Hoja 3:-

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Ej 2:- Encontrar la familia conjugada natural y hallar la distribución a posteriori:

a) $(X_1 \dots X_n)$ m.a.s. $X \sim Poisson(\theta)$ $\theta > 0$

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad x = 0, 1, 2 \dots$$

$$f(x_1 \dots x_n|\theta) = \prod_{i=1}^n \frac{e^{-\theta}\theta^x}{x!} = e^{-n\theta}\theta^{\sum_{i=1}^n x_i} \cdot \frac{1}{\prod_{i=1}^n x_i!}$$

Veamos que si: $\pi(\theta) \sim Gamma(a,p) \Rightarrow \pi(\theta|x_1 \dots x_n) \sim Gamma(a_1, p_1)$ con a_1, p_1 por determinar.

$$\pi(\theta) = \frac{a^p}{\Gamma(p)} e^{-a\theta}\theta^{p-1} \qquad \text{con } a,p > 0 \quad \theta > 0$$

$$\pi(\theta|x_1 \dots x_n) = \frac{\pi(\theta) \cdot f(x_1 \dots x_n|\theta)}{\int_0^\infty \pi(\theta) \cdot f(x_1 \dots x_n|\theta)\, d\theta} = \frac{\cancel{\frac{a^p}{\Gamma(p)}} e^{-a\theta}\theta^{p-1} e^{-n\theta}\theta^{\sum_{i=1}^n x_i} \cancel{\frac{1}{\prod_{i=1}^n x_i!}}}{\int_0^\infty \cancel{\frac{a^p}{\Gamma(p)}} e^{-a\theta}\theta^{p-1} e^{-n\theta}\theta^{\sum x_i} \cancel{\frac{1}{\prod_{i=1}^n x_i!}}\, d\theta} =$$

$$= \frac{e^{-\theta(a+n)}\theta^{p+\sum_{i=1}^n x_i - 1}}{\frac{\Gamma(p+\sum x_i)}{(a+n)^{p+\sum x_i}} \underbrace{\int_0^\infty \frac{(a+n)^{p+\sum x_i}}{\Gamma(p+\sum x_i)} e^{-\theta(a+n)}\theta^{p+\sum x_i - 1}\, d\theta}_{\text{función de densidad de una v.a. } X \sim Gamma(a+n,\, p+\sum x_i)}} = \frac{(a+n)^{p+\sum x_i}}{\Gamma(p+\sum x_i)} e^{-\theta(a+n)}\theta^{p+\sum_{i=1}^n x_i - 1}$$

$$\Rightarrow \pi(\theta|x_1 \dots x_n) \sim Gamma(a+n,\, p+\sum_{i=1}^n x_i)$$

b) $(X_1 \dots X_n)$ m.a.s. $X \sim Gamma(\theta, 1) = Exp(\theta)$

$$f(x|\theta) = \theta e^{-x\theta} \qquad \text{con } x > 0, \ \theta > 0$$

Veamos que la familia conjugada natural es la Gamma.

$$\pi(\theta) = \frac{a^p}{\Gamma(p)} e^{-a\theta}\theta^{p-1} \qquad \text{con } \theta > 0, \ a,p > 0.$$

$$f(x_1 \dots x_n|\theta) = \prod_{i=1}^n \theta e^{-x\theta} = \theta^n e^{-\theta\sum_{i=1}^n x_i}$$