Ejercicio 5.2.- Determina la F asociada a while $\neg(x=0)$ do $x:=x-1$.

Considera las siguientes funciones parciales de State $\hookrightarrow$ State:

$g_1\, s = \text{undef} \ \forall s$ $\qquad$ $g_2\, s = \begin{cases} s[x\to 0] & \text{si } sx \geq 0 \\ \text{undef} & \text{si } sx<0 \end{cases}$

$g_3\, s = \begin{cases} s[x\to 0] & \text{si } sx \geq 0 \\ s & \text{si } sx<0 \end{cases}$ $\qquad$ $g_4\, s = s[x\to 0] \ \forall s.$

$g_5\, s = s \ \forall s.$

Determina cuales de estas funciones son puntos fijos de F.

En primer lugar

$Fg = \text{cond}(\mathcal{B}[\![\neg(x=0)]\!], g\circ \mathcal{S}[\![x:=x-1]\!], \mathcal{S}[\![\text{skip}]\!])$, luego g_i es punto fijo de $F \Leftrightarrow Fg_i = g_i \Leftrightarrow Fg_i\, s = g_i\, s \ \forall s.$

$Fg_i\, s = \text{cond}(\mathcal{B}[\![\neg(x=0)]\!], g\circ \mathcal{S}[\![x:=x-1]\!], \mathcal{S}[\![\text{skip}]\!])\, s =$

$= \begin{cases} (g_i\circ \mathcal{S}[\![x:=x-1]\!])(s) & \text{si } \mathcal{B}[\![\neg(x=0)]\!]\, s = tt \\ \mathcal{S}[\![\text{skip}]\!]\, s & \text{si } \mathcal{B}[\![\neg(x=0)]\!]\, s = ff \end{cases} =$

$= \begin{cases} g_i(\mathcal{S}[\![x:=x-1]\!]\, s) & \text{si } sx\neq 0 \\ id(s) & \text{si } sx=0 \end{cases} = \begin{cases} g_i(s[x\mapsto (sx)-1]) & \text{si } sx\neq 0 \\ s & \text{si } sx=0. \end{cases}$

Por tanto, para g_1, dado un s tal que $sx=0$ se tiene que

$Fg_1\, s = \begin{cases} g_1(s[x\mapsto (sx)-1]) & \text{si } sx\neq 0 \\ s & \text{si } sx=0 \end{cases} = s \neq \text{undefined} = g_1\, s.$

$\Rightarrow g_1$ no es un punto fijo de F.

Para g_2. Sea $s \in State$.

Si $sx > 0$

$$F g_2\, s = \begin{cases} g_2(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} \overset{sx>0}{=} g_2(\overbrace{s[x \to (sx)-1]}^{s'}) \overset{s'x = sx-1 \geq 0}{=}$$

$$= s'[x \to 0] = s[x \to (sx)-1][x \to 0] \overset{\text{Se prueba}}{=} s[x \to 0] \overset{sx \geq 0.}{=} g_2 s$$

Si $sx = 0$

$$F g_2\, s = \begin{cases} g_2(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} \overset{sx=0}{=} s \overset{sx=0}{=} s[x \to 0] \overset{sx \geq 0.}{=} g_2\, s$$

Si $sx < 0$

$$F g_2\, s = \begin{cases} g_2(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} \overset{sx \neq 0}{=} g_2(\overbrace{s[x \mapsto (sx)-1]}^{s'}) \overset{s'x = sx-1 < -1 < 0}{=} \text{undefined} =$$

$$\underset{sx<0.}{=} g_2\, s$$ Por tanto $F g_2 = g_2$ y g_2 es un punto fijo de F.

Para g_3: Sea s, $sx < 0$.

$$F g_3\, s = \begin{cases} g_3(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} = g_3(\overbrace{s[x \mapsto (sx)-1]}^{s'}) \overset{s'x = (sx)-1 < -1 < 0}{=} s' =$$

$$= s[x \mapsto (sx)-1] \neq s \underset{sx<0}{=} g_3\, s$$

Obviamente $s' = s[x \to (sx)-1]$ y s no son iguales porque $s'x = s[x \to (sx)-1]\, x = sx - 1 < sx$. Por tanto g_3 no es punto fijo de F.

Para g_4: Sea $s \in State$.

Si $sx \neq 0$

$$F g_4\, s = \begin{cases} g_4(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} \overset{sx \neq 0}{=} g_4(s[x \mapsto (sx)-1]) = s[x \mapsto sx-1][x \to 0] = s[x \to 0] = g_4\, s.$$

Si $sx = 0$

$$F g_4\, s = \begin{cases} g_4(s[x \mapsto (sx)-1]) & \text{si } sx \neq 0 \\ s & \text{si } sx = 0 \end{cases} = s \overset{sx=0}{=} s[x \to 0] = g_4\, s.$$

Por tanto g_4 es un punto fijo de F.

Para g_5 Sea s / $sx \neq 0$.

$$F g_5\, s = \begin{cases} g_5(s[x \mapsto (sx)-1]) & si\ sx \neq 0 \\ s & si\ sx = 0 \end{cases} \overset{sx \neq 0}{\downarrow}{=} g_5(s[x \mapsto (sx)-1]) =$$

$$= s[x \longmapsto (sx)-1] \neq s = g_5\, s.$$

Esto es así porque $s[x \mapsto sx-1]\, x = sx - 1 \neq sx$

Por tanto g_5 no es un punto fijo de F.

Ejercicio 5.3.- Considera el siguiente fragmento del código del factorial while $\neg(x=1)$ do $(y := y * x, x := x-1)$: Determina la función F asociada y al menos dos puntos fijos de F distintos.

La función F es $F g = cond(\mathcal{B}[\![\neg(x=1)]\!], g \circ \mathcal{S}[\![y:=y*x;\ x:=x-1]\!], id)$.

Vamos a encontrar g_1 y g_2 tales que $F g_i = g_i$, es decir, $F g_i\, s = g_i\, s \quad \forall s \in State$.

$$F g_i\, s = cond(\mathcal{B}[\![\neg(x=1)]\!], g_i \circ \mathcal{S}[\![y:=y*x;\ x:=x-1]\!], id)\, s =$$

$$= \begin{cases} (g_i \circ \mathcal{S}[\![y:=y*x;\ x:=x-1]\!])(s) & si\ \mathcal{B}[\![\neg(x=1)]\!]\, s = tt \\ id(s) & si\ \mathcal{B}[\![\neg(x=1)]\!]\, s = ff \end{cases} =$$

$$= \begin{cases} g_i(\mathcal{S}[\![y:=y*x;\ x:=x-1]\!]\, s) & si\ sx \neq 1 \\ s & si\ sx = 1. \end{cases} =$$

$$= \begin{cases} g_i\left(\left(\mathcal{S}[\![x := x-1]\!] \circ \mathcal{S}[\![y := y*x]\!]\right)(s) \right) & \text{si} \quad sx \neq 1 \\ s & \text{si} \quad sx = 1 \end{cases} =$$

$$= \begin{cases} g_i\left(\mathcal{S}[\![x := x-1]\!] \left(\mathcal{S}[\![y := y*x]\!]\, s \right) \right) & \text{si} \quad sx \neq 1 \\ s & \text{si} \quad sx = 1 \end{cases} =$$

$$= \begin{cases} g_i\left(\mathcal{S}[\![x := x-1]\!] \left(\overbrace{s[y \mapsto (sy)*(sx)]}^{s'} \right) \right) & \text{si} \quad sx \neq 1 \\ s & \text{si} \quad sx = 1 \end{cases} =$$

$$= \begin{cases} g_i\left(s'[x \mapsto s'x - 1] \right) & \text{si} \quad sx \neq 1 \\ s & \text{si} \quad sx = 1 \end{cases}.$$

Como $s'x = s[y \mapsto (sy)*(sx)]\, x \overset{x \neq y}{=} sx$ esto queda como.

$$F\, g_i\, s = \begin{cases} g_i\left(s[y \to (sy)*(sx)][x \mapsto sx-1] \right) & \text{si} \quad sx \neq 1 \\ s & \text{si} \quad sx = 1. \end{cases}$$

Sea $$g_1\, s = \begin{cases} s[x \to 1][y \mapsto (sy)*(sx)!] & \text{si} \quad sx \geq 1 \\ \text{undefined} & \text{si} \quad sx < 1. \end{cases}$$

Sea $s \in State$.

$\boxed{\text{Si } sx > 1}$

$$F\, g_1\, s = \begin{cases} g_1\left(s[y \to (sy)*(sx)][x \to sx-1] \right) & \text{si} \quad sx \geq 1 \\ s & \text{si} \quad sx < 1 \end{cases} \overset{sx > 1}{=} g_1\Big(\underbrace{s[y \to (sy)*(sx)][x \to sx-1]}_{s'} \Big) =$$

$\Big($Como $s'x = s[y \to (sy)*(sx)][x \to sx-1]\, x = sx - 1 \overset{sx>1}{\geq} 1 \Rightarrow g_1(s') = s'[x \to 1][y \to (s'y)*(s'x)!]\Big)$

$$= s'[x \to 1][y \to (s'y)*(s'x)!] \overset{*}{=} s'[x \to 1][y \to (sy)*(sx)*(sx-1)!] =$$

$s'y = s[y \to (sy)*(sx)][x \to (sx)-1]\, y = (sy)*(sx)$

$= s'[x \to 1][y \to (sy) * (sx)!] = s[y \to (sy) * (sx)][x \to sx - 1][x \to 1][y \to (sy) * (sx)!] =$

$= s[x \to 1][y \to (sy) * (sx)!] \underset{\substack{\uparrow \\ sx > 1.}}{=} g_1\, s$

Si $sx = 1$|

$$F g_1\, s = \begin{cases} g_1 (s[y \to (sy) * (sx)][x \to sx - 1] & \text{si } sx \neq 1 \\ s & \text{si } sx = 1 \end{cases} \overset{sx = 1}{\underset{\downarrow}{=}} s$$

$$s \overset{?}{=} g_1\, s \overset{sx \neq 1}{\underset{\downarrow}{=}} s[x - 1][y \to (sy) * (sx)!] \underset{\substack{\uparrow \\ sx = 1}}{=} s[x \to 1][y \to sy] = s[x \to 1] \underset{\Leftarrow sx = 1}{\parallel} s$$

Luego $F g_1\, s = g_1\, s$ cuando $sx = 1$.

Si $sx < 1$|

$$F g_1\, s = \begin{cases} g_1 (s[y \to (sy) * (sx)][x \to sx - 1]) & \text{si } sx \neq 1 \\ s & \text{si } sx \leq 1. \end{cases} \overset{sx < 1}{\underset{\downarrow}{=}}$$

$$= g_1 (\underbrace{s[y \to (sy) * (sx)][x \to sx - 1]}_{s'}) \underset{\uparrow}{=} \text{undefined} \underset{\substack{\uparrow \\ sx < 1.}}{=} g_1\, s$$

$s'x = s[y \to (sy) * (sx)][x \to sx - 1]x = sx - 1 \underset{sx < 1}{<} 1$

Por tanto $\forall s \in State$, $F g_1\, s = g_1\, s$, es decir $F g_1 = g_1$ y g_1 es un punto fijo de F.

Sea $g_2\, s = s[x \to 1][y \to (sy) * (sx)!]$ $\forall s$.

Veamos que g_2 también es un punto fijo de F.

Sea $s \in State$

Si $sx \neq 1$|

$$F\, g_2\, s = \begin{cases} g_2\,(s[y \mapsto (sy)*(sx)][x \mapsto sx-1]) & si\ sx \neq 1 \\ s & si\ sx = 1 \end{cases} \overset{sx\neq 1}{=} g_2(\underbrace{s[y \mapsto (sy)*(sx)][x \mapsto sx-1]}_{s'})$$

$$= s'[x \to 1][y \to (s'y) * (s'x)!] \underset{\uparrow}{=} s'[x \to 1][y \to (sy)*(sx)*(sx-1)!] =$$

$$s'x = s[y \to (sy)*(sx)][x \to sx-1]\, x = sx - 1$$
$$s'y = s[y \to (sy)*(sx)][x \to sx-1]\, y = (sy)*(sx)$$

$$= s'[x \to 1][y \to (sy)*(sx)!] = s[y \to (sy)*(sx)][x \to sx-1][x \to 1][y \to (sy)*(sx)!] =$$

$$= s[x \to 1][y \to (sy)*(sx)!] = g_2\, s$$

Si $sx = 1$|

$$F\, g_2\, s = \begin{cases} g_2\,(s[y \to (sy)*(sx)][x \mapsto sx-1]) & si\ sx \neq 1 \\ s & si\ sx = 1 \end{cases} \overset{sx=1}{=} s \overset{?}{=} g_2\, s.$$

$$g_2\, s = s[x \to 1][y \to (sy)*(sx)!] \underset{\overset{\uparrow}{sx=1}}{=} s[x \to 1][y \to sy] = s[x \to 1] \underset{\overset{\uparrow}{sx=1}}{=} s \quad \checkmark$$

Por tanto $F g_2 s = g_2 s$ $\forall s \in State$, es decir, $F g_2 = g_2$ y g_2 es un punto fijo de F.