Ejercicio 3.- Sea $(X_1 - - X_n)$ m.a.s. Demostrar que

$S^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$ es un estimador insesgado para estimar la varianza poblacional.

Vamos a calcular $E[S^2]$.

$$E[S^2] = E\left[\frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})^2\right] = \frac{1}{n-1} \sum_{j=1}^{n} E[(X_j - \bar{X})^2].$$

$$E[(X_j - \bar{X})^2] = E[X_j^2 - 2X_j \bar{X} + \bar{X}^2] = E[X_j^2] - 2E[X_j \bar{X}] + E[\bar{X}^2] =$$

$$= Var(X_j) + E[X_j]^2 - 2E[X_j \bar{X}] + Var(\bar{X}) + E[\bar{X}]^2 =$$

$$= Var(X) + E[X]^2 + \frac{Var(X)}{n} + E[X]^2 - \frac{2}{n} E\left[X_j \sum_{i=1}^{n} X_i\right].$$

Ahora $E\left[X_j \sum_{i=1}^{n} X_i\right] = E\left[X_j^2 + \sum_{\substack{i \neq j \\ i=1}}^{n} X_i X_j\right] = E[X_j^2] + \sum_{\substack{i \neq j \\ i=1}}^{n} E[X_i X_j] =$

$$= Var(X_j) + E[X_j]^2 + \sum_{\substack{i \neq j \\ i=1}}^{n} E[X_i] E[X_j] = Var(X) + E(X)^2 + (n-1) E[X]^2 =$$

X_i y X_j indep.

$$= Var(X) + n E[X]^2$$

Sustituyendo

$$E[(X_j - \bar{X})^2] = Var(X) + E[X]^2 + \frac{Var(X)}{n} + E[X]^2 - \frac{2}{n}\left(Var(X) + n E[X]^2\right) =$$

$$= Var(X)\left(1 + \frac{1}{n} - \frac{2}{n}\right) + E[X]^2\left(1 + 1 - \frac{2}{n} \cdot n\right) = Var(X) \frac{n-1}{n}$$

Y volviendo al principio

$$E[S^2] = \frac{1}{n-1} \sum_{j=1}^{n} E[(X_j - \bar{X})^2] = \frac{1}{n-1} \sum_{i=1}^{n} Var(X) \frac{n-1}{n} = Var(X),$$ es decir,

la varianza poblacional. Por tanto el sesgo de S^2 para estimar $Var(X)$ es $b_\theta(S^2) = E[S^2] - Var(X) = Var(X) - Var(X) = 0$, es decir, S^2 es insesgado.