $(a_1,b_1)\times\cdots\times(a_m,b_m)$.

4.4.8 Negligible Sets in $\mathbb{R}^n$

$Y \subset \mathbb{R}^n$ is negligible if $\exists$ finitely many $\gamma_i: X_i \to \mathbb{R}^n$ parametrized m_i-sets with $m_i < n$, s.t. $Y \subset \bigcup \gamma_i(x_i)$.

For example:

n = 1: $Y \subset$ union of finitely many points.

n = 2: $Y \subset$ union of finitely many images of parametrized curves.

If $Y \subset \mathbb{R}^n$ closed and negligible then $\int_Y f(x)\mathrm{d}x = 0 \quad \forall f$.

4.5 Improper Integrals

For $X \subset \mathbb{R}^n$ non-compact and $f: X \to \mathbb{R}$ s.t $\int_K f(x)\mathrm{d}x \quad \forall K \subset X$ where K is compact. For a sequence of regions $X_k, k = 1,\dots$ s.t.: • x_k is compact $\forall k$ • $x_k \subset x_{x+1}$ • $\bigcup_{k=1}^{\infty} x_k = X$. Then $\int_X f(x)\mathrm{d}x = \lim_{n\to\infty}\int_{X_n} f(x)\mathrm{d}x$ if the limit exists.

4.6 Change of Variables

For $X = X_0 \cup A, Y = Y_0 \cup B$ where: • $X, Y \subset \mathbb{R}^n$ • X, Y compact • X_0, Y_0 open • A, B negligible . Let $\gamma: X \to Y \in C^1$ bijective and $\det J_p(x) \neq 0 \quad \forall x \in X_0$. Let $Y = \gamma(X)$ and suppose $f: Y \to \mathbb{R}$ continuous. Then $\int_Y f(y)\mathrm{d}y = \int_X f(\gamma(x))|\det J_p(x)|\mathrm{d}x$.

4.6.1 Special Coordinates

Polar Coordinates • $f: [0,\infty)\times[0,2\pi) \to \mathbb{R}^2$ • $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ • $\det J_f(r,\theta) = r$ • $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$

Cylindrical Coordinates

- $f: [0,\infty)\times[0,2\pi)\times\mathbb{R} \to \mathbb{R}^3$
- $(0,\theta,z) \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$
- $\det J_f = r$
- $\mathrm{d}x\mathrm{d}y z = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$

Spherical Coordinates

- $f: [0,\infty)\times[0,2\pi)\times[0,\pi) \to \mathbb{R}^3$
- $(r,\theta,\varnothing) \mapsto \begin{pmatrix} r\cos\theta\sin\varnothing \\ r\sin\theta\sin\varnothing \\ r\cos\varnothing \end{pmatrix}$
- $\det J_f = -r^2\sin\varnothing$
- $\mathrm{d}x\mathrm{d}y z = r^2\sin(\varnothing)\mathrm{d}r\mathrm{d}\theta z\varnothing$

4.6.2 Area in $n = 3$

Given a surface in $S = \{(x,y,z) \in \mathbb{R}^3\}$ by $f: \mathbb{R}^2 \to \mathbb{R}$, then $\mathrm{Area}(S) = \int\int\sqrt{1+(\partial_x f(x,y))^2+(\partial_y f(x,y))^2}\mathrm{d}x\mathrm{d}y$ for $X \subset \mathbb{R}^2$.

- Can also be used to calculate the length of the arc of a curve.

R: Calculate Arc Length of γ

Given curve $\gamma[a,b] \to \mathbb{R}$, calculate its length.

1. Calculate $\gamma'(t)$
2. Evaluate $int_a^b\sqrt{1+|\gamma'(t)|^2}\mathrm{d}t$

4.7 Green's Formula

For:

1. $X \subset \mathbb{R}^2$:
   - X is compact
   - X is always on the left hand side of the target vector to the boundary
2. Curve $\gamma: [a,b] \to \mathbb{R}^2$ forming the boundary (denoted as ∂X) of X:
   - γ is closed: $\gamma(a) = \gamma(b)$
   - γ is simple: $\nexists a < s < t < b$ s.t. $\gamma(s) = \gamma(t)$ (i.e. γ has no cycles)
3. Vector field $f: X \to \mathbb{R}^2$:
   - $f \in C^1$
   - $f = (f_1, f_2)$ has components f_1, f_2.
   - $\partial_x i, \partial_y i, i = 1,2$ exist and are continuous ($\implies$ curl f exists and is continuous).

Then $\int\int_X \underbrace{(\partial_x f_2 - \partial_y f_1)}_{\text{curl } f}\mathrm{d}x\mathrm{d}y = \int_\gamma f\mathrm{d}s$.

- A region can be the union of k simple closed curves: $\gamma = \bigcup_{i=1}^{k} \gamma_i$
  - Then $\int\int_X \text{curl } f \mathrm{d}x\mathrm{d}y = \sum_{k=1}^{k} \int_{\gamma_i} f \mathrm{d}s$.

Usage

1. Calculate area of a region as a line integral.
2. Calculate line integral if the double integral of curl f looks simpler.

R: Calculate line integral as double integral

Given $f = (f_1, f_2) : X \to \mathbb{R}^2 \in C^1$ for which both partial derivatives exists and curve $\gamma : [a, b] \to \mathbb{R}^2$ which forms the boundary of X and is closed and simple.

Goal, calculate $\int_\gamma f \mathrm{d}s$

1. Check if X lies on the left of γ. If it is not, make it do that.
2. Calculate partial derivatives $\partial_x f_2$ and $\partial_y f_1$.
3. Calculate curl $f = \partial_x f_2 - \partial_y f_1$
4. Calculate $\int\int_X \text{curl } f \mathrm{d}x\mathrm{d}y$

R: Calculate area enclosed by curve

Given curve $\gamma : [a, b] \to \mathbb{R}^2$ which is closed and simple.

Goal, find the area of X which is enclosed by the curve.

1. Select $f = (0, x)$ or $f = (-y, 0)$ or anything else with curl $f = 1$.
2. Area$(X) = \int_{\partial X} f(x)\mathrm{d}x = \int\int_X 1\mathrm{d}x\mathrm{y}$.

R: Curve goes in wrong direction

Given curve $\gamma : [a, b] \to \mathbb{R}^2$ which is closed and simple but the enclosed area is on the right.

- If curve is symmetric w.r.t. x-axis
  - $-\gamma$ parametrizes curve in opposite direction.
  - $\int_{-\gamma} f \mathrm{d}s = -\int_\gamma f \mathrm{d}s$

Other cases:

- Given a two dimensional integral $\int\int_X g(x, y)\mathrm{d}x\mathrm{y}$ which we want to evaluate. If we can find $f = (f_1, f_2)$ with curl $f = g$ then we can use $\int\int_X g\mathrm{d}x\mathrm{d}y = \int\int_X \text{curl } f\mathrm{d}x\mathrm{y} = \int_{\partial X = \gamma} f\mathrm{d}x$