$$= \int_0^{2\pi}\left(-2\cos^2\theta\int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr - 4\operatorname{sen}\theta\cos\theta\int_0^1\frac{r^3}{\sqrt{1-r^2}}\,dr + 3\cos\theta\int_0^1\frac{r^2}{\sqrt{1-r^2}}\,dr + \int_0^1\left(4r-2r\sqrt{1-r^2}\right)dr\right)d\theta$$

Calculamos por separado

$$\int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr \underset{\uparrow}{=} \int_0^{\pi/2}\frac{\operatorname{sen}^3 x}{\cancel{\cos x}}\cancel{\cos x}\,dx = \int_0^{\pi/2}\operatorname{sen}x\,(1-\cos^2 x)\,dx =$$

$\operatorname{sen}x = r$
$\cos x\,dx = dr$
$r=1 \Rightarrow x=\pi/2$
$r=0 \Rightarrow x=0$

$$= \int_0^{\pi/2}\operatorname{sen}x\,dx - \int_0^{\pi/2}\operatorname{sen}x\cos^2 x\,dx = -\cos x\Big]_0^{\pi/2} + \frac{\cos^3 x}{3}\Big]_0^{\pi/2} =$$

$$= -0 + 1 + 0 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr \underset{\uparrow}{=} \int_0^{\pi/2}\frac{\operatorname{sen}^2 x}{\cancel{\cos x}}\cancel{\cos x}\,dx = \int_0^{\pi/2}\frac{1}{2}-\frac{\cos 2x}{2}\,dx = \frac{\pi}{4} - \cancelto{0}{\frac{\operatorname{sen}2x}{4}\Big]_0^{\pi/2}} = \frac{\pi}{4}$$

$\operatorname{sen}x = r$
$\cos x\,dx = dr$
$r=1 \Rightarrow x=\pi/2$
$r=0 \Rightarrow x=0$

$$\int_0^1 \left(4r - 2r\sqrt{1-r^2}\right)dr = 2 + \frac{(1-r^2)^{3/2}}{3/2}\Big]_0^1 = 2 + \frac{2}{3}(0-1) = 2-\frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow \iint_{S_2} \nabla f\cdot d\vec{S} = \int_0^{2\pi} -2\cos^2\theta\cdot\frac{2}{3}\,d\theta - \int_0^{2\pi} 4\cdot\frac{2}{3}\cdot\operatorname{sen}\theta\cos\theta\,d\theta + \int_0^{2\pi}3\cos\theta\cdot\frac{\pi}{4}\,d\theta + \int_0^{2\pi}\frac{4}{3}\,d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta - \frac{4}{3}\cancelto{0}{\operatorname{sen}^2\theta\Big]_0^{2\pi}} + \frac{3\pi}{4}\cancelto{0}{\operatorname{sen}\theta\Big]_0^{2\pi}} + \frac{8\pi}{3} =$$

$$= \frac{8\pi}{3} - \frac{4\pi}{3} - \cancelto{0}{\frac{\operatorname{sen}2\theta}{3}\Big]_0^{2\pi}} = \frac{4\pi}{3}$$