$$\frac{1}{2} = F(t) = \int_{X_{(n)}}^{t} \frac{1-2n}{\theta^{2n}} \cdot \frac{1}{1-X_{(n)}^{1-2n}}\, d\theta = \frac{1-2n}{1-X_{(n)}^{1-2n}} \int_{X_{(n)}}^{t} \theta^{-2n}\, d\theta =$$

$$= \frac{1-2n}{1-X_{(n)}^{1-2n}} \cdot \left. \frac{\theta^{1-2n}}{\cancel{1-2n}} \right]_{X_{(n)}}^{t} = \frac{1}{1-X_{(n)}^{1-2n}} \cdot \left(t^{1-2n} - X_{(n)}^{1-2n} \right)$$

$$\Rightarrow t^{1-2n} = \frac{1-X_{(n)}^{1-2n}}{2} + X_{(n)}^{1-2n} = \frac{1+X_{(n)}^{1-2n}}{2} \Leftrightarrow t = \left(\frac{1+X_{(n)}^{1-2n}}{2} \right)^{\frac{1}{1-2n}}$$

Ej 4.- $(X_1 - - X_n)$ m.a.s. don $X \sim f(x|\theta) = \theta x^{\theta-1}$ $\quad x \in (0,1),\ \theta > 0$

Hallar la familia conjugada natural y la distribución a posteriori.

$$f(x_1 -- x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta\, x_i^{\theta-1}\, I_{(0,1)}(x_i) = \theta^n \left(\prod_{i=1}^{n} x_i \right)^{\theta-1} \cdot I_{(0,\infty)}(x_{(1)}) \cdot I_{(-\infty,1)}(x_{(n)})$$

$$= \theta^n \exp\left\{ (\theta-1) \operatorname{Ln}\left(\prod_{i=1}^{n} x_i \right) \right\} I_{(0,\infty)}(x_{(1)}) \cdot I_{(-\infty,1)}(x_{(n)}) =$$

$$= \theta^n \exp\left\{ \theta \sum_{i=1}^{n} \operatorname{Ln}(x_i) - \operatorname{Ln}\left(\prod_{i=1}^{n} x_i \right) \right\} I_{(0,\infty)}(x_{(1)}) \cdot I_{(-\infty,1)}(x_{(n)}) =$$

$$= \theta^n \cdot \exp\left\{ \theta \sum_{i=1}^{n} \operatorname{Ln}(x_i) \right\} \cdot \exp\left\{ -\operatorname{Ln}\left(\prod_{i=1}^{n} x_i \right) \right\} I_{(0,\infty)}(x_{(1)})\, I_{(-\infty,1)}(x_{(n)})$$

Parece razonable aventurarse a afirmar que la familia conjudada natural es la Gamma. Comprobémoslo.

Supongamos que $\pi(\theta) \sim$ Gamma (a,p) con $a, p > 0$ conocidos.

$$\pi(\theta|x_1 -- x_n) = \frac{\pi(\theta)\, f(x_1 -- x_n|\theta)}{\int_0^\infty \pi(\theta)\, f(x_1 -- x_n|\theta)\, dt}$$

Realizamos de nuevo el mismo comentario que en el ejercicio anterior. Si $(x_{(1)}, x_{(n)}) \notin (0,1)$ no tiene sentido esta construcción.

$$\pi(\theta|x_1 -- x_n) = \frac{\cancel{\frac{a^p}{\Gamma(p)}}\, e^{-a\theta} \theta^{p-1} \cdot \theta^n e^{\theta \sum \operatorname{Ln}(x_i)} \cdot \cancel{e^{-\operatorname{Ln}(\prod x_i)}} \cdot \cancel{I_{(0,\infty)}(x_{(1)}) \cdot I_{(-\infty,1)}(x_{(n)})}}{\int_0^\infty \cancel{\frac{a^p}{\Gamma(p)}}\, e^{-a\theta} \theta^{p-1} \theta^n e^{\theta \sum \operatorname{Ln} x_i}\, \cancel{e^{-\operatorname{Ln} \prod x_i}}\, \cancel{I_{(0,\infty)}(x_{(1)}) \cdot I_{(-\infty,1)}(x_{(n)})}\, d\theta} =$$