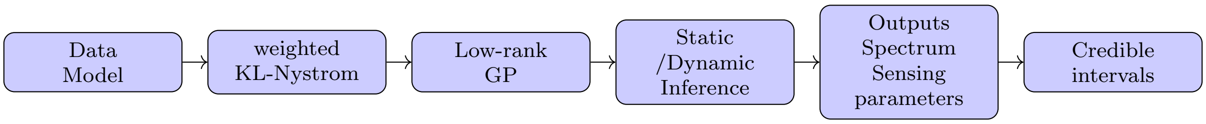
Estimation of Wideband PSD Parameters



PSD Estimation via KL + GP

Data model. Let $\{f_i\}_{i=1}^{M}$ be a frequency grid with quadrature weights $\{w_i\}_{i=1}^{M}$ (w.r.t. the Lebesgue measure unless otherwise specified). For wideband spectrum sensing, the default choice is a uniform grid of weights since FFT-based measurements naturally produce uniform frequency grids with spacing Δf, that is $w_i{=}\Delta f$, $\forall i$. This avoids scaling errors in high-dimensional problems and ensures $\sum_{i=1}^{M} w_i{=}M\Delta f{=}B_{\text{total}}$. Trapezoidal weights, i.e., $w_i{=}(f_{i+1}-f_{i-1})/2$ with boundary adjustments, are preferred when high accuracy is needed for narrow-band features or non-uniform grids.

Over a given frequency grid, assume $\boldsymbol{y}$ be a linearized/averaged surrogate of PSD data $|X(f)|^2$, then the observed subsampled measurements are modelled as:

$$\boldsymbol{y} = \boldsymbol{\Theta}\,\boldsymbol{s} + \boldsymbol{\varepsilon}, \quad \boldsymbol{y} \in \mathbb{R}^N, \qquad \boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\varepsilon}), \tag{1}$$

where $\boldsymbol{s}{\in}\mathbb{R}^M$ is the (unknown) PSD over the frequency grid, $\boldsymbol{\Theta}{\in}\mathbb{R}^{N\times M}$ is a known linear sampling/sensing operator ($N{\leq}M$ – subsampling), and $\boldsymbol{\varepsilon}{\sim}\mathcal{N}(0, \Sigma_\epsilon)$ is the noise term, having covariance, $\Sigma_\epsilon{\in}\mathbb{R}^{N\times N}$. A common special case is homoscedastic noise with $\Sigma_\epsilon{=}\sigma^2\boldsymbol{I}_N$.

KL basis construction. Choose a positive definite kernel $k_f{:}\mathcal{F}\times\mathcal{F}{\to}\mathbb{R}$ on frequency. Form the weighted Gram matrix with entries:

$$\boldsymbol{K}_w[ij] = k_f(f_i, f_j)\sqrt{w_i w_j}, \quad i, j = 1, \ldots, M. \tag{2}$$

Compute its eigendecomposition (retain $R{\ll}M$ dominant modes):

$$\boldsymbol{K}_w\,\boldsymbol{v}_n = \lambda_n\,\boldsymbol{v}_n, \quad \lambda_1{\geq}\cdots{\geq}\lambda_R{>}0, \quad n = 1, \ldots, R \tag{3}$$

Define discrete eigenfunctions on the grid

$$\phi_n(f_i) \approx \boldsymbol{v}_n[i]/\sqrt{w_i}, \qquad i = 1, \ldots, M. \tag{4}$$

For off-grid evaluation $f{\in}\mathcal{F}$ (Nyström extension):

$$\phi_n(f) \approx \frac{1}{\lambda_n}\sum_{j=1}^{M} k_f(f, f_j)\,w_j\,\phi_n(f_j). \tag{5}$$

KL feature matrix and low-rank field model. Define the KL feature matrix $\boldsymbol{\Phi}_{\text{KL}}{\in}\mathbb{R}^{M\times R}$ by its entries

$$(\boldsymbol{\Phi}_{\text{KL}})_{in} := \sqrt{\lambda_n}\,\phi_n(f_i), \qquad i = 1, \ldots, M,\ n = 1, \ldots, R. \tag{6}$$

Approximate the PSD on the grid by the rank-R KL expansion

$$\boldsymbol{s} \approx \boldsymbol{\Phi}_{\text{KL}}\,\boldsymbol{\xi}, \qquad \boldsymbol{\xi} \in \mathbb{R}^R, \tag{7}$$

with prior $\boldsymbol{\xi}{\sim}\mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_R)$. This induces the (approximate) truncated Mercer/KL covariance on $\boldsymbol{s}$:

$$\mathrm{Cov}(\boldsymbol{s}) \approx \boldsymbol{\Phi}_{\text{KL}}\boldsymbol{\Phi}_{\text{KL}}^\top = \sum_{n=1}^{R}\lambda_n\,\boldsymbol{\phi}_n\boldsymbol{\phi}_n^\top, \tag{8}$$

where $\boldsymbol{\phi}_n := [\phi_n(f_1), \ldots, \phi_n(f_M)]^\top \in \mathbb{R}^M$. The measurement model becomes

$$\boldsymbol{y} \approx \boldsymbol{\Theta}\boldsymbol{\Phi}_{\text{KL}}\,\boldsymbol{\xi} + \boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\varepsilon}). \tag{9}$$

[Single-window] Static Bayesian linear regression. Let $\mathbf{A}{:=}\boldsymbol{\Theta}\,\boldsymbol{\Phi}_{\mathrm{KL}}{\in}\mathbb{R}^{N\times R}$ so that $\boldsymbol{y}{=}\boldsymbol{A}\,\boldsymbol{\xi}{+}\boldsymbol{\varepsilon}$. With $\boldsymbol{\xi}{\sim}\mathcal{N}(\mathbf{0},\boldsymbol{I}_R)$ and $\boldsymbol{\varepsilon}{\sim}\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}_\varepsilon)$, the posterior is

$$\boldsymbol{\Sigma}_{\xi|y} = \Big(\boldsymbol{I}_R + \boldsymbol{A}^\top \boldsymbol{\Sigma}_\varepsilon^{-1}\boldsymbol{A}\Big)^{-1}, \tag{10a}$$

$$\boldsymbol{\mu}_{\xi|y} = \boldsymbol{\Sigma}_{\xi|y}\,\boldsymbol{A}^\top \boldsymbol{\Sigma}_\varepsilon^{-1}\boldsymbol{y}. \tag{10b}$$

Within a single window, the PSD posterior (on the grid) is Gaussian with

$$\boldsymbol{\mu}_{s|y} = \boldsymbol{\Phi}_{\mathrm{KL}}\,\boldsymbol{\mu}_{\xi|y}, \tag{11a}$$

$$\boldsymbol{\Sigma}_{s|y} = \boldsymbol{\Phi}_{\mathrm{KL}}\,\boldsymbol{\Sigma}_{\xi|y}\,\boldsymbol{\Phi}_{\mathrm{KL}}^\top. \tag{11b}$$

[Multiple-windows] Dynamic GP prior on coefficients. For windows $t{=}1,\ldots,T$, let $\boldsymbol{y}_t{=}\boldsymbol{\Theta}_t\,\boldsymbol{s}_t + \boldsymbol{\varepsilon}_t$ and assume $\boldsymbol{s}_t{\approx}\boldsymbol{\Phi}_{\mathrm{KL}}\boldsymbol{\xi}_t$ with $\boldsymbol{\xi}_t{\in}\mathbb{R}^R$.

Place independent GP priors across time on coefficients:

$$\xi_n(\cdot) \sim \mathcal{GP}(0, k_n(\cdot,\cdot)), \qquad n = 1,\ldots,R, \tag{12}$$

e.g., Matérn kernels ($\nu \in \{1/2, 3/2, 5/2\}$).

Use state-space realizations for scalable inference. Particularly, for $\nu{=}1/2$ (Ornstein–Uhlenbeck state-space form), the discrete-time model for step Δt_t is

$$\xi_{n,t} = a_{n,t}\,\xi_{n,t-1} + \eta_{n,t}, \quad a_{n,t} = e^{-\alpha_n \Delta t_t}, \tag{13a}$$

$$\eta_{n,t} \sim \mathcal{N}\Big(0,\ q_n \frac{1-e^{-2\alpha_n\Delta t_t}}{2\alpha_n}\Big), \tag{13b}$$

The stationary initial prior is $\xi_{n,1}{\sim}\mathcal{N}(0, q_n/(2\alpha_n))$, with hyperparameters $\alpha_n, q_n{>}0$. This state-space representation enables efficient $\mathcal{O}(TR^2)$ inference via Kalman filtering/smoothing

In the following case of $\nu{\in}\{3/2, 5/2\}$, use standard p-dimensional Markov state embeddings with known $(\boldsymbol{F}_n, \boldsymbol{L}_n, \boldsymbol{Q}_n)$ (omitted for brevity).

The observation equation is

$$\boldsymbol{y}_t = \boldsymbol{H}_t\,\boldsymbol{\xi}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{H}_t := \boldsymbol{\Theta}_t\,\boldsymbol{\Phi}_{\mathrm{KL}}, \qquad \boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\varepsilon,t}). \tag{14}$$

Run a R-dimensional Kalman filtering with diaganonal transition $\boldsymbol{A}{=}\,\mathrm{diag}(a_{n,t})$ and diagonal process noise $\boldsymbol{Q}_t{=}\,\mathrm{diag}(\cdot)$, using Rauch–Tung–Striebel smoothing across t and across modes n, and producing smoothed $\boldsymbol{\mu}_{\xi|y,t}$ and $\boldsymbol{\Sigma}_{\xi|y,t}$, and reconstruct

$$\boldsymbol{\mu}_{s|y,t} = \boldsymbol{\Phi}_{\mathrm{KL}}\,\boldsymbol{\mu}_{\xi|y,t}, \qquad \boldsymbol{\Sigma}_{s|y,t} = \boldsymbol{\Phi}_{\mathrm{KL}}\,\boldsymbol{\Sigma}_{\xi|y,t}\,\boldsymbol{\Phi}_{\mathrm{KL}}^\top. \tag{15}$$

Mostly, state-space representations reformulate GP regression using Kalman filtering theory, achieving $\mathcal{O}(n)$ complexity for linear-time Markovian processes (plus the cost to handle $\boldsymbol{\Sigma}_{\epsilon,t}$).

Outputs: Spectral Parameter Estimation

Spectral parameter estimation from the reconstructed PSD. Let $\{f_i\}_{i=1}^M$ be the frequency grid with quadrature weights $\{w_i\}_{i=1}^M$ (for a uniform grid, $w_i = \Delta f$). From the KL+GP inference, assume the (approx.) Gaussian posterior

$$\boldsymbol{s} \mid \boldsymbol{y} \ \sim\ \mathcal{N}\big(\boldsymbol{\mu}_{s|y},\, \boldsymbol{\Sigma}_{s|y}\big), \qquad \mu_{s|y,i} := \boldsymbol{\mu}_{s|y}[i], \quad \sigma_{s|y,i} := \sqrt{\boldsymbol{\Sigma}_{s|y}[i,i]}.$$

All feature computations assume PSD values are in *linear* power units (not dB); if inputs are in dB, convert via $s^{\mathrm{lin}} = 10^{s^{\mathrm{dB}}/10}$ before applying the definitions below.

PSD credible bands

Band statistic: Let $B\subset\{1,\ldots,M\}$ be an index set (band), and define a linear band statistic $p_B{:=}\boldsymbol{c}_B^\top\boldsymbol{s}$, $\boldsymbol{c}_B\in\mathbb{R}^M$. Two common choices for $\boldsymbol{c}_B$ are:

$$\text{(Average PSD over } B) \quad c_B[i] = \begin{cases} \dfrac{w_i}{\sum_{j\in B} w_j}, & i \in B, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{(Total band power over } B) \quad c_B[i] = \begin{cases} w_i, & i \in B, \\ 0, & \text{otherwise.} \end{cases}$$

The average PSD choice, p_B, has units of power/Hz; the total power choice, p_B, has units of power. Note that $\sum_{i=1}^M w_i s_i$ approximates $\int s(f)\,df$.

Pointwise PSD posterior and credible intervals. For a grid index i,

$$s_i \mid \boldsymbol{y} \sim \mathcal{N}\big(\mu_{s|y,i},\, \sigma^2_{s|y,i}\big), \quad \mu_{s|y,i} := \mu_{s|y}[i], \quad \sigma_{s|y,i} := \sqrt{\Sigma_{s|y}[i,i]}.$$

A $(1-\alpha)$ pointwise credible interval is

$$\mu_{s|y,i} \;\pm\; z_{1-\alpha/2}\,\sigma_{s|y,i}, \quad z_p := \Psi^{-1}(p),$$

where $\Psi(\cdot)$ is the standard normal CDF.

Band statistic posterior and credible intervals. Because p_B is linear in $\boldsymbol{s}$, under the Gaussian posterior for $\boldsymbol{s}$ we have

$$p_B \mid \boldsymbol{y} \sim \mathcal{N}\big(\mu_B,\, \sigma_B^2\big), \quad \mu_B := \boldsymbol{c}_B^\top \mu_{s|y}, \quad \sigma_B^2 := \boldsymbol{c}_B^\top \Sigma_{s|y}\, \boldsymbol{c}_B.$$

A $(1-\alpha)$ credible interval for p_B is $\mu_B \;\pm\; z_{1-\alpha/2}\,\sigma_B$.

Occupancy probabilities (per-bin and per-band). Given thresholds τ_i (per-bin) and τ_B (per-band), the posterior occupancy probability is defined as:

$$\pi_i := \Pr\big(s_i > \tau_i \mid \boldsymbol{y}\big) = 1 - \Psi\left(\frac{\tau_i - \mu_{s|y,i}}{\sigma_{s|y,i}}\right),$$

$$\pi_B := \Pr\big(p_B > \tau_B \mid \boldsymbol{y}\big) = 1 - \Psi\left(\frac{\tau_B - \mu_B}{\sigma_B}\right).$$

Decision rule: declare bin i (or band B) occupied if the corresponding posterior probability $\pi_i{\in}[0,1]$ exceeds a target (e.g., 0.5 or another application-driven threshold).

Threshold calibration under noise-only (example). Let s_{nf} denote the (approximately white) *noise-floor PSD level* (in linear units, e.g., W/Hz) under a noise-only reference; s_{nf} is a property of the background spectrum $\boldsymbol{s}$ and should not be confused with the *measurement-error* covariance $\boldsymbol{\Sigma}_\varepsilon$ in $\boldsymbol{y}{=}\boldsymbol{\Theta}\boldsymbol{s}{+}\boldsymbol{\varepsilon}$. For the average-PSD statistic over B with $|B|$ bins, under a noise-only reference model with (white) noise PSD s_{nf} and an independent-bin normal approximation,

$$\tau_B \;\approx\; s_{\text{nf}}\left(1 + \sqrt{\frac{2}{|B|}}\, Q^{-1}\big(P_{\text{fa}}\big)\right), \qquad Q(x) := 1 - \Psi(x),$$

which yields Pr(false alarm)$\approx P_{\text{fa}}$. A consistent per-bin choice is obtained by setting $|B|{=}1$:

$$\tau_i \;\approx\; s_{\text{nf}}\Big(1 + \sqrt{2}\,Q^{-1}\big(P_{\text{fa},i}\big)\Big).$$

Calibrate analogously for the total-power statistic if that definition of $\mathbf{c}_B$ is used.

plug-in rule: Replace $\pi_i \geq \pi_0$ by $\mu_{s|y,i} > \tau_i$.

Peak frequency. A simple (single-peak) estimate is

$$i_\star \;:=\; \arg\max_{1\le i\le M}\; \mu_{s|y,i}, \qquad \widehat{f}_{\text{peak}} \;:=\; f_{i_\star}.$$

If you want the peak *within* occupied bins, use $i_\star := \arg\max_{i\in\widehat{\mathcal{O}}} \mu_{s|y,i}$ (when $\widehat{\mathcal{O}} \neq \emptyset$).

Spectral centroid (center frequency). The power-weighted centroid (optionally restricted to occupied bins) is

$$\widehat{f}_c \;:=\; \sum_{i\in\widehat{\mathcal{O}}} f_i\, w_i\, \mu_{s|y,i} \Big/ \sum_{i\in\widehat{\mathcal{O}}} w_i\, \mu_{s|y,i}, \qquad \text{(use } \widehat{\mathcal{O}} = \{1,\dots,M\} \text{ if no occupancy gating).}$$

Occupied bandwidth. If $\widehat{\mathcal{O}} \neq \emptyset$, define the occupied support endpoints

$$\widehat{f}_{\min} := \min_{i\in\widehat{\mathcal{O}}} f_i, \qquad \widehat{f}_{\max} := \max_{i\in\widehat{\mathcal{O}}} f_i, \qquad \widehat{B}_{\text{occ}} := \widehat{f}_{\max} - \widehat{f}_{\min}.$$

For multiple disjoint emitters, compute *connected components* of $\widehat{\mathcal{O}}$ (contiguous index runs) to obtain $\{\widehat{B}^{(k)}_{\text{occ}}\}_k$ and peaks per component.

Uncertainty-aware alternative (expected occupied bandwidth). For a uniform grid ($w_i = \Delta f$), a soft estimate is

$$\mathbb{E}[B_{\text{occ}} \mid \boldsymbol{y}] \;\approx\; \sum_{i=1}^{M} \pi_i\, \Delta f \quad \text{(or } \sum_i \pi_i w_i \text{ for non-uniform grids).}$$

SNR from reconstructed PSD (noise-floor estimated from unoccupied bins). Let π_N be a small cutoff (e.g., $\pi_N = 0.1$) and define a noise-bin set

$$\widehat{\mathcal{N}} \;:=\; \{\, i :\; \pi_i \le \pi_N \,\}.$$

Estimate the (approximately white) noise floor (PSD level) by a weighted average

$$\widehat{s}_n \;:=\; \sum_{i\in\widehat{\mathcal{N}}} w_i\, \mu_{s|y,i} \Big/ \sum_{i\in\widehat{\mathcal{N}}} w_i.$$

Then estimate *signal power above the noise floor* over occupied bins as

$$\widehat{P}_{\text{sig}} \;:=\; \sum_{i\in\widehat{\mathcal{O}}} w_i\,\big(\mu_{s|y,i} - \widehat{s}_n\big)_+, \qquad (x)_+ := \max(x,0),$$

and the corresponding noise power in the same support as

$$\widehat{P}_{\text{noise}} \;:=\; \widehat{s}_n \sum_{i\in\widehat{\mathcal{O}}} w_i.$$

Finally, define the SNR (linear and dB):

$$\widehat{\text{SNR}} \;:=\; \frac{\widehat{P}_{\text{sig}}}{\widehat{P}_{\text{noise}}}, \qquad \widehat{\text{SNR}}_{\text{dB}} \;:=\; 10\log_{10}\big(\widehat{\text{SNR}}\big).$$

If a calibrated noise-floor PSD level s_{nf} is already available (e.g., from hardware calibration or empty-band measurements), the value $\widehat{s}_n := s_{\text{nf}}$ can be set and skip $\widehat{\mathcal{N}}$.

(0)Credible intervals for these features. To propagate posterior uncertainty, draw samples $\boldsymbol{s}^{(m)} \sim \mathcal{N}(\boldsymbol{\mu}_{s|y}, \boldsymbol{\Sigma}_{s|y})$ (or from the KL-latent posterior) and compute

$$\{f^{(m)}_{\text{peak}}, f^{(m)}_c, B^{(m)}_{\text{occ}}, \text{SNR}^{(m)}\}_{m=1}^{M}$$

Report empirical quantiles (e.g., $2.5\% - 97.5\%$) as credible intervals.