I. E. S. " SAN ISIDRO "

Asignatura.................................... Fecha ....

Alumno/a.................................... Curso.............. Nº..........

Apellidos Nombre

Calificación

10.- Demostrar las siguientes igualdades:

a) $\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln\left|2\operatorname{sen}\frac{\theta}{2}\right| \quad (0 < |\theta| \leq \pi)$

c) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\theta}{n} = -\ln\left(2\cos\frac{\theta}{2}\right) \quad (0 < \theta < \pi)$

b) $\sum_{n=1}^{\infty} \frac{\operatorname{sen}(n\theta)}{n} = \frac{\pi - \theta}{2} \quad (0 < \theta < 2\pi)$

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\operatorname{sen}(n\theta)}{n} = \frac{\theta}{2} \quad (-\pi < \theta < \pi)$.

Para sumar estas series vamos a hacer uso de las series de Fourier. Si tenemos una función $f(t)$ de variable real t integrable en un cierto intervalo $[t_0 - T/2, t_0 + T/2]$ podemos expresar el valor de la función en ese intervalo como $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \operatorname{sen}\left(\frac{2n\pi t}{T}\right)$

donde $a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\operatorname{sen}\left(\frac{2n\pi}{T}t\right)dt.$$

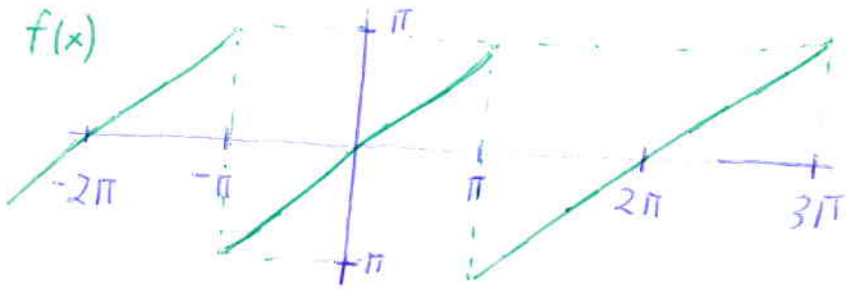
Empezamos por el apartado d).

Consideramos la función $\hat{f}(x) = x$ con $x \in (-\pi, \pi)$

La extendemos de manera periódica en $\mathbb{R}$ de tal forma que

$f(x) = x$ con $x \in (2n\pi - \pi, 2n\pi + \pi)$. Por tanto

$f(x)$ es una función periódica de periodo $T = 2\pi$ ...



Se puede comprobar que x es una función impar

$f(-x) = -x$ $\quad -x \in (2n\pi - \pi, 2n\pi + \pi) \Leftrightarrow x \in (-2n\pi - \pi, -2n\pi + \pi)$

$-f(x) = -x$ $\quad x \in (2n'\pi - \pi, 2n'\pi + \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \underset{\uparrow}{=} 0$$

$g(x) = x\cos(nx)$ es impar

$x = u \quad dx = du$

$\operatorname{sen}(nx)dx = dv \quad -\frac{\cos(nx)}{n} = v$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\operatorname{sen}(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\operatorname{sen}(nx)\,dx =$$

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$$= \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n}\Big]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right) = \frac{2}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \cancel{\frac{\operatorname{sen}(nx)}{n^2}}\Big]_0^{\pi}\right)$$

$$= -\frac{2}{n}\cos(n\pi) \underset{\uparrow}{=} -\frac{2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}$$

$$\cos(n\pi) = \left\{\begin{array}{l} 1 \text{ si } n \text{ par} \\ -1 \text{ si } n \text{ impar}\end{array}\right\} = (-1)^n$$

Por tanto en $x \in (-\pi, \pi)$

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\operatorname{sen}(nx) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\operatorname{sen}(nx)$$

$$\Longleftrightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\operatorname{sen}(nx) = \frac{x}{2} \quad \text{con } x \in (-\pi, \pi)$$

Para el apartado b) consideramos la función

$\hat{f}(x) = \pi - x$ con $0 < x = 2\pi$.

La extendemos en $\mathbb{R}$ de manera periódica para que

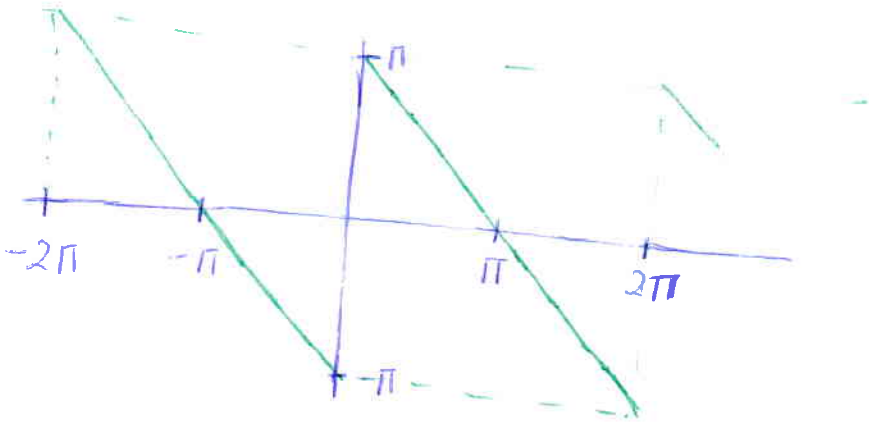
$f(x) = \pi - x$ con $x \in (2n\pi, 2\pi + 2n\pi)$. Esta es una función periódica de periodo 2π. Veamos que es impar

$f(-x) = \pi + x \qquad \text{con} \quad -x \in (2n\pi, 2n\pi + 2\pi)$

$-f(x) = -\pi - x \qquad \text{con} \quad x \in (2n'\pi, 2n'\pi + 2\pi)$

Como f es periódica de periodo 2π

$-f(x) = -f(x+2\pi) = -\pi + (x+2\pi) = -\pi + x + 2\pi = \pi + x = f(-x)$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \underset{\substack{\uparrow \\ f \text{ impar}}}{=} 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \underset{\substack{\uparrow \\ g(x) = f(x)\cos(nx) \text{ es impar}}}{=} 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}(nx)\,dx \overset{\downarrow}{=} \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)\,\text{sen}(nx)\,dx =$$

$$= \frac{2}{\pi}\pi\int_0^{\pi} \text{sen}(nx)\,dx - \frac{2}{\pi}\int_0^{\pi} x\,\text{sen}(nx)\,dx = \left[\begin{matrix} x = u & dx = du \\ \text{sen}(nx)dx = dv & -\frac{\cos(nx)}{n} = v \end{matrix}\right]$$

$$= 2\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} - \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\Bigg) =$$

$$= \frac{2}{n}\left(1-\cos(n\pi)\right) - \frac{2}{\pi}\left(-\frac{\pi\cos(n\pi)}{n} + 0 + \cancel{\left[\frac{\operatorname{sen}(nx)}{n^2}\right]_0^{\pi}}^{0}\right) =$$

$$= \frac{2}{n} - \frac{2\cos(n\pi)}{n} + \frac{2\cos(n\pi)}{n} = \frac{2}{n}$$

Por tanto en $x\in(0,2\pi)$

$$\pi - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\operatorname{sen}(nx) =$$

$$= 0 + \sum_{n=1}^{\infty} 0 + \frac{2}{n}\operatorname{sen}(nx) = 2\sum_{n=1}^{\infty}\frac{\operatorname{sen}(nx)}{n}$$

$$\Longrightarrow \frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{\operatorname{sen}(nx)}{n} \quad \text{si } x\in(0,2\theta)$$

Para el apartado a) hacemos lo propio

Sea $f(x)$ la extension periódica de $-\ln\left|2\operatorname{sen}\frac{\theta}{2}\right|$ en $-\pi<\theta<\pi$, salvo $\theta=0$.

Esta función es par ya que

$$f(-x) = -\ln\left|2\operatorname{sen}\left(\frac{-x}{2}\right)\right| = -\ln\left|-2\operatorname{sen}\left(\frac{x}{2}\right)\right| = -\ln\left|2\operatorname{sen}\left(\frac{x}{2}\right)\right| =$$

$$= f(x)$$

Entonces

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \underset{\substack{\uparrow \\ f \text{ función par}}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \; ;$$

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} -\ln\left|2\operatorname{sen}\frac{x}{2}\right| dx = \int_0^{\pi} -\ln\left(2\operatorname{sen}\frac{x}{2}\right) dx =$$

$$-\int_0^{\pi} \ln 2\,dx - \int_0^{\pi} \ln\left(\operatorname{sen}\frac{x}{2}\right)dx \; ;$$

$$\int_0^{\pi} \ln\left(\operatorname{sen}\frac{x}{2}\right) dx \underset{\uparrow}{=} 2\int_0^{\pi/2} \ln(\operatorname{sen} u)\,du \underset{\uparrow}{=} -\pi \ln 2$$

$\frac{x}{2} = u$

$\frac{dx}{2} = du$

$x = \pi \Rightarrow u = \pi/2$

$x = 0 \Rightarrow u = 0$

Nos lo creemos de momento y luego lo probamos $*_1$

$$\Longrightarrow \int_0^{\pi} f(x)\,dx = -\pi \ln 2 - (-\pi \ln 2) = 0 \quad \Longrightarrow a_0 = 0$$

Como la función es par todos los términos b_n que multiplican a los $\operatorname{sen}(nx)$ serán 0 ya que

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(nx)\,dx \quad \text{y} \quad g(x) = f(x)\operatorname{sen}(nx) \text{ es impar}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \underset{\substack{\downarrow \\ g(x) \text{ es impar.}}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} -\ln 2 \cos nx\,dx + \frac{2}{\pi}\int_0^{\pi} h(x)\cdot\cos nx\,dx$$

Nótese que $\int_0^{\pi} \ln 2 \cdot \cos(nx)dx = \ln 2 \cdot \left.\frac{\operatorname{sen}(nx)}{n}\right]_0^{\pi} = 0$

$$\int_0^{\pi} h(x)\cos(nx)dx = -\int_0^{\pi} \ln(\operatorname{sen}(\tfrac{x}{2}))\cos(nx)\,dx \underset{\uparrow}{=}$$

$$\left[\begin{array}{ll} \ln(\operatorname{sen}\frac{x}{2}) = u & \cos(nx)dx = dv \\ \frac{1}{2}\frac{\cos\frac{x}{2}}{\operatorname{sen}\frac{x}{2}}dx = du & \frac{\operatorname{sen}(nx)}{n} = v \end{array}\right.$$

$$= -\left.\frac{\ln(\operatorname{sen}\frac{x}{2})\cdot\operatorname{sen}(nx)}{2n}\right]_0^{\pi} + \frac{1}{2n}\int_0^{\pi} \operatorname{cotg}(\tfrac{x}{2})\cdot\operatorname{sen}(nx)\,dx =$$

$$= 0 + \frac{1}{2n}\lim_{x\to 0^+} \ln(\operatorname{sen}\tfrac{x}{2})\cdot\operatorname{sen}(nx) + \frac{1}{2n}\int_0^{\pi} \operatorname{cotg}(\tfrac{x}{2})\cdot\operatorname{sen}(nx)\,dx =$$

$$= 0 + \frac{\pi}{2n}$$

(Queda demostrar que $\ln(\operatorname{sen}\frac{x}{2})\cdot\operatorname{sen}(nx) \xrightarrow[x\to 0^+]{} 0$ $*_2$

y $\int_0^{\pi} \operatorname{cotg}(\frac{x}{2})\cdot\operatorname{sen}(nx)\,dx = \pi$ $*_3$)

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0 + \frac{2}{\pi}\cdot\frac{\pi}{2n} = \frac{1}{n}$$

Por tanto $-\ln\left|2\operatorname{sen}\frac{x}{2}\right| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\operatorname{sen}(nx) =$

$$= 0 + \sum_{n=1}^{\infty}\frac{1}{n}\cos(nx) \iff \sum_{n=1}^{\infty}\cos(nx) = -\ln\left|2\operatorname{sen}\frac{x}{2}\right| \quad 0<|x|\leq\pi$$

$*1 \quad \int_0^{\pi/2} Ln(\text{sen } u)\,du = -\frac{\pi}{2} Ln 2$

Sea $\quad I = \int_0^{\pi/2} Ln(\text{sen } u)\,du \underset{\uparrow}{=} \int_0^{\pi/2} Ln\left(\cos\left(\frac{\pi}{2}-u\right)\right) du \underset{\uparrow}{=}$

$\text{sen } u = \cos\left(\frac{\pi}{2}-u\right)$

$\frac{\pi}{2} - u = x$

$-du = dx$

$u = \pi/2 \Rightarrow x = 0$

$u = 0 \Rightarrow x = \pi/2$

$= \int_{\pi/2}^{0} -Ln(\cos x)\,dx = \int_0^{\pi/2} Ln(\cos x)\,dx$

$\Rightarrow 2I = I + I = \int_0^{\pi/2} Ln(\text{sen } x)\,dx + \int_0^{\pi/2} Ln(\cos x)\,dx =$

$= \int_0^{\pi/2} Ln(\text{sen}x \cos x)\,dx = \int_0^{\pi/2} Ln\left(\frac{1}{2}\text{sen } 2x\right) dx =$

$= \int_0^{\pi/2} Ln\left(\frac{1}{2}\right) dx + \int_0^{\pi/2} Ln(\text{sen } 2x)\,dx = \left[\begin{array}{l} 2x = u \\ 2dx = du \\ x = 0 \Rightarrow u = 0 \\ x = \pi/2 \Rightarrow u = \pi \end{array}\right]$

$= -Ln 2 \cdot \frac{\pi}{2} + \frac{1}{2}\int_0^{\pi} Ln(\text{sen } u)\,du \underset{\uparrow}{=}$

$Ln(\text{sen } u)$ es simétrica respecto a $\pi/2$

$= -Ln 2 \frac{\pi}{2} + \frac{2}{2}\int_0^{\pi/2} Ln(\text{sen } u)\,du = -Ln 2 \cdot \frac{\pi}{2} + I$

$\Rightarrow 2I = -Ln 2 \cdot \frac{\pi}{2} + I \iff \boxed{I = -Ln 2 \cdot \frac{\pi}{2}}$

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$*_2$ $\lim\limits_{x \to} Ln(sen\frac{x}{2})\cdot sen(nx) = 0$

$$\lim_{x\to 0^+} Ln(sen\tfrac{x}{2})\cdot sen(nx) = \lim_{x\to 0^+} \frac{Ln(sen\frac{x}{2})}{\frac{1}{sen(nx)}} \underset{=}{\overset{\frac{\infty}{\infty}}{\downarrow}}$$

$\frac{\infty}{\infty}$ IND; Aplicamos la Regla de L'Hôpital

$$= \lim_{x\to 0^+} \frac{\frac{\cos\frac{x}{2}}{sen\frac{x}{2}}\cdot\frac{1}{2}}{-\frac{n}{sen^2(nx)}} = \lim_{x\to 0^+} \frac{1}{2n}\cdot\frac{sen^2(nx)\cdot\cos\frac{x}{2}}{sen\frac{x}{2}} \underset{=}{\overset{\frac{0}{0}}{\downarrow}}$$

$\frac{0}{0}$ IND; Aplicamos la Regla de L'Hôpital.

$$= \lim_{x\to 0^+} \frac{1}{2n}\cdot\frac{2n\,sen(nx)\cos(nx)\cdot\cos(\frac{x}{2}) + sen^2(nx)\cdot(-sen\frac{x}{2})\cdot\frac{1}{2}}{\cos(\frac{x}{2})\cdot\frac{1}{2}} = 0$$

$*_3$ $\displaystyle\int_0^{\pi} cotg(\tfrac{x}{2})\cdot sen(nx)\,dx = \pi$

Para probar esto hay que ver que $1+2\sum_{k=1}^{n}\cos(kx) = \frac{sen((n+\frac{1}{2})x)}{sen\frac{x}{2}}$ $*_4$

Asumiendo que esto es cierto

$$1+2\sum_{k=1}^{n}\cos(kx) = \frac{sen((n+\frac{1}{2})x)}{sen\,x/2} = \frac{1}{sen\,x/2}\cdot\left(sen(nx)\cos\tfrac{x}{2} + \cos(nx)\,sen\tfrac{x}{2}\right) =$$

$$= sen(nx)\,cotg\tfrac{x}{2} + \cos(nx)$$

$$\Rightarrow cotg(\tfrac{x}{2})\,sen(nx) = 1+2\sum_{k=1}^{n-1}\cos(kx) + \cos(nx).$$

Si integramos en ambos lados:

$$\int_0^{\pi} \cotg\left(\frac{x}{2}\right)\cdot \operatorname{sen}(nx)\,dx = \int_0^{\pi}\left(1+2\sum_{k=1}^{n-1}\cos kx + \cos nx\right)dx =$$

$$= \int_0^{\pi} dx + 2\sum_{k=1}^{n-1}\int_0^{\pi}\cos kx\,dx + \int_0^{\pi}\cos nx\,dx = \pi$$ ya que

la integral de todos los cosenos es 0 porque $\int_0^{\pi}\cos kx\,dx = \left.\frac{\operatorname{sen} kx}{k}\right]_0^{\pi} = 0$

*4

$$1+2\sum_{k=1}^{n}\cos kx = 1+2\cdot\frac{1}{\operatorname{sen}\frac{x}{2}}\sum_{k=1}^{n}\operatorname{sen}\frac{x}{2}\cdot\cos kx =$$

$$= 1+\frac{1}{\operatorname{sen}\frac{x}{2}}\sum_{k=1}^{n}2\operatorname{sen}\frac{x}{2}\cos kx =$$

$$= 1+\frac{1}{\operatorname{sen}\frac{x}{2}}\sum_{k=1}^{n}\left(\operatorname{sen}\left(kx+\frac{x}{2}\right)-\operatorname{sen}\left(kx-\frac{x}{2}\right)\right) \overset{=}{1}$$ Suma telescópica

$$= 1+\frac{1}{\operatorname{sen}\frac{x}{2}}\left(\operatorname{sen}\left(nx+\frac{x}{2}\right)-\operatorname{sen}\frac{x}{2}\right) = 1+\frac{\operatorname{sen}\left(nx+\frac{x}{2}\right)}{\operatorname{sen}\frac{x}{2}} - 1 =$$

$$= \frac{\operatorname{sen}\left(\frac{n+1}{2}x\right)}{\operatorname{sen}\frac{x}{2}}$$

Bibliografía

*1 Video youtube : Improper Integral of ln(sinx) from 0 to pi/2 : MIT Integration Bee (4)

*3 math.stackexchange.com

*4 math.stackexchange.com