Como $L(\sigma^2|x_1 \dots x_n)$ es creciente en $\sigma^2 \in \left(0, \frac{\sum x_i^2}{n}\right)$

se tiene que

$$\sup_{\sigma^2 \leq \sigma_0^2} \{f(x_1 \dots x_n|\sigma^2)\} = \begin{cases} L\left(\frac{\sum x_i^2}{n} \Big| x_1 \dots x_n\right) & \text{si } \sigma_0^2 \geq \frac{\sum x_i^2}{n} \\ L(\sigma_0^2 | x_1 \dots x_n) & \text{si } \sigma_0^2 < \frac{\sum x_i^2}{n} \end{cases}$$

Por tanto $\lambda(x_1 \dots x_n)$ queda como

$$\lambda(x_1 \dots x_n) = \frac{\sup_{\sigma^2 \leq \sigma_0^2} \{f(x_1 \dots x_n|\sigma^2)\}}{\sup_{\sigma^2 > 0} \{f(x_1 \dots x_n|\sigma^2)\}} = \begin{cases} 1 & \text{si } \frac{\sum x_i^2}{n} \leq \sigma_0^2 \\ \dfrac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} e^{-\frac{\sum x_i^2}{2\sigma_0^2}}}{\left(\frac{n}{2\pi \sum_{i=1}^n x_i^2}\right)^{n/2} e^{-n/2}} & \text{si } \frac{\sum x_i^2}{n} > \sigma_0^2 \end{cases}$$

Simplificando

$$\lambda(x_1 \dots x_n) = \begin{cases} 1 & \text{si } \frac{\sum x_i^2}{n} \leq \sigma_0^2 \\ \left(\frac{\sum x_i^2}{n\sigma_0^2}\right)^{n/2} e^{-\frac{n}{2\sigma_0^2}\left(\frac{\sum x_i^2}{n} - \sigma_0^2\right)} & \text{si } \frac{\sum x_i^2}{n} > \sigma_0^2 \end{cases}$$

Si denotamos por $t(x_1 \dots x_n) = \sum_{i=1}^n x_i^2$ nos damos cuenta que

$$\lambda(x_1 \dots x_n) = f(t(x_1 \dots x_n))$$

$$\text{con } f(t) = \begin{cases} 1 & \text{si } t \leq n\sigma_0^2 \\ \left(\frac{t}{n\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}(t - n\sigma_0^2)} & \text{si } t > n\sigma_0^2 \end{cases}$$

Podemos analizar esta función para ver que está sucediendo

$$f'(t) = \frac{n}{2}\left(\frac{t}{n\sigma_0^2}\right)^{n/2 - 1} \frac{1}{n\sigma_0^2} e^{-\frac{1}{2\sigma_0^2}(t - n\sigma_0^2)} + \left(\frac{t}{n\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}(t - n\sigma_0^2)} \cdot \left(-\frac{1}{2\sigma_0^2}\right)$$