Unknown Mean

We need to establish equality between the following two objective function calculation methods, labelled J_{std} for the standard version, and J_{eff} for the efficient version.

$$\begin{aligned} J_{std} &= \frac{1}{2}\mathbf{s}^T\mathbf{G}\mathbf{s} + \frac{1}{2}\left(\left(\mathbf{y}-\mathbf{h}\left(\mathbf{s}\right)\right)^T\mathbf{R}^{-1}\left(\mathbf{y}-\mathbf{h}\left(\mathbf{s}\right)\right)\right) \\ J_{eff} &= \frac{1}{2}\xi^T\left(\mathbf{H}\mathbf{Q}\mathbf{H}^T\right)\xi + \frac{1}{2}\left(\left(\mathbf{y}-\mathbf{h}\left(\mathbf{s}\right)\right)^T\mathbf{R}^{-1}\left(\mathbf{y}-\mathbf{h}\left(\mathbf{s}\right)\right)\right) \end{aligned}$$

For these two formulations to be equivalent, it is only necessary for the following to hold (removing the $\frac{1}{2}$ terms

$$\xi^T\left(\mathbf{H}\mathbf{Q}\mathbf{H}^T\right)\xi = \mathbf{s}^T\mathbf{G}\mathbf{s} \tag{1}$$

Recalling that

$$\mathbf{G} = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Q}^{-1} \tag{2}$$

and

$$\mathbf{s} = \mathbf{X}\beta + \mathbf{Q}\mathbf{H}^T\xi \tag{3}$$

We can expand the right-hand term of Equation 1 incorporating both Equations 2 and 3 into four terms

$$\begin{aligned} \mathbf{s}^T\mathbf{G}\mathbf{s} &= \beta^T\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\beta \\ &+ \xi^T\mathbf{H}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{H}^T\xi \\ &- \beta^T\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\beta \\ &- \xi^T\mathbf{H}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{Q}\mathbf{H}^T\xi \end{aligned} \tag{4}$$

$$\tag{5}$$

Simplifying all inverses, immediately the first and third terms cancel. Furthermore, the second term

$$\xi^T\mathbf{H}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{H}^T\xi = \xi^T\mathbf{H}\mathbf{Q}\mathbf{H}\xi \tag{6}$$

which is our goal from Equation 1.

Finally, then, for Equation 1 to hold, the fourth term from Equation 4 must completely cancel. To evaluate this, we first simplify through inverses resulting in our new goal

$$\xi^T \mathbf{H}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{H}^T\xi = 0 \tag{7}$$

Now, we must expand this using the definition

$$\xi = \mathbf{Q}_{yy}^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{X}\beta\right) \tag{8}$$

where $\mathbf{Q}_{yy}^{-1}{=}\left(\mathbf{HQH^T} + \mathbf{R}\right)^{-1}$ is substituted because early inspection indicates that $\mathbf{Q}_{yy}^{-1}$ cannot be reduced further.

Substituting Equation 8 into Equation 7 is accomplished by first making some temporary definitions, noting the property that for an arbitrary symmetix matrix $\mathbf{A}$, $\left(\mathbf{A}^{-1}\right)^T = \mathbf{A}^{-1}$.

First

$$\begin{aligned}\xi &= \mathbf{Q}_{yy}^{-1}\mathbf{y} - \mathbf{Q}_{yy}^{-1}\mathbf{H}\mathbf{X}\beta \\ \xi^T &= -\beta^T\mathbf{X}^T\mathbf{H}^T\mathbf{Q}_{yy}^{-1} + \mathbf{y}^T\mathbf{Q}_{yy}^{-1}\end{aligned} \tag{9}$$

With these definitions, we can assign

$$\begin{aligned}A &= \mathbf{Q}_{yy}^{-1}\mathbf{y} \\ B &= \mathbf{Q}_{yy}^{-1}\mathbf{H}\mathbf{X}\beta \\ C &= \beta^T\mathbf{X}^T\mathbf{H}^T\mathbf{Q}_{yy}^{-1} \\ D &= \mathbf{y}^T\mathbf{Q}_{yy}^{-1} \\ E &= \mathbf{H}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{H}^T\end{aligned} \tag{10}$$

Returning to Equation 7, substituting the assignations in Equation 10

$$\begin{aligned}\xi^T \mathbf{H}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{H}^T\xi &= \left(-C + D\right)E\left(A - B\right) \\ &= \left(-CE + DE\right)\left(A - B\right) \\ &= -CEA - DEA + DEA + DEB \\ &= DEB - CEA\end{aligned} \tag{11}$$

To conclude, we can expand Equation 11 using Equation 10

$$\begin{aligned} DEB - CEA &= \mathbf{y}^T\mathbf{Q}_{yy}^{-1}\mathbf{H}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{H}^T\mathbf{Q}_{yy}^{-1}\mathbf{H}\mathbf{X}\beta \\ &\quad -\beta^T\mathbf{X}^T\mathbf{H}^T\mathbf{Q}_{yy}^{-1}\mathbf{H}\mathbf{X}\left(\mathbf{X}^T\mathbf{Q}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{H}^T\mathbf{Q}_{yy}^{-1} \\ &= 0 \end{aligned} \tag{12}$$

Equation 12 is correct by virtue of the fact that both terms are equal scalars.

Diffuse Mean

To expand the above simplification of the objective function to cases with diffuse information about the mean, we make the following substitution

$$\mathbf{Q} = \mathbf{Q}_{ss} + \mathbf{X}\mathbf{Q}_{\beta\beta}\mathbf{X}^T. \tag{13}$$

By virtue of this new version of $\mathbf{Q}$ having the same dimensions as in the previous derivation, by direct substitution, the equation for J_{eff} can be used directly as

$$J_{eff} = \frac{1}{2}\xi^T\left(\mathbf{H}\left(\mathbf{Q} + \mathbf{X}\mathbf{Q}_{\beta\beta}\mathbf{X}^T\right)\mathbf{H}^T\right)\xi + \frac{1}{2}\left(\left(\mathbf{y} - \mathbf{h}\left(\mathbf{s}\right)\right)^T\mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{h}\left(\mathbf{s}\right)\right)\right) \tag{14}$$