Parking Functions and Volumes of Polytopes

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Iowa State University

AMS Fall Central Section
Parking Functions and their Generalizations
October 18, 2025

Overview

- Parking Functions
  - Classical
  - $\mathbf{x}$-parking functions
  - Parking completions
- Polytopes
  - Classical
  - Pitman-Stanley
  - Volumes
- (My) Problem
- Product/Payoff (i.e., Result)

Definitions

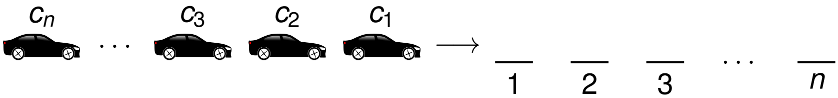

Figure: Parking Function Illustration.

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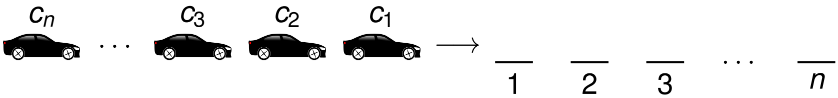

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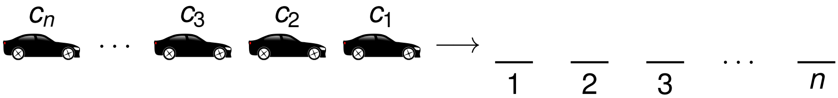

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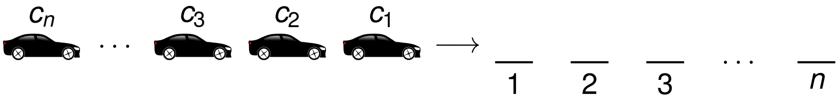

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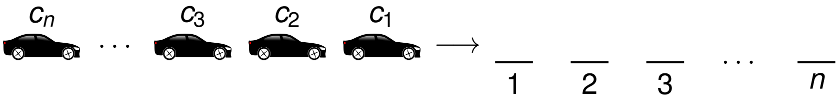

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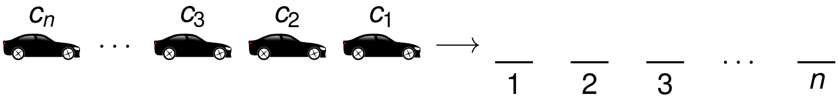

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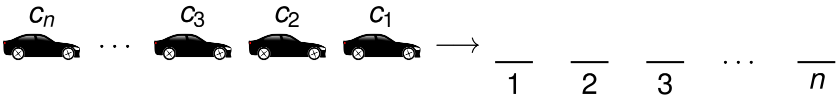

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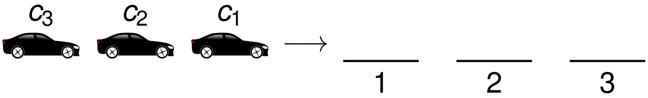

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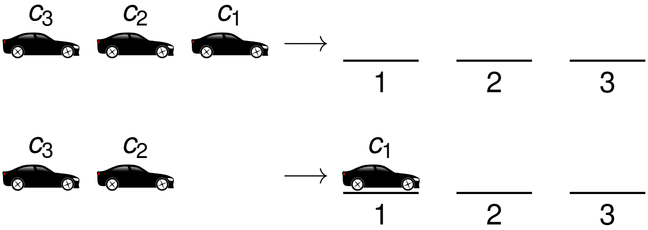

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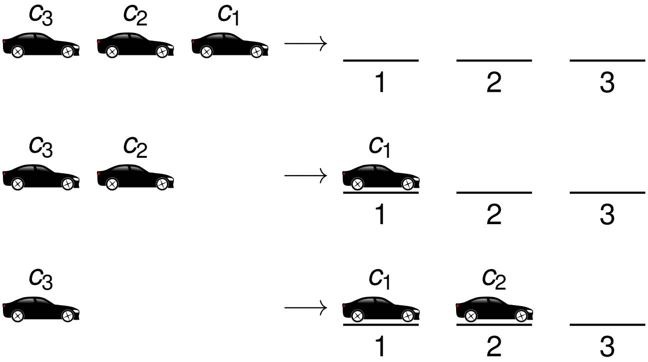

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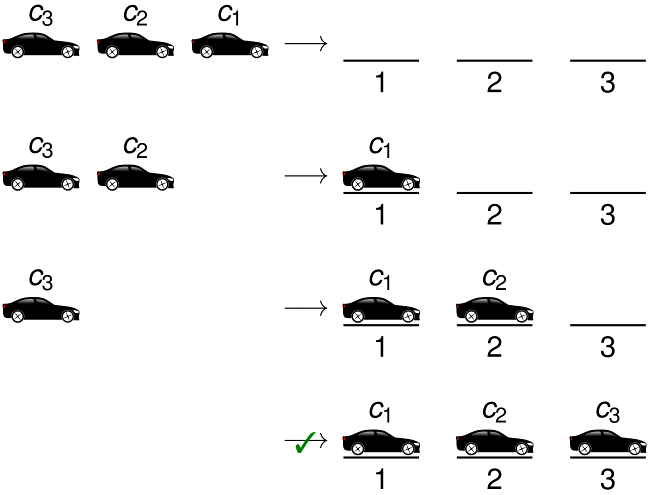

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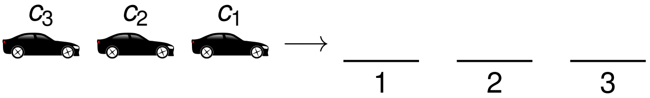

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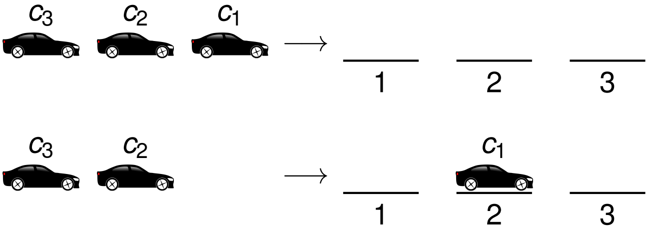

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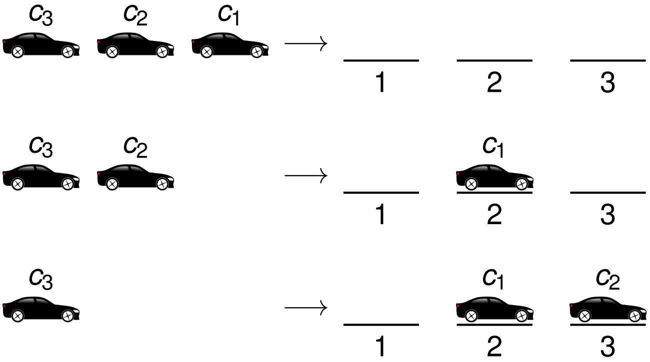

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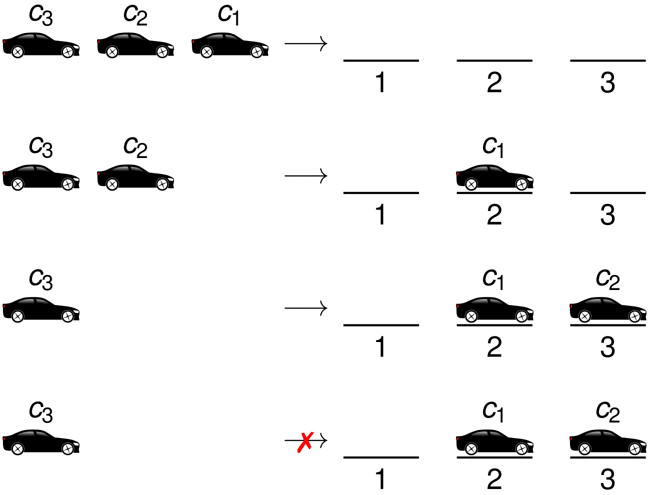

Examples and Enumeration

Theorem (N. Shales, 2018)

The number of parking functions of length n is $|PF_n| = \sum_{i=1}^{n} i \binom{n-1}{i-1} |PF_{i-1}| \cdot |PF_{n-i}|$.

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Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Publications!

Figure: *Honk! Honk!*, Part 1: An Introduction to Parking Functions (Hadaway, 2021)

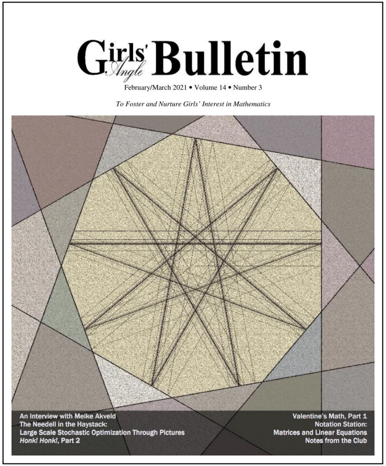

Figure: *Honk! Honk!*, Part 2: An Introduction to Parking Functions (Hadaway, 2021)

Another Characterization

Definition

A **parking function** is a sequence of integers $\alpha = (a_1, a_2, \ldots, a_n)$ such that its weakly increasing rearrangement $\alpha^{\uparrow} := \mathbf{b} = (b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$ for all $i \in [n]$.

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For $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, define an **x-parking function** to be a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers whose increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq x_1 + x_2 + \cdots + x_i$. Let $P_n(\mathbf{x})$ denote the number of **x**-parking functions.

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$$b_1 = 1 \leq 2, b_2 = 3 \leq 2 + 1 = 3, \text{ and } b_3 = 3 \leq 2 + 1 + 1 = 4,$$

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DETOUR

Polytopes

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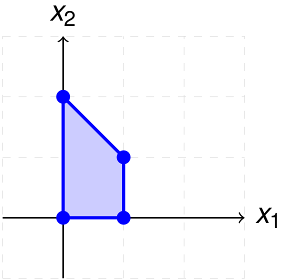

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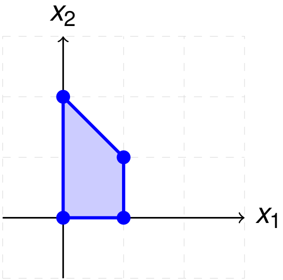

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_1 &\leq 1, \text{ and} \\ x_1 + x_2 &\leq 1 + 1 = 2. \end{aligned}$$

Figure: Illustration of $\mathrm{PS}_2((1,1))$

Pitman-Stanley Polytope

Definition

Given a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, the **Pitman-Stanley polytope** is

$$\mathrm{PS}_n(\mathbf{a}) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid \mathbf{x} \preceq \mathbf{a}\}.$$

Definition

We say **x is less than a in dominance order**, denoted $\mathbf{x} \preceq \mathbf{a}$, if

$$\begin{aligned} x_1 &\leq a_1 \\ x_1 + x_2 &\leq a_1 + a_2 \\ &\vdots \\ x_1 + x_2 + \cdots + x_n &\leq a_1 + a_2 + \cdots + a_n. \end{aligned}$$

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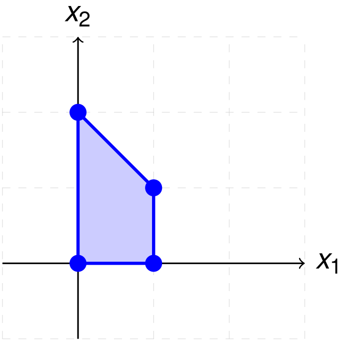

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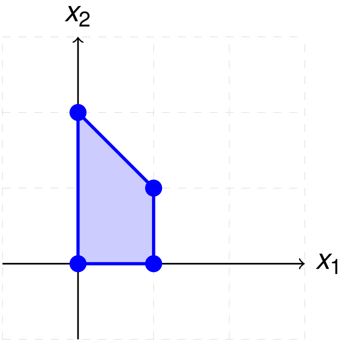

Figure: Illustration of $\mathrm{PS}_2((1,1))$

(here, yes boundary!)

- $\operatorname{dvol} \mathrm{PS}_2((1,1)) = 5$
- $\operatorname{Evol} \mathrm{PS}_2((1,1)) = \frac{3}{2}$
- $\operatorname{nvol} \mathrm{PS}_2((1,1)) = 2! \cdot \frac{3}{2} = 3$

Volume of Pitman-Stanley polytope

Theorem (Stanley-Pitman, 2002)

With $P_n(\mathbf{x})$ *and* $\operatorname{Evol}\operatorname{PS}_n(\mathbf{x})$ *as above, we have*

$$P_n(\mathbf{x}) = \sum_{(a_1,a_2,\ldots,a_n)\in PF_n} x_{a_1}x_{a_2}\cdots x_{a_n} = n!\cdot\operatorname{Evol}\operatorname{PS}_n(\mathbf{x}).$$


LANES
MERGE

Gap in the Literature

Adeniran, et al. (2022) show that the number of parking completions is

$$|\operatorname{PC}_n(\mathbf{t})| = \sum_{\ell \in L_n(\mathbf{t})} \binom{n-m}{\ell} \prod_{j=1}^{m+1} (\ell_j + 1)^{(\ell_j - 1)} \tag{1}$$

where $L_n(\mathbf{t}) = \{\ell = (\ell_1, \ell_2, \ldots, \ell_{m+1}) \in \mathbb{N}^{(m+1)}$ such that $\ell_1 + \ell_2 + \cdots + \ell_j \geq t_j - j$ for all $j \in [m]$ and $\ell_1 + \ell_2 + \cdots + \ell_{m+1} = n - m$.

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$$\mathrm{P}_N(\mathbf{x}) = N! \sum_{\mathbf{k} \in K_N} \prod_{i=1}^{N} \frac{x_i^{k_i}}{k_i!} = \sum_{\mathbf{k} \in K_N} \binom{N}{\mathbf{k}} x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N} \tag{2}$$

where $K_N = \{\mathbf{k} \in \mathbb{N}^N$ such that $\sum_{i=1}^{j} k_i \geq j$ for all $j \in [N-1]$ and $\sum_{i=1}^{N} k_i = N\}$.

Main Result

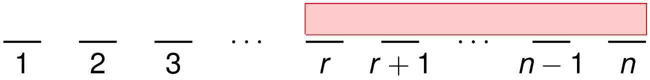

Figure: Parking Completion Street Diagram.

Theorem (Hadaway, 2025+)

Fix $n \in \mathbb{N}$. If

- $\mathbf{t} = ()$,
- $\mathbf{t} = (1, 2, \ldots, n)$,
- $\mathbf{t} = (n)$, *or*
- $\mathbf{t} = (r, r+1, \ldots, n)$ *for* $r \in [n]$,

then $\mathbf{t}$ *determines* $\mathbf{x}$, *and* $|\,\mathrm{PC}_n(\mathbf{t})| = \mathrm{P}_n(\mathbf{x})$.

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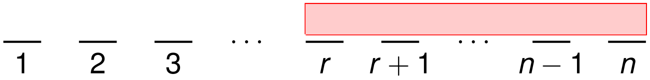

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Sketch.

Start with $\mathbf{t}$. Take the (ordered) complement $\mathbf{u}$. Take consecutive differences to find $\mathbf{x}$. For these cases, $\mathbf{x} = (1, 1, \ldots, 1)$. Then, do algebra. □

Example

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Example

Let $n = 4$. Start with $\mathbf{t} = ()$.

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Example

Let $n = 4$. Start with $\mathbf{t} = ()$. Then, $\mathbf{u} = (1, 2, 3, 4)$, so
$\mathbf{x} = (1, 2-1, 3-2, 4-3) = (1, 1, 1, 1)$.

Example

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Let $n = 4$. Start with $\mathbf{t} = ()$. Then, $\mathbf{u} = (1, 2, 3, 4)$, so $\mathbf{x} = (1, 2-1, 3-2, 4-3) = (1, 1, 1, 1)$. Observe

$$PC_4((1)) = (4+1)^{(4-1)} =$$

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Let $n = 4$. Start with $\mathbf{t} = ()$. Then, $\mathbf{u} = (1, 2, 3, 4)$, so $\mathbf{x} = (1, 2-1, 3-2, 4-3) = (1, 1, 1, 1)$. Observe

$$PC_4\big((1)\big) = (4+1)^{(4-1)} = P_4\big((2, 1, 1)\big).$$

Example 2

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Example

Let $n = 6$. Start with $\mathbf{t} = (4, 5, 6)$.

Example 2

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Example

Let $n = 6$. Start with $\mathbf{t} = (4, 5, 6)$. Then, $\mathbf{u} = (1, 2, 3)$,

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Example

Let $n = 6$. Start with $\mathbf{t} = (4, 5, 6)$. Then, $\mathbf{u} = (1, 2, 3)$, so
$\mathbf{x} = (1, 2-1, 3-2) = (1, 1, 1)$.

Example 2

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Example

Let $n = 6$. Start with $\mathbf{t} = (4,5,6)$. Then, $\mathbf{u} = (1,2,3)$, so $\mathbf{x} = (1, 2-1, 3-2) = (1,1,1)$. Observe

$$PC_6((4,5,6)) = 4^{(4-2)} =$$

Example 2

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

The number of parking functions of length n is $|PF_n| = (n+1)^{n-1}$.

Example

Let $n = 6$. Start with $\mathbf{t} = (4,5,6)$. Then, $\mathbf{u} = (1,2,3)$, so $\mathbf{x} = (1, 2-1, 3-2) = (1,1,1)$. Observe

$$PC_6\big((4,5,6)\big) = 4^{(4-2)} = (3+1)^{3-1} = P_4\big((1,1,1)\big).$$

Thank you!!!

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