Lecture 7: Counting Lattice Paths

Date: February 11th, 2026 **Scribe:** Baihan Liu

1 Definition of Lattice Paths

Let P be a lattice path:

$$P: \ (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n),$$

where $x_i, y_i \in \mathbb{Z}$ for all $i \in [n]_0$.

A step is the vector

$$[x_i - x_{i-1},\, y_i - y_{i-1}].$$

A north step, denoted N, is the vector $[0, 1]$. An east step, denoted E, is the vector $[1, 0]$.

We will focus on lattice paths that involve only N and E steps.

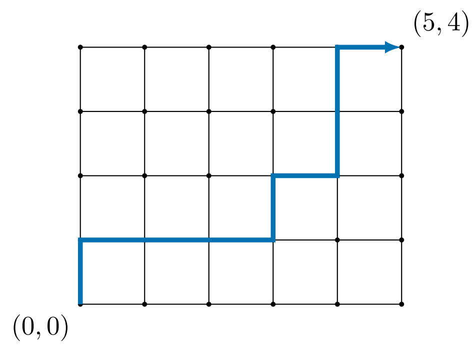


Figure 1: A lattice path P: `NEEENENNE`.

2 Catalan Numbers and Dyck Paths

A Dyck path of semilength n is a lattice path from $(0, 0)$ to (n, n) such that it never goes below the main diagonal $y = x$.

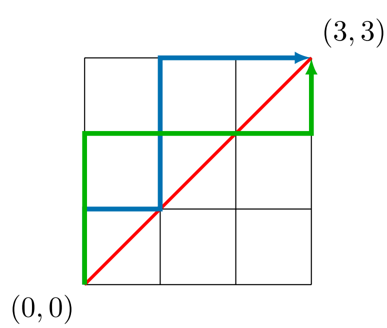


Figure 2: Dyck path (Blue, NENNEE), a path that is not a Dyck path (Green, NNEEEN), diagonal $y = x$ (red).

Question. Let $D(n)$ be the set of Dyck paths of semilength n. What is $|D(n)|$?
We let $C_n := |D(n)|$, where C_n is the nth Catalan number, with the convention that $C_0 = 1$.

2.1 Recurrence Relation

Lemma 2.1. *For* $n \geq 1, C_n = \sum_{j=1}^{n} C_{j-1}C_{n-j}$.

Proof. Let $P : v_0, v_1, \ldots, v_{2n} \in D(n)$, where $v_0 = (0,0)$ and $v_{2n} = (n,n)$. While the endpoint is fixed on the diagonal, P may also touch the diagonal $y = x$ several times along the way. Let j be the first return of P to the diagonal, i.e., j is the smallest $j > 0$ such that $v_{2j} = (j,j)$. This is an example of extremality. For example, in Figure 3, $j = 3$ is the smallest $j > 0$ with $v_6 = (3,3)$.

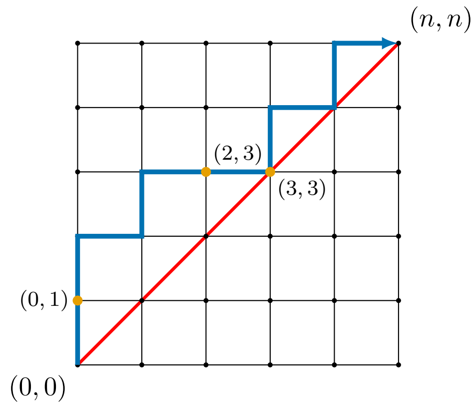


Figure 3: A Dyck path P : `NNENEENENE` (blue).

P can be decomposed as $P = P_1 + P_2$, where P_1 is a Dyck path from $(0,1)$ to $(j-1,j)$ and P_2 is a Dyck path from (j,j) to (n,n).

To see why P_1 is a Dyck path, observe that the first step in P must be $(0,1)$, and the last step before the first return is $(1,0)$. Moreover, by our extreme choice of j, from $(0,1)$ to $(j-1,j)$ the path does not touch the line $y = x$.

Thus, $P_1 \in D(j-1)$, and $P_2 \in D(n-j)$. For a fixed j, there are $|D(j-1)||D(n-j)| = C_{j-1}C_{n-j}$ different ways of building P.

j can be anywhere in $j \in \{1, 2, \ldots, n\}$. These options for first return are mutually exclusive and collectively exhaustive, so we can use sum principle over choice of j. Thus,

$$C_n = \sum_{j=1}^{n} C_{j-1}C_{n-j}, \qquad n \geq 1.$$

□

2.2 Closed-form Formula

Lemma 2.2. *For* $n \geq 1, C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. There are $\binom{2n}{n}$ different lattice paths from $(0,0)$ to (n,n). Some of these are Dyck paths, which we called "good paths". Some of these are not Dyck paths, which we called "bad paths".

The total number of paths is equal to the number of good paths plus the number of bad paths:

$$\binom{2n}{n} = |G| + |B|.$$

$$C_n = |G| = \binom{2n}{n} - |B|.$$

Thus, to find the number of "good paths", we only need to find the number of "bad paths".

A path P is bad, i.e., $P \in B$, if it crosses below the diagonal $y = x$ at least once. Thus, it touches the off-diagonal $y = x - 1$ at least once, as shown in Figure 4.

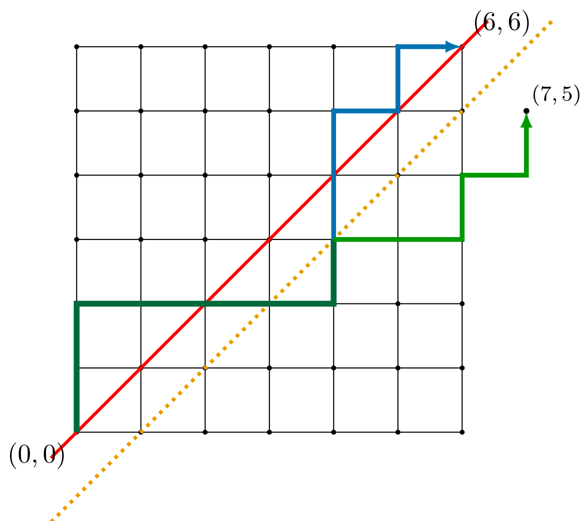


Figure 4: Bad path P(blue, NNEEEEENNNENE), $\phi(P)$ (green, NNEEEEENEEENEN), diagonal $y = x$ (red), offset line $y = x - 1$ (orange dotted).

The reflection of (n, n) across $y = x - 1$ is $(n + 1, n - 1)$. Let $(i + 1, i)$ be the last lattice point of P on $y = x - 1$, where $i \in \{0, 1, 2, \dots, n\}$. The subpath of P from $(i + 1, i)$ to (n, n) is the final portion of P.

Let $\phi : B \longrightarrow \mathcal{P}((0, 0), (n + 1, n - 1))$ by reflecting the final portion of $P \in B$ after final intersection with $y = x - 1$. For example, in Figure 4, the green path is the image of the blue path under ϕ. In the step encoding, the final portion NNENE is reflected to EENEN.

ϕ turns $P \in B$, which touches $y = x-1$, into a $(0, 0)$ to $(n+1, n-1)$ path that touches $y = x-1$ at the same points.

Conversely, every path from $(0, 0)$ to $(n+1, n-1)$ crosses the off-diagonal $y = x-1$, since $(0, 0)$ and $(n + 1, n - 1)$ are on different sides of it. Reflecting the final portion of the path produces a path from $(0, 0)$ to (n, n) that intersects $y = x - 1$, hence lies in B.

Reflecting the final portion twice returns it to its original position. Thus, B and paths from $(0, 0)$ to $(n + 1, n - 1)$ are in bijection. In fact, ϕ is an involution on $B \cup \mathcal{P}((0, 0), (n + 1, n - 1))$. Therefore,

$$|B| = |\mathcal{P}((0, 0), (n + 1, n - 1))| = \binom{n+1+n-1}{n+1} = \binom{2n}{n+1}$$

$$C_n = \binom{2n}{n} - |B| = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = (1 - \frac{n}{n+1})\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

□

2.3 Final Project

One final project idea is to explore more Catalan objects. Stanley's book is a useful reference.

n	0	1	2	3	4	5	...
C_n	1	1	2	5	14	42	...

Table 1: Catalan numbers

3 Fibonacci Number

Let $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$.

n	0	1	2	3	4	...
F_n	1	1	2	3	5	...

Table 2: Fibonacci Number

A combinatorial interpretation: Let P_n be a path of length n. Consider the set of all possible "matchings" (ways of pairing/not pairing adjacent nodes) in P_n.

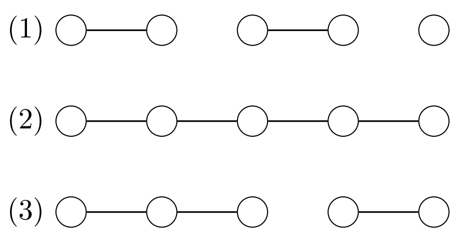
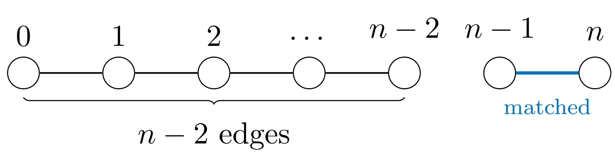


Figure 5: Some possible "matchings" in P_4.

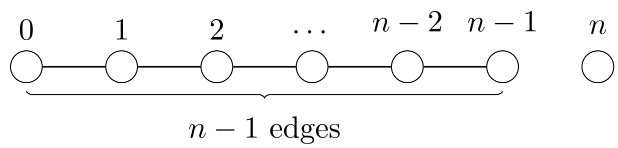
Claim: F_n is the number of different matchings in P_n, $n \geq 2$.

Proof. For $n \geq 2$,

1. If we match $\{n-1, n\}$, then $n-1$ cannot be matched again. There are F_{n-2} matching options for nodes $0, 1, 2, \ldots, n-2$.



2. If we do not match n with $n-1$. $n-1$ is free to be matched with $n-2$. There are F_{n-1} matchings options for nodes $0, 1, 2, \ldots, n-1$.



By assumptions, $F_n = F_{n-1} + F_{n-2}$. Proof holds inductively.

$\square$

Remark 3.1. *The answer so far for F_n is a recursion formula, which is less nice than our best answer for C_n, which is a closed-form formula.*