Ejercicio 1

Si $L(\theta,t) = |t-\theta|$, entonces $\hat{\theta}_B$ es la mediana de $\pi(\theta|x_1,\dots,x_n)$

$$PEP(L(\theta,t)) = E(L(\theta,t)|x_1,\dots,x_n)$$

$$= \int_{\textcircled{H}} |t-\theta|\, \pi(\theta|x_1;\dots;x_n)\, d\theta$$

$$= \int_{\textcircled{H}} (t-\theta)\, I_{(-\infty,t]}(\theta)\, \pi(\theta|x_1;\dots,x_n)\, d\theta$$

$$- \int_{\textcircled{H}} (t-\theta)\, I_{(t,\infty)}(\theta)\, \pi(\theta|x_1,\dots,x_n)\, d\theta$$

$$= \int_{-\infty}^{t} (t-\theta)\, \pi(\theta|x_1,\dots,x_n)\, d\theta - \int_{t}^{\infty} (t-\theta)\, \pi(\theta|x_1,\dots,x_n)\, d\theta$$

$$\frac{d}{dt} PEP(L(\theta,t)) = 1\cdot \underbrace{\cancel{(t-t)}}_{0}\, \pi(t|x_1,\dots,x_n) + \int_{-\infty}^{t} 1\cdot\pi(\theta|x_1\dots x_n)\, d\theta$$

$$+ 1\cdot \underbrace{\cancel{(t-t)}}_{0}\, \pi(t|x_1;\dots,x_n) - \int_{t}^{\infty} 1\cdot \pi(\theta|x_1,\dots,x_n)\, d\theta$$

$$= F(t) - (1-F(t)) = 2F(t) - 1$$

$$2F(t) - 1 = 0 \Rightarrow F(t) = 1/2$$

$$\frac{d^2}{dt^2} PEP(L(\theta,t)) = 2\,\pi(t \mid x_1,\dots,x_n) > 0$$

$\hat{\theta}_B = t$ tal que $F(t) = 1/2$

$\hat{\theta}_B$ es la mediana a posteriori

Ejercicio 2

$(X_1,\dots,X_N)$ m.a.s. de $X \sim B(n,p)$, n conocido.

$$f_\theta(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad p \in (0,1)$$

$$f_\theta(x_1,\dots,x_N) = \prod_{j=1}^{N} \binom{n}{x_j} \cdot p^{\sum_{j=1}^{N} x_j} (1-p)^{nN - \sum_{j=1}^{N} x_j}$$

$\pi(p) \sim \text{Beta}(a,b)$

$$\pi(p) = \frac{1}{B(a,b)} p^{a-1} (1-p)^{b-1}$$

$$\pi(p \mid x_1,\dots,x_N) \propto p^{\sum_{j=1}^{N} x_j} (1-p)^{nN - \sum_{j=1}^{N} x_j} \cdot p^{a-1} (1-p)^{b-1}$$
$$= p^{\sum_{j=1}^{N} x_j + a - 1} (1-p)^{nN - \sum_{j=1}^{N} x_j + b - 1}$$

$$\pi(p \mid x_1,\dots,x_N) \sim \text{Beta}\left(\sum_{j=1}^{N} x_j + a,\; nN - \sum_{j=1}^{N} x_j + b\right)$$

$$E(p|x_1,\dots,x_N) = \frac{\sum_{j=1}^{N} x_j + a}{\sum_{j=1}^{N} x_j + a + nN - \sum_{j=1}^{N} x_j + b}$$

$$= \frac{\sum_{j=1}^{N} x_j + a}{a + nN + b}$$

Si $a = b = 1 \rightarrow \pi(p)$ es no informativa y

$$E(p|x_1,\dots,x_N) = \frac{\sum_{j=1}^{N} x_j + 1}{nN + 2}$$

El estimador de máxima verosimilitud es

$$\hat{p}_{MV} = \frac{\sum_{j=1}^{N} x_j}{nN}$$