$$= \frac{1}{a-b} \cdot f(a) \cdot 2\pi i \,\mathrm{Ind}(\gamma; a) + \frac{1}{b-a} f(b) 2\pi i \,\mathrm{Ind}(\gamma; b) =$$

$$= \frac{f(a)-f(b)}{a-b} \cdot 2\pi i$$

Si f está acotada tomando $b=0$

$$f(a) - f(0) = \frac{a}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-a)z}\, dz$$

$$\Rightarrow \left| f(a) - f(0) \right| = \frac{|a|}{|2\pi i|} \left| \int_{|z|=R} \frac{f(z)}{(z-a)|z|}\, dz \right| \leq \frac{|a|}{2\pi} \int_{|z|=R} \frac{|f(z)|}{|z-a||z|}\, dz$$

$$\leq \frac{|a| \cdot M}{2\pi} \cdot \int_{|z|=R} \frac{1}{|z-a| \cdot R}\, dz \leq \frac{|a|\, M}{2\pi\, R} \int_{|z|=R} \frac{1}{|z|-|a|}\, dz =$$

$$= \frac{|a| M}{2\pi\, R} \cdot \frac{1}{R-|a|} \int_{|z|=R} 1\, dz = 0 \;\rightarrow\; f(a) = f(0) \quad \forall a,\ |a|<R \text{ y } \forall R$$

$\boxed{5.-}$ Desarrolla en serie de Taylor

$\boxed{a)}$ e^z en $z_0 = 1$

$f(z) = e^z$ holomorfa en $D(1;R)$ $\forall R>0$.

$f^{n)}(z) = e^z$ $\forall n \in \mathbb{N}$.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{n)}(1)}{n!} (z-1)^n \Leftrightarrow e^z = \sum_{n=0}^{\infty} \frac{e}{n!} (z-1)^n \quad \forall z \in D(1,R)\ \forall R>0.$$

$\boxed{b)}$ $(3z+1)^{-1}$ en $z_0 = -2$.

$$f(z) = \frac{1}{3z+1} \quad \text{holomorfa en } D(-2; |-2+\tfrac{1}{3}|) = D(-2; \tfrac{5}{3})$$

$$f'(z) = -(3z+1)^{-2} \cdot 3 = -3(3z+1)^{-2}$$

$$f^{2)}(z) = 2 \cdot 3 \cdot (3z+1)^{-3} \cdot 3 = 2! \cdot 3^2 \cdot (3z+1)^{-3}$$

$$f^{3)}(z) = -3! \cdot (3z+1)^{-4} \cdot 3^3$$

Veamos que $f^{n)}(z) = (-1)^n \cdot n! \cdot 3^n \cdot (3z+1)^{-n-1}$

Los casos base ya están.

El paso inductivo. Supuesto cierto para k.

$$f^{k+1)}(z) = (f^{k)}(z))' = ((-1)^k \cdot k! \cdot 3^k (3z+1)^{-k-1})' =$$

$$= (-1)^k k! \cdot 3^k \cdot (3z+1)^{-k-2} \cdot (-k-1) \cdot 3 = (-1)^{k+1} (k+1)! \cdot 3^{k+1} \cdot (3z+1)^{-k-2}.$$

$$\Rightarrow f(z) = \frac{1}{3z+1} = \sum_{n=0}^{\infty} \frac{f^{n)}(-2)}{n!} (z+2)^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}\, 3^n \cdot (-5)^{-n-1}}{\cancel{n!}} (z+2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n}{(-5)^{n+1}} (z+2)^n =$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{(-1) \cdot 3}{-5}\right)^n \cdot (z+2)^n = -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n (z+2)^n \quad \forall z \in D(-2, \tfrac{5}{3})$$

$\boxed{c)}$ $\cos^2\left(\frac{iz}{2}\right)$ en $z_0 = 0$

$f(z) = \cos^2\left(\frac{iz}{2}\right)$ es holomorfa en $D(0; R)$ $\forall R > 0$.

$$\cos^2\theta + \operatorname{sen}^2\theta = \left(\frac{e^{it} + e^{-it}}{2}\right)^2 + \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 = \frac{e^{2it} + e^{-2it} + 2 - e^{2it} - e^{-2it} + 2}{4} = 1$$

$$\cos^2\theta - \operatorname{sen}^2\theta = \left(\frac{e^{it} + e^{-it}}{2}\right)^2 - \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 = \frac{e^{2it} + e^{-2it} + 2 + e^{2it} + e^{-2it} - 2}{4} =$$

$$= \frac{e^{2it} + e^{-2it}}{2} = \cos 2\theta$$

I. E. S. " SAN ISIDRO "

Asignatura.. Fecha ....

Alumno/a .... Curso.................. Nº..........

Apellidos Nombre

Calificación

$$\Rightarrow 2\cos^2\theta = 1 + \cos 2\theta \Leftrightarrow \cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

$$\Rightarrow f(z) = \cos^2\left(\frac{iz}{2}\right) = \frac{1}{2} + \frac{\cos\left(2\cdot\frac{iz}{2}\right)}{2} = \frac{1}{2} + \frac{\cos(iz)}{2} = \frac{1}{2} + \frac{\cosh(z)}{2}$$

$$f'(z) = \frac{\operatorname{senh} z}{2}$$

$$f^{2)}(z) = \frac{\cosh z}{2}$$

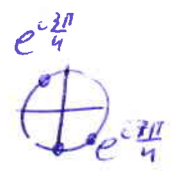
$$f^{3)}(z) = \frac{\operatorname{senh} z}{2}$$

$$\Rightarrow f^{n)}(z) = \begin{cases} \dfrac{\operatorname{senh} z}{2} & \text{si } n \text{ impar} \\ \dfrac{\cosh z}{2} & \text{si } n \text{ par.} \end{cases} \quad ; \quad f^{n)}(0) = \begin{cases} \dfrac{\operatorname{senh}(0)}{2} = 0 & \text{si } n \text{ impar} \\ \dfrac{\cosh(0)}{2} = \dfrac{1}{2} & \text{si } n \text{ par.} \end{cases}$$

$$\Rightarrow f(z) = \cos^2\left(\frac{iz}{2}\right) = \sum_{n=0}^{\infty} \frac{f^{n)}(0)}{n!}(z-0)^n = 1 + \sum_{\substack{n=1 \\ n \text{ par}}}^{\infty} \frac{f^{n)}(0)}{n!} z^n + \underbrace{\cancel{\sum_{\substack{n=1 \\ n \text{ impar}}}^{\infty} \frac{f^{n)}(0)}{n!} z^n}}_{} \, 0$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{2n}}{2n!}$$

7.- Desarrolla en serie de potencias las siguientes funciones y halla el radio de convergencia



a) $f(z) = \dfrac{z}{i+z^2}$ $\quad z_0 = 0$ y $z_0 = 1$. $\qquad z^2 = -i$

$f(z) = \dfrac{z}{i+z^2}$ es holomorfa en el disco $D(0,1)$ y en $D(1, |1-e^{i\frac{7\pi}{4}}|)$

$\shortparallel$

$D(1, \sqrt{2-\sqrt{2}})$

$$\frac{z}{i+z^2} = \frac{z}{(z-e^{i\frac{3\pi}{4}})(z-e^{i\frac{7\pi}{4}})} = \frac{A}{z-e^{i3\pi/4}} + \frac{B}{z-e^{i\frac{7\pi}{4}}}$$

$$\Updownarrow$$

$$z = A(z-e^{i\frac{7\pi}{4}}) + B(z-e^{i3\pi/4})$$