Probability and Parking

Based off of joint work with Pamela E. Harris, Thiago Holleben,
J. Carlos Martínez Mori, Keith Sullivan, and Per Wagenius

From the 2024 Graduate Research Workshop In Combinatorics

The Classical Parking Process

In 1966 Konheim and Weiss [5] introduced a now classical parking protocol.

- Fix $n \in \mathbb{N} := \{1,2,3,\ldots\}$ and consider a sequence of n cars, each given a label in the set $[n] := \{1,2,\ldots,n\}$ on the labeled parking lot $[n]$.
- Initially, each car records their preferred parking spot in the preference vector $\alpha := (\alpha_1, \ldots, \alpha_n) \in [n]^n$ with α_i being the preference of car i.
- The cars enter the street from left to right and attempt to park in increasing order of their labels according to the following rules.

The Classical Parking Process

The Rules:

- On its turn to park, if car i's preferred spot is unoccupied it is able to park right away in spot α_i.
- Otherwise, the car continues on to find the nearest open parking spot to the right and parks there.
- If there is no such spot, the car exits the street and is unable to park.

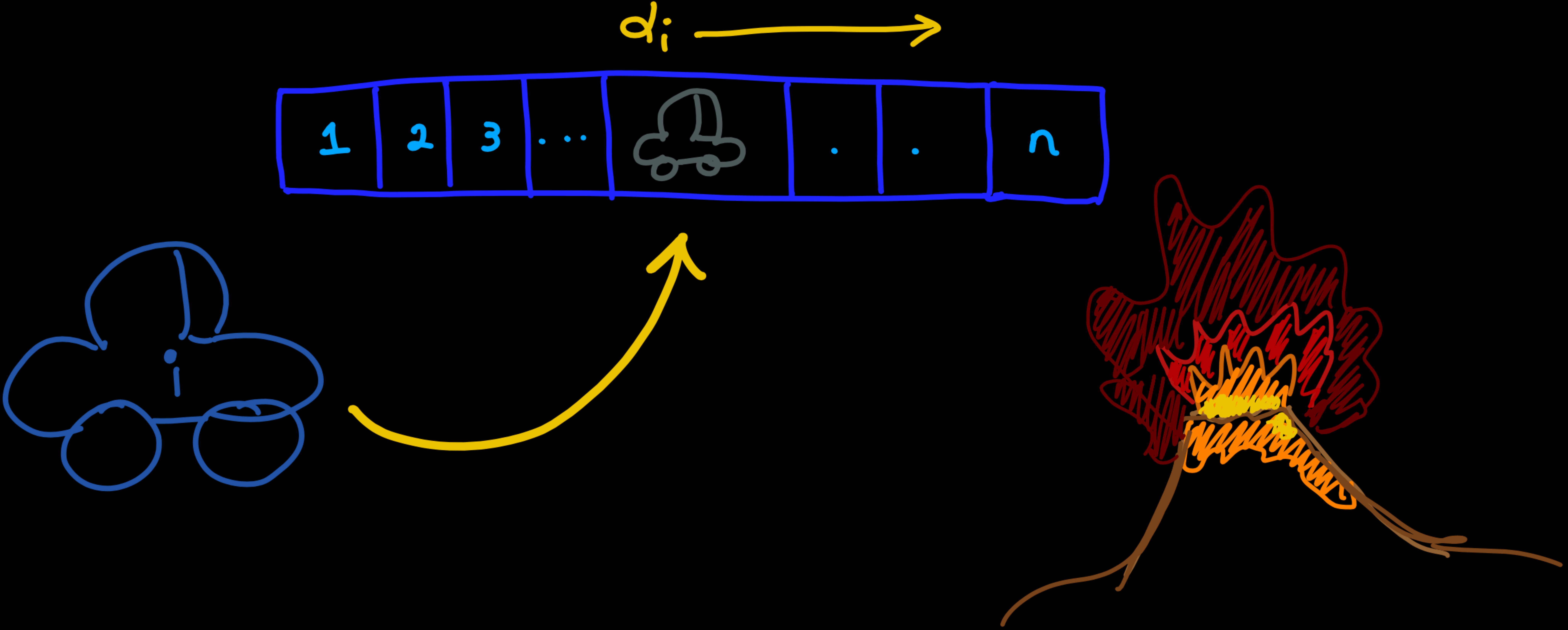
Definition:

— A preference vector $\alpha \in [n]^n$ which allows all of the cars to park (i.e. none of the cars leave the parking lot) under this classical parking protocol is called a *Parking Function* (denoted by PF_n).

— The tuple recording where each car parks is called the *outcome permutation*.

—Parking Functions are objects of central interest within enumerative combinations community. This began with Konheim and Weiss proving that $\left|\mathrm{PF}_n\right| = (n+1)^{n-1}$.

$(\alpha_1, \alpha_2, \dots, \alpha_n)$



Examples

Some simple, but illustrative examples of parking functions are:

- $(1,1,1,1,\ldots,1)$
- $\sigma \in S_n$
- $(2,1,3,4,4,4,7)$

An example of a preference vector which is not a parking function is:

- $(n, n, n, n, \ldots, n)$
- $(2,1,\underline{4},4,4,4,7)$

The Classical Parking Process

The Rules:

- On its turn to park, if car i's preferred spot is unoccupied it is able to park right away in spot α_i.
- Otherwise, the car continues on to find the nearest open parking spot to the right and parks there.
- If there is no such spot, the car exits the street and is unable to park.

Definition:

— A preference vector $\alpha \in [n]^n$ which allows all of the cars to park (i.e. none of the cars leave the parking lot) under the classical parking protocol is called a *Parking Function* (denoted by PF_n).

— The tuple recording where each car parks is called the *outcome permutation*.

—Parking Functions are objects of central interest within enumerative combinations community. This began with Konheim and Weiss proving that $\left|\mathrm{PF}_n\right| = (n+1)^{n-1}$.

The Probabilistic Parking Protocol

We introduce a probabilistic parking protocol which is a generalization of **IDLA** introduced by Diaconis and Fulton [2], and later studied by Lawler, Bramson and Griffeath [6], Ben Arous and Ramírez [4], and Gravner and Quastel [1] (amongst others).

As in the classical parking protocol of Konheim and Weiss, we still have n labeled cars parking on the points of the line segment $[n]$.

The case of successful parking is still the same:

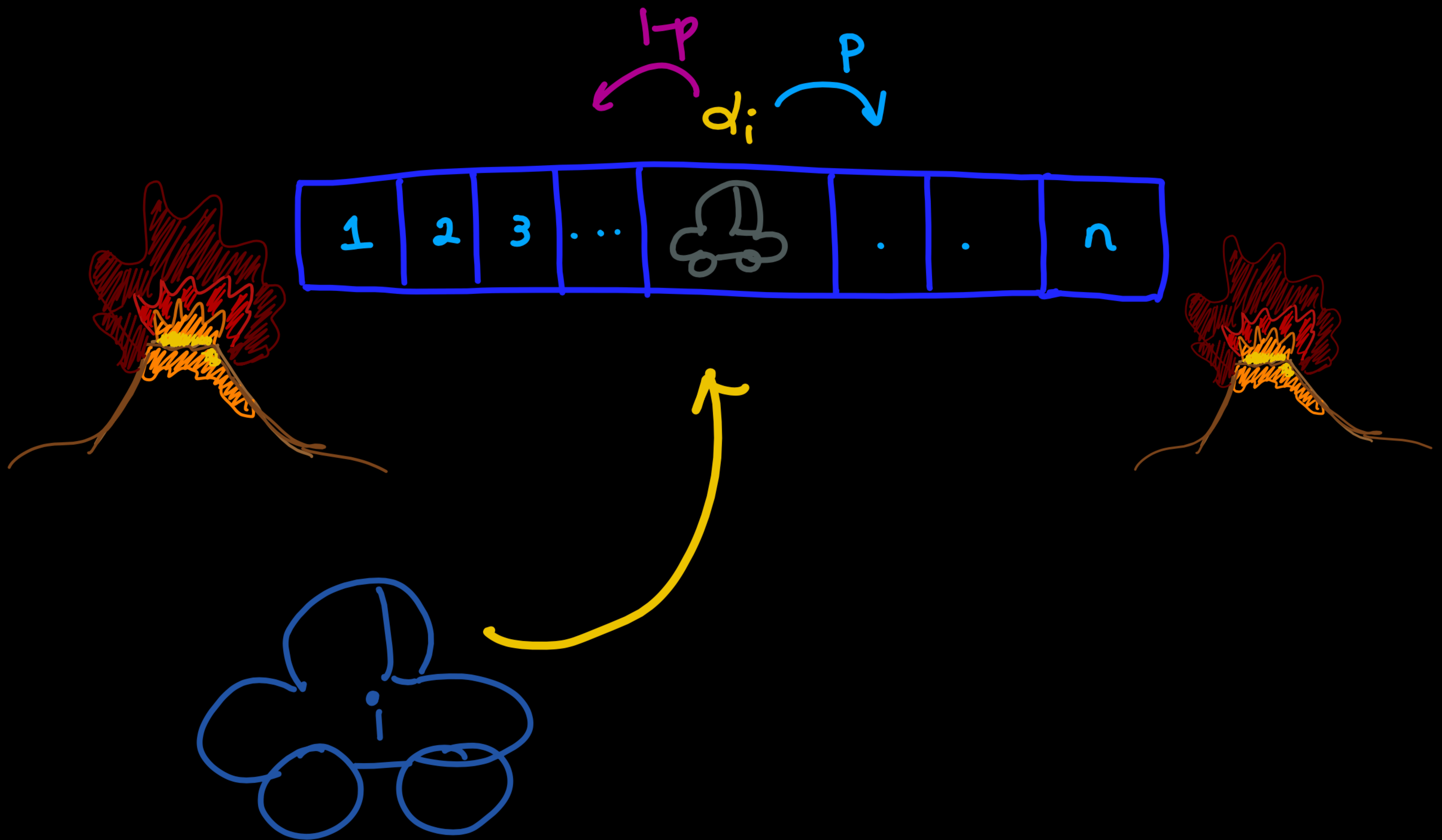
- On its turn to park, if car i's preferred spot is unoccupied it is able to park right away in spot α_i.

Otherwise, if spot α_i is occupied:

- Car i performs a p-biased random walk starting from spot α_i, where it takes a step forward with probability p, a step backwards with probability $q := 1 - p$. The continues its random walk until parking in the first empty spot it encounters (if any).

Fix a probability p.

$(\alpha_1, \alpha_2, \dots, \alpha_n)$



Boundary Conditions

In particular, we consider two rules applied to steps made from the *boundaries (i.e. in spots* 1 and n).

- **Open Boundaries:** This case is the same as in the classical protocol. If a car is in position n (respectively 1) and attempts to take a step to the right (respectively left), the car leaves the lot and fails to park.
- **Unbounded Model:** In this case, the car is free to wander around the infinite set $\mathbb{Z}$ but is only allowed to park on the line segment $[n]$.
- **Periodic Boundary Condition**: The probabilistic parking process takes place on the circle $\mathbb{Z}/n\mathbb{Z}$. (Varin [10])
- **Closed Boundaries**: The probabilistic parking process takes place on the line segment $[n]$, and if a car attempts to take a step to the right from position n (or a step to the left from position 1) the move is canceled and the car remains in place.

Boundary Conditions

Open Boundaries: In this case the boundary conditions are the same as in the classical protocol. If a car is in position n (respectively 1) and attempts to take a step to the right (respectively left), the car leaves the lot and fails to park.

Today I will be focusing on the Open Boundary Case.

Notice that, the case of $p = 1$ is equivalent to the classical parking protocol of Konheim and Weiss, while $q = 1$ is equivalent to running the classical protocol "backwards" from right to left on the segment.

The Question of Interest

Recall: In the original parking protocol by definition PF_n is the set of preference vectors allowing all cars to park, and the *outcome permutation* is fixed.

In our case, however, for each vector $\alpha \in [n]^n$ there is some probability that it allows all of the cars to park, and the process induces a distribution over permutations. Thus, our questions of interest are:

What is the probability that the probabilistic parking protocol started with preference vector α parks in outcome permutation σ, for fixed $\alpha \in [n]^n$ and $\sigma \in S_n$

If we assume all of the cars are able to park, how long will this process take?

A similar question was answered for the Unbounded Model using algebraic methods by Nadeau [5] *and Nadeau and Tewari* [6]

Results

We answer this question for the set of parking functions whose *outcome permutation* (in the sense of the classical parking process) is the identity permutation. We use $\mathrm{PF}_n(\mathrm{id})$ to represent this set.

We actually have a succinct characterization for these parking functions:

Lemma 1.2. *A preference vector* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [n]^n$ *satisfies* $\alpha \in \mathrm{PF}_n(\mathrm{id})$ *if and only if its entries satisfy* $\alpha_i \leq i$ *for all* $i \in [n]$.

Moreover, notice that this **also** characterizes the subset of vectors where the cars park in order of the identity permutation in our **probabilistic parking process**, assuming all of the cars are able to park.

Probabilistic Parking Started With $\alpha \in \mathrm{PF}_n\left(\mathrm{id}\right)$

Theorem 4.1 *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{PF}_n(\mathrm{id})$*. Under the probabilistic parking protocol with open boundaries and initial preferences given by* α*, the probability that all of the cars are able to park is given by*

$$\Pr\left[X_\alpha = 1\right] = \begin{cases} \prod_{k=1}^{n} \frac{\alpha_k}{k} & \text{if} \quad p = \frac{1}{2} \\ \displaystyle\prod_{k=1}^{n} p^{k-\alpha_k}\left(\frac{p^{\alpha_k} - q^{\alpha_k}}{p^k - q^k}\right) & \text{otherwise}. \end{cases}$$

In particular, notice that when $p = q = \frac{1}{2}$ *this is simply the product of all of the parking preferences normalized by* $\frac{1}{n!}$*.*

Probabilistic Parking Started With $\alpha \in \mathrm{PF}_n\,(\mathbf{id})$

Theorem 4.4 *Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{PF}_n(\mathrm{id})$ the expected time for all of the cars to park under the probabilistic parking protocol with open boundaries is given by*

$$\mathbb{E}\left[\tau_\alpha \mid X_\alpha = 1\right] = \begin{cases} \frac{1}{3}\sum_{i=1}^{n}(i^2 - \alpha_i^2) = \frac{2n^3+3n^2+n}{18} - \frac{1}{3}\sum_{i}^{n}\alpha_i^2 & \text{if} \quad p = \frac{1}{2} \\ \frac{1}{p-q}\sum_{i=1}^{n} \frac{i\left(1+\left(\frac{q}{p}\right)^{i}\right)}{\left(1-\left(\frac{q}{p}\right)^{i}\right)} - \frac{\alpha_i\left(1+\left(\frac{q}{p}\right)^{\alpha_i}\right)}{\left(1-\left(\frac{q}{p}\right)^{\alpha_i}\right)} & \text{otherwise}\,. \end{cases}$$

Proof Sketch

- We rely heavily on the fact that our outcome permutation is fixed to be the identity permutation, and the *chain rule of probability*.
- Let A_i denote the event that the i^{th} car to arrive parks in spot i. Using the chain rule we have the probability that the first k cars park, for $k \in [n]$ is given by:

$$
\begin{aligned}
\Pr\left[\bigcap_{i=1}^{k} A_i\right] &= \Pr\left[A_1\right] \cdot \Pr\left[A_2 \mid A_1\right] \cdot \Pr\left[A_3 \mid A_1 \cap A_2\right] \dots \\
&= \prod_{i=1}^{k} \Pr\left[A_i \mid \cap_{j=1}^{i-1} A_j\right]
\end{aligned}
$$

Proof Sketch

- When we begin our probabilistic parking protocol with a parking function $\alpha \in \mathrm{PF}_n\left(\mathrm{id}\right)$, for any car i, for $i \in [n]$ the term

$$\Pr\left[A_i \mid \cap_{j=1}^{i-1} A_j\right]$$

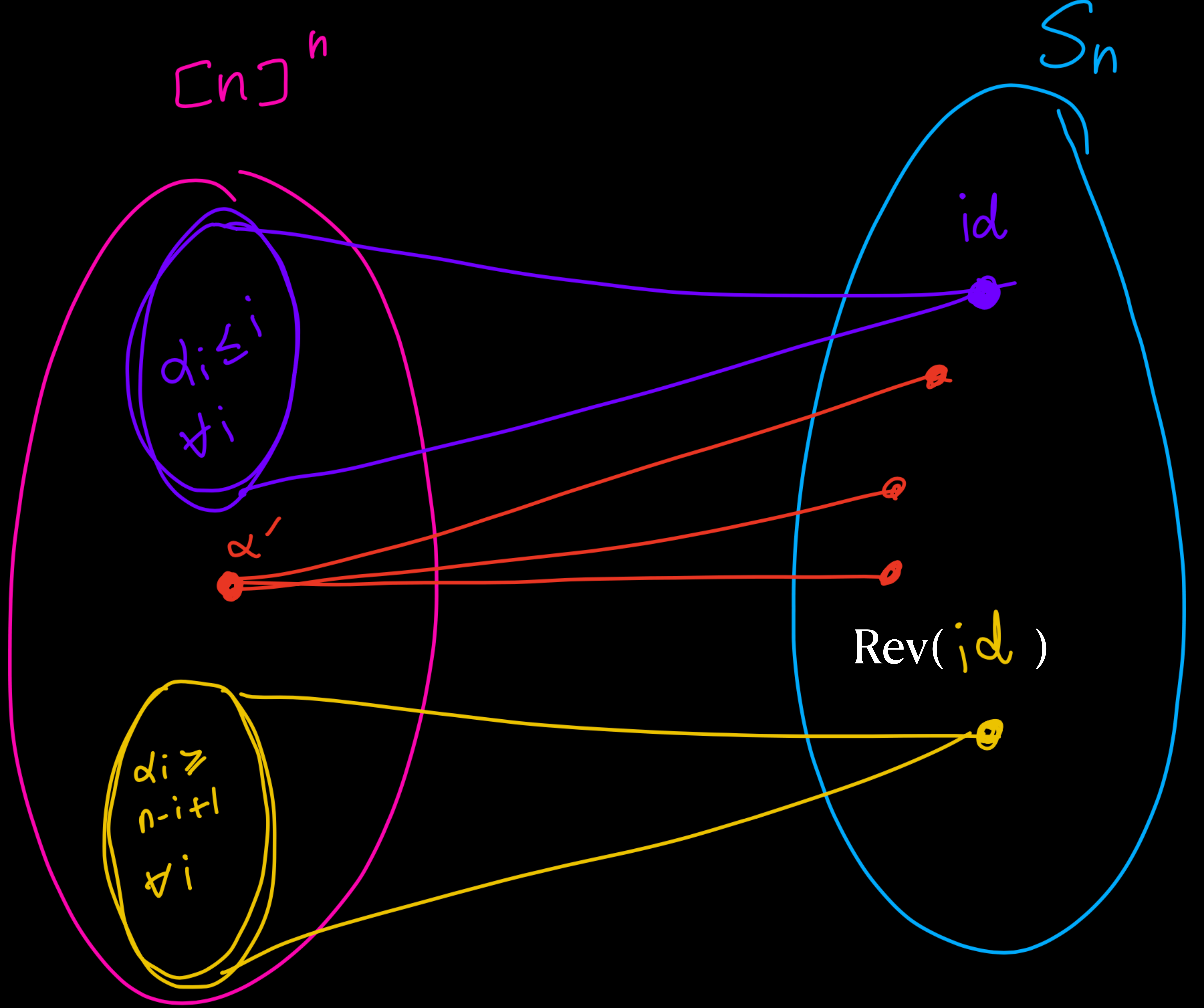
- Is equivalent to the probability car i's random walk hits position i before hitting "position 0" when starting from position α_i.
- This is an instance of the classical and well studied *Gambler's Ruin problem*.

Other Results

- In addition, we have analogous results for the Unbounded Model. The results for the Unbounded model similarly rely on gambler's ruin computations (on $\mathbb{Z}$) by McMullen [7].
  - When starting with $\alpha \in \mathrm{PF}_n\left(\mathrm{id}\right)$ if $p \in \left[\frac{1}{2}, 1\right]$ then car i parks with probability going to 1 as, with probability going to 1 the car will hit position i in a finite amount of time.
- In addition, we also show that the parking process with open boundaries is *negatively correlated.*
  - That is, we show that an earlier car parking only makes it less likely that later cars will be able to park. (This is probably intuitive, as the number of free spots will only decrease).
- We also include some statistics of weakly increasing parking functions, such as the distribution and expected value of the last entry. In addition we give the distribution of the set of *lucky cars* of a weakly increasing parking function, where *a* car is *lucky* if it is able to park in its preferred spot immediately (regardless of the protocol).

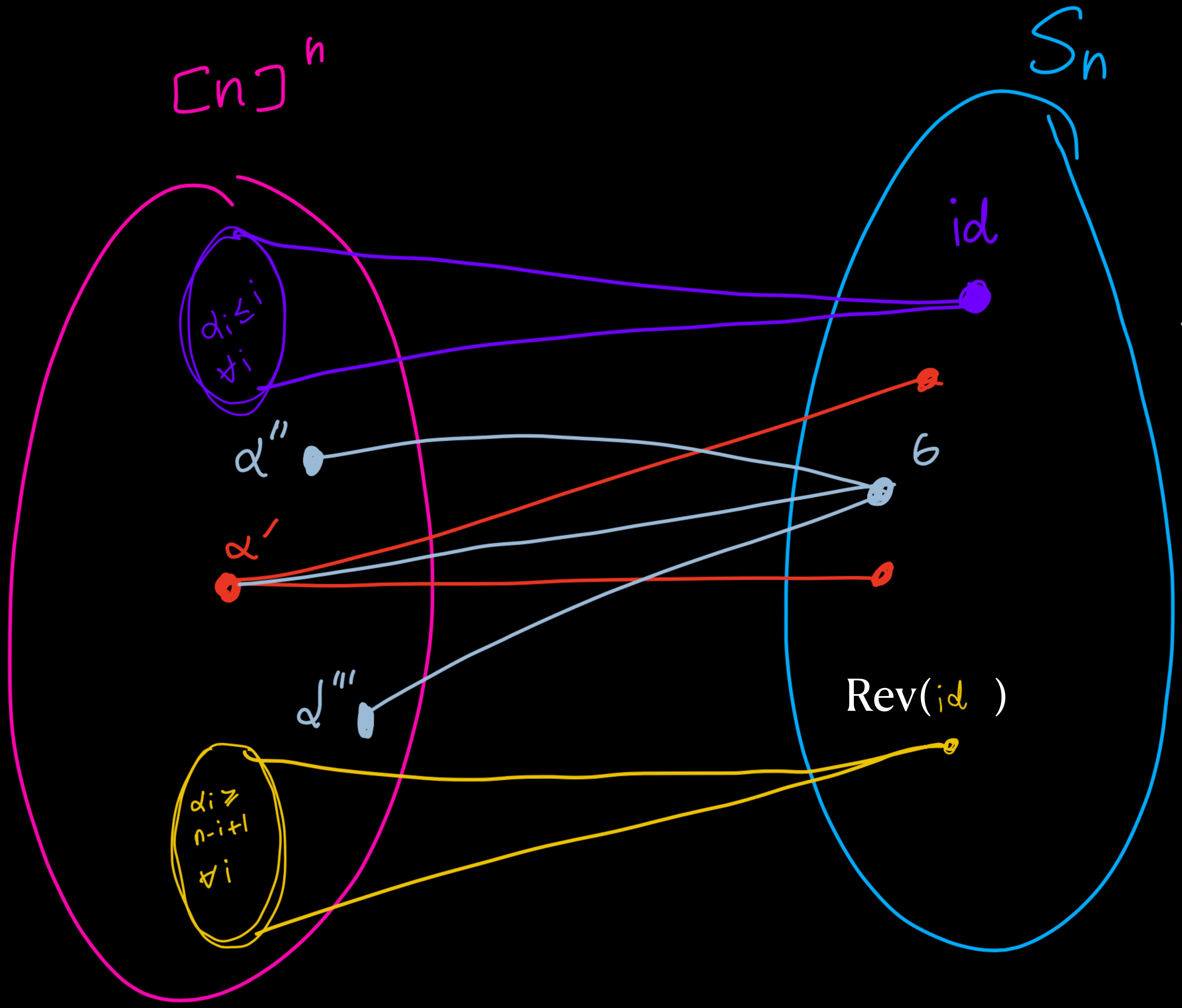
My Favorite Open Problems

- **Understanding the distribution of the outcome permutation**
- What if each car is only allowed k steps of a random walk?
  (This makes the process on the cycle more interesting)
- What if p depends on which site you are on, or depending on which car is trying to park?



Fix $\alpha \in [n]^n$ and $p \in (0,1)$

$\Pr(\mathrm{Out}(\alpha)=\sigma)$?



Fix σ. and $p \in (0,1)$

How many α map to this σ?

My Favorite Open Problems

- Understanding the distributions of the outcome permutations.
- **What if each car is only allowed *k* steps of a random walk?**
  (This makes the process on the cycle more interesting)
- What if p depends on which site you are on, or depending on which car is trying to park?

Boundary Conditions

- **Periodic Boundary Condition**: The probabilistic parking process takes place on the circle $\mathbb{Z}/n\mathbb{Z}$.
- **Closed Boundaries**: The probabilistic parking process takes place on the line segment $[n]$, and if a car attempts to take a step to the right from position n (or a step to the left from position 1) the move is canceled and the car remains in place.

- In the case of the periodic boundary (and assuming reasonable values of n and p in the closed boundary case) all of the cars will be able to park with probability 1.

My Favorite Open Problems

- Understanding the distributions of the outcome permutations.
- What if each car is only allowed k steps of a random walk?
  (This makes the process on the cycle more interesting)
- **What if p depends on which site you are on, or depending on which car is trying to park?**

Bibliography

[1] Ben Arous, Gerard, and Alejandro F. Ramírez. "Asymptotic survival probabilities in the random saturation process." *Annals of probability* (2000): 1470-1527.

[2] Diaconis, Persi, and William Fulton. "A growth model, a game, an algebra, Lagrange inversion, and characteristic classes." *Rend. Sem. Mat. Univ. Pol. Torino* 49, no. 1 (1991): 95-119.

[3] Durmić, Irfan, Alex Han, Pamela E. Harris, Rodrigo Ribeiro, and Mei Yin. "Probabilistic Parking Functions." The Electronic Journal of Combinatorics (2023): P3-18.

[4] Gravner, Janko, and Jeremy Quastel. "Internal DLA and the Stefan problem." Annals of probability (2000): 1528-1562.

[5] Konheim, Alan G., and Benjamin Weiss. "An Occupancy Discipline and Applications." SIAM Journal on Applied Mathematics 14, no. 6 (1966): 1266–74. http://www.jstor.org/stable/2946240.

[6] Lawler, Gregory F., Maury Bramson and David Griffeath. "Internal Diffusion Limited Aggregation." Ann. Probab. 20 (4) 2117 - 2140, October, 1992. https://doi.org/10.1214/aop/1176989542

[7] McMullen, Matthew. 2020. "A Drunken Walk in Las Vegas: Catalan Convolutions and Gambling." Mathematics Magazine 93 (1): 62–68. doi:10.1080/0025570X.2020.1684798.

[8] Nadeau, Philippe. "Bilateral parking procedures." 2025. hal-04262134v2

[9] Nadeau, Philippe, and Vasu Tewari. "Remixed Eulerian Numbers." Forum of Mathematics, Sigma 11 (2023): e65. https://doi.org/10.1017/fms.2023.57.

[10] Varin, Zoé. "The golf model on $\mathbb{Z}/n\mathbb{Z}$ and on $\mathbb{Z}$." *arXiv preprint arXiv:2401.13380* (2024).