The Defective Parking Space and Defective Kreweras Numbers

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Parking Functions and Defective Parking Functions

Parking Functions

Parking Functions

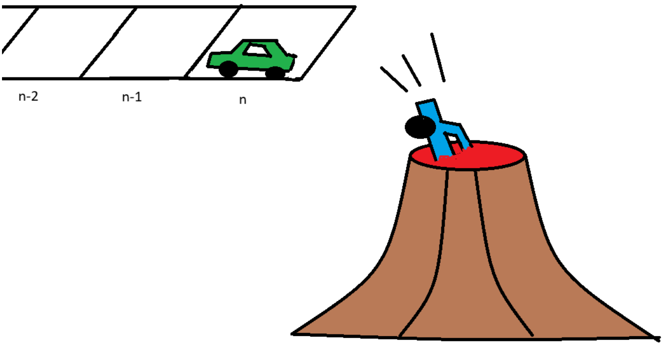

Figure: A failed parking function

Defective Parking Functions

What if we allow some cars to fail to park? What if the number of spots differs from the number of cars?

- A **defective parking function** of defect d, with m cars and n spots is a tuple where d cars fail to park.

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- What about $[n+1]^m$?

Definition

$$\mathrm{DPF}_{m,n,d} = \{\alpha \in [n+1]^m \mid m \text{ cars, } n \text{ spots, } d \text{ cars fail to park}\}$$

Defective Parking Functions Example

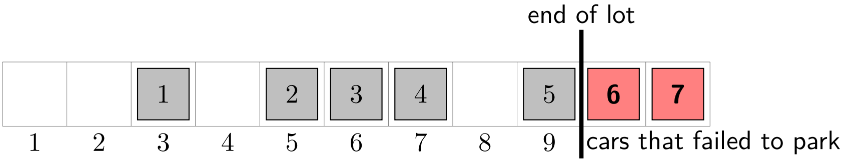


Figure: Parking position of cars with preference list $(3, 5, 5, 6, 9, 9, 10) \in \mathrm{DPF}^{\uparrow}_{7,9,2}$.

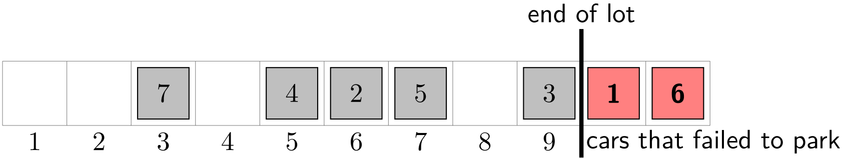


Figure: Parking position of cars with preference list $(10, 6, 9, 5, 5, 9, 3) \in \mathrm{DPF}^{\uparrow}_{7,9,2}$.

Cars can prefer spot 10!

$\mathfrak{S}_m$ invariance

Theorem (GHMOQRWW '24)

Defective parking functions are invariant under the action of $\mathfrak{S}_m$ permuting indices.

Corollary

The space $\mathsf{DPark}_{m,n} = \mathbb{C}[n+1]^m$ *is an* $\mathfrak{S}_m$ *module graded by defect* d*; that is, we can define*

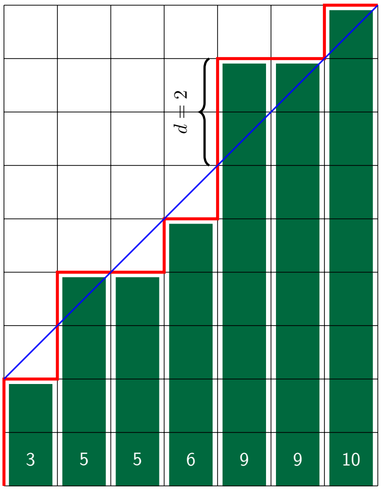
$$\mathsf{DPark}_{m,n}^{(d)} = \mathbb{C}\{\mathbf{x} : \mathrm{dft}(\mathbf{x}) = d\}$$

such that

$$\mathsf{DPark}_{m,n} = \bigoplus_{d=0}^{m} \mathsf{DPark}_{m,n}^{(d)}$$

as an $\mathfrak{S}_m$ *module.*

Computing defect



Computing defect algebraically

Proposition (GHMOQRWW '24)

For weakly increasing $\mathbf{p} \in [n+1]^m$,

$$\mathrm{dft}(\mathbf{p}) = \max\{0, \max_i\{p_i - i - (n-m)\}\}$$

Example

$$\mathbf{p} = (3,5,5,6,9,9,10) \in [9+1]^7$$

$$\begin{aligned} \mathrm{dft}(\mathbf{p}) = \max\{&0, 3-1-2, 5-2-2, 5-3-2, \\ &6-4-2, 9-5-2, 9-6-2, 10-7-2\} \\ = \max\{&0,0,1,0,0,2,1,1\}. \end{aligned}$$

Thus, $\mathbf{p} \in \mathrm{DPF}_{7,9,2}$.

Representation Theory of $\mathfrak{S}_m$

$\mathfrak{S}_m$ modules

Definition

A (complex) $\mathfrak{S}_m$ module is a vector space V over $\mathbb{C}$ equipped with a linear action of $\mathfrak{S}_m$ on V.

Example

- The trivial module: $\mathbb{C}$ where $\sigma \cdot v = v$.

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- The defining module: $\mathbb{C}^m$ where $\sigma \cdot e_i = e_{\sigma(e)}$.
- The (left) regular module: $\mathbb{C}[\mathfrak{S}_m]$ where $\sigma \cdot \tau = \sigma\tau$.

Complete reducibility

Definition

An $\mathfrak{S}_m$ module V is **reducible** if $V \cong V_1 \oplus V_2$ as $\mathfrak{S}_m$ modules - otherwise **irreducible**.

Fact

Irreducible $\mathfrak{S}_m$ modules S_λ are naturally indexed by partitions $\lambda \vdash m$.

Theorem (Maschke, 1898)

Every $\mathfrak{S}_m$ module V is uniquely expressible as a sum of irreducible submodules, i.e.

$$V = \bigoplus_{\lambda \vdash m} a_\lambda S_\lambda.$$

How to summarize the irreducible decomposition of a module?

Symmetric functions

Definition

A formal power series in $\mathbb{C}[x_1, x_2, \ldots]$ is **symmetric** if it is invariant under the action of $\mathfrak{S}$ permuting indices.

Example

Yes: $\sum_{i<j} x_i x_j$

No: $x_1 + \sum_{i<j} x_i x_j$

No: $x_1 x_2 + x_1 x_3 + x_2 x_3$

Proposition

The set of symmetric functions forms a commutative, associative $\mathbb{C}$-algebra under addition and multiplication called Sym. Sym is graded by degree.

Bases

Fact

Some bases for the mth graded part of Sym include:

- Monomial basis $\{m_\lambda\}_{\lambda \vdash m}$,
- Schur basis $\{s_\lambda\}_{\lambda \vdash m}$,
- Complete homogeneous basis $\{h_\lambda\}_{\lambda \vdash m}$.

Change of basis matrices are well-known.

Summarizing $\mathfrak{S}_m$ modules as symmetric functions

- Any $\mathfrak{S}_m$ module V can be reduced to a sum of irreducible modules:

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$$\mathrm{ch}\left(\bigoplus_{\lambda \vdash m} a_\lambda S_\lambda\right) = \sum_{\lambda \vdash m} a_\lambda s_\lambda,$$

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    - Isomorphism of $\mathbb{C}$-algebras!
- We are characterizing a module using a symmetric function - we will use the h basis.

Why h?

Example

- Let $\mathfrak{S}_3$ act on (a, a, b) by permutation.

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- Let $\mathfrak{S}_3$ act on (a, a, b) by permutation.
- This generates an $\mathfrak{S}_3$ module $V = \mathbb{C}[(a, a, b), (a, b, a), (b, a, a)]$.

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- This generates an $\mathfrak{S}_3$ module $V = \mathbb{C}[(a, a, b), (a, b, a), (b, a, a)]$.
- We have

$$\mathrm{ch}(V) = h_{21}$$

The graded version

Definition

An $\mathfrak{S}_m$ module V is **graded** if it can be expressed as the sum

$$V = \bigoplus_{d \geq 0} V^{(d)}$$

of graded parts. The **graded Frobenius transform** has

$$\mathrm{ch}_t(V) = \sum_{d \geq 0} \mathrm{ch}\left(V^{(d)}\right) t^d.$$

The Defective Parking Space

Frobenius transform of $\mathsf{Park}_{n,n}$

Theorem (Stanley, 1997, [3])

We have

$$\mathrm{ch}(\mathsf{Park}_{n,n}) = \sum_{\lambda \vdash n} \mathrm{Krew}(\lambda) h_\lambda$$

Frobenius transform of $\mathsf{Park}_{n,n}$

Theorem (Stanley, 1997, [3])

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Definition

The **Kreweras number** $\mathrm{Krew}(\lambda)$ of a partition $\lambda \vdash m$ with k parts is

$$\mathrm{Krew}(\lambda) = \frac{1}{m+1} \binom{m+1}{m+1-k, \mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_m(\lambda)},$$

where μ is for μultiplicity. $\mathrm{Krew}(\lambda)$ counts weakly increasing parking functions of multiplicity type λ.

Kreweras numbers example

$\lambda = 4$	$\lambda = 31$	$\lambda = 22$	$\lambda = 211$	$\lambda = 1111$
1111	1112	1122	1123	1234
	1113	1133	1124	
	1114		1134	
	1222		1223	
			1224	
			1233	

Table: $\mathrm{Krew}(4) = 1$, $\mathrm{Krew}(31) = 4$, etc.

Thus

$$\mathrm{ch}(\mathsf{Park}_{4,4}) = h_4 + 4h_{31} + 2h_{22} + 6h_{211} + h_{1111}.$$

Defective Kreweras Numbers

Definition

For $\lambda \vdash m$, the **defective Kreweras numbers** $\mathrm{Krew}_{\delta,n}(\lambda)$ is the number of weakly increasing elements $\mathbf{x}$ of $[n+1]^m$ with multiplicity type λ and $\max_i(x_i - i) = \delta$.

- δ is *predefect*: Defect is $\delta + (m-n)$.

Defective Kreweras numbers example

$m = 4, n = 3, \delta = 1$:

$\lambda = 4$	$\lambda = 31$	$\lambda = 22$	$\lambda = 211$	$\lambda = 1111$
2222	1333	1144	1244	
	2333	2233	1344	
	2223	2244	2344	
	2224		1334	
			2334	
			2234	

Table: $\mathrm{Krew}_{2,3}(4) = 1$, $\mathrm{Krew}_{2,3}(31) = 4$, etc.

Corresponding defect is $\delta + (m - n) = 2$, so

$$\mathrm{ch}(\mathsf{DPark}_{4,3}^{(2)}) = h_4 + 4h_{31} + 3h_{22} + 6h_{211}.$$

The formula

Theorem (GHMOQRWW '24)

We have

$$\mathrm{ch}_t(\mathsf{DPark}_{n,n}) = \sum_{\lambda \vdash n} \sum_{d \geq 0} \mathrm{Krew}_{d,n}(\lambda) t^d h_\lambda,$$

which generalizes to

$$\mathrm{ch}_t(\mathsf{DPark}_{m,n}) = \sum_{\lambda \vdash m} \left(\sum_{\delta=0}^{n-m} \mathrm{Krew}_{\delta,n}(\lambda) + \sum_{\delta > n-m} \mathrm{Krew}_{\delta,n}(\lambda) t^{\delta-(n-m)} \right) h_\lambda.$$

Example

- $\mathrm{ch}_0(\mathsf{DPark}_{m,n}) = \mathrm{ch}(\mathsf{Park}_{m,n})$.
- $\mathrm{ch}_1(\mathsf{DPark}_{m,n}) = \mathrm{ch}((\mathbb{C}^{n+1})^{\otimes m})$, where $\mathfrak{S}_m$ acts by permuting tensor factors. Think "forgetting" defect.

A conjecture

Conjecture (GHMOQRWW '24)

For $n \geq d+m-1$ and $\lambda \vdash m$ of length k,

$$\mathrm{Krew}_{d,n}(\lambda) = \frac{m+dk}{m+d} \frac{1}{m+d+1} \binom{m+d+1}{m+d+1-k, \mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_m(\lambda)}$$

where $\mu_i(\lambda)$ is the number of parts of λ of size i.

Conclusion

Acknowledgements

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Further Reading

[0] F. Bergeron, Algebraic combinatorics and Coinvariant Spaces, CMS Treatises in Mathematics, AK Peters, 2009

[1] B. E. Sagan, The symmetric group, Second, Graduate Texts in Mathematics, Springer-Verlag, New York, 2001

[2] R. P. Stanley, Enumerative combinatorics. Vol. 2, Cambridge University Press, Cambridge, 2024

[3] R. P. Stanley, Parking functions and noncrossing partitions, Research Paper 20, approx. 14, The Wilf Festschrift, 1997

Conclusion

Thank you! Questions?

(a) Link to preprint on ArXiv.

(b) Link to my website.