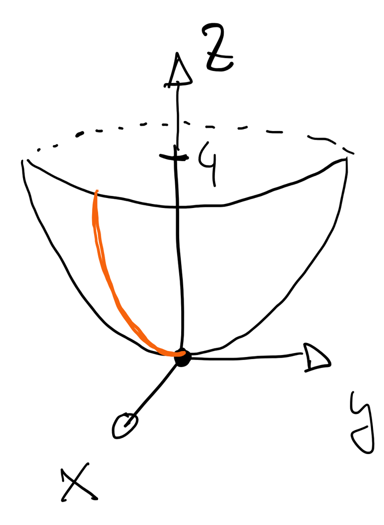
23-04-CI

jueves, 23 de abril de 2020 17:00

Ejercicio 1: Calcular el área de la superficie S del paraboloide $z = x^2+y^2$ delimitado por $0<z<4$



Vamos a calcular el área de $\hat{S} = S \setminus \{(x,y,z) \in S : y=0, x \geq 0\}$

parametrizamos $\hat{S}$ mediante

$$\Phi(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta, r^2)$$

$$\Phi : (0,2) \times (0,2\pi) \longrightarrow \mathbb{R}^3$$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \operatorname{sen}\theta, 2r)$$

$$\frac{\partial \Phi}{\partial \theta} = (-r\operatorname{sen}\theta, r\cos\theta, 0)$$

$$\frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \operatorname{sen}\theta & 2r \\ -r\operatorname{sen}\theta & r\cos\theta & 0 \end{vmatrix} =$$

$$= -2r^2\cos\theta\,\vec{i} - 2r^2\operatorname{sen}\theta\,\vec{j} + r\,\vec{k}$$

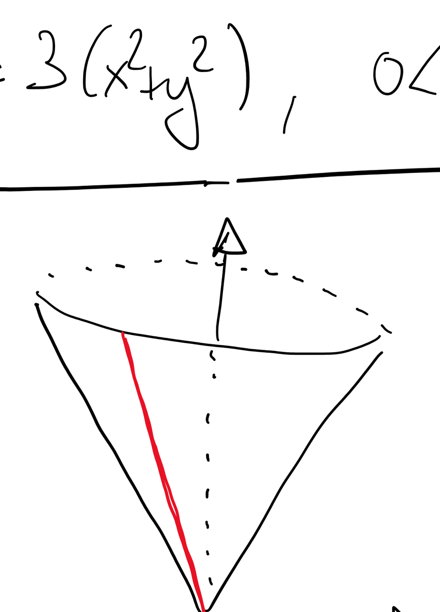
$$\left\| \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} \right\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$\text{Área}(S) = \text{Área}(\hat{S}) = \int_0^2 \int_0^{2\pi} r\sqrt{4r^2+1}\, d\theta\, dr$$

$$= 2\pi \int_0^2 r\sqrt{4r^2+1}\, dr = \frac{\pi}{6}\left[(4r^2+1)^{3/2}\right]_0^2 =$$

$$= \frac{\pi}{6}\left[17^{3/2} - 1\right]$$

Ejercicio 2: Calcular $\iint_S f\, dS$ con $f(x,y,z) = x^2+y^2$, S la superficie del cono $z^2 = 3(x^2+y^2)$, $0<z<3$



$\hat{S} = S \setminus \{(x,y,z) \in S : y=0, x \geq 0\}$

$$\iint_S f\, dS = \iint_{\hat{S}} f\, dS$$

parametrizamos $\hat{S}$ por

$$\Phi(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta, \sqrt{3}r)$$

$$\Phi : (0,\sqrt{3}) \times (0,2\pi) \longrightarrow \mathbb{R}^3$$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \operatorname{sen}\theta, \sqrt{3})$$

$$\frac{\partial \Phi}{\partial \theta} = (-r\operatorname{sen}\theta, r\cos\theta, 0)$$

$$\frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \operatorname{sen}\theta & \sqrt{3} \\ -r\operatorname{sen}\theta & r\cos\theta & 0 \end{vmatrix} =$$

$$-\sqrt{3}r\cos\theta\,\vec{i} - \sqrt{3}r\operatorname{sen}\theta\,\vec{j} + r\,\vec{k}$$

$$\int_S f\, dS =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} f(r\cos\theta, r\operatorname{sen}\theta, \sqrt{3}r) \left\| \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} \right\| dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2 \cdot \sqrt{3r^2+r^2}\, dr\, d\theta =$$

$$= 2\pi \int_0^{\sqrt{3}} 2r^3\, dr = \pi\left[r^4\right]_0^{\sqrt{3}} = 9\pi$$

Ejercicio 3: Sea S la porción de superficie cilíndrica

$$\Phi \rightsquigarrow \begin{cases} x = 3\cos\theta \\ y = 3\operatorname{sen}\theta \\ z = z \end{cases} \qquad \begin{matrix} \theta \in (0,\pi) \\ z \in (-3,3) \end{matrix}$$

$\vec{F}(x,y,z) = (x,y,z)$. Hallar el flujo de $\vec{F}$ a través de S

$$\frac{\partial \Phi}{\partial \theta} = (-3\operatorname{sen}\theta, 3\cos\theta, 0)$$

$$\frac{\partial \Phi}{\partial z} = (0, 0, 1)$$

$$\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\operatorname{sen}\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= 3\cos\theta\,\vec{i} + 3\operatorname{sen}\theta\,\vec{j}$$

$$(F \circ \Phi) \cdot \left(\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial z}\right) =$$

$$(3\cos\theta, 3\operatorname{sen}\theta, z) \cdot (3\cos\theta, 3\operatorname{sen}\theta, 0) = 9$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi}\int_{-3}^{3} 9\, dz\, d\theta = 54\pi$$