One Dimensional Wave Equation I

- Standard problem:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)\,(x \in [0,X], t \in [0,T])$$

where

$$\text{(i) } u(x,0) = I(x),\ u_t(x,0) = V(x),\ \text{(IC)}$$

$$\text{(ii) } u(0,t) = 0,\ u(L,t) = 0.\ \text{((fixed) BC)}$$

- Space-time mesh
  - $x = x_0, x_1, x_2, ..., x_{N_x}$ where $x_i = i\triangle x$ $(i = 0, 1, 2, ..., N_x)$ and $x_{N_x} = L$.

One Dimensional Wave Equation II

- $t = t_0, t_1, t_2, ..., t_{N_t}$ where $t_n = n\triangle t$ $(n = 0, 1, 2, ..., N_t)$ and $t_{N_t} = T$.
  - $i = 0$ and $i = N_x$ represent two boundaries.
- Let $u_i^n = u(i\triangle x, n\triangle t)$.
- Discretization of $u_{tt} = cu_{xx}$ by centered finite difference method

$$u_{tt} \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\triangle t^2}$$
$$u_{xx} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\triangle x^2}$$

$$(n = 0, 1, 2, ..., N_t, i = 0, 1, 2, ..., N_x)$$

One Dimensional Wave Equation III

Therefore,

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\triangle t^2} = c\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\triangle x^2}$$
$$(n = 0, 1, 2, ..., N_t, i = 0, 1, 2, ..., N_x)$$

Rearranging,

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \qquad (1)$$
$$(n = 0, 1, 2, ..., N_t, i = 0, 1, 2, ..., N_x)$$

where $C = c\triangle t/\triangle x$. This is the basic difference equation of motion.

One Dimensional Wave Equation IV

- Proceed in two steps
  - 1st step
    - Determine u_i^0 for $i = 0, 1, \ldots, N_x$.
    - Determine u_i^1 for all interior points ($i = 1, 3, \ldots, N_x - 1$).
    - Determine u_i^n for $n = 2, 3, \ldots, N_t$ and for all interior points.
  - 2nd step
    - Determine all boundary points ($i = 0, N_x$) for $n = 0, 1, 2, 3, \ldots, N_t$)
- Time step 0: u_i^0

One Dimensional Wave Equation V

- Use the first intial condition

$$u_i^0 = I(i\triangle x)\,(i = 0, \ldots, N_x)$$

- Time step 1: u_i^1
  - The equation of motion (equation (1)) needs to be modified because

$$u_i^1 = -u_i^{-1} + 2u_i^0 + C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

  (when $n = 1$) involves u_i^{-1}, which does not exist.
  - Use the second initial condition $u_t(0, x) = V(x)$ for $i = 1, 2, ..., N_x - 1$ to remedy this problem.

One Dimensional Wave Equation VI

- Discretization of $u_t(0,x) = V(x)$ by centered finite difference method

$$u_t \approx \frac{u_i^1 - u_i^{-1}}{2\triangle t} = V(x)\,(i = 1, 2, ..., N_x - 1)$$

Rearranging,

$$u_i^{-1} = u_i^1 - 2\triangle t V(x)\,(i = 1, 2, ..., N_x - 1).$$

Substituting the above in (1),

$$u_i^1 = u_i^0 + 0.5C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \triangle t V(x) \qquad (i = 1, 2, ..., N_x - 1). \tag{2}$$

One Dimensional Wave Equation VII

- Time step 2 onward: u_i^n $(n = 2, 3, \ldots, N_t)$
  - Use equation (1) for $i = 1, 2, ..., N_x - 1.$
- Boundary conditions for all t
  - $u_0^n = 0,$
  - $u_{N_x}^n = 0 \quad (n = 0, 1, 2, ..., N_t)$

Reflecting (Neuman) Boundary Problem I

- Reflecting boundary condition:

$$u_x(0,t) = 0 \text{ and/or } u_x(L,t) = 0.$$

- Discretization of $u_x(0,t) = 0$.

$$u_x \approx \frac{u_1^n - u_0^n}{\triangle t} \, (n = 0, 1, 2, ..., N_t)$$

Rearranging,

$$u_0^n = u_1^n \, (n = 0, 1, 2, ..., N_t).$$

Reflecting (Neuman) Boundary Problem II

Similarly,

$$u^n_{N_x} = u^n_{N_x-1}\,(n = 0, 1, 2, ..., N_t).$$

Open Boundary Problem I

- Open boundary condition:

$$u_t(0,t) - cu_x(0,t) = 0$$
$$u_t(L,t) + cu_x(L,t) = 0$$

- Discretization of $u_t(0,t) - cu_x(0,t) = 0$.

$$u_t(0,t) - cu_x(0,t) \approx \frac{u_0^n - u_0^{n-1}}{\triangle t} - c\frac{u_1^n - u_0^n}{\triangle x} = 0.$$

$$(n = 1, 2, ..., N_t)$$

Open Boundary Problem II

Rearranging,

$$(\triangle x + c\triangle t)u_0^n - \triangle x u_0^{n-1} - c\triangle t u_1^n = 0.$$
$$(n = 1, 2, ..., N_t)$$

Or

$$u_0^n = \frac{\triangle x u_0^{n-1} + c\triangle t u_1^n}{\triangle x + c\triangle t} \; (n = 1, 2, ..., N_t)$$

Similarly, it turns out that

$$u_{N_x}^n = \frac{\triangle x u_{N_x}^{n-1} + c\triangle t u_{N_x-1}^n}{\triangle x + c\triangle t} \; (n = 1, 2, ..., N_t)$$

Open Boundary Problem III

- Note that u_0^0 and $u_{N_x}^0$ are already specified at time step 0;

Periodic Boundary Problem I

- Periodic boundary condition:

$$u(0,t) = u(L,t).$$

Or

$$u_0^n = u_{N_x}^n.$$

Two Dimensional Wave Equation I

- Standard problem:

$$\begin{aligned} u_{tt}(x,y,t) =& c^2(u_{xx}(x,y,t) + u_{yy}(x,y,t)) \\ & (x \in [0,X], y \in [0,Y], t \in [0,T]) \end{aligned}$$

where

$$\begin{aligned} &\text{(i) } u(x,y,0) = I(x,y),\ u_t(x,y,0) = V(x,y),\ \text{(IC)} \\ &\text{(ii) } u(0,y,t) = 0,\ u(X,y,t) = 0, \\ &\quad u(x,0,t) = 0, u(x,Y,t) = 0.\ \text{((fixed) BC)} \end{aligned}$$

Analytic Solution I

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\mu_m x \sin\nu_n y (B_{mn}\cos\lambda_{mn}t \\ + B^*_{mn}\sin\lambda_{mn}t),$$

where

$$\mu_m = \frac{m\pi}{X}, \nu_n = \frac{n\pi}{Y}, \lambda_{mn} = c\sqrt{\mu_m^2+\nu_n^2}$$

$$B_{mn} = \frac{4}{XY}\int_0^X\int_0^Y I(x,y)\sin\frac{m\pi}{X}x\sin\frac{n\pi}{Y}y\,dy\,dx$$

$$B^*_{mn} = \frac{4}{XY\lambda_{mn}}\int_0^X\int_0^Y V(x,y)\sin\frac{m\pi}{X}x\sin\frac{n\pi}{Y}y\,dy\,dx$$

Numerical Solution I

- Discretization

$$\frac{u_{i,j}^{n-1} - 2u_{i,j}^{n} + u_{i,j}^{m+1}}{\triangle t^2} = c^2\Big(\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{m+1}}{\triangle x^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{m}}{\triangle y^2}\Big)$$

$$(n = 0, 1, 2, \ldots, N_t, i = 0, 1, 2, \ldots, N_x, j = 0, 1, 2, \ldots, N_y).$$

Numerical Solution II

Let $\triangle x = \triangle y = \triangle$. Then,

$$u_{i,j}^{n+1} = -u_{i,j}^{n-1} + 2u_{i,j}^{n} + C^2(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n})$$

$(n = 0, 1, 2, \ldots, N_t, i = 0, 1, 2, \ldots, N_x, j = 0, 1, 2, \ldots, N_y)$.

where $C = c\triangle t/\triangle x$. This is the basic difference equation of motion.

- Proceed in two steps
  - 1st step

Numerical Solution III

- Determine $u_{i,j}^{0}$ for $i = 0, 1, \ldots, N_x$. and $j = 0, 1, \ldots, N_y$
- Determine $u_{i,j}^{1}$ for all interior points $(i = 1, 2, \ldots, N_x - 1)$ and $j = 1, 2, \ldots, N_y - 1$.
- Determine $u_{i,j}^{n}$ for $n = 2, 3, \ldots, N_t$ and for all interior points.

- 2nd step
  - Determine all boundary points $(i = 0, N_x$ and $j = 0, N_y)$ for $n = 0, 1, 2, 3, \ldots, N_t)$

- Time step 0: $u_{i,j}^{0}$
  - Use the first intial condition

Numerical Solution IV

$$u^0_{i,j} = I(i\triangle, j\triangle)\,(i = 0, \ldots, N_x)$$

- Time step 1: $u^1_{i,j}$
    - The equation of motion (equation (1)) needs to be modified because

$$u^1_{i,j} = -u^{-1}_{i,j} + 2u^0_{i,j} + C^2(u^0_{i+1,j} + u^0_{i-1,j} + u^0_{i,j+1} + u^0_{i,j-1} - 4u^0_{i,j})$$

(when $n = 1$) involves $u^{-1}_{i,j}$, which does not exist.

Numerical Solution V

- Use the second initial condition $u_t(0, x, y) = V(x, y)$ for $i = 1, 2, ..., N_x - 1$ and $j = 1, 2, ..., N_y - 1$to remedy this problem.
- Discretization of $u_t(0, x, y) = V(x, y)$ by centered finite difference method

$$u_t \approx \frac{u_i^1 - u_i^{-1}}{2\triangle t} = V(x)\,(i = 1, 2, ..., N_x - 1)$$

Rearranging,

$$u_i^{-1} = u_i^1 - 2\triangle t V(x)\,(i = 1, 2, ..., N_x - 1).$$

Numerical Solution VI

Substituting the above in (1),

$$u_i^1 = u_i^0 + 0.5C^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \triangle tV(x) \qquad (i = 1, 2, ..., N_x - 1). \tag{3}$$

- Time step 2 onward: $u_{i,j}^n$ $(n = 2, 3, \ldots, N_t)$
  - Use equation (1) for $i = 1, 2, ..., N_x - 1$.
- Boundary conditions for all t
  - $u_{0,j}^n = 0$ and $u_{N_x,j}^n = 0$,
  - $u_{i,0}^n = 0$ and $u_{i,N_y}^n = 0$ and $(n = 0, 1, 2, ..., N_t)$