Lectures on Climate to Mathematicians

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Chapter 1

Snowball Earth and Climate Modeling

Mathematics required: *Taylor series expansion, or tangent approximation.*
Mathematics developed: *Bifurcation diagram, multiple equilibrium branches, linear stability, slope-stability theorem*.

1. Introduction

Figure 1. Snowball earth.

We currently live in a rare period of warm climate over most of the globe, although there are still permanent glaciers over Antarctica, Greenland, Northern Canada and Russia. Over millions of years in the earth's history, mass ice sheets repeatedly advanced over continental land masses from polar to temperate latitudes, and then retreated. We are actually still in the midst of a long cooling period which started three million years ago, punctuated by short *interglacial* periods of warmth, lasting about 20,000 years. We are in, and probably near the end of, one of these interglacial respites. *Ice ages* were a norm rather than an exception at least during much of the time our species evolved into modern

Figure 2. Glaciers are currently found only on continental land masses near the poles. Taken from http://teachers.sduhsd.k12.ca.us

men. The harsh conditions might have played a role in the evolution of our brain as humans struggled to survive in the cold climate.

If we look further back into the paleo-climate record, we find some notable long periods of *equable* climate, where the planet was ice-free and warm conditions prevailed over the globe. (The word, *equable,* means *even*, and refers to the lack of temperature contrast between the equator and the pole during this period.) The Eocene, some 50 millions years ago, is one such period. An earlier one is the Cretaceous, some 65 to 140 million years ago. During the Eocene, alligators and flying lemurs were found on Ellesmere Island (at paleo-latitude 78^{o} N), tropical plants, which are intolerant of even episodic frost, were thriving on Spitsbergen (at paleo-latitude 79^{o} N), and trees grew in both the Artic and the Antarctic. During the Cretaceous, palm trees grew in the interior of the Asian continent near paleo-latitude 60^{o} N.

Still further back in time, about 600-800 million years ago, the earth was probably covered entirely by ice. This scenario of global glaciation (in particular over the equator) was first proposed by Brian Harland at Cambridge University, who called it the *Great Glaciation* (read the *Scientific American* article, Harland and Rudwick (1964)). The term *snowball earth* was coined later by a Caltech geologist, Joseph Kirschvink in 1992. In her popular 2003 book, *Snowball Earth, The Story of a Maverick Scientist and His Theory of the Global Catastrophe That Spawned Life as We Know It*, writer Gabrielle

Walker follows the adventures of a field geologist Paul Hoffman of Harvard University, as he pieces together the evidence supporting a dramatic theory: about 600 million years ago our earth was entirely covered by ice. The evidence was laid out by Hoffman and his colleagues in their 1998 paper in *Science* and further described in a *Scientific American* article in 2000 by Hoffman and Schrag. That such a scenario is possible and inevitable was predicted by the Russian climate theorist, Mikhail Budyko, in the 1960s using a simple climate model which now bears his name. According to Budyko's model, however, once the earth is completely covered by ice, it is irreversible. Life would die off on land and in the oceans because sunlight could not reach across the thick ice sheets. Hoffman's evidence shows furthermore that there was a dramatic end to the snowball earth. The abrupt end of the last episode, in an inferno with torrential acid rain, actually led to an explosion of diversity of multi-cellular life-forms, called the "Cambrian Explosion". We will present first Budyko's model and then discuss some later theory which may explain how the earth escaped from its icy tomb.

A climate model should be able to account for all three types of climate: ice-covered globe, ice-free globe and partially ice-covered globe, and explain transitions among such drastically different climates under solar inputs which has not fluctuated by more than 6% in hundreds of millions of years of earth's history.

2. Simple Climate Models

The simplest type of climate models was the Energy Balance Models pioneered in 1969 separately by M.I. Budyko of the State Hydrological Institute in Leningrad and W. D. Sellers of the University of Arizona at Tucson, Arizona. These models try to predict the latitudinal distribution of surface temperature T, based on the concept that the energy the earth receives from the sun's radiation must balance the radiation the earth is losing to space by reemission at its temperature T. They also take into account the reflection of solar radiation back to space, the so-called the *albedo* effect, by the ice and snow and by the clouds. For an ice-free planet, these models tend to give an annual average temperature at the equator of 46^o C, and -43^o C at the pole. (This is much warmer at the equator and much colder at the pole than our current values of 27^o C and -13^o C, at the two locations respectively.) The cold temperature at high latitudes is inconsistent with the prior assumption of an ice-free planet. Water must freeze at polar latitudes. Allowing for the formation of ice makes the problem much more interesting. Since ice reflects sunlight back to space more than land or ocean surfaces, the earth is actually losing more heat---with less absorption of the sun's radiation---than when there is no ice cover. So it gets colder, consequently more ice forms, and the ice sheet advances equatorward. On the other hand, if, for some reason, the solar radiation is increased, ice melts a little near the ice edge, exposing more land, which absorbs more solar radiation, which makes the earth warmer, and more ice melts. The ice sheet retreats poleward. The albedo effect serves to amplify any small changes to solar radiation that the earth receives in its orbit. And since such orbital changes are known to be really small, the inherent instability of the ice-albedo feedback mechanism is therefore of much interest to climate scientists.

There are some minor differences in the model of Budyko as compared to that of Sellers. We will first discuss the Budyko model, as it has a simpler form of transport which we can analyze using simple mathematics.

2.1. Incoming solar radiation

The incoming solar radiation at the top of the earth's atmosphere is written as *Q· s(y),* where $y=\sin\theta$, with θ being the latitude. The latitudinal distribution function *s(y)* is normalized so that its integral with respect to y from 0 to 1 is unity. *Q* then is the overall (integrated) total solar input into the atmosphere-ocean system. Its magnitude is 343 watts per square meter at present.

[*More geometrical details if you are interested*: Let *S(r)* be the radiation impinging on a unit area of a disk facing the sun. The value of *S(r)* for *r* at the mean earth-sun distance is called the *solar constant*. It was once thought to be unchanging, as implied by the name, but we now know that it changes on many time scales. The solar constant, along with detailed information on the solar radiation at various wavelengths of light and energy, has been measured by satellite since 1979. So we know the solar constant at the top of our atmosphere. At the mean earth-sun distance it is about 1372 watts per square meters at present. Various parts of the earth receive more or less of the solar energy. On an annual average, the equator is closer to the sun than the poles, and so it receives more. The earth's rotational axis tilts (about 23.5° at present) from the normal to the elliptical plane of the earth's orbit. In January, the Southern Hemisphere is closer to the sun than the Northern Hemisphere, and vice versa in July. So there is a seasonal as well as a latitudinal variation of the incoming radiation. We shall consider here annual averages, and deal with latitudinal variation only. The analytical formula for this, obtained from astronomical and geometrical calculations, is known, but is complicated to write down. Nevertheless it has been tabulated; see Chylek and Coakley (1975). The rate of solar energy input per unit earth area is usually written in the form: *Q· s(y)*, where $y=\sin\theta$, with θ being the latitude. The total solar energy input is obtained by integrating over the surface of the earth of radius *a*:

$$\int_{-\pi/2}^{\pi/2} Qs(\sin\theta)2\pi a\cos\theta a d\theta = 4\pi a^2 Q\int_0^1 s(y)dy = 4\pi a^2 Q\,,$$

if the function *s(y)* is normalized so that its integral from 0 to 1 is unity. The above integrated solar input should be equal to the solar flux intercepted by an area of the circular disk of the earth seen by the sun: $S\,\pi a^2$. Therefore $Q = S/4=343$ watts per square meter. The function *s(y)* is uniformly approximated to within 2% by North (1975) to be:

$$s(y)\cong 1-S_2P_2(y),\ \text{where}\ S_2=0.482\ \text{and}\ P_2(y)=(3y^2-1)/2\,,$$

for the present obliquity of the earth's orbit. The obliquity is the angle between the earth's axis of rotation and the normal to the plane of its orbit around the sun. We shall consider *s(y)* as known in our model to follow.]

2.2. Albedo

A fraction of the solar radiation is reflected back to space without being absorbed by the earth. Let $\alpha(y)$ denote the fraction reflected; α is called the *albedo* (from the Latin word *albus*, for whiteness. The word *albino* comes from the same root.). The amount absorbed by the earth per unit area is therefore

$$Q\, s(y)\, (1- \alpha(y)). \tag{1}$$

2.3. Outward radiation

Let $I(y)$ be the rate of energy emission by the earth per unit area. It is temperature dependent; the warmer the planet the higher its rate of energy emission. It is given by:

$$I=A + B\, T, \tag{2}$$

where T is the surface temperature in ^{o} C. The constants A and B are chosen empirically based on the present climate. They are $A=205$ watts per square meter, and $B=1.95$ watts per square meter per ^{o} C.

[*Some details of physics*: The earth's reemission of absorbed solar radiation is different from the reflection of solar radiation. The reflection of solar radiation occurs at the wavelength of the incident radiation, which contains mostly the ultraviolet and visible parts of the spectrum, without any transformation of the energy. In reemission, the earth-atmosphere-ocean system heats up after the absorption of the incoming solar radiation. From space the planet appears as a warm sphere which is radiating its energy to space at a rate characteristic of its surface temperature. For the temperatures we are considering the re-radiation occurs mostly at infrared wavelengths. A well-known law, the Stefan-Boltzmann Law, states that for a black body (without an atmosphere), the rate of energy emission per unit surface area, $I(y)$, is proportional to the 4$^{\text{th}}$ power of the surface temperature T. It is written in the form:

$$I(y)=\sigma T^4,$$

with T in absolute temperature and σ=5.6686x 10^{-8} watts per square meter per ^{o} K^4 being the Stefan-Boltzmann constant. The earth is not a black body. In particular its atmosphere has several greenhouse gases, such as carbon dioxide and water vapor, which trap the infrared emission from the surface. This effect is taken into account in this simple model by multiplying σ by an emissivity fraction $\delta < 1$. A further difficulty is the nonlinear dependence of T, and this is dealt with in these simple models by linearization (tangent approximation) about 0 degree C, which is $T_0= 273\,^o$ K. Thus,

$$I(y)= \delta\sigma\, T^4 \cong \delta\sigma\, T_0^{\ 4}[1+4(T- T_0)/ T_0] =A + B\, (T- T_0),$$

The second step above is a linear tangent approximation to the function T^4. This approximation then leads to (2). However, according to this tangent approximation, the constants should be related as: $A=\delta\sigma T_0{}^4$, and $B/A=4/T_0$. Different values of A and B have been used by various authors and they don't necessarily satisfy this relationship between A and B, because the δ may depend on T. Since I(y) can now be measured directly by satellites as Outgoing-Longwave-Radiation (OLR), one can directly fit a straight line to the measured data and obtain the parameters A and B. There is a very good correlation between OLR and the surface temperature, and it can be fitted to a straight line. This procedure gives A=202 watts per square meters and B=1.90 watts per square meters per degree C. See Graves *et al* (1993).]

2.4 Ice dynamics

Ice forms from pure water when temperature is below 0^o C. However, permanent glaciers cannot be sustained until the annually averaged temperature is much colder, especially over ocean. (If the annually averaged temperature over land is 0^o C, it means that during summer the temperature is above freezing and the ice melts. Over salt water even during winter, ice cannot form until the surface temperature is much colder.) In the models of Budyko and Sellers the prescription is for an ice sheet to form when $T< T_c = -10\,^o C$.

Let y_s be the location of the ice line, so that poleward of this latitude the earth is covered with ice and is ice free equatorward of this location. The albedo is higher in the ice-covered part of the earth; Budyko took the following form for $\alpha(y)$:

$$\alpha(y) = \{ \begin{matrix} \alpha_2 = 0.62, y > y_s \\ \alpha_1 = 0.32, y < y_s \end{matrix} . \qquad (3)$$

At the ice boundary the temperature is taken to be T_c , i.e.

$$T(y_s)= T_c.$$

Following Lindzen (1990) we assume the albedo at the ice edge to be the average of that on the ice side and on the ice free side:

$$\alpha(y_s) =\alpha_0=(\alpha_1 + \alpha_2)/2 = 0.47.$$

2.5. Transport

When hot fluid is place next to a cold fluid, heat is often exchanged in a way so as to make the differences in temperatures less. In ordinary fluids, such as water or air, this heat exchange is accomplished either through conduction or convection. Convection, involving the overturning of the fluid, which can carries the heat from the hot spot to the cold spot, is often the more effective of the two mechanisms. Heat is transported by earth's atmosphere-ocean system in a number of ways. In the tropical atmosphere, rapid vertical convection and the presence of a north-south overturning circulation (called the Hadley circulation) tend to smooth out the north-south temperature gradient. In the extra-tropical atmosphere, large-scale waves, in the form of cyclones, anticyclones and storms, also act to transport heat from where it is warm to where it is colder. A detailed description of these processes will require a complex dynamical model involving many scales of motion. In the simple model of Budyko, the transport processes are lumped into a simple relaxation term for the rate of change of heat energy due to dynamical transport:

$$D(y) = C(\overline{T} - T), \tag{4}$$

where $\overline{T} = \frac{1}{2}\int_{-1}^{1} T(y)dy$ is the globally averaged temperature. The simple form in (4) satisfies the constraint that transport only moves heat from hot to cold spots while having no effect on the globally integrated temperature. If the local temperature at a particular latitude is greater than the global mean, heat will be taken out of that latitude. Conversely, if the local temperature is colder than the global mean, that latitude will gain heat. The empirical parameter C was assumed by Budyko to be $2.4B$ so that the solution can fit the present climate when the radiative parameters are taken to be the current climate values. Held and Suarez (1974) discussed how C and B can be evaluated from radiation and temperature measurements, and suggested a value of $C=2.1B$. Using the more updated value of solar constant measured after 1979 using satellites, we choose C=1.6B (see later calculation of the present date ice line location.)

2.6 The model equation.

We now construct a model equation. Our equation should say that the rate of change of earth's surface temperature should be equal to that due to the net absorption of solar energy input minus that due to earth's outward radiation, plus the heat gained or lost from transport. Thus:

$$R\frac{\partial}{\partial t}T = F(T) \tag{5}$$

where $F(T)=Q\, s(y)\,(1-\alpha(y)) - I(y) + D(y)$.

The dependence of F on y and t are not displayed for convenience.

[Although we use the partial derivative with respect to t in Eq.(5) because T depends on both y and t, you can treat it the same as an ordinary derivative for all practical purposes. This is because there is no y-derivative in that equation; we can therefore treat y as another parameter, instead of as a second independent variable.]
The parameter R on the left-hand side of (5) is the heat capacity of the earth, which is mostly determined by the heat capacity of the atmosphere and oceans. It is needed so that RT will have the dimension of energy, since the right-hand side contains energies. Some authors use the value R=0.634 watts per square meter per ^{0}C, but we will not need to specify a value. This time-dependent version of the Budyko equation was first used by Held and Suarez (1974), and further analyzed by many later authors, including Frideriksen (1976).

On an annual mean basis, the problem is symmetric about the equator, and so we will only need to consider the case of $y \geq 0$ after we assume the symmetry condition across the equator: *dT/dy=0 at y=0*. Under this symmetry condition, the global mean temperature is the same as the hemispherically averaged temperature. i.e.

$$\overline{T} = \int_0^1 T(y)dy$$

An equation governing the evolution of the global mean temperature can be obtained by integrating Eq. (5) hemispherically. It is

$$R\frac{d}{dt}\overline{T} = Q(1-\overline{\alpha}) - A - B\overline{T}\ , \tag{6}$$

where $\overline{\alpha} = \int_0^1 s(y)\alpha(y)dy = \alpha_1\int_0^{y_s} s(y)dy + \alpha_2\int_{y_s}^1 s(y)dy$.

$\overline{\alpha}$ =α_1 for an ice-free globe; $\overline{\alpha}$ =α_2 for an ice-covered globe. For an earth partially covered by ice with the ice line at y_s, it is

$$\overline{\alpha} = \alpha_2 + (\alpha_1 - \alpha_2)y_s[1 - 0.241(y_s^2 - 1)].$$

For the present ice line, located at y_s=0.95 (corresponding to 72 ^{o} N), we have $\overline{\alpha}$ =0.33, and is close to the ice-free albedo of 0.32.

3. The equilibrium solutions

We shall first seek the equilibrium solution T^* of Eq.(5) by setting its right-hand side to zero:

$$F(T^*)=Q\ s(y)\ (1-\ \alpha(y)) - (A+BT^*) + C(\overline{T}\ -T^*\ )=0. \tag{7}$$

This time-independent equation was first studied by Budyko. There are multiple equilibrium solutions depending on the extent of ice cover on the globe. The global mean temperature at equilibrium can be obtained directly by setting the right hand side of Eq. (6) to zero:

$$\overline{T}^* = [Q(1-\overline{\alpha}) - A]/B\,. \tag{8}$$

Substituting (8) into (7), we obtain the equilibrium solution at y:

$$\begin{aligned} T(y)^* &= [C\overline{T}^* + Qs(y)(1-\alpha(y)) - A]/(B+C) \\ &= \frac{Q}{B+C}[s(y)(1-\alpha(y)) + \frac{C}{B}(1-\overline{\alpha})] - \frac{A}{B} \end{aligned} \tag{9}$$

The location of the ice line is determined by evaluating (9) at y_s , where T=T_c :

$$T_c = \frac{Q}{B+C}[s(y_s)(1-\alpha(y_s)) + \frac{C}{B}(1-\bar{\alpha})] - \frac{A}{B}. \tag{10}$$

This equation yields the location of the ice line as a function of the solar constant *4Q*. [Instead of solving (10), a cubic equation in y_s as a function of Q, one can alternatively solve for Q as a function of y_s, which is much easier.] [This equation is valid for $0< y_s<1$. When the ice line is at the equator, it cannot move any more equatorward even for higher Q. Similarly for the iceline at the pole, it cannot move any more poleward for smaller value of Q.]

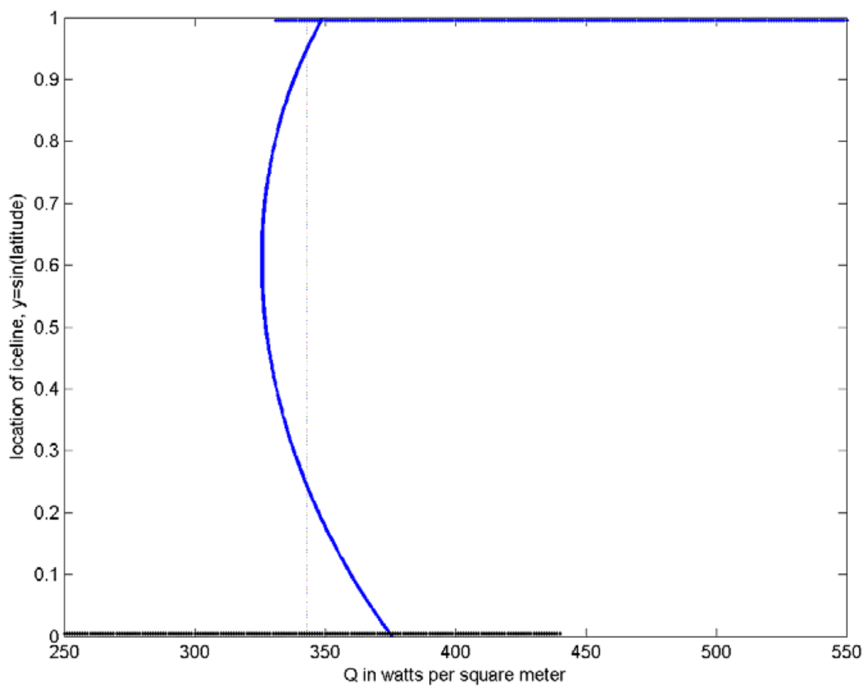

Figure 3: The location of the iceline, y_s, as a function of Q, obtained by evaluating for Q in Eq.(10) for various y_s. The vertical dotted line indicates the present climate at Q=343. At this value of Q there are four possible locations of the ice line; the present location at y_s =0.95 is one of the four possibilities. The top horizontal line is for the ice-free solution, while the lower horizontal line is for the snowball earth solution.

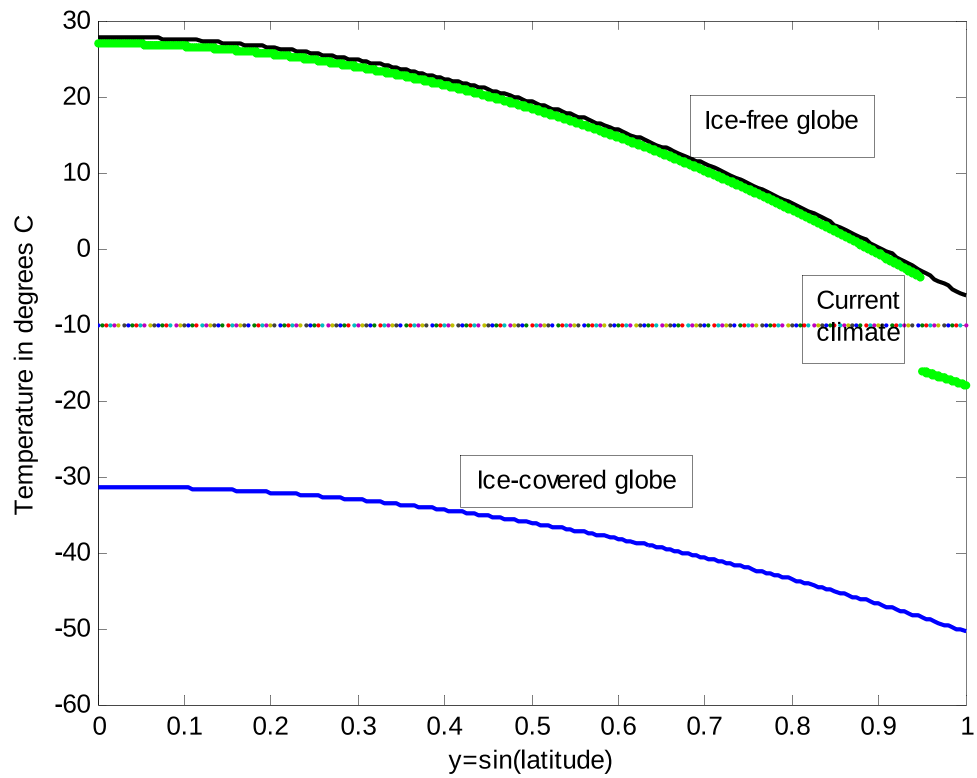

Figure 4. Equilibrium temperature as a function of y for current climate parameters (Q=343, A=202, B=1.90 and C=1.6B). Note that for the same solar constant as in the current climate, it is possible to have an ice-free globe (top curve), an ice-covered globe (bottom curve), and the current climate, which is a partially ice-covered with the ice line at y=0.95 (the intermediate, discontinuous curve).

3.1 Ice-free globe

We first investigate the possibility of an ice-free solution. In that case, $\alpha(y) = \alpha_1 = 0.32$ everywhere. The solution in (9) becomes:

$$T(y)^* = \frac{Q\cdot(1-\alpha_1)}{B+C}[s(y)+\frac{C}{B}]-\frac{A}{B}. \qquad (11)$$

This solution is plotted in Figure 4.

In order for it to be a self-consistent solution for an ice-free globe, solution (11) must be everywhere greater than T_c, including at the pole, the location of the minimum temperature. This condition is obtained by setting $T(1)^* > T_c$, thus yielding a restriction on the magnitudes of Q as a function of A, B and C as:

$$Q > \frac{(B+C)(T_c + A/B)}{(1-\alpha_1)(s(1)+C/B)}$$

For the parameter values given previously for the present climate, i.e. $A=202$ watts per square meter, $B=1.90$ watts per square meter per ^{o} C, and $C=1.6B$. The condition that the polar temperature $T(1)^*$ in this ice-free scenario must be greater than T_c yields the condition on the mean solar energy input of Q:

$Q > 330$ watts per square meter.

Since our earth currently receives Q=343 watts per square meter, this scenario of an ice-free globe is a distinct alternative climate under the present conditions provided that we can show that this equilibrium solution is stable. In such a climate, the globally averaged temperature is a warm 16 ^{o} C:

$$\overline{T}^* = [Q(1-\alpha_1) - A]/B = [343(0.68) - 202]/1.9 = 16^o\ C.$$

3.2 Ice-covered globe

Similar to the previous section, we can investigate the possible solution for a completely ice-covered earth by setting the albedo to $\alpha(y) = \alpha_2 = 0.62$ everywhere.

The equilibrium temperature solution is, from Eq.(9):

$$T(y)^* = \frac{Q\cdot(1-\alpha_2)}{B+C}[s(y) + \frac{C}{B}] - \frac{A}{B}. \qquad (12)$$

This solution is plotted in Figure 4.

Again, to be consistent with the prior assumption of an ice-covered globe, the temperature must everywhere be less than T_c, including at the equator, the location of maximum temperature. Using the same parameters as in our current climate, we find that a completely glaciated globe is a possibility if the solar input drops below a threshold value given by:

$$Q < \frac{(B+C)(T_c + A/B)}{(1-\alpha_2)(s(0) + C/B)}.$$

$Q < 441$ watts per square meter.

Since we currently receive even less than this threshold value----our current Q is 343 watts per square meter----our earth might alternatively be totally ice-covered, if this equilibrium turns out to be stable. In such a climate, the globally averaged temperature is a frigid minus 38 ^{o} C:

$$\overline{T}^* = [Q\cdot(1-\alpha_2) - A]/B = [343(0.38) - 202]/1.9 = -38^o\ C.$$

3.3 Partially ice covered

The more general solution is a globe partially covered by ice. The mathematics is slightly more involved, but still straightforward. To find the global mean temperature we can either use (8) or evaluate Eq.(7) at the ice edge. The latter procedure yields:

$$\overline{T}^* = A/C + (1 + B/C)T_c - Q \cdot s(y_s)(1 - \alpha_0)/C, \qquad (13)$$

where $\alpha_0 = \alpha(y_s)$.

Solving Eq.(7) separately for the ice-covered part and the ice-free part of the globe, we find:

$$T(y)^* = T_1(y) \equiv [Q \cdot (1 - \alpha_1)\, s(y) + C\overline{T}^*]/(B + C) \quad \text{for } y < y_s$$
$$T(y)^* = T_2(y) \equiv [Q \cdot (1 - \alpha_2)\, s(y) + C\overline{T}^*]/(B + C) \quad \text{for } y > y_s$$

We substitute (13) into these expressions, and find that we can write the above solution in the following compact form (due to Frederiksen (1976)):

$$T_i(y) = T_c + \frac{Q}{B + C}[s(y)(1 - \alpha_i) - s(y_s)(1 - \alpha_0)], \quad i = 0, 1, 2. \qquad (14)$$

For Q=343 watts per square meter and for the ice line located at 72 degrees of latitude, (14) gives the temperature distribution for our "current" climate in this simple model. This is plotted in Figure 4. The globally averaged temperature of this equilibrium solution is, from either (8) or (13)

$$\overline{T}^* = [343(1 - 0.33) - 202]/1.9 = 15\,^{o}\,C,$$

which is quite close to the observed global mean temperature currently.

3.4 Multiple equilibria

We see from the above results that there exist multiple equilibria under the same set of parameter values. For example, for the same solar forcing, the current one, we can have either the current climate with a global mean temperature of 15 ^{o} C, or a completely ice-covered earth with a global mean temperature of -38 ^{o} C or a completely ice-free earth with a global mean temperature of 16 ^{o} C. These multiple branches of solutions are plotted in Figure 5, for the global mean equilibrium temperature as a function of the solar constant. Some branches are stable while others are not. The stability of these equilibria is discussed next.

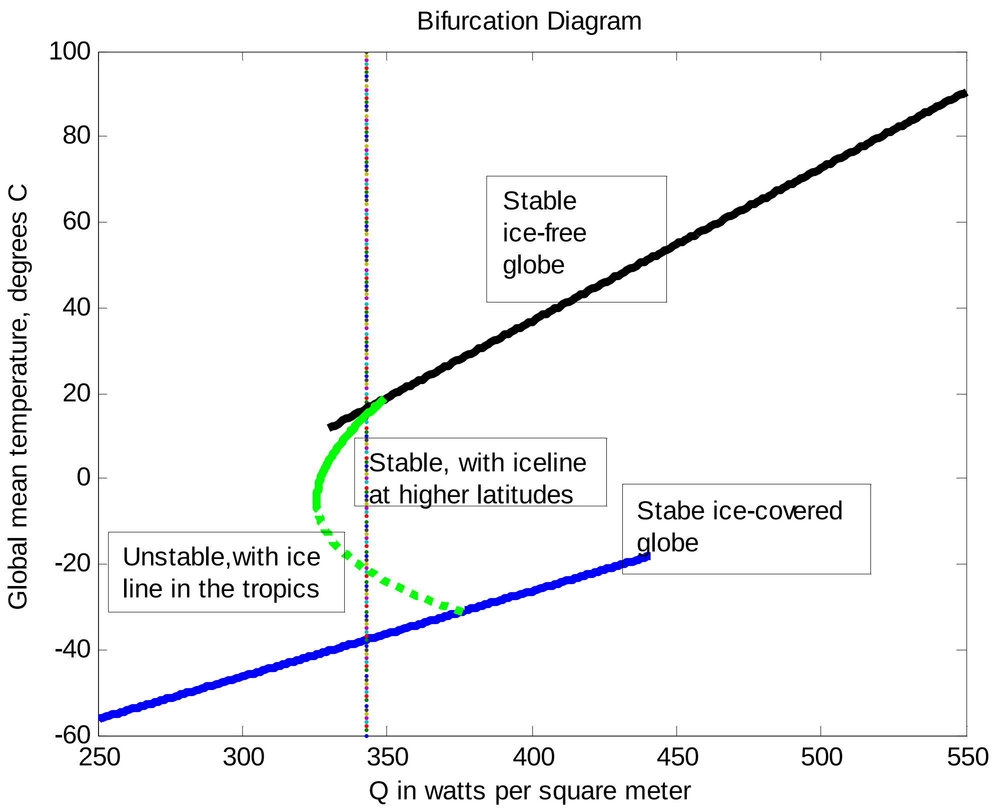

Figure 5. Bifurcation diagram of global mean equilibrium temperature vs Q, which is ¼ of the solar constant. The current Q=343 is indicated with a vertical dotted line. At this value of Q there are four equilibrium solutions, the top one is ice free, the lower one is ice covered, and the middle two have partial ice cover. The current climate has the ice line at high latitudes, and there is another equilibrium with the ice line in the tropics (which turns out to be unstable)

4. Stability

Let Eq. (6) be written as

$$R\frac{d}{dt}\overline{T} = G(\overline{T}),$$

where we have symbolically let, $G(\overline{T}) = Q(1-\overline{\alpha}) - A - B\overline{T}$.

The equilibrium solutions are given by the zeroes of *G* and have been denoted with an asterisk. We perturb the temperature slightly from that equilibrium while holding all the parameters (such as *Q*, *A* and *B*) the same as in the equilibrium solution. We write

$$\overline{T} = \overline{T}^* + u(t).$$

We furthermore expand *G* in a Taylor series and drop terms of order u^2 and higher (This process of approximating a nonlinear function by a linear function is called *linearization*):

$$G(\overline{T}) = G(\overline{T}^* + u) \approx G(\overline{T}^*) + \frac{dG}{d\overline{T}}(\overline{T}^*)u = \frac{dG}{d\overline{T}}(\overline{T}^*)u,$$

where

$$\frac{dG}{d\overline{T}}(\overline{T}^*) = -B - Q\frac{d\overline{\alpha}}{d\overline{T}}(\overline{T}^*)$$

from the expression for G. Furthermore, differentiating the equilibrium solution (8) with respect to itself, we get:

$$B = \frac{dQ}{d\overline{T}^*}(1-\overline{\alpha}) - Q\frac{d\overline{\alpha}}{d\overline{T}^*}.$$

(Since the equilibrium solution $\overline{T}^*$ is a function of Q, Q therefore depends on $\overline{T}^*$.)
Therefore

$$\frac{dG}{d\overline{T}}(\overline{T}^*) = -(1-\overline{\alpha})\frac{dQ}{d\overline{T}^*}$$

4.1. The slope-stability theorem.

Thus the time-dependent equation governing the temperature perturbation is:

$$R\frac{d}{dt}u(t) = -\gamma u(t),$$

where we have let $\gamma \equiv (1-\overline{\alpha})\frac{dQ}{d\overline{T}^*}$. Its sign depends on the sign of $\frac{dQ}{d\overline{T}^*}$.
The solution to the equation above is

$$u(t) = u(0)\exp\{-\frac{\gamma}{R}t\}. \tag{15}$$

So the small perturbation will decay in time if γ is positive. In this case the equilibrium is stable to small perturbations. If γ is negative, the small perturbation will grow larger; the equilibrium is unstable to small perturbations. We now have obtained the so-called *slope-stability theorem* (Cahalan and North (1979)):

$\frac{dQ}{d\overline{T}^*}$>0: stable

$\frac{dQ}{d\overline{T}^*}$<0: unstable.

This result was first obtained by Budyko (1972) using intuitive arguments.

One can examine the bifurcation diagram in Figure 5, and see that the equilibrium solution branch with the positive slope is stable, while that with the negative slope is unstable. With the slope of the bifurcation diagram yielding information on the stability of the equilibrium solution, this diagram is thus seen to be doubly useful. We see that the branch for the ice-free solution and the branch for the ice-covered solution have positive slopes, and therefore we conclude that these two scenarios are stable. For a globe partially covered by ice, it appears that once the ice sheet gets large enough it becomes unstable. Our current climate with the ice sheet at high latitudes is stable. (Some of you may have taken a course on dynamical systems theory and have learned that in a bifurcation diagram stable equilibrium points have to be separated by an unstable equilibrium. You may therefore think that there is something wrong with our result that in the bifurcation diagram depicted in Figure 5, the stable ice-free equilibrium is next to the stable equilibrium with the ice line located at high latitudes, without an unstable equilibrium in between. Our result is not wrong. The explanation, for those interested, is that our solution was constructed with different values of y_s. In fact you can view Figure 5 as composed of many solutions each with a different value of y_s.)

Alternatively, one can differentiate the equilibrium temperature with respect to Q and obtain the slope analytically. This will be done in the following two subsections. They can be skipped if you are satisfied with the numerical/graphical solution depicted in Figure 5.

4.2 The stability of the ice-free and ice-covered globe.

Examining the bifurcation diagram in Figure 5, we see that both the ice-free globe and the ice-covered globe correspond to stable equilibria. One can also show explicitly, by differentiating (8) with respect to Q, that :

$$\frac{d\overline{T}^*}{dQ} = \frac{(1-\alpha_i)}{B} > 0,$$

for either the ice-free case ($i=1$) or the ice-covered case (i=2), thus satisfying the condition for stability. Note that this stability condition is independent of many factors affecting the current climate and independent of *C,* hence our parameterization of dynamical transport.

4.3. Stability and instability of the partially ice-covered globe.

For the case of a partially ice-covered globe, we again differentiate Eq. (8) with respect to Q, but this time we note that $\overline{\alpha}$ is a function of y_s, which depends on Q:

$$B\frac{d\overline{T}^*}{dQ} = (1-\overline{\alpha}) + Q\cdot(-\frac{d\overline{\alpha}}{dy_s})\frac{dy_s}{dQ}. \tag{16}$$

We know

$$\frac{d\overline{\alpha}}{dy_s} = -(\alpha_2 - \alpha_1)[1 - 0.482y_s - 0.241(y_s^{\ 2} - 1)]$$

is always negative. This is consistent with our intuition that as the ice sheet retreats poleward, exposing darker surfaces, the overall albedo of the earth will decrease. It then follows that if dy_s/dQ is positive (that is, the ice line would retreat poleward with an increase of solar constant), (16) will be positive and the equilibrium solution will be stable. Differentiating (10) with respect to y_s, we find:

$$0=\frac{dQ}{dy_s}[s(y_s)(1-\alpha_0)+\frac{C}{B}(1-\overline{\alpha})]+Q[-3\cdot 0.482y_s(1-\alpha_0)-\frac{C}{B}\frac{d\overline{\alpha}}{dy_s}].$$

This can be used to solve:

$$\frac{1}{Q}\frac{dQ}{dy_s}=[1.45y_s(1-\alpha_0)+\frac{C}{B}\frac{d\overline{\alpha}}{dy_s}]/[\ s(y_s)(1-\alpha_0)+\frac{C}{B}(1-\overline{\alpha})]. \tag{17}$$

Substituting (17) into (16), we find:

$$B\frac{d\overline{T}^*}{dQ}=(1-\overline{\alpha})+(-\frac{d\overline{\alpha}}{dy_s})\frac{[s(y_s)(1-\alpha_0)+\frac{C}{B}(1-\overline{\alpha})]}{[1.45y_s(1-\alpha_0)+\frac{C}{B}\frac{d\overline{\alpha}}{dy_s}]}.$$

Therefore the decay rate in (15) is:

$$\gamma\equiv(1-\overline{\alpha})\frac{dQ}{d\overline{T}^*}=\frac{[1.45y_s(1-\alpha_0)+\frac{C}{B}\frac{d\overline{\alpha}}{dy_s}]/B}{(-\frac{d\overline{\alpha}}{dy_s})s(y_s)(1-\alpha_0)+1.45y_s(1-\alpha_0)(1-\overline{\alpha})}. \tag{18}$$

So γ changes sign when the numerator in the above expression changes sign. This occurs when

$$1.45(1-\alpha_0)y_s=\frac{C}{B}(-\frac{d\overline{\alpha}}{dy_s}). \tag{19}$$

[Note that the radiative equilibrium solution, obtained by setting the dynamical transport *C* to zero, yields a positive numerator. Hence (18) is always positive for that solution. The radiative equilibrium solution is stable wherever the ice line. It is the dynamical transport which destabilizes the ice-albedo feedback.] In the presence of nonzero transport, *C*, there are two roots to the quadratic equation (19), one positive and one negative. The positive root is

$$y_s=-[1+3\frac{(1-\alpha_0)}{(\alpha_2-\alpha_1)}\frac{B}{C}]+\sqrt{[1+3\frac{(1-\alpha_0)}{(\alpha_2-\alpha_1)}\frac{B}{C}]^2+5.15}\approx 0.56.$$

i.e. about 34^{o} latitude. (18) is positive if the ice line is located poleward of this latitude, and the equilibrium solution is stable.

Luckily our present climate, with the ice line located at 72^{o} latitude, is stable, according to this simple model. One way to gauge how complacent we can be is to ask: How many percent can Q change from the value of our current climate before our climate is moved from the stable equilibrium to the unstable equilibrium? In other words, how much must Q change to move the ice line from 72 to 34 degrees of latitude? This problem is left to Exercise 5. You will be surprised by how small this value is. Once at 34 degrees, the ice albedo feedback will initiate a runaway freeze.

Since the stability property depends critically on dynamical transport, and our treatment of transport is admittedly very crude, the above result may change with better models. Nevertheless, about the same conclusions were obtained by North (1975) using a model with diffusive heat transport (see Exercise 6), including the result that the ice-free globe, the ice-covered globe and the present climate are stable, and that it becomes unstable when the ice sheet advances to near the tropics. [Some more recent general circulation models incorporating detailed atmospheric circulations and ice dynamics appear to show that a very narrow band of water on the equator may remain ice-free even when our simple model predicts a snowball earth This open water might have provided a refuge for multicellular animals. See Hide *et al* (2000).]

4.4. How does a snowball earth end?

If for some reason the ice sheet advances past 34 degrees, the solution would become unstable. The ice sheet will then advance all the way to the equator, reaching the stable equilibrium of a snowball earth. Considering the fact that the sun's output 600 million years ago was 6% less than the present value, we see that the possibility of the ice sheet advancing into the tropics and then all the way to the equator is rather real.

Once the earth is completed glaciated, the above simple analysis suggests that it would remain so. The global temperature would plummet to less than -42^{o} C (remember that the sun was 6% fainter 600 million years ago). The earth could not escape its ice-encased tomb unless the solar constant increased by more than 40% (from Q= 322 to Q> 450 watts per square meter), which we know had not happen.

On the other hand, for the same solar input, the atmosphere could have warmed up by increasing its greenhouse effect, which lowers its emissivity δ. This has the effect of lowering the parameters A and B, which are here calibrated using the present value of emissivity. Caldeira and Kastings (1992) investigated the effect of varying amounts of carbon dioxide concentration in the atmosphere, measured by its partial pressure, pCO_2 , on the Outgoing-Longwave-Radiation: $I=A+BT$. Using results from 2000 runs of radiative equilibrium calculation with different carbon dioxide partial pressures, they fitted the constants A and B as a function of $\varphi \equiv \ln(pCO_2 /(pCO_2)ref)$, where $(pCO_2)ref$ is a reference value corresponding to the present value of CO_2 at 300 parts per million:

$$A = -326.4 + 9.161\varphi - 3.164\varphi^2 + 0.5468\varphi^3 \quad \text{watts per square meter}$$
$$B = 1.953 - 0.04866\varphi + 0.01309\varphi^2 - 0.002577\varphi^3 \quad \text{watts per square meter per } {}^{o}\text{K}.$$

Setting φ=0 should give close to our current value of A and B.[Note that the authors used degrees K for their T instead of our degree C, and so one should add 273B to their A to convert into our A.]

Let g be the factor that A and B must be reduced by from its current value so that

Q/g>441 watts per square meter.

Therefore g must be less than 73% if Q is at 322 watts per square meters. It was estimated that the needed carbon dioxide concentration in the atmosphere would have been 400 times the present concentration!

5. Evidence of a snowball earth and its fiery end.

Brian Harland of Cambridge Universtiy was the first to suggest, in the early 1960s, that the earth experienced a "Great Infra-Cambrian Glaciation" 600 million years ago. He came to this conclusion by noting that glacial deposits where found in rocks dated to that period (called the Neoproterpzoic period by geologists) across virtually every continent on earth. In particular, Harland found glacial deposits within types of marine sedimentary strata characteristics of low latitudes. There has not been evidence of ice at sea level at the equator again since that time. Today we find glaciers at equator only more than 5000 meters above sea level, above the Andes and Mt. Kilimanjaro. It came down to no lower than 4000 meters above sea level during the last ice age.

In 1992, Joseph Kirschvink of Caltech suggested that carbon dioxide supplied by volcanoes might have been what was needed for the earth to escape from its icy tomb, which otherwise might have been permanent as we have inferred from the Budyko model. In an ice-covered earth, the normal process of removing carbon dioxide from the air would have been absent, while the input from volcanoes continued. There would not have been any evaporation and thus no rain or even snow in such a cold climate. Rain acts in our present climate to wash carbon dioxide from the air. The weathering of silicate rocks on land converts carbon dioxide to bicarbonate, which, when washed to the oceans, becomes carbonate sediments. It was estimated by Hoffman *et al* (1998) that with such a removal process shut down, it would have taken the volcanoes ten million years to build up the carbon dioxide level in the air---400 times the present level----to initiate a hyper greenhouse which was capable of melting the snowball earth.

From their field observations of rock cliffs in Namibia and elsewhere, Paul Hoffman and his colleagues found that Neoproterozoic glacial deposits are almost always capped by carbonate rocks, which typically form in warm water, and the transition from glacial deposits to the cap carbonates is abrupt, in perhaps a few thousand years. In their 1998 *Science* article, Hoffman *et al* pieced these and other (isotopic) evidence together, and suggested that the cap carbonate sediments must have formed in the aftermath of the snowball earth: rain in an atmosphere high in carbon dioxide would be in the form of acid rain, accelerating the erosion of rocks on exposed land. The sediments were then deposited at the bottom of the shallow seas, forming the observed cap carbonate rocks. Thus there is evidence of both a snowball earth and its abrupt end in a hot greenhouse.

Figure 6 : Daniel Shrag (left) and Paul Hoffman (right) point to a layer of abrupt cap carbonate rocks above a layer of glacial marine dropstones in Namibia. Taken from http://www-eps.harvard.edu/people/faculty/hoffman/Snowball-fig11.jpg

6. Exercises

1. *Radiative equilibrium temperature*
   Determine the radiative equilibrium temperature distribution as a function of y for the current climate, with the ice line located at y_s=0.95. The radiative equilibrium solution is the solution of Eq. (7) with no dynamical transports.
   (a) Plot such a solution. Is such a temperature distribution consistent with an ice edge located at y=0.95? Why?
   (b) If y_s is not fixed at the present value, but is allowed to vary so that the temperature is greater than T_c to the south of the ice edge and greater to the north of the ice edge. Where would such a location be?

2. *Stability of radiative equilibrium temperature*
   Determine the stability of the radiative equilibrium solution to small perturbations. Does your result apply to finite perturbations?

3. *Stabilizing effect of dynamics*.

(a) Calculate how low must the solar input Q be for the onset of ice under radiative equilibrium.
(b) Do the same calculation as in (a) except now with transport C nonzero.
(c) Based on the results in (a) and (b), do you think the effect of dynamical transport of heat is stabilizing or destabilizing to the climate? Why do you think this is so? How can you reconcile this result with the known destabilizing effect of dynamics when the ice line moves past the midlatitudes?

4. *Unfreezing the snowball earth.*
If the earth is completely ice covered, what must the total solar input (Q) be increased to in order for the ice to melt at the equator. At that higher Q, is the partially ice covered climate stable? What is the eventual climate at that value of solar input?

5 *Sensitivity of our current climate*
to measure the sensitivity of our current climate to the catastrophe of a runaway freeze, calculate the percentage change in Q which is needed to move the ice line from its present location of 72 degrees to the unstable latitude of 34 degrees.

6. *Diffusive dynamical transport model*
A better form for the dynamical transport of heat from one latitude to the other is that of a diffusive process (see North (1975). His model of transport of heat is

$D(y) = \mu a^2 \Delta T,$
where μ is an empirical diffusion coefficient. The Laplacian operator in spherical coordinates is:

$$\Delta = \frac{1}{a^2}\frac{d}{dy}(1-y^2)\frac{d}{dy}.$$

When integrated over the globe the effect of transport should be zero. The radius of the earth is *a.*

(a) Find the equilibrium solution $\overline{T}^*(y)$, for an ice-free globe. Assume a power series solution of the form: $\overline{T}^*(y) = a_0 + a_1 y + a_2 y^2 + \ldots$.
[Note that powers of y higher than 2 are not needed; neither are odd powers of y.] In addition, find the consistency condition on Q so that the temperature at the pole is above that for glaciation (T_c).

(b) Repeat (a) but for an ice-covered globe. In this case find the consistency condition on Q so that the temperature at the equator is lower than that for glaciation.

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Chapter 2.

Mathematics required: *Taylor series expansion. Chapter 1.*

The Global Warming Controversy

1. Introduction

We now turn to a problem closer to our time. We have already gained some sense from discussions on paleo-climate that our planet has a very sensitive climate system. Small radiative perturbations can lead to dramatic changes in our climate through feedback processes. We have discussed the ice-albedo feedback process in relation to the onset of *snowball earth*, and mentioned briefly the greenhouse effect of carbon dioxide from cumulative emissions by volcanoes in deglaciating the planet. We now want to study the greenhouse effect more quantitatively.

Carbon dioxide is but one of many greenhouse gases naturally occurring in our atmosphere, the others being methane, nitrous oxide, and more importantly water vapor. These greenhouse gases are what are responsible for our current global temperature of 15 degrees C. Without them, our global temperature would have been a chilly -17 degrees C. Prior to the Industrial Revolution, carbon dioxide concentration is probably around 280 parts per million of air, but has since increased rapidly. In U.S. carbon dioxide constitutes about 80% of all anthropogenic emissions of greenhouse gases, and is currently increasing at the rate of 2% per year.

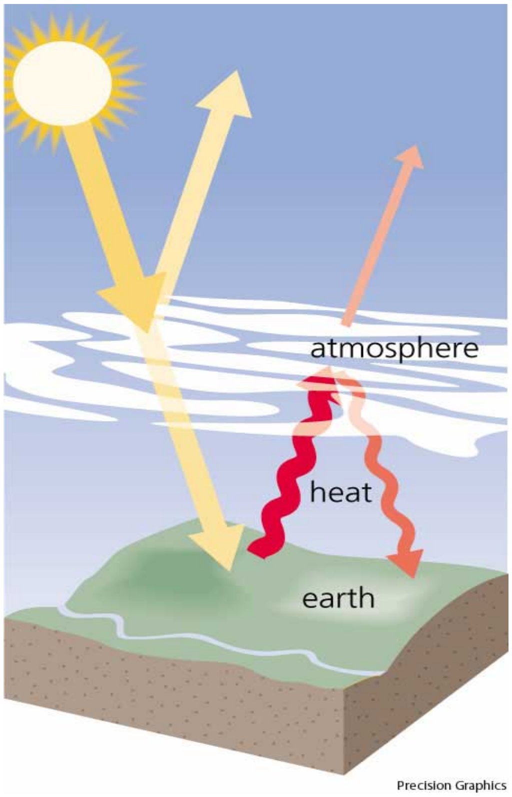
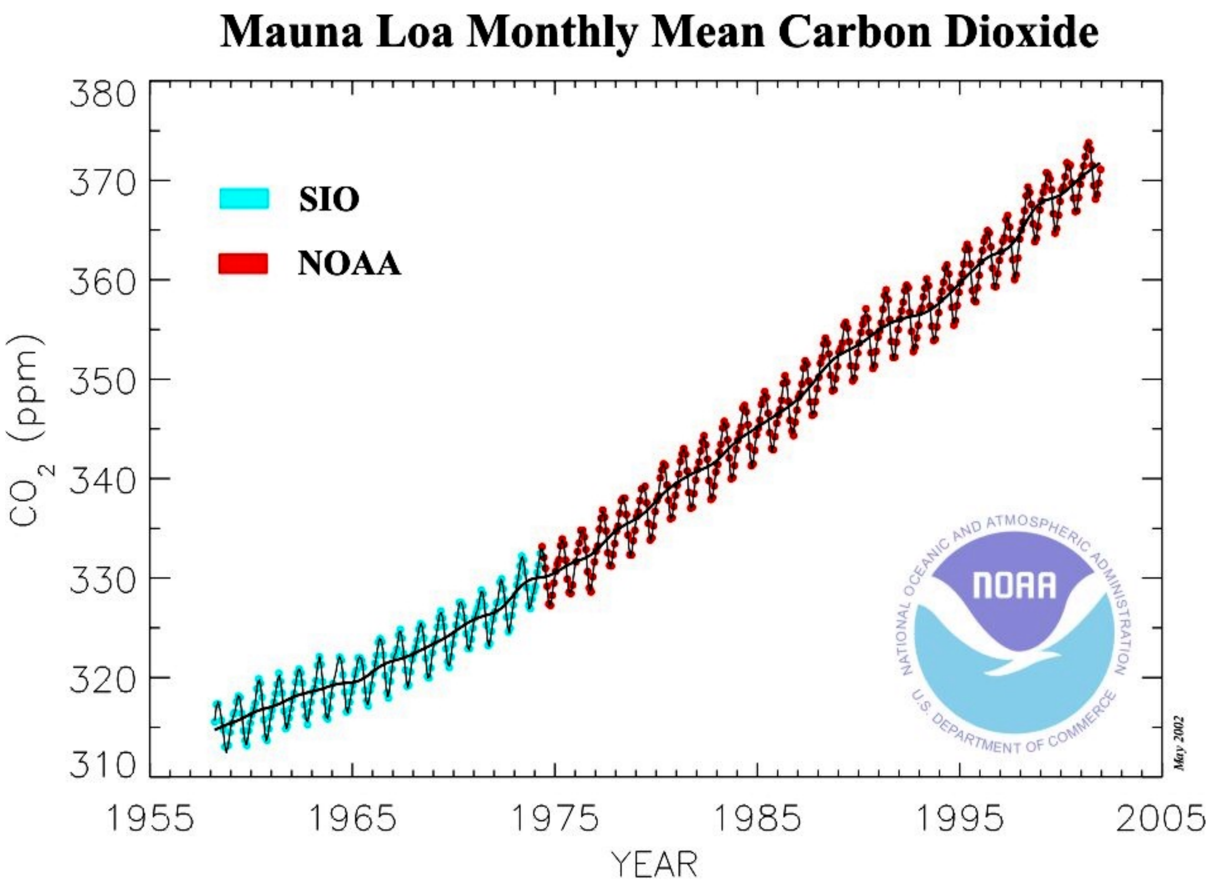


Figure 1. An atmosphere with greenhouse gases such as carbon dioxide traps more of the outgoing radiation from earth's reemission, increasing the warming.
Taken from http://www.yourdictionary.com/images/ahd/jpg/A4grneff.jpg



Atmospheric carbon dioxide monthly mean mixing ratios. Data prior to May 1974 are from the Scripps Institution of Oceanography (SIO, blue), data since May 1974 are from the National Oceanic and Atmospheric Administration (NOAA, red). A long-term trend curve is fitted to the monthly mean values. Principal investigators: Dr. Pieter Tans, NOAA CMDL Carbon Cycle Greenhouse Gases, Boulder, Colorado, (303) 497-6678, ptans@cmdl.noaa.gov, and Dr. Charles D. Keeling, SIO, La Jolla, California, (616) 534-6001, cdkeeling@ucsd.edu.

Figure 2: Measurement of atmospheric carbon dioxide at Mauna Loa in Hawaii. The vertical axis is its concentration in parts per million of air. The horizontal axis is the year.

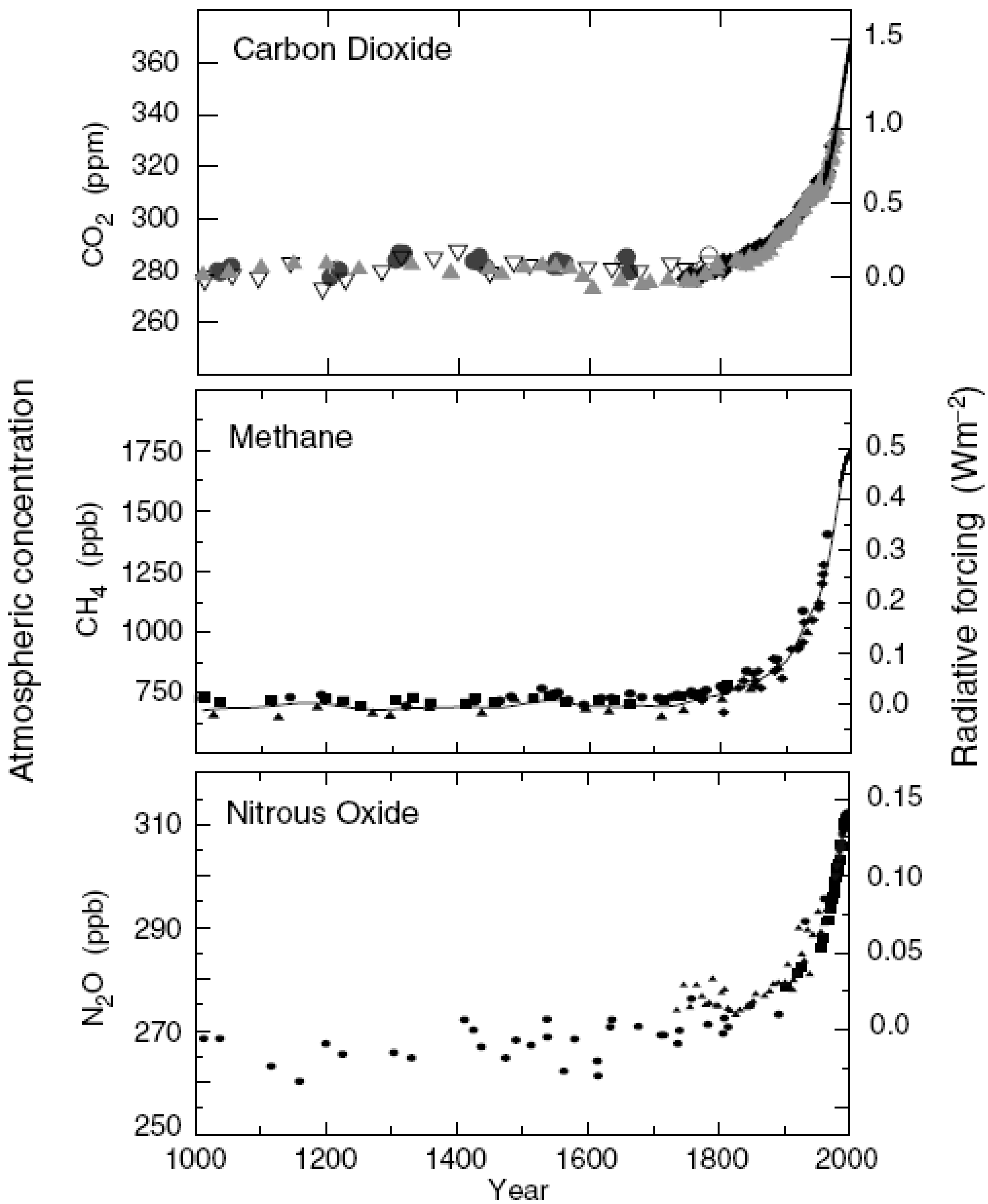


Figure 3. From IPCC (2001): Records of atmospheric concentration of carbon dioxide, methane and nitrous oxide for the past 1000 years. Ice core and firn data from several sites in Antarctica and Greenland (shown by different symbols) are supplemented by direct atmospheric samples over the past few decades. The estimated radiative forcing from these greenhouse gas changes are indicated on the right-hand scale.

There is no controversy concerning the fact that the carbon dioxide concentration in the atmosphere is increasing steadily. Measurements at the pristine mountain top of Mauna Loa shows in Figure 2 a steady increase from 310 parts per million of air in 1958 to our current concentration of 375 ppm. (There is a pronounced seasonal cycle in the carbon dioxide emissions, as plants suck up more carbon dioxide during the summer growing season. In fall the decay of leaves releases some carbon dioxide back to the atmosphere. We are interested in the annually averaged value, denoted by the black line through the seasonal fluctuations.) There is even evidence from ice cores (see Figure 3) that the atmospheric carbon dioxide concentration hovered around 280 ppm for over a thousand years prior to 1800, and then increased rapidly since.

Like a greenhouse, which admits short-wave radiation of the sun through its glass, but traps within the greenhouse the infra-red re-emission from inside of the house, the greenhouse gases in the atmosphere warms the lower atmosphere of the earth by keeping in more of the infra-red reemission from the ground. It has been estimated that this is equivalent to an additional net radiative heating (δF) of the lower atmosphere of 3.7 watts per square meter (more specifically $\delta F = \delta Q\ (1-\alpha) \sim 3.7$ watts per square meter) (Hansen et al, 2005).

The controversy centers around the following quantitative question: If the carbon dioxide concentration in the atmosphere is doubled, say, from its pre-industrial value of 280 ppm, how much warmer will the global temperature be? This question can be phrased either as an equilibrium response or as a time-dependent response. A back-of-envelope equilibrium response can be obtained from:

$$\delta T \sim \delta F/B \sim 3.7/1.9 \sim 1.9 \text{ degrees C.}$$

Yet the range of model predictions of global warming due to a doubling of carbon dioxide is $\delta T \sim 1.5$ to 4.5 degrees C. Despite intense efforts of hundreds of climate modelers and two Inter-government Panels on Climate Change (IPCC (1990) and IPCC(2001)), this large range of uncertainty remains almost unchanged for more than two decades. The latest IPCC report (IPCC(2007)) concluded that warming of less than 1.5 C is very unlikely, and suggested a likely range of 2.0-4.5 C. While a warming of a globe of 4.5 degrees C may be alarming and would be a cause of concern and call for action, an eventual warming of 2 degrees may be more benign to human society. Given the high cost of the proposed remedy (involving drastic curbs on the burning of fossil fuels) to nations' industrial production and development, the large scientific uncertainty from model predictions fuels political debates on whether nations should undertake immediate action to curb carbon dioxide emission despite the cost.

In the next section we will use the simple climate model developed so far to try to understand the source of the uncertainty, and discuss a possible way to reduce it. There is even greater uncertainty concerning the time-dependent solution, because it involves the thermal inertial of the atmosphere and oceans. There is however a greater need to understand the time-dependent solution, because it is more relevant.

2. A simple equation for climate perturbation

We again use the time-dependent annually averaged energy balance climate model of Held and Suares (1974) governing the near-surface atmospheric temperature *T(y,t)*:

$$R\frac{\partial}{\partial t}T = Qs(y)(1-\alpha(y)) - (A+BT) + \nabla\cdot(HeatFlux)\,, \qquad (1)$$

where Q is ¼ of the solar constant, s(y) its distribution with respect to latitude, globally normalized to unity, y=sin (latitude), and α(y) is the albedo-- the fraction of the sun's radiation reflected back to space by clouds and surface. ($A+BT$) is the linearized form of the infrared emission of the earth to space fitted from observational data on outgoing longwave radiation (Graves et al, 1993), with $A=202$ *watts* m^{-2}, and $B=1.90$ *watts* m^{-2} C^{-1} for our current climate. The parameter R in Eq.(22) represents the thermal capacity of the atmosphere-ocean climate system. Its value is uncertain and that has prevented the use of the time-dependent version of this equation. Dynamical transport of heat is written in the more general form of a divergence of heat fluxes. Nevertheless its global average vanishes. We will consider here the global average to Eq. (1), which is:

$$R\frac{\partial}{\partial t}\overline{T} = Q(1-\overline{\alpha}) - (A + B\overline{T}), \tag{2}$$

where an overbar denotes global average, and $\overline{\alpha} = \frac{1}{2}\int_{-1}^{1}\alpha(y)s(y)dy$ is the weighted global average albedo. The overbar is henceforth dropped for convenience.

Considering small radiative perturbation δQ in $Q=Q_0+\delta Q$, the equation governing the small temperature perturbation can be obtained from the first variation of the above equation. We write

$T = T_0 + \delta T$,

where T_0 is the unperturbed temperature and δT is the perturbation temperature response to the perturbation in heating δQ. Linearizing Eq. (2) (using a Taylor series expansion in T about T_0) then leads to the following perturbation equation:

$$R\frac{\partial}{\partial t}\delta T = (1-\alpha)\delta Q - B\delta T - (\frac{\partial}{\partial T}A)_0\delta T - (T\frac{\partial}{\partial T}B)_0\delta T - (Q\frac{\partial}{\partial T}\alpha)_0\delta T .$$

This can be rewritten as

$$\begin{aligned}
&B\tau\frac{\partial}{\partial t}\delta T = (1-\alpha)\delta Q - B\delta T / g,\\
&where\\
&\tau \equiv R / B\\
&g = 1/(1-f),\\
&f = f_1 + f_2,\\
&f_1 = (-T\frac{\partial}{\partial T}B - \frac{\partial}{\partial T}A)_0\\
&f_2 = (-Q\frac{\partial}{\partial T}\alpha)_0
\end{aligned} \tag{3}$$

The factor g is the controversial climate gain; it amplifies any equilibrium response to radiative perturbation by a factor g (see below). f_1 represents the water-vapor feedback and f_2 represents ice and snow albedo feedback. Cloud feedback has effects in both f_1 and f_2 .

The water-vapor feedback factor is potentially the largest and therefore the most controversial. The cloud feedback is the most uncertain; even its sign is under debate. Note that various feedback processes can be superimposed in f (but not in g).

Water-vapor feedback:
When the surface warms, it is natural to expect that there will be more evaporation and hence more water vapor will be present in the atmosphere. Since water vapor is a natural greenhouse gas, one expects that the initial warming may be amplified, i.e. the factor g should be greater than 1. In one of the earliest models of global warming in 1967, Manabe and Weatherald of Princeton's Geophysical Fluid Dynamics Laboratory made the simplifying assumption that the relative humidity of the atmosphere remains unchanged when the atmosphere warms. This is in effect saying that the atmosphere can hold more water vapor if it is warmer. The presence of this additional greenhouse gas (i.e. water vapor) would amplify the initial warming, and double it. That is, the climate gain factor is $g_1 \sim 2$ due to water-vapor feedback alone. This implies a feedback factor of $f_1 \sim 0.5$. This result appears to have stood the test of time. Most modern models yield water-vapor amounts consistent with this prediction. However, that most models tend to have similar water-vapor feedback factors does not necessarily mean that they are all correct; there are still controversies.

Cloud feedback:
Cloud tops reflect visible sunlight back to space. Therefore, more clouds imply higher albedo and cooling. However, clouds also behave like greenhouse gases in trapping infra-red radiation from below. Clouds are actually the second most important greenhouse gas, after water vapor but ahead of carbon dioxide. The cancellation of the albedo effect and the greenhouse effect of the clouds differs in different climate models, and represents the greatest uncertainty in these models. As a consequence, even the sign of the cloud feedback is uncertain, although typical values in some climate models are around ~0.1 for the f factor. It is probably close to zero.

Ice-snow albedo feedback:
As the surface warms, snow or ice melts, exposing the darker surface underneath, thus lowering the albedo and increasing the absorption of sun's radiation. This is a positive feedback process and is probably more important at high latitudes than at low latitudes. It may explain the higher sensitivity of the polar latitudes to global warming. On a globally averaged basis it is probably between 0.1 to 0.2 for the f factor.

Total climate gain:
Adding all the feedback processes yields $f \sim 0.7$ in most climate models. This then yields a climate gain factor of

$$g=1/(1-f)\sim 3.$$

As noted previously, this number is uncertain.

Using observation to infer climate gain:
The sun's radiation is observed to vary slightly over an 11-year cycle. This is related to the appearance of darker sunspots on the surface of the sun. Sunspots have been

observed since ancient times (mostly in the Orient, because of the belief by ancient Greeks and Roman Catholics that the Sun should be an unblemished perfect sphere) , but an accurate measurement of its radiative variation was not available until recently, when starting in 1979 satellites can measure the solar constant *S* above the earth's atmosphere. It is found that the solar constant varies by about 0.1% over a solar cycle. The atmosphere's response near the surface to this solar cycle variation has also been measured to be about 0.2 degrees C on a global average. This information can be used to infer a climate gain factor. This is left as an exercise (in Exercises 1 and 2). This leads to $g \sim 3$.

3. Equilibrium Solutions

3.1 Equilibrium global warming

Setting the time derivative to zero, the steady state solution to (3) is:

$$(\delta T)_{eq} = \frac{(1-\alpha)\delta Q}{B} g = \frac{\delta F}{B} g \equiv (\delta T)_{eq}^{0} g. \tag{4}$$

The solution shows prominently the climate gain factor *g* in amplifying the equilibrium response to a given radiative forcing. For an "adjusted radiative forcing" due to doubling CO_2 of $\delta F=(1-\alpha)\delta Q=3.7$ *watts* m^{-2}, the expected global warming is $(\delta T)_{eq}^{0} = \delta F/B = 1.9$ *C* without the amplifying factor, but could be close to 6 C with the amplifying factor of g=3.

The range of current model predictions of 1.5-4.5 C indicates that the various models have different feedback mechanisms and that their climate gain factors have an uncertainty by a factor of about 3.

3.2 Uncertainty in the climate gain factor

Roe and Baker (2007) at University of Washington recently pointed out in a *Science* article why this large uncertainty in the climate gain factor persists so stubbornly despite intense efforts in trying to reduce it for three decades. Suppose there is an uncertainty in the feedback factor *f*. This uncertainty then translates into the climate gain factor *g* through their relationship:

$$g = \frac{1}{1-f}.$$

The uncertainty in a climate that is more sensitive, i.e. having a value of *f* closer to 1, will be amplified much more. As a result, the high end of the predicted warming, e.g. for $(\delta T)_{eq}$ >4.5 C, is highly uncertain. We cannot reasonably rule out warmings of 8 or 12 C. This is illustrated graphically in Figure 4, taken from their paper. Mathematically, let $f = \overline{f} + \sigma$, where σ is the small uncertainty about a mean value $\overline{f}$. Then

$$g = \frac{1}{1-\overline{f}-\sigma} \simeq \frac{1}{(1-\overline{f})} + \frac{\sigma}{(1-\overline{f})^2},$$

via a Taylor series expansion. Then the climate gain factor g is uncertain about its mean value of $\bar{g} = \frac{1}{(1-\bar{f})}$ by $\delta g = \frac{\sigma}{(1-\bar{f})^2} = \bar{g}^2\sigma$. For $\bar{g}\sim 3$, the uncertainty in g is about a factor of 10 larger than the uncertainty in f. Consequently, trying to narrow the uncertainty in the feedback processes (lowering σ) by better understanding the physics involved will help lower the overall uncertainty, but not by much in g especially for a sensitive climate like ours. That is why it is useful to use other independent observations to constrain g directly (see Exercises 1 and 2).

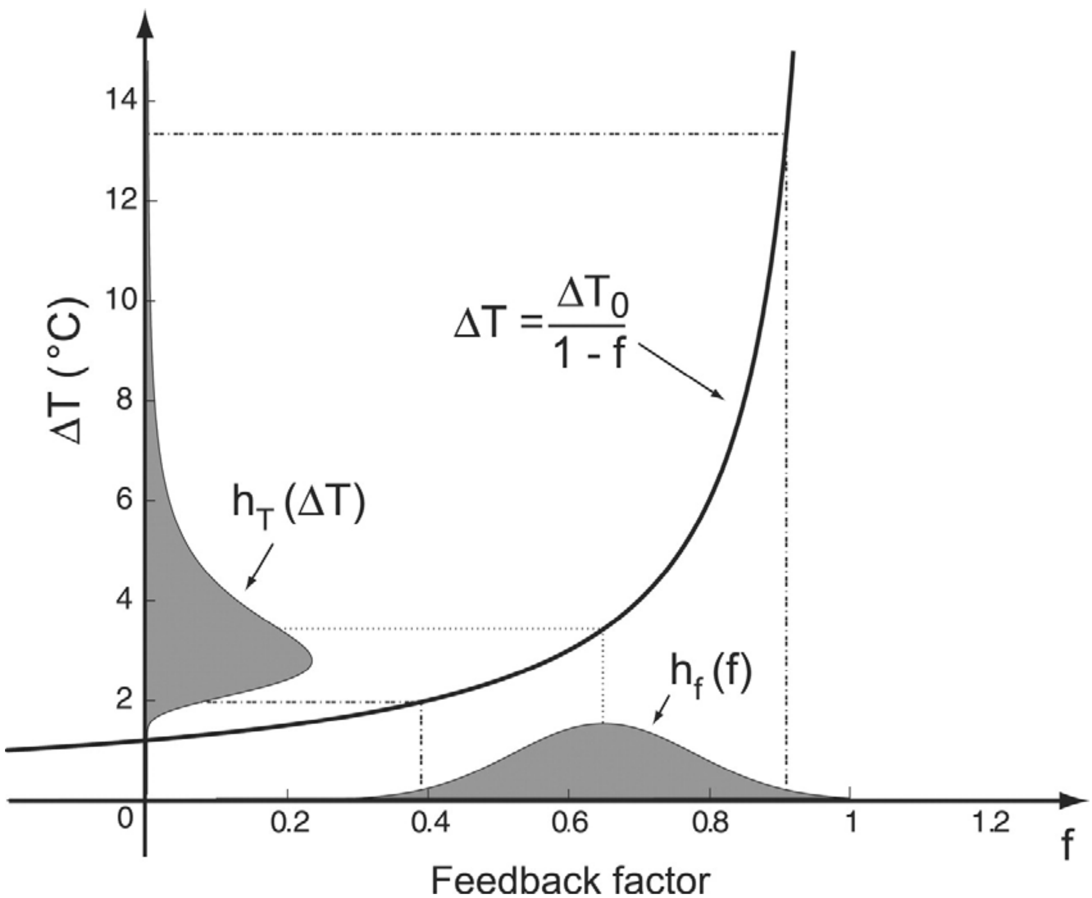


Figure 4. A plot of the equilibrium warming due to a doubling of CO_2 in the y-axis, vs the feedback factor f. ΔT_0 is the sensitivity in the absence of feedbacks. If the mean estimate of the total feedbacks is substantially positive, any distribution in the uncertainty $h_f(f)$ will lead to a highly skewed distribution in ΔT. For the purposes of illustration, a normal distribution in $h_f(f)$ is shown with a mean of 0.65 and a standard deviation of 0.13, typical to that obtained from feedback studies of GCMs. The dot-dashed lines represent 95% confidence intervals on the distributions. Note that values of $f \geq 1$ imply an unphysical, catastrophic runaway feedback. Taken from Roe and Baker (2007).

4. Time-dependent solution

4.1 Transient warming

Growth phase:

As a model for the increase in greenhouse gases we assume that their radiative forcing increases linearly since 1800, the year we call $t=0$:

$$\delta F(t)= bt, \text{ for } t>0$$

and $$\delta F(t)= 0, \text{ for } t<0. \tag{5}$$

This is the model considered by Hartmann (1994). It leads to a linearly increasing global warming. Given the recent accelerated warming, a model giving rise to an exponentially increasing temperature may be more appropriate. That model is discussed in Exercises 3 and 4. (Note however that from quantum mechanics the radiative heating by carbon dioxide varies as the logarithm of its concentration, and so the greenhouse gas has to increase faster than exponential to cause an exponentially increasing radiative heating.) Staying with Eq. (5), the solution to the time-dependent equation (3) is now obtained. In view of the form of the forcing term, we assume the solution to consist of homogeneous plus particular solution. The particular solution is of the form:

$$\delta T_{particular}= at-c,$$

and the homogeneous solution is of the form:

$$\delta T_{\text{homogeneous}}= c\exp\{-t/(g\tau)\}.$$

The two c's are of opposite sign so as to satisfy the initial condition that the total temperature perturbation is zero at t=0. The constants a and c are found by substituting this assumed solution into Eq. (3). This yields, for the sum of homogeneous plus particular solution:

$$\delta T(t) = \frac{b\cdot(t-g\tau)}{B}g + \frac{bg\tau}{B}g\exp\{-\frac{t}{g\tau}\} \text{ for } t>0. \tag{6}$$

The last term diminishes in importance after an initial transient period. Then the solution can be written in the following more interesting form:

$$\delta T(t) \simeq \frac{\delta F(t-g\tau)}{B}g, \quad \text{for } t>g\tau. \tag{7}$$

It looks just like the equilibrium solution (4), except that it is evaluated at time t using the value of the radiative forcing at time $t-g\tau$. We call this the quasi-equilibrium solution with delay. The delay is given by $g\tau$.

Curbs in effect:

Suppose at some $t=t_s$ in the future all nations decide to implement a curb on emission of greenhouse gases. For simplicity we assume that the emission curbs are such that the concentration of the greenhouse gases in the atmosphere remains constant:

$$\delta F(t)=constant \;\; for \;\; t>t_s. \tag{8}$$

The solution for the constant forcing can be found using a particular solution which is a constant. This constant is found by substituting this trial particular solution into Eq. (3), yielding:
$\delta T_{particular} = \delta F(t_s)g/B$. The homogeneous solution is the same as before: $\delta T_{homogeneous} = c\exp\{-t/(g\tau)\}$, but now the constant c needs to be evaluated so that the solution at t_s matches that from Eq. (6). This yields:

$$\delta T(t) = \frac{\delta F(t_s)g}{B}(1-(\frac{g\tau}{t_s})\exp\{-\frac{t-t_s}{g\tau}\}) . \tag{9}$$

We see that eventually the warming will approach the equilibrium value predicted by Eq. (4), and that warming will be amplified by the climate gain factor g. However, it takes a time longer than $g\tau$ to reach that equilibrium. We have now a conclusion which is consistent with what other scientists have found using more complex computer model simulations, and is rather general:

> *The more sensitive the climate response (the larger climate gain factor), the larger will the global warming at equilibrium be. However it also takes longer to reach that equilibrium.*

Next we will try to determine how long is “long”.

4.2 Thermal inertia of the atmosphere-ocean system

Before we can gain any insight from the time-dependent solution, we need to estimate the thermal capacity $R = B\tau$ of the atmosphere-ocean system. This is very uncertain because we do not know how deep the warming would penetrate into the ocean. If the response of the climate system involves deep ocean circulations, the climate response time may be order of centuries. This is currently a subject of intense study using state-of-the-art coupled atmosphere-ocean general circulation computer models.

Because of the inertia, the radiative budget of our climate system at present is not balanced. That is, the earth currently receives more solar energy (in the first term on the right-hand side of Eq. (3)) than it radiates back to space (in the second term in that same equation). This radiative imbalance is measured and estimated in 2003 to be 0.85+-0.15 watts per square meter by Hansen et al (2005). The imbalance is due to the thermal inertia of our climate system. This we modeled by the left-hand side of Eq. (3). Since the right hand side of Eqs.(3) represents the difference between radiative input and output of the earth, the left hand side can be estimated from this measured imbalance, yielding, for 2003 values:

$$R\frac{\partial}{\partial t}\delta T \approx 0.85 \text{ watts m}^{-2} .$$

The earth has warmed globally by 0.6 C+-0.2 C is from 1880 to 2003 (IPCC 2001; Hansen et al, 2001). The time-like quantity τ can now be assigned a value:

$$\tau \equiv R/B \approx 0.85/[(1.90)(0.6/123)] \approx 90 \text{ years.}$$

Taking the extremes of the error bars in the parameter inputs, this estimate of τ can range from 57 years to 162 years.

As can be seen in the time-dependent solution (7), the lag time for the climate system response is not τ; instead it is $g\tau$, which is ~170-490 years for a climate gain factor of g~3.

> *It takes probably more than 200 years after the greenhouse gases have been curbed for our climate system to reach the predicted equilibrium! If the carbon dioxide is doubled and maintained at that level for 200 years, we will reach a global warming of about 6 degrees C. In the meantime, that predicted equilibrium warming is less relevant.*

This estimate of τ depends somewhat on our assumption of linear increase of temperature and greenhouse heating. An exponentially increasing model is considered in Exercises 3 and 4.

5. Transient Climate Response (TCR) vs Equilibrium Climate Sensitivity (ECS)

IPCC has adopted a peculiar way of defining climate sensitivity. It is defined as the globally averaged increase in temperature when CO_2 concentration in the atmosphere is doubled from its pre-industrial value, and has units of degrees of C. The Equilibrium Climate Sensitivity (ECS) for a model is the difference in temperature between two equilibrium runs, one with double the CO_2 than the other. Since it is at equilibrium, there is no transfer of heat between the atmosphere and the oceans, the ocean part of the model can be simplified, and it is often replaced by a slab. The Transient Climate Response (TCR) is the transient warming at the time when CO_2 is doubled in a scenario where CO_2 is increased at the rate of 1% per year. At a compound rate of increase of 1% per year, it takes about 70 years to reach doubling of the CO_2 concentration. In these transient runs, the ocean is colder than the atmosphere and so there is heat flux into the oceans. It is expected that the TCR is lower than the ECS, and the actual value is critically dependent on the ocean part of the model.

The simple model presented above, lacking an explicit representation of the ocean heat transport, should not be used quantitatively for climate sensitivity studies, especially for transient climate responses. It nevertheless can be used to illustrate the issues involved.

At doubled CO_2, the radiative heating change is $\delta F=3.7$ watts m^{-2}. The equilibrium warming is given by (4)

$$(\delta T)_{eq} = \frac{\delta F}{B} g$$

So ECS is essentially a measure of the climate gain factor g. It is independent of the ocean inertia.

It is a little more complicated for transient climate response. The simplest case is that of exponentially increasing temperature discussed in exercise 3, where

$$\delta F(t) = a\exp\{-bt\}$$
$$\delta T(t) = \frac{\delta F(t)}{B} g \frac{1}{1+bg\tau}.$$

Therefore the ratio of ECS and TCR at the time of doubling of CO_2 is given by:

$$\frac{ECS}{TCR} = 1 + bg\tau \sim 3,$$

since b=1% per year, and $g\tau$~200 years. So for an ECS of 6 C, the TCR is approximately 2 C. This ratio depends on the ocean inertia, and varies in different models. The larger the inertia, the smaller the transient warming response is relative to the equilibrium response for the same instantaneous heating. The TCR in the latest IPCC models range from 1.2 to 2.6 C.

The case of linearly increasing temperature is left to exercise 5.

6. Model deficiency

The model we have used in this chapter is the so-called zero-dimensional energy balance model. The greatest deficiency lies in the fact that there is no vertical distribution of radiative heating and its absorption, and the atmosphere and the ocean are lumped together. For long-term secular changes in heating, the heat should be conducted deeper and deeper into the lower and colder layer of the ocean, while for a periodic phenomenon such as the solar cycle, only the upper few tens of meters of the ocean above the thermocline are involved. The thermal inertia of the upper ocean is much smaller than the deeper ocean, and yet here there is only one thermal inertia. You will find as you go through the exercises that the thermal inertia is different for different phenomena. This deficiency can be overcome by introduce an ocean layer below the atmospheric layer, connected by a heat flux. Such a model can be found in Hansen *et al* (1985).

7. Exercises

1. Eleven-year solar cycle

The sun's radiant output fluctuates on an 11-year periodic cycle, which is modeled by: $Q=Q_0+\delta Q$, where $\delta Q(t)=a\cos(\omega t)$, with $\omega=2\pi/$(11 years). Solve the time-dependent equation (24) for the periodic temperature response of the atmosphere near the surface, $\delta T(t)$. Show that it can be written in the form:

$$\delta T(t) = \frac{\delta Q(t-\Delta)\cdot(1-\alpha)g/B}{\sqrt{1+\varepsilon^2}},$$

where

$\varepsilon=g\omega\tau$, and $\omega\Delta=\tan^{-1}(\varepsilon)$. Δ is the time lag of the response, and the factor in the denominator gives the reduction in amplitude from the equilibrium value because of the periodic nature of the response.

2. Climate gain inferred from climate's response to solar cycle

The variability of the sun's radiation through the 11-year solar cycle has been measured since 1979 by earth orbiting satellites. We know that the solar constant varies by 0.08% from solar minimum to solar maximum. Referring to the solution in Exercise 1, we know that $2a/Q_0=0.08\%$. So $2a(1-\alpha)=0.18$ watts per square meter. The atmosphere's temperature response is found to lag only slightly (by about 1 year) and its magnitude is measured near the surface to be about 0.2 degrees C on a global average from minimum to maximum. Use these values to deduce the climate gain factor *g*.

3. Time-dependent global warming

We consider the scenario of a period of radiative perturbation growing with rate *b*, i.e.

$\delta F(t) = a \exp(bt)$ *for* $-\infty < t < t_s$,

before a policy action to curb the growth at a future time $t = t_s$:

$\delta F(t) = \delta F(t_s)$ *for* $t > t_s$.

By solving Eq.(3), show that the atmosphere's response to this forcing, subject to the initial condition $\delta T(-\infty)=0$, is:

$$\delta T(t) = \frac{\delta F(t)}{B} g \frac{1}{(1+\gamma)} \quad \text{for } t < t_s$$

and

$$\delta T(t) = \frac{\delta F(t_s)}{B} g [1 - \exp\{-\frac{(t-t_s)}{g\tau}\} \frac{\gamma}{1+\gamma}] \qquad \text{for } t > t_s,$$

where $\gamma = b(R/B)g$.

4. Asymptotic limits of the global warming solution.

The nature of the solution obtained in Exercise 3 depends on the non-dimensional quantity $\gamma = b(R/B)g = bg\tau$. This is a measure of how fast the forcing is increasing relative to the natural response time of the atmosphere-ocean system.

To help understand the exact solution we next consider the solution in different asymptotic limits with respect to γ.

a. *The slow growth limit,* $\gamma =<< 1$: Show that the solution is given approximately by:

$$\delta T = \frac{\delta F(t)}{B} g \quad \text{for } t<t_s \text{ and for } t>t_s,$$

which is in the same form as the equilibrium solution, except with instantaneous forcing. We call this the quasi-equilibrium solution.

b. *The rapid growth limit,* $\gamma >> 1$. Show that the solution becomes:

$$\delta T(t) = \frac{\delta F(t)}{B} \cdot \frac{1}{b\tau} \qquad \text{for } t<t_s.$$

The surprising result is that the climate's response to rapid radiative forcing is independent of the climate gain factor *g*.

c. *The rapid growth limit,* $\gamma >> 1$. Show that for $t>t_s$, the time-dependent solution is independent of γ, and that the time scale for approach to equilibrium is given by $g\tau$.

5. Climate sensitivity.
Find the ratio ECS/TCR for the case of linearly increasing radiative heating given by (5), and show that this ratio is about 3. [Hint: t~70 years, $t/g\tau$~1/3. Since $t<g\tau$, we cannot use the simpler form of (7). However, since $t/g\tau$ is small, you can expand the exponential in (6) to three terms.]

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