$$\Rightarrow 2\cos^2\theta = 1 + \cos 2\theta \Leftrightarrow \cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

$$\Rightarrow f(z) = \cos^2\left(\frac{iz}{2}\right) = \frac{1}{2} + \frac{\cos\left(2\cdot\frac{iz}{2}\right)}{2} = \frac{1}{2} + \frac{\cos(iz)}{2} = \frac{1}{2} + \frac{\cosh(z)}{2}$$

$$f'(z) = \frac{\operatorname{senh} z}{2}$$

$$f^{2)}(z) = \frac{\cosh z}{2}$$

$$f^{3)}(z) = \frac{\operatorname{senh} z}{2}$$

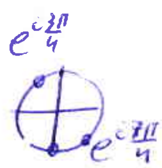
$$\Rightarrow f^{n)}(z) = \begin{cases} \frac{\operatorname{senh} z}{2} & \text{si } n \text{ impar} \\ \frac{\cosh z}{2} & \text{si } n \text{ par.} \end{cases} ; \quad f^{n)}(0) = \begin{cases} \frac{\operatorname{senh}(0)}{2} = 0 & \text{si } n \text{ impar} \\ \frac{\cosh(0)}{2} = \frac{1}{2} & \text{si } n \text{ par.} \end{cases}$$

$$\Rightarrow f(z) = \cos^2\left(\frac{iz}{2}\right) = \sum_{n=0}^{\infty} \frac{f^{n)}(0)}{n!}(z-0)^n = 1 + \sum_{\substack{n=1 \\ n \text{ par}}}^{\infty} \frac{f^{n)}(0)}{n!} z^n + \cancelto{0}{\sum_{\substack{n=1 \\ n \text{ impar}}}^{\infty} \frac{f^{n)}(0)}{n!} z^n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{2n}}{2n!}$$

$\boxed{7.-}$ Desarrolla en serie de potencias las siguientes funciones y halla el radio de convergencia

$\boxed{a)}$ $f(z) = \dfrac{z}{i+z^2}$ $\quad z_0 = 0$ y $z_0 = 1$. $\qquad z^2 = -i$



$f(z) = \dfrac{z}{i+z^2}$ es holomorfa en el disco $D(0,1)$ y en $D(1, |1-e^{i\frac{7\pi}{4}}|)$

$$\shortparallel$$

$$D(1, \sqrt{2-\sqrt{2}})$$

$$\frac{z}{i+z^2} = \frac{z}{(z-e^{i\frac{3\pi}{4}})(z-e^{i\frac{7\pi}{4}})} = \frac{A}{z-e^{i3\pi/4}} + \frac{B}{z-e^{i\frac{7\pi}{4}}}$$

$$\Uparrow$$

$$z = A(z-e^{i\frac{7\pi}{4}}) + B(z-e^{i3\pi/4})$$

$$\boxed{\begin{array}{l} e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \\ e^{i\frac{7\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \end{array}}$$

Para $z = e^{i\frac{3\pi}{4}} \Rightarrow e^{i\frac{3\pi}{4}} = A\left(e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}}\right)$

$\Updownarrow$

$e^{i\frac{3\pi}{4}} = -e^{i\frac{7\pi}{4}}$

$$A = \frac{e^{i\frac{3\pi}{4}}}{e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}}} = \frac{e^{i\frac{3\pi}{4}}}{2e^{i\frac{3\pi}{4}}} = \frac{1}{2}$$

Para $z = e^{i\frac{7\pi}{4}} \Rightarrow e^{i\frac{7\pi}{4}} = B\left(e^{i\frac{7\pi}{4}} - e^{i\frac{3\pi}{4}}\right) \Leftrightarrow$

$$\Leftrightarrow B = \frac{e^{i\frac{7\pi}{4}}}{e^{i\frac{7\pi}{4}} - e^{i\frac{3\pi}{4}}} = \frac{e^{i7\pi/4}}{2e^{i7\pi/4}} = \frac{1}{2}$$

$$\Rightarrow f(z) = \frac{z}{z^2+i} = \frac{1/2}{z - e^{\frac{3\pi}{4}i}} + \frac{1/2}{z - e^{\frac{7\pi}{4}i}} = \frac{1}{2}\left(z - e^{\frac{3\pi}{4}i}\right)^{-1} + \frac{1}{2}\left(z - e^{\frac{7\pi}{4}i}\right)^{-1}$$

$$\Rightarrow f'(z) = \frac{1}{2}\cdot(-1)\cdot\left(z - e^{\frac{3\pi}{4}i}\right)^{-2} + \frac{1}{2}(-1)\left(z - e^{\frac{7\pi}{4}i}\right)^{-2}$$

$$f''(z) = \frac{1}{2}(-1)\cdot(-2)\left(z - e^{\frac{3\pi}{4}i}\right)^{-3} + \frac{1}{2}(-1)(-2)\left(z - e^{\frac{7\pi}{4}i}\right)^{-3}$$

En general $f^{n)}(z) = \frac{1}{2}(-1)^n n!\left(z - e^{\frac{3\pi}{4}i}\right)^{-n-1} + \frac{1}{2}(-1)^n n!\left(z - e^{\frac{7\pi}{4}i}\right)^{-n-1}$

Los casos base ya están y el paso inductivo

$$f^{n+1)}(z) = \frac{\partial}{\partial z}\left(\frac{1}{2}(-1)^n n!\left(z - e^{\frac{3\pi}{4}i}\right)^{-(n+1)} + \frac{1}{2}(-1)^n n!\left(z - e^{\frac{7\pi}{4}i}\right)^{-(n+1)}\right) =$$

$$= \frac{1}{2}(-1)^n n!\left(z - e^{\frac{3\pi}{4}i}\right)^{-(n+1)-1}\cdot(-n-1) + \frac{1}{2}(-1)^n n!\left(z - e^{\frac{7\pi}{4}i}\right)^{-(n+1)-1}(-n-1) =$$

$$= \frac{1}{2}(-1)^{n+1}(n+1)!\left(z - e^{\frac{3\pi}{4}i}\right)^{-(n+1)-1} + \frac{1}{2}(-1)^{n+1}(n+1)!\left(z - e^{\frac{7\pi}{4}i}\right)^{-(n+1)-1}$$

$$f^{n)}(0) = \frac{1}{2}(-1)^n n!\left[\frac{1}{\left(-e^{\frac{3\pi}{4}i}\right)^{n+1}} + \frac{1}{\left(-e^{\frac{7\pi}{4}}\right)^{n+1}}\right] = \frac{(-1)^n n!}{2}\left[\frac{(-1)^{n+1}}{\left(e^{\frac{3\pi}{4}i}\right)^{n+1}} + \frac{1}{\left(e^{\frac{3\pi}{4}i}\right)^{n+1}}\right.$$

I. E. S. " SAN ISIDRO "

Asignatura .... Fecha ....

Alumno/a .... Curso .... Nº ....

Apellidos Nombre

Calificación

$$= \begin{cases} (n \text{ par}) = \dfrac{n!}{2}\left[\dfrac{-1}{(e^{\frac{3\pi}{4}i})^{n+1}} + \dfrac{1}{(e^{\frac{3\pi}{4}i})^{n+1}}\right] = 0 \\ (n \text{ impar}) \end{cases}$$

$$= -\frac{n!}{2}\left[\frac{1}{(e^{\frac{3\pi}{4}i})^{n+1}} + \frac{1}{(e^{\frac{3\pi}{4}i})^{n+1}}\right] = -\frac{n!}{(e^{\frac{3\pi}{4}i})^{n+1}}$$

$$\Longrightarrow f(z) = \frac{z}{z^2+i} = \sum_{n=0}^{\infty} \frac{f^{n)}(0)}{n!}(z-0)^n = \sum_{\substack{n=0 \\ n \text{ impar}}}^{\infty} - \frac{\cancel{n!}}{(e^{\frac{3\pi}{4}i})^{n+1}} \cdot \frac{z^n}{\cancel{n!}} =$$

$$= -\frac{1}{e^{\frac{3\pi}{4}i}} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(e^{\frac{3\pi}{4}i})^{2n+1}} \qquad \forall z \in D(0,1)$$

$$f^{n)}(1) = \frac{(-1)^n n!}{2}\left[\left(1-e^{\frac{3\pi}{4}}\right)^{-n-1} + \left(1+e^{\frac{3\pi}{4}}\right)^{-n-1}\right]$$

$$\Longrightarrow f(z) = \frac{z}{z^2+i} = \sum_{n=0}^{\infty} \frac{f^{n}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2\, n!}\left[\left(1-e^{\frac{3\pi}{4}}\right)^{-n-1} + \left(1+e^{3\pi/4}\right)^{-n-1}\right](z-1)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(1-e^{3\pi/4 i})^{n+1}} + \frac{1}{(1+e^{3\pi/4 i})^{n+1}}\right)(z-1)^n =$$

$$= \frac{1}{2(1-e^{\frac{3\pi i}{4}})} \sum_{n=0}^{\infty} \left(\frac{z-1}{e^{3/4 i}-1}\right)^n + \frac{1}{2(1+e^{3\pi/4 i})} \sum_{n=0}^{\infty} \left(\frac{1-z}{e^{3/4 i}+1}\right)^n \qquad \forall z \in D(1, \sqrt{2-\sqrt{2}})$$