# Chapter 2

# (Part 2)

# Functions

# Functions

- Let $X$ and $Y$ be nonempty sets
- A function $f$ from $X$ to $Y$ is a relation from $X$ to $Y$ having the properties.
  - The domain of $f$ is $X$
  - If $(x,y), (x,y') \in f$, then $y=y'$

(e.g. f(1)=b, f(2)=b is a function, but f(1)=a, f(1)=b is NOT a function)

# Functions

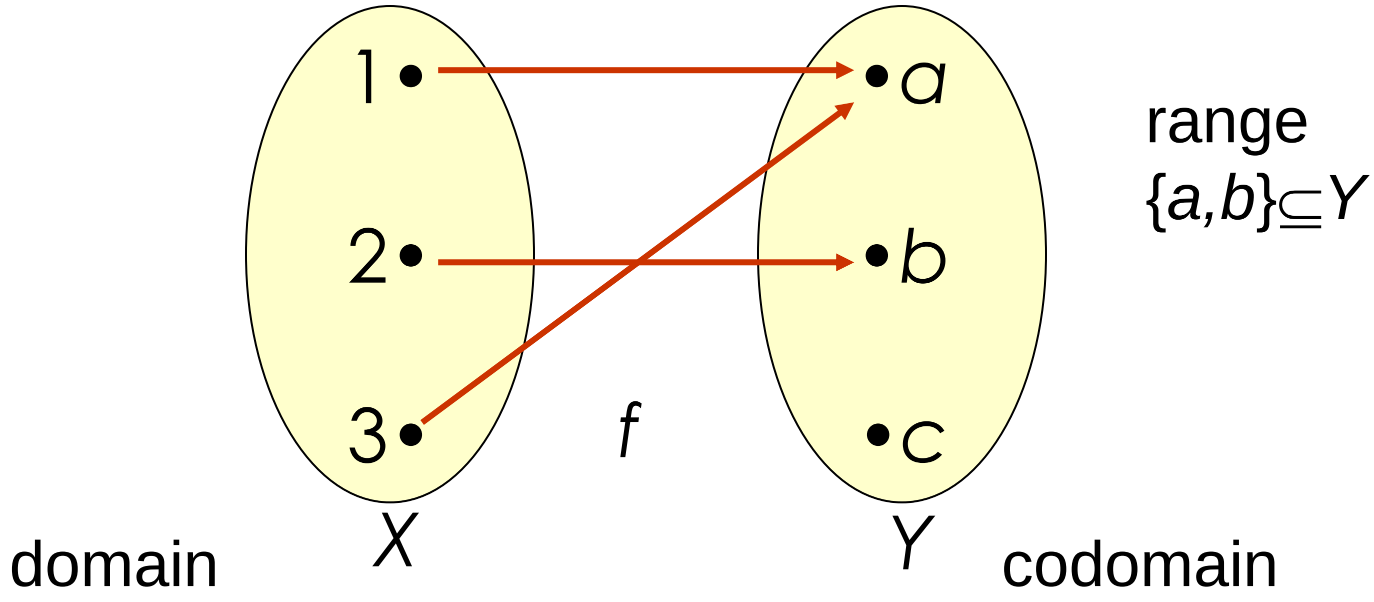
- A function from $X$ to $Y$ is denoted, $f : X \rightarrow Y$
- The domain of $f$ is the set $X$.
- The set $Y$ is called the codomain or target of $f$.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

# example

- The relation, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$ is a function from X to Y.
- The domain of $f$ is $X$
- The range of $f$ is $\{a, b\}$

# example

- $f = \{ (1,a), (2,b), (3,a) \}$



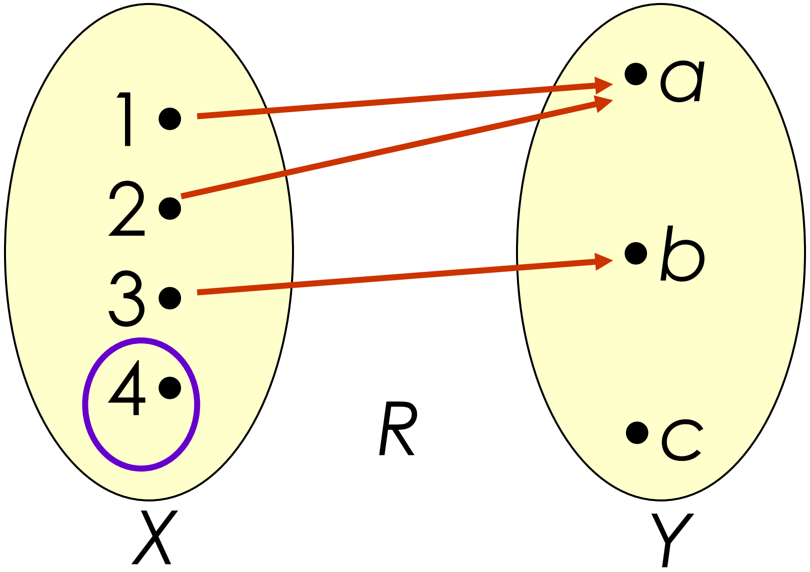
# example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from $X$ to $Y$.

- The domain of $R$, $\{1,2,3\}$ is not equal to $X$.

# example

$R = \{(1,a), (2,a), (3,b)\}$




There is no arrow from 4

# example

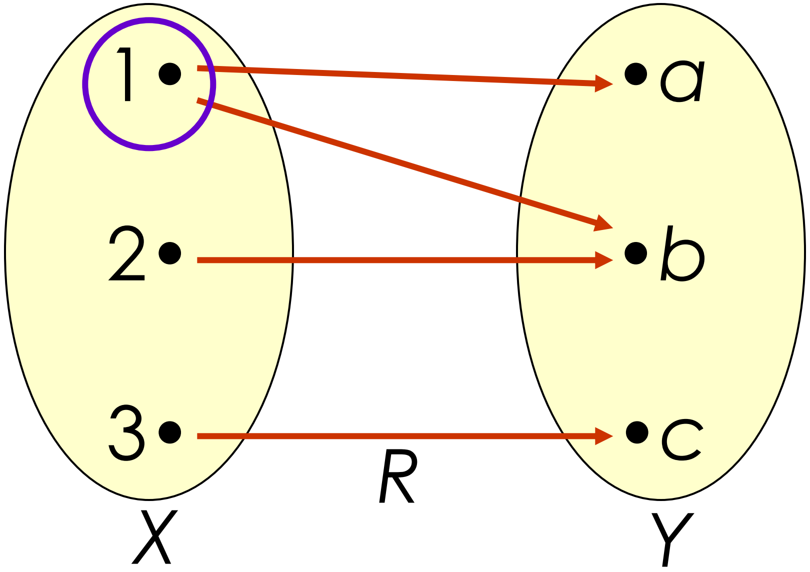
- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y

- (1,a) and (1,b) in R but $a \neq b$.

# example

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



# example

- For the function, $f = \{(1,a), (2,b), (3,a)\}$
- We may write

  $f(1)=a$, $f(2)=b$, $f(3)=a$
- Notation $f(x)$ is used to define a function

# example

- $f(x) = x^2$

  $f(2) = 4$,  $f(-3.5) = 12.25$,  $f(0) = 0$

  $f = \{(x, x^2) \mid x \text{ is a real number}\}$

# One-to-one

- A function $f$ from $X$ to $Y$, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.
- For all $x_1$, $x_2$ , if $f(x_1) = f(x_2)$, then $x_1=x_2$.

  $\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$

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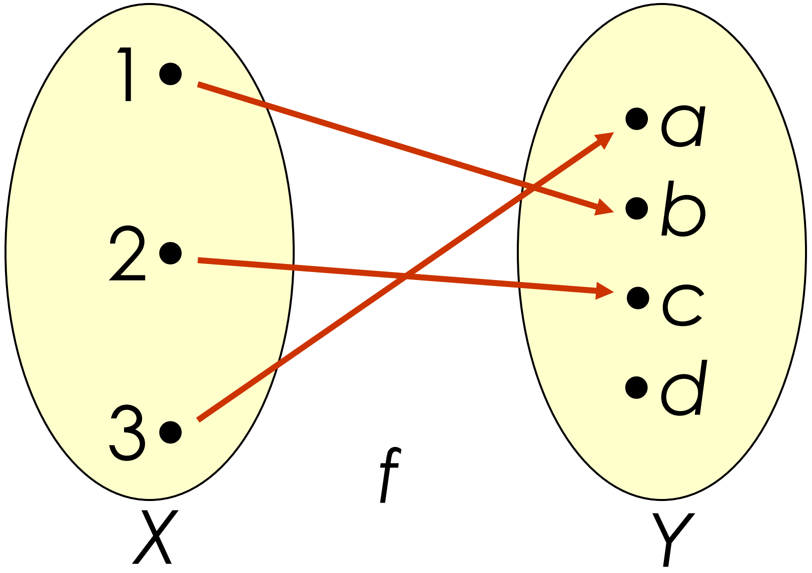
# example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c, d \}$ is one-to-one.

Each element in *Y* has at most one arrow pointing to it



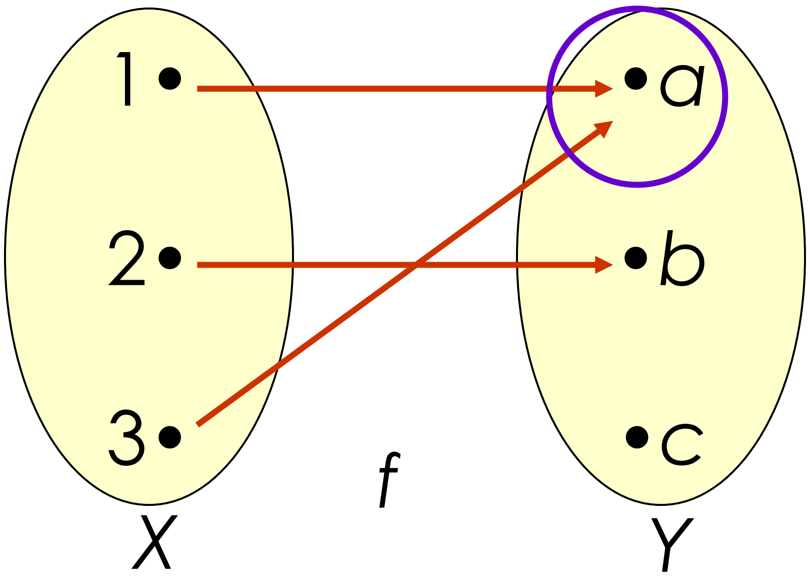
# example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c\}$ is NOT one-to-one.

- $f(1) = a = f(3)$

# example

$f = \{ (1,a), (2,b), (3,a) \}$




*a* has 2 arrows pointing to it

# example

Show that the function,

$$f\,(n) = 2n+1$$

on the set of positive integers is one-to-one.

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# example

- For all positive integer, $n_1$ and $n_2$ if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ ( -1)

  $2n_1 = 2n_2$ (÷2)

  $n_1 = n_2$

- This shows that $f$ is one-to-one.

# example

- Show that the function,

$$f(n) = 2^n - n^2$$

from the set of positive integers to the set of integers is NOT one-to-one.

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# example

- Need to find 2 positive integers, $n_1$and $n_2$
  $n_1 \neq n_2$ with f ($n_1$) = f ($n_2$).

- Trial and error,

  $$f\,(2) = f\,(4)$$

- $f$ is not one-to-one.

# Onto

- If $f$ is a function from $X$ to $Y$ and the range of $f$ is $Y$, $f$ is said to be onto $Y$

  (or an onto function or a surjective function)

- For every $y \in Y$, there exists at least one $x \in X$ such that $f(x)=y$

$$\forall y \in Y \ \exists x \in X \ ( f(x)=y )$$

# example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$
  from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$
  is one-to-one and onto $Y$.

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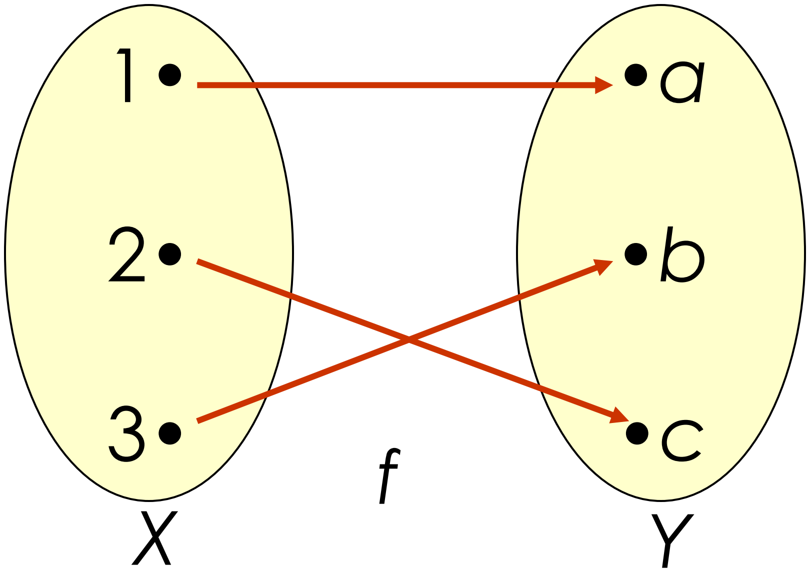
# example

- $f = \{ (1,a), (2,c), (3,b) \}$

**One-to-one**
Each element in *Y* has at most one arrow




**Onto**
Each element in *Y* has at least one arrow pointing to it

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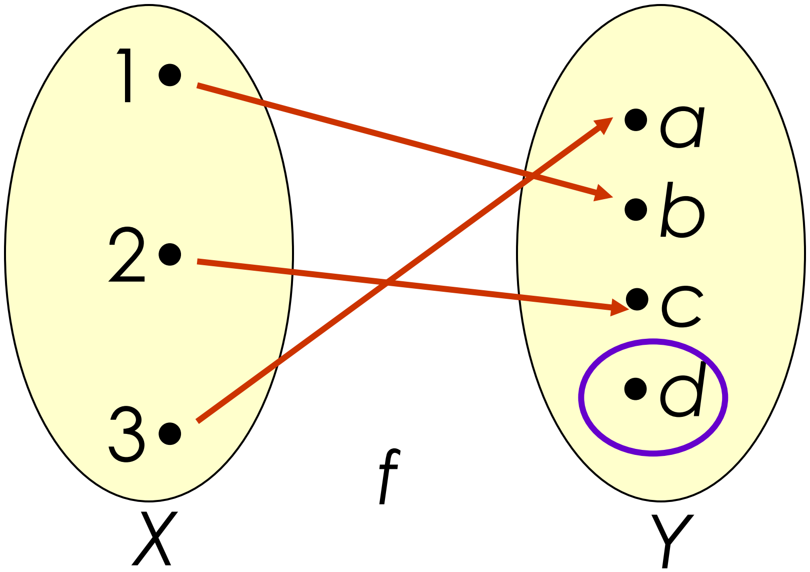
# example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto $Y = \{a, b, c, d\}$

- It is onto $\{a, b, c\}$

# example

$f = \{ (1,b), (3,a), (2,c) \}$




**not onto**
no arrow pointing to *d*

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# Bijection

- *f* is called one-to-one correspondence (or bijective or bijection) if *f* is both one-to-one and onto.

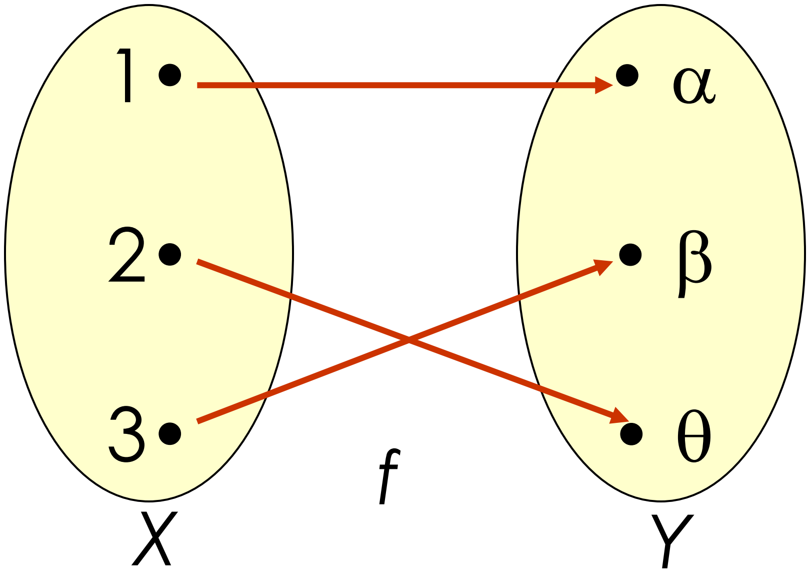
# example

- The function, $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{\alpha, \beta, \theta \}$ is one-to-one and onto $Y$.

- The function $f$ is a bijection

# example

$f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$




One-to-one and onto *Y*
-**bijection**

# exercise

Determine which of the relations *f* are functions from the set *X* to the set *Y*.

a) $X = \{ -2, -1, 0, 1, 2 \}$ , $Y = \{ -3, 4, 5 \}$ and
$f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

b) $X = \{ -2, -1, 0, 1, 2 \}$ , $Y = \{ -3, 4, 5 \}$ and
$f = \{ (-2,-3), (1,4), (2,5)\}$

c) $X = Y = \{ -3, -1, 0, 2 \}$ and
$f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

In case any of these relations are functions, determine if they are one-to-one, onto *Y*, and/or bijection.

# Solutions

Determine which of the relations $f$ are functions from the set $X$ to the set $Y$.

a) $X = \{ -2, -1, 0, 1, 2 \}$ , $Y = \{ -3, 4, 5 \}$ and

$f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

Function

The domain of $f$ is $X$

$(x,y)$ & $(x,y') \in f$ where $y \neq y'$ *(does not exist)*

Onto

Not one-to-one

# exercise

Determine which of the relations *f* are functions from the set *X* to the set *Y*.

b) $X = \{ -2, -1, 0, 1, 2 \}$ , $Y = \{ -3, 4, 5 \}$ and
$f = \{ (-2,-3), (1,4), (2,5)\}$

Not a function

The domain is not X

# Solutions

Determine which of the relations $f$ are functions from the set $X$ to the set $Y$.

c) $X = Y = \{ -3, -1, 0, 2 \}$ and

$f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

Not a function

$(-3,-1), (-3,0) \in f$, but $-1 \neq 0$

# Inverse function

- Let $f: X \to Y$ be a function.
- The inverse relation $f^{-1} \subseteq Y \times X$ is a function from $Y$ to $X$, if and only if $f$ is both one-to-one and onto Y.

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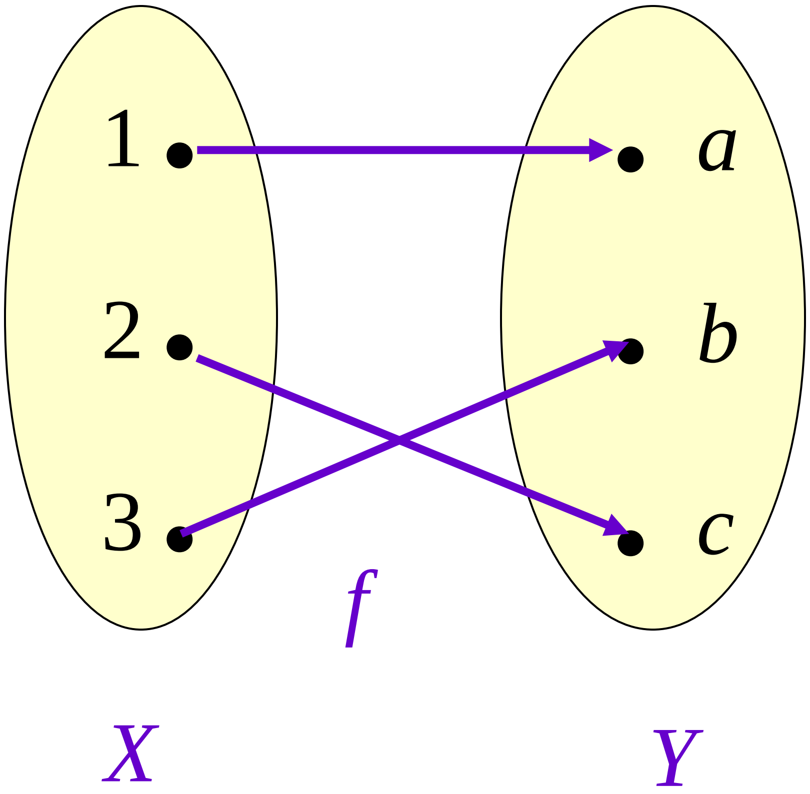
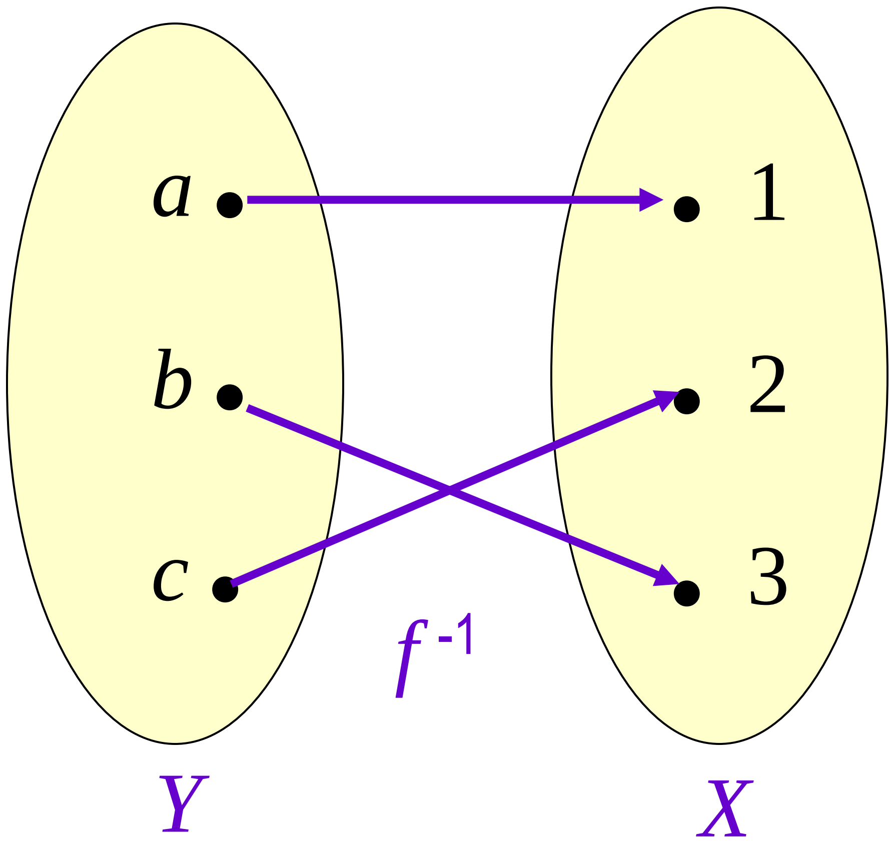
# example

- $f = \{(1,a),(2,c),(3,b)\}$
- $f^{-1} = \{(a,1),(c,2),(b,3)\}$



# example

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# example

- The function, $f(x) = 9x + 5$ for all $x \in R$
  ($R$ is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, $f^{-1}$ exists.
- Let $(y, x) \in f^{-1}$, $f^{-1}(y) = x$

  $(x,y) \in f$, $\quad y = 9x + 5$

  $x = (y-5)/9$

  $f^{-1}(y)=(y-5)/9$

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# exercise

- Find each inverse function.

  a) $f(x) = 4x + 2, \ x \in R$

  b) $f(x) = 3 + (1/x), \ x \in R$

# Solution

- Find each inverse function.

a) $f(x) = 4x + 2,\ x \in R$

$4x + 2 = y$

$4x = y - 2$

$x = (y-2)/4$

$f^{-1}(y) = ((y-2)/4$, or $f^{-1}(x) = ((x-2)/4$

# Solution

- Find each inverse function.

b) $f(x) = 3 + (1/x),\ x \in R$

$3 + (1/x) = y$

$1/x = y - 3$

$1 = x(y-3)$

$1/(y-3) = x$

$f^{-1}(y) = 1/(y-3)$ or $f^{-1}(x) = 1/(x-3)$

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# Composition

- Suppose that $g$ is a function from $X$ to $Y$ and $f$ is a function from $Y$ to $Z$.

- The composition of $f$ with $g$,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f\,(g(x))$$

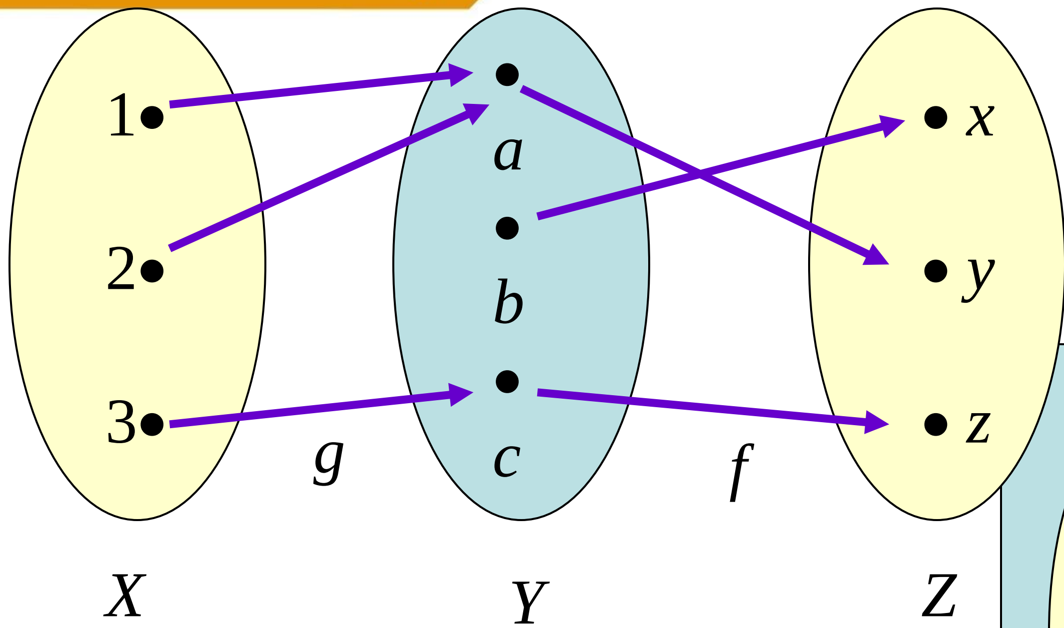
from $X$ to $Z$

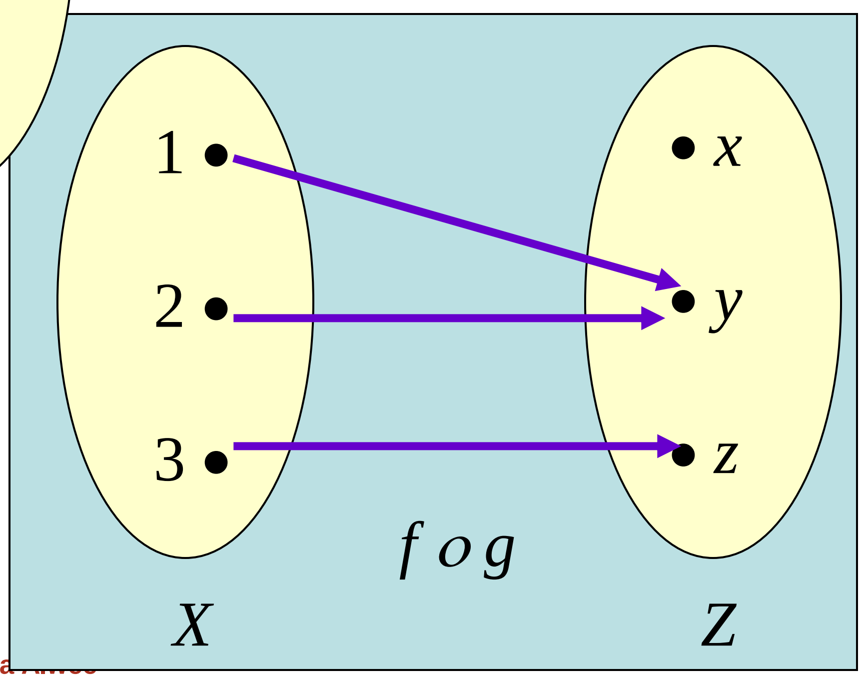
## example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$
  a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ and
  $$f = \{ (a,y), (b,x), (c,z) \}$$
  a function from $Y$ to $Z = \{ x, y, z \}$

- The composition function from $X$ to $Z$ is the function
  $$f \circ g = \{ (1,y), (2,y), (3,z) \}$$

# example



# example

- $f(x) = \log_3 x$ and $g(x) = x^4$

$$f(g(x)) = \log_3 (x^4)$$

$$g(f(x)) = (\log_3 x)^4$$

Note: $f \circ g \neq g \circ f$

# example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$(g o f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

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# example

- Composition sometimes allows us to decompose complicated functions into simpler functions.

- example

$$f(x) = \sqrt{\sin 2x}$$

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$

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# exercise

- Let $f$ dan $g$ be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$ ,      $g(n) = 2^n$

- Find the compositions

  a) $f \circ f$

  b) $g \circ g$

  c) $f \circ g$

  d) $g \circ f$

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# Solution

- Let $f$ dan $g$ be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$ , $g(n) = 2^n$

- Find the compositions

  a) $f \circ f$

  $f(f(n)) = f(n^2) = (n^2)^2 = n^4$

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# Solution

- Let $f$ dan $g$ be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$ , $g(n) = 2^n$

- Find the compositions

  b) $g \circ g$

$$g(g(n)) = g(2^n) = 2^{2^n}$$

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# Solution

- Let $f$ dan $g$ be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$ ,  $g(n) = 2^n$

- Find the compositions

  c) $f \circ g$

  $f(g(n)) = f(2^n) = (2^n)^2$

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# Solution

- Let $f$ dan $g$ be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$ , $g(n) = 2^n$

- Find the compositions

  d) $g \circ f$

$g(f(n)) = g(n^2) = 2^{n^2}$

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