Wigner's Cusp Anomalies in Multimode Systems

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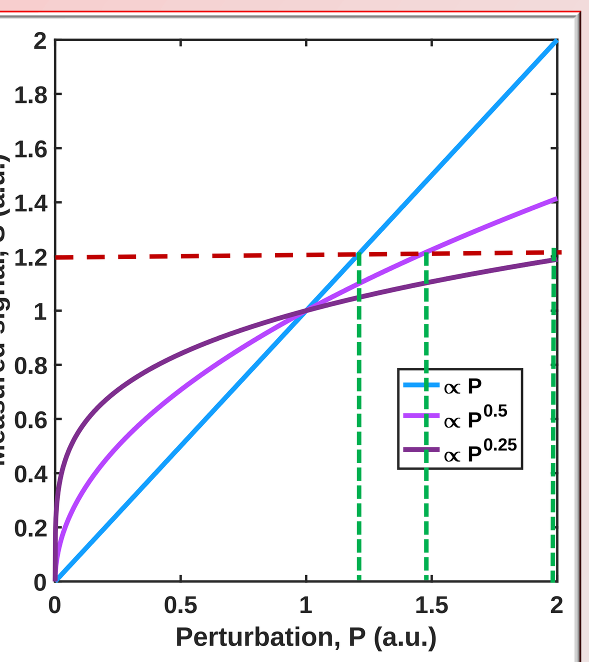
Abstract

Most sensors operate on a linear response principle (LRP) which implies a fundamental bound on the dynamic range of a sensor: the ratio between the maximum and minimum perturbations a sensor can measure. To overcome this limit a nonlinear sensing scheme is required. We have recently identified that such sensors can be created based on physical sensing platforms which operate at the vicinity of Wigner's cusp anomalies (WCAs). These are square-root singularities of the differential scattering cross-section around the energy threshold of a newly opened channel. It can manifest itself in a range of frameworks including nuclear reactions, scattering of a quantum particle in the proximity of a step potential, and reflectance of a monochromatic wave from the intersection of two dielectric media (the first having a greater refractive index than the second) about the critical angle. The WCAs also can be implemented in a multimode system, where each mode has a threshold associated with a specific type of perturbations allowing the independent and simultaneous detection of a large number of various perturbations. Our effort aims to achieve a multi-faceted sensor-platform with extreme sensitivity and large dynamic range, which is capable of measuring multiple independent observable quantities at once.

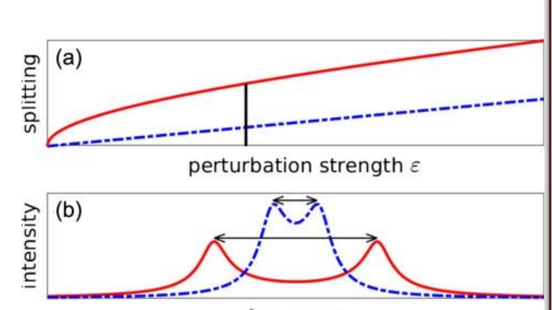
Motivation

Typical sensors utilize an LRP, where the measured signal is directly related to the experienced perturbation. In this scheme any increase in dynamic range comes at the cost of sensitivity to small perturbations. However, if a sublinear response is achieved this constraint no longer exists. Sensing at the sublinear level allows for enhanced sensitivity to small perturbations and an increased dynamic range. The plot on the right displays the differences in sensing capabilities between various schemes.

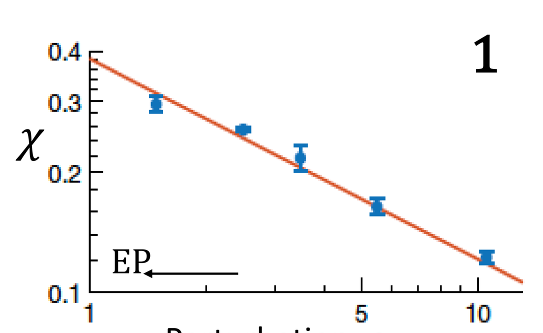


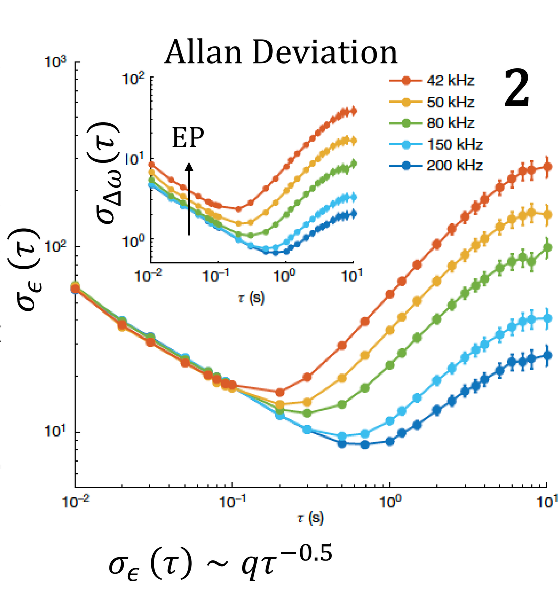
$$\hat{H}_0 = \begin{pmatrix} \omega_0 + i\alpha & g \\ g & \omega_0 - i\alpha \end{pmatrix}.$$

$$E_\pm = \omega_0 \pm \sqrt{g^2 - \alpha^2}.$$

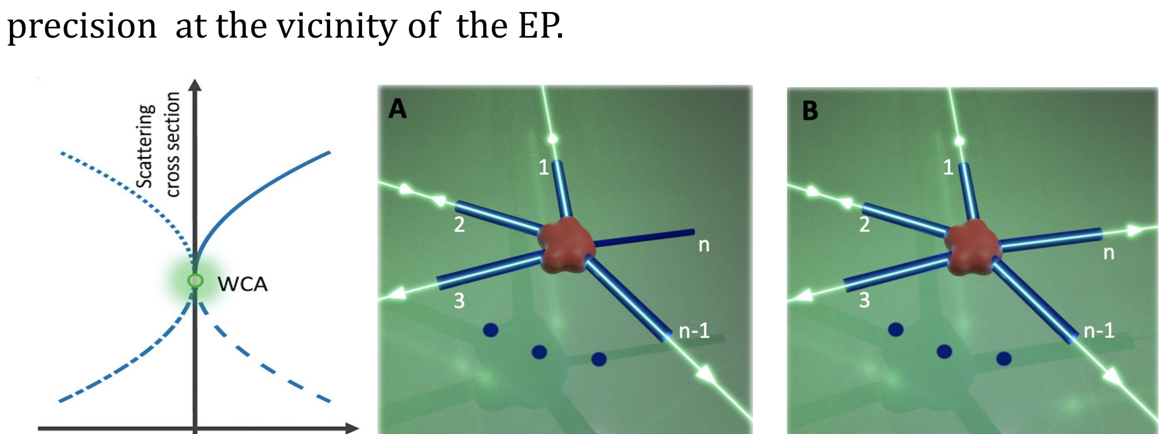


Prominent examples of platforms that allow sublinear sensing are systems operating near exceptional points (EPs). Such systems can be described with a Non-Hermitian Hamiltonian shown above with pair of eigenfrequencies $E_\pm$. If $g = \alpha$ eigenfrequencies and eigenvectors coalesce, forming an EP. In such a platform, sensitivity χ of eigenfrequencies splitting $\Delta\omega = E_+ - E_-$ with respect to perturbation ϵ from the EP demonstrates $1/\sqrt{\epsilon}$ behavior. It was recently shown [1] that the error in measured frequency splitting $\sigma_{\Delta\omega}$ increases at the vicinity of EP. This error is a consequence of quantum noise effects which result in an emission linewidth broadening, which leads to increase in $\sigma_{\Delta\omega}$. It appears that at the vicinity of EP $\sigma_{\Delta\omega}$ increases exactly at the same rate as the sensitivity χ. Therefore, the error in the measurement of observable $\sigma_\epsilon = \frac{\sigma_{\Delta\omega}}{\chi}$ remains the same and is independent of the proximity to the EP, resulting in no improvement of precision at the vicinity of the EP.



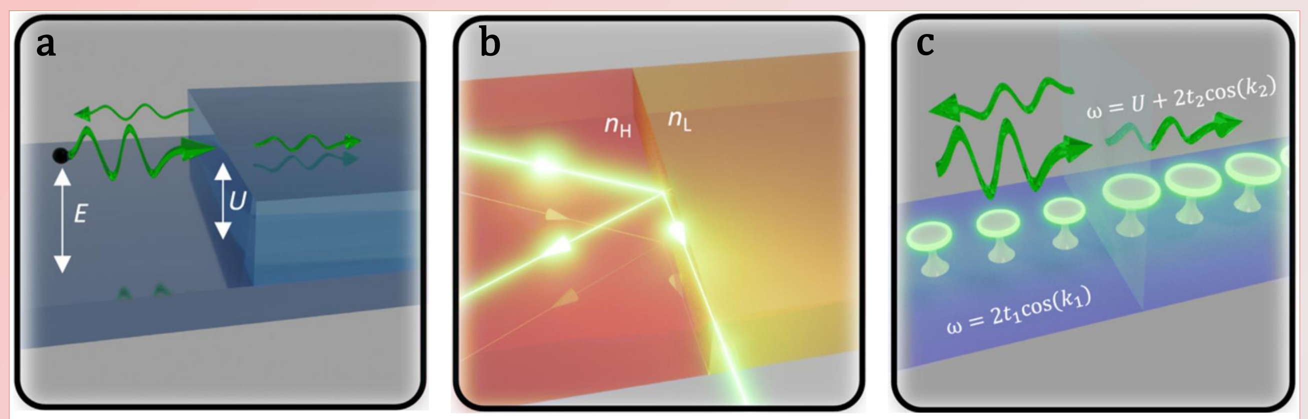


$\sigma_\epsilon(\tau) \sim q\tau^{-0.5}$



In order to overcome these potential limitations of EP-based sublinear sensing we propose another mechanism, Wigner's Cusp Anomalies (WCAs). These cusps are square root singularities of the differential scattering cross-section around the energy threshold of a newly opened channel (see figure above, where channel n experiences a close-to-open transition). Unlike at the EPs, these cusps are not impacted by quantum noise effects.

Examples



While Wigner's cusps were first discussed in a framework of nuclear physics, they appear in many other settings. For example, figure **a** displays a quantum step potential of height U and incident energy E, the cusp occurs in reflectance/transmittance in the vicinity of E > U. Another example, (figure **b)** is a dielectric interface between two media with different refractive idexies $n_H > n_L$, where the WCA occurs once the angle of incidence is lower than the critical angle. In photonics, WCAs also occur at the interface between two coupled resonator optical waveguides (CROW), with different central frequencies.

Multimode Systems

The ability to connect multiple channels (waveguides) allows for the formation of a platform with multiple cusps, which could be used in the creation of a sensor that is hyper-sensitive to multiple independent observables at the same time. In order to investigate a system with multiple unique cusps, we studied a multimode setting. Specifically, we utilized a double step potential. In this setup we can sense two independent observables at once: transmittance from the first to the second medium, and transmittance from the first to the third medium.

We solve the Schrödinger equation in each domain to obtain:

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$$
$$\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}$$
$$\psi_3(x) = Ee^{ik_3x} + Fe^{-ik_3x}$$

with $k_1 = \sqrt{E}$,

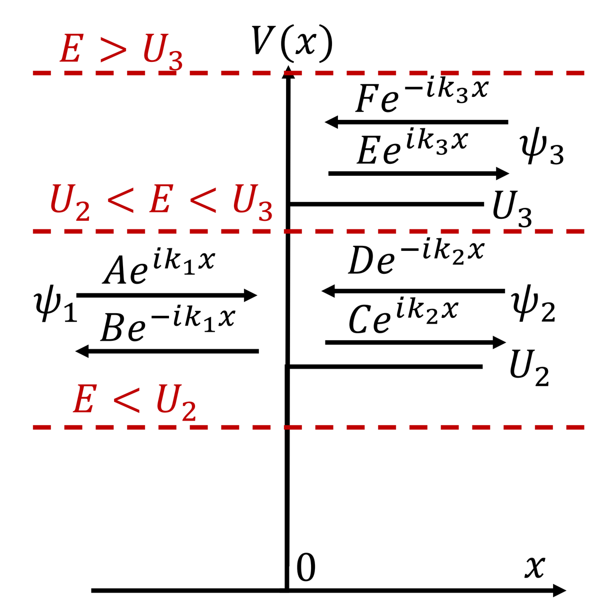
$k_2 = \sqrt{E - U_2}$ for $E > U_2$, and

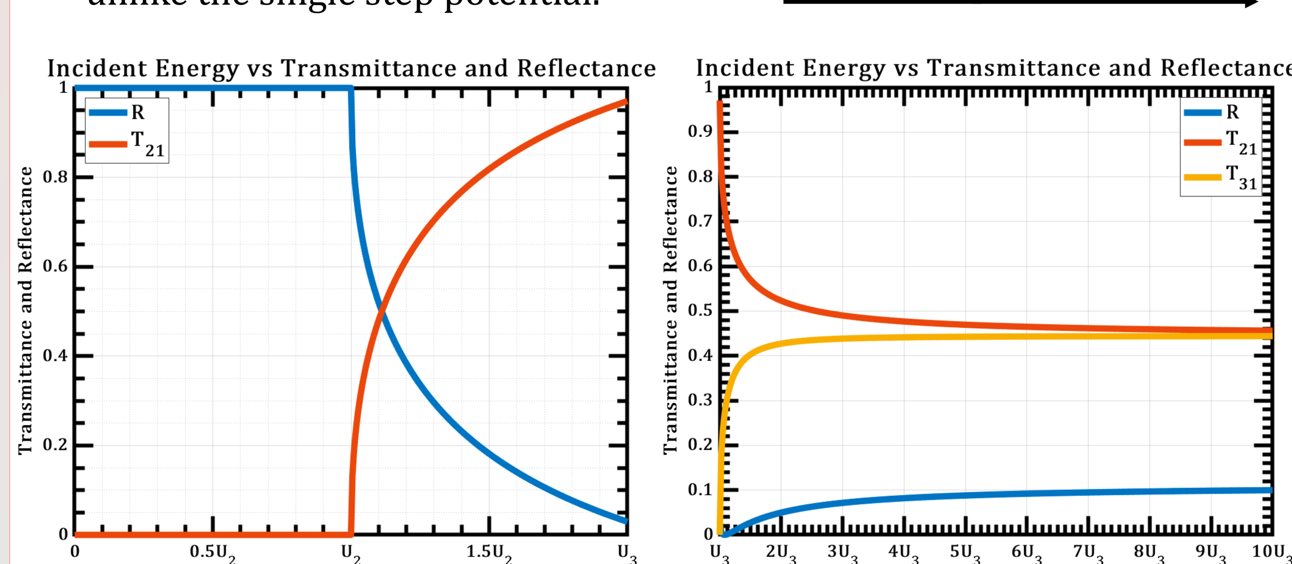
$k_3 = \sqrt{E - U_3}$ for $E > U_3$

In this scenario

$$\psi_1'(0) = \psi_2'(0) + \psi_3'(0),$$

unlike the single step potential.





In the case of $E < U_2$ we see total reflection, just as in the single step potential case. When $U_2 < E < U_3$ the reflectance cusp is given by $R = 1 - \frac{4\sqrt{U_2}}{U_3}\sqrt{\varepsilon}$, and the transmittance cusp is given by $T_{21} = \frac{4\sqrt{U_2}}{U_3}\sqrt{\varepsilon}$ and $\varepsilon = U_2 + \delta, \delta \ll 1$.

When $E > U_3$ the reflectance cusp is given by

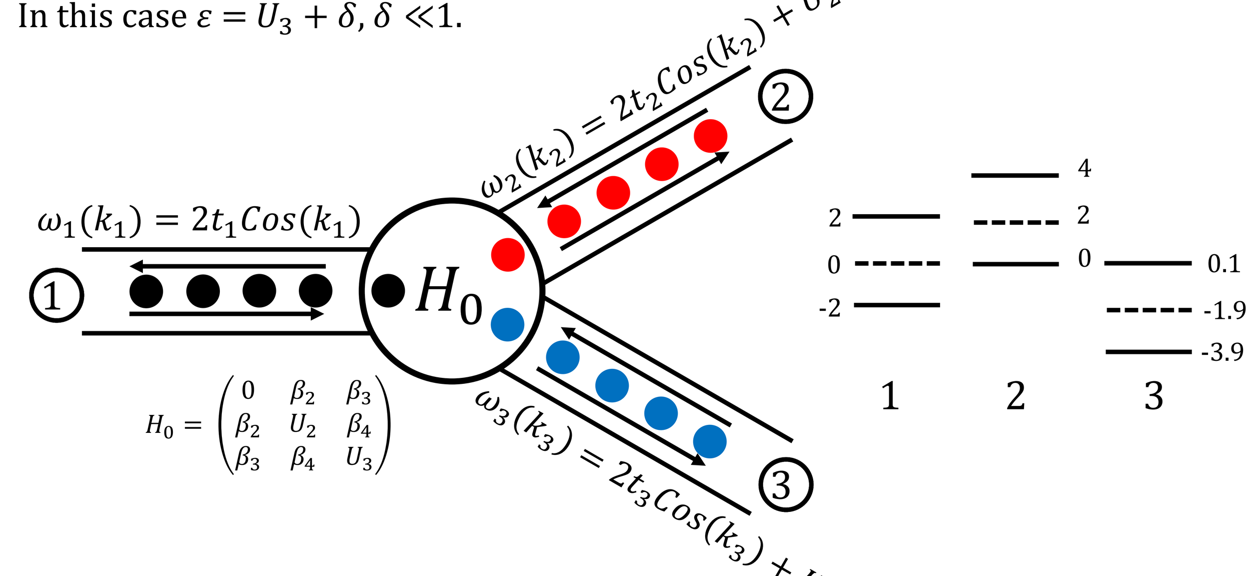
$$R = \frac{U_2^2}{\left(\sqrt{U_3} + \sqrt{U_3 - U_2}\right)^4} - \frac{4U_2\sqrt{U_3}}{\left(\sqrt{U_3} + \sqrt{U_3 - U_2}\right)^4}\sqrt{\varepsilon}$$

The transmittance cusps are given by

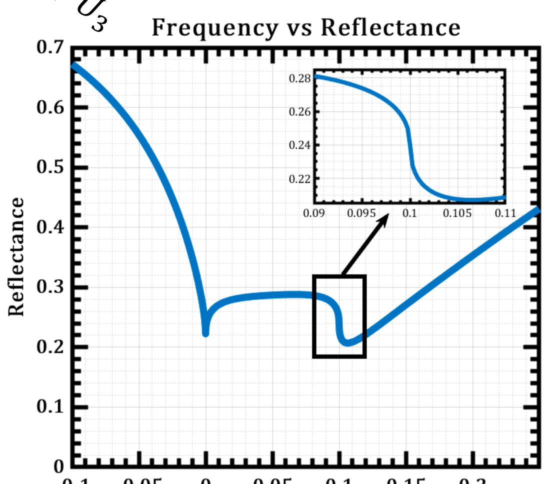
$$T_{21} = \frac{4\sqrt{U_3}\sqrt{U_3 - U_2}}{\left(\sqrt{U_3} + \sqrt{U_3 - U_2}\right)^2} - \frac{8\sqrt{U_3}\sqrt{U_3 - U_2}}{\left(\sqrt{U_3} + \sqrt{U_3 - U_2}\right)^3}\sqrt{\varepsilon}$$

$$T_{31} = \frac{4\sqrt{U_3}}{\left(\sqrt{U_3} + \sqrt{U_3 - U_2}\right)^2}\sqrt{\varepsilon}$$

In this case $\varepsilon = U_3 + \delta, \delta \ll 1$.

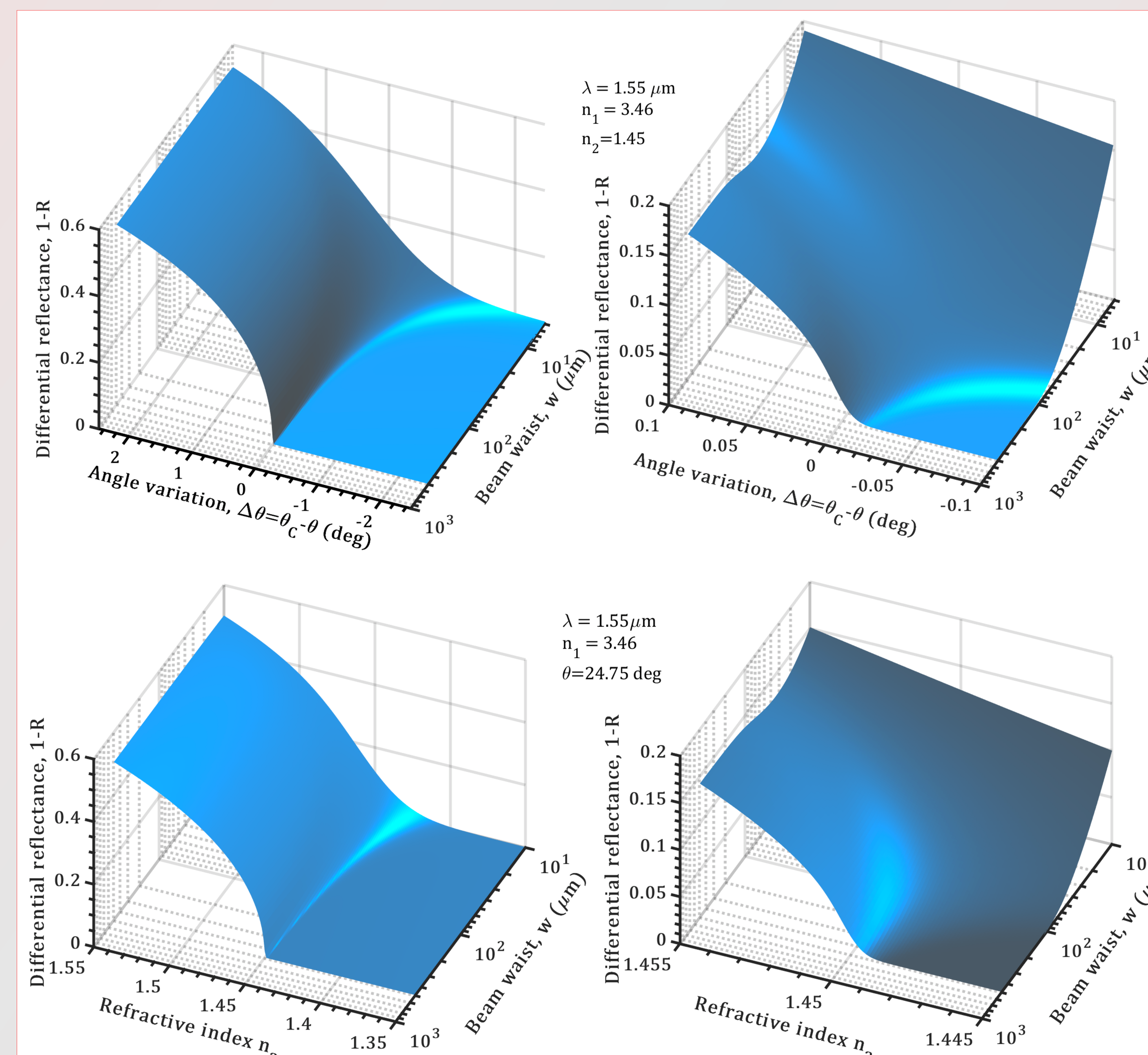


We also studied transmission in a system consisting of three CROW arrays using the tight binding model. In this setup we are also able to sense multiple different observables at once. At the same time, in this system we see multiple double-sided cusps, which open many possibilities for sensing. The double-sided cusp centered at ω= 0.1 Hz is an S-bend cusp and can be especially useful for sensing. This cusp provides a way to sense hyper-sensitively in both the positive and negative direction. Such a feature would be best suited for measurables that may be positive or negative such as acceleration, rotation, etc.



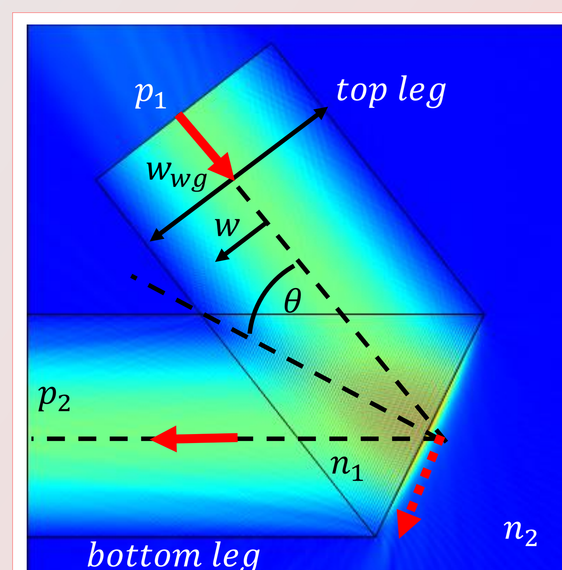
The plot above is the result of the configuration with the following parameters: β_2= -8, β_3= -8, β_4= 0, t_1= -1, t_2= -1, t_3= -1, U_2= 2, U_3= -1.9

Numerical Calculations



The data above is a numerical simulation of a gaussian beam incident on an interface between two dielectric media with refractive indices n_1 and n_2. We observe that differential reflectance behaves as a square root with variations to incident angle and refractive index. We also see a "washing out" of the cusp as the beam waist decreases. The cusps are less distinct for narrower beam waists because beam waist and divergence are inversely related, as given by the relation $D = \frac{2\lambda}{w\pi}$.

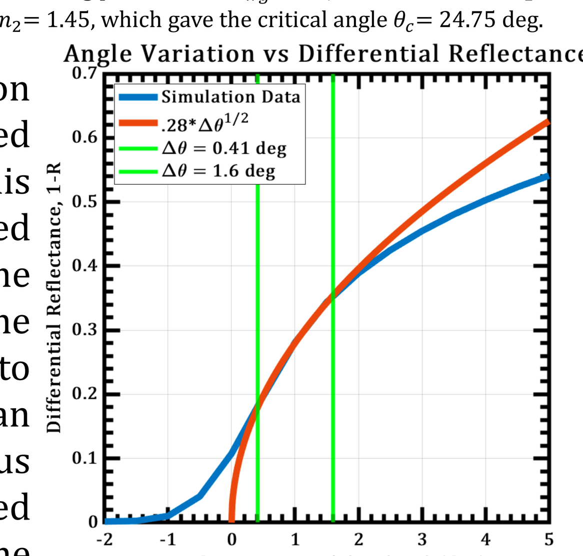
Simulations



Parameters

w – beam waist
w_{wg} – waveguide width
θ – angle of incidence
n_1 – refractive index of the core
n_2 – refractive index of the cladding
p_1 – port 1, point of wave excitation
p_2 – port 2
λ – wavelength

The plot below is the result of the configuration with the following parameters: w_{wg}= 40 μm, λ= 1550 nm, $n_1 = \sqrt{12}$, n_2= 1.45, which gave the critical angle θ_c= 24.75 deg.



The image above displays the silicon multimode waveguide that we used to conduct our simulations. In this image the waveguide is positioned so that the wave is incident at the critical angle. We constructed the simulation in a way that allows us to rotate the top leg so that we can perform calculations for various incident angles. The wave is excited at port 1 and is reflected off the interface where the waveguide meets the surrounding medium. We then measure the reflected wave at port 2. Our numerical calculations show that the cusp in differential reflectance becomes significantly "washed out" for beam waists less than approximately 100 μm. However, we still observe the prevalence of the square root behavior in our simulations with smaller beam waists.

Conclusion

- WCAs provide square-root nonlinear sensitivity
- In contrast to EP sensors, WCA sensors measure the differential transmittance, not the frequency split
- WCA sensors do not suffer from excessive quantum noise effects associated with a broadening of the emission peaks near EP, which masks the sensitivity enhancement
- With WCA one can create multimode systems, which allow for enhanced sensing of multiple independent observables
- WCA sensing allows "directional" sensing with S-bend cusps
- The sharpness of the WCA cusp is typically fundamentally limited by the divergence of the incident beam
- Increasing the beam waist reduces the range where the WCA cusp is "washed out" and sensitivity is reduced
- Now we are working on optimization of the waveguide model which would allow us to experimentally demonstrate the on-chip sensor with an enhanced sensitivity

1. R. Kononchuk, J. Feinberg, J. Knee, T. Kottos. Enhanced avionic sensing based on Wigner's cusp anomalies. *Sci.Adv.* **7**, eabg8118 (2021).
2. Lai, YH., Lu, YK., Suh, MG. *et al.* Observation of the exceptional-point-enhanced Sagnac effect. *Nature* **576**, 65-69 (2019).