

# Parking Completions and Volumes of Polytopes

Kimberly P. Hadaway

Iowa State University

AMS Fall Central Section  
Parking Functions and their Generalizations  
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# Overview

- Parking Functions
  - ▶ Classical
  - ▶  $x$ -parking functions
  - ▶ Parking completions
- Polytopes
  - ▶ Classical
  - ▶ Pitman-Stanley
  - ▶ Volumes
- (My) Problem
- Product/Payoff (i.e., Result)

## Definitions

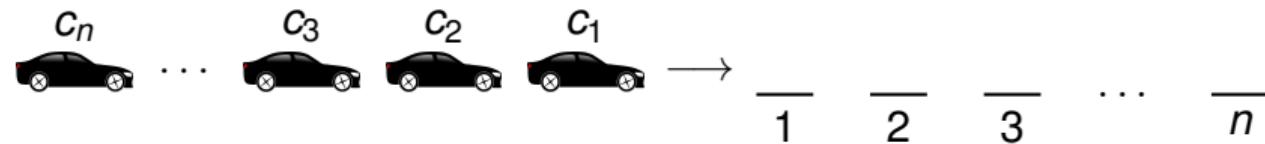


Figure: Parking Function Illustration.

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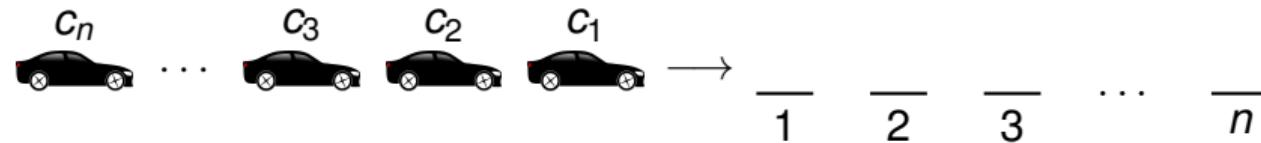


Figure: Parking Function Illustration.

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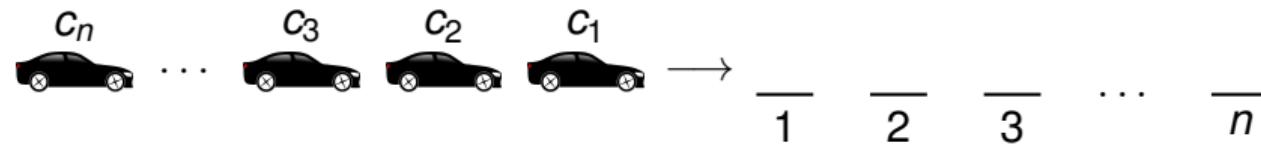


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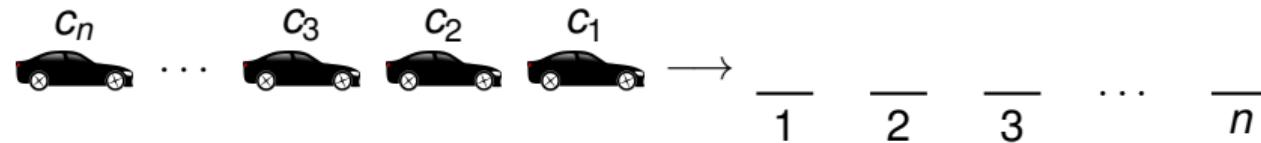


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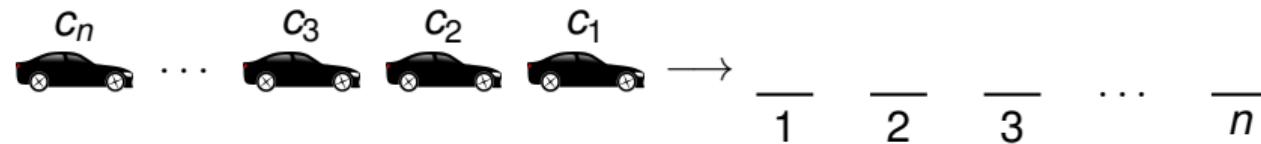


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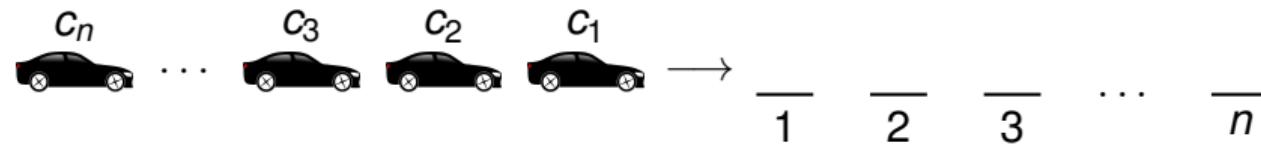


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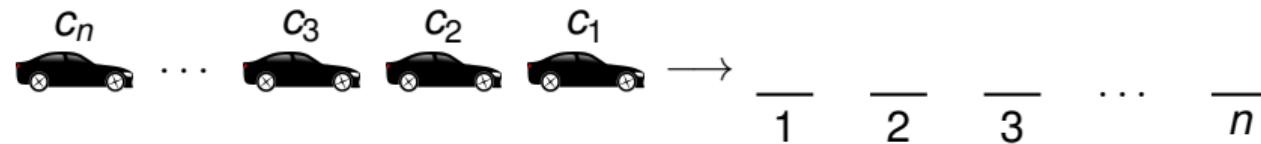
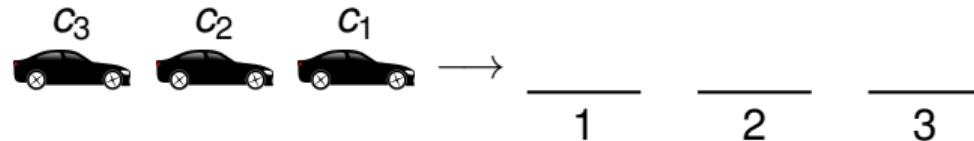


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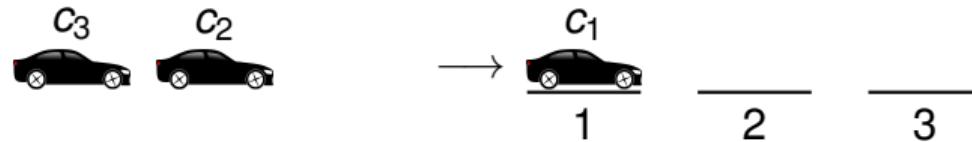
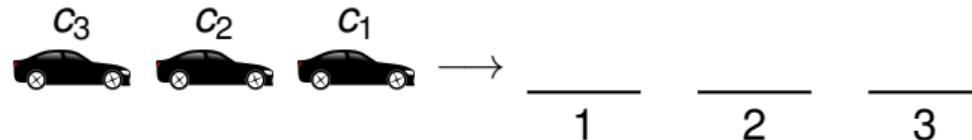
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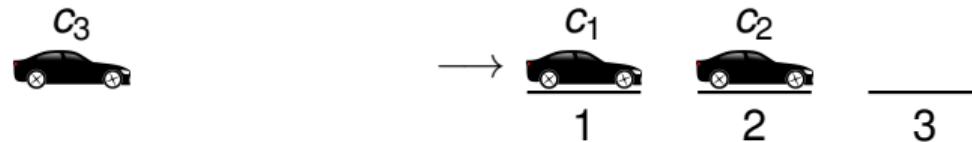
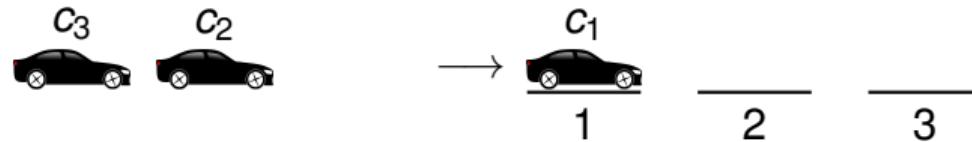
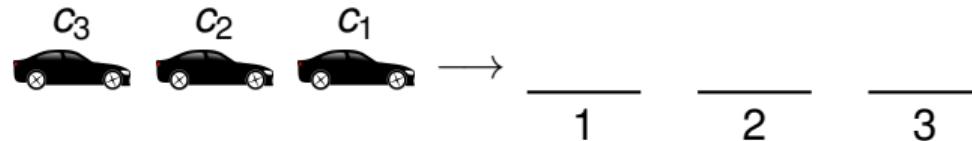
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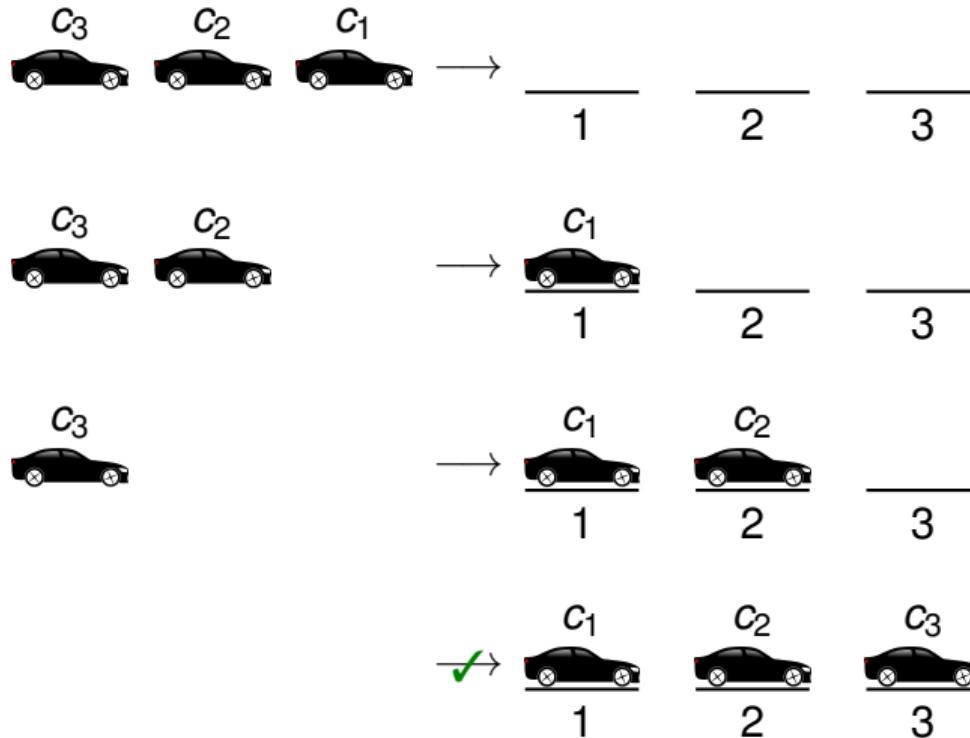
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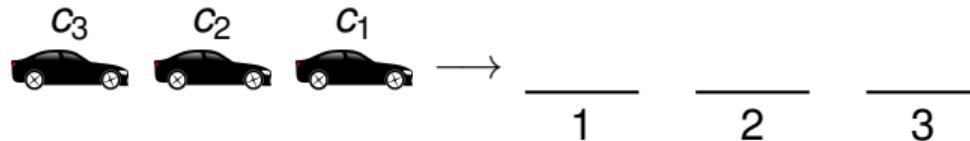
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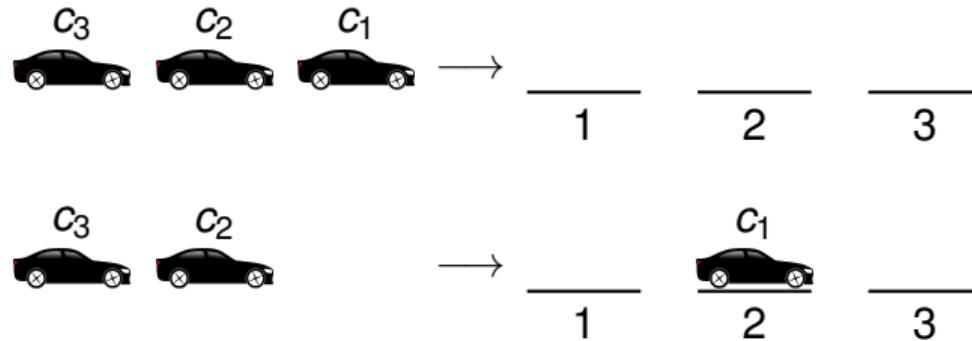
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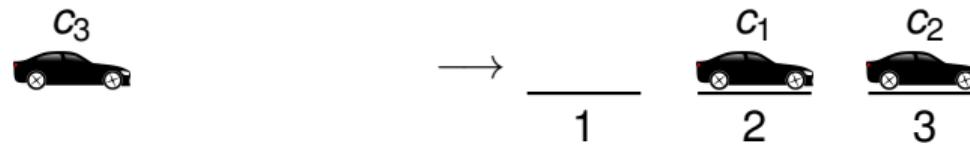
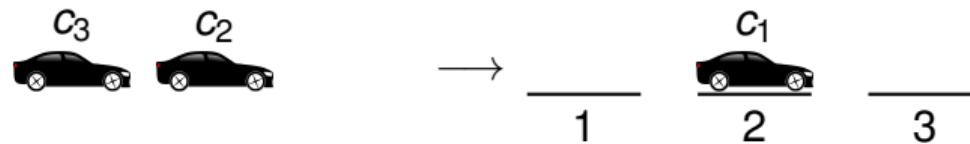
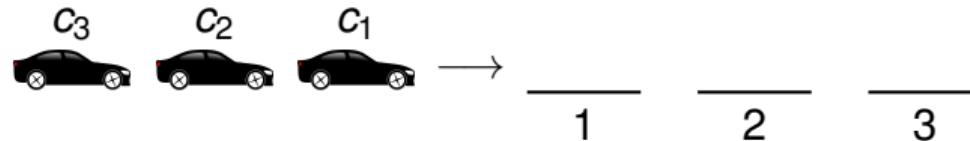
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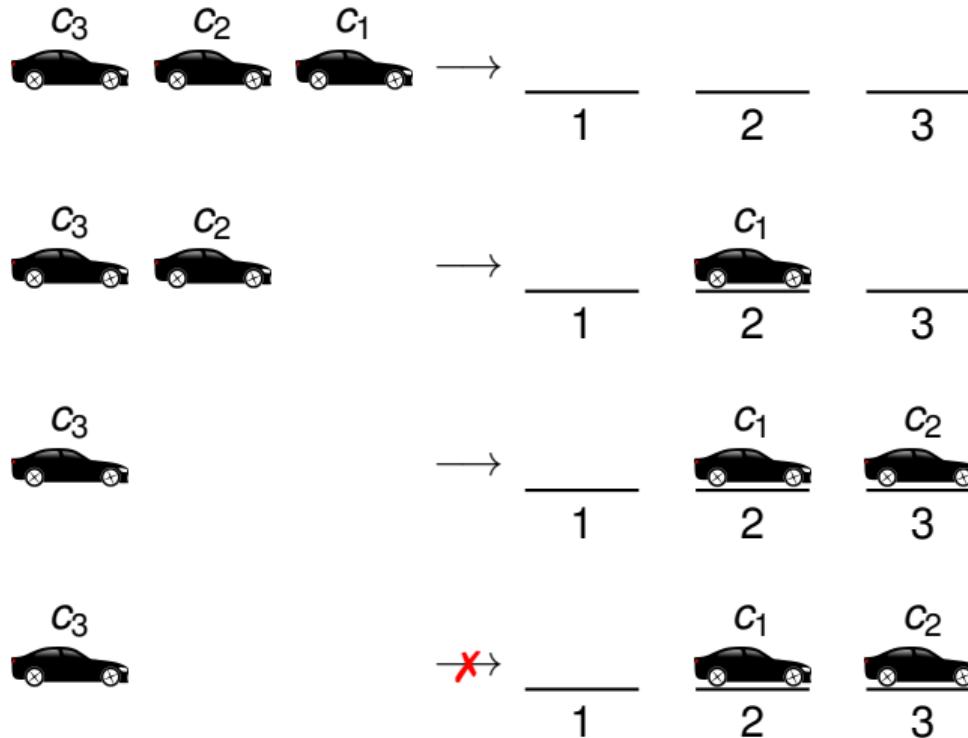
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## Examples and Enumeration

Theorem (N. Shales, 2018)

*The number of parking functions of length  $n$  is  $|PF_n| = \sum_{i=1}^n i \binom{n-1}{i-1} |PF_{i-1}| \cdot |PF_{n-i}|$ .*

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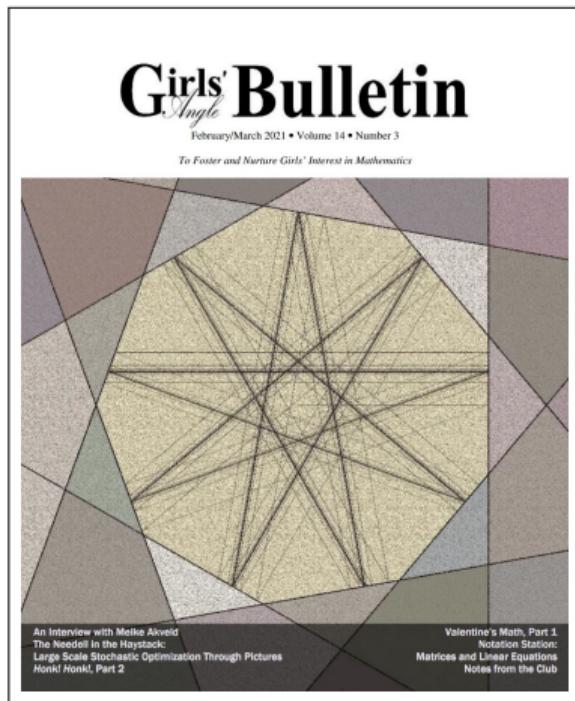
Theorem (Pyke, 1959; Konheim and Weiss, 1966)

*The number of parking functions of length  $n$  is  $|PF_n| = (n+1)^{n-1}.$*

# Publications!



**Figure:** *Honk! Honk!, Part 1: An Introduction to Parking Functions* (Hadaway, 2021)



**Figure:** *Honk! Honk!, Part 2: An Introduction to Parking Functions* (Hadaway, 2021)

## Another Characterization

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A **parking function** is a sequence of positive integers  $\alpha = (a_1, a_2, \dots, a_n)$  such that its weakly increasing rearrangement  $\alpha^\uparrow := \mathbf{b} = (b_1, b_2, \dots, b_n)$  satisfies  $b_i \leq i$  for all  $i \in [n]$ .

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For  $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ , define an  **$\mathbf{x}$ -parking function** to be a sequence  $(a_1, a_2, \dots, a_n)$  of positive integers whose increasing rearrangement  $b_1 \leq b_2 \leq \dots \leq b_n$  satisfies  $b_i \leq x_1 + x_2 + \dots + x_i$ . Let  $P_n(\mathbf{x})$  denote the number of  $\mathbf{x}$ -parking functions.

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# Generalization — parking completions

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Let  $\mathbf{t} = (t_1, t_2, \dots, t_m)$  denote which spots are taken (in increasing order). Then,  $\mathbf{c} = (c_1, c_2, \dots, c_{n-m})$  is a **parking completion** if all cars (whose preferences are in  $\mathbf{c}$ ) can park. Let  $PC_n(\mathbf{t})$  denote the set of parking completions of  $\mathbf{t}$  in  $[n]$ .

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- car 1 parks in spot 3,
- car 2 parks in spot 2, and
- car 3 parks in spot 4,

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# Polytopes

## Definition

A **polytope**  $P$  in  $\mathbb{R}^n$  with integer vertices is:

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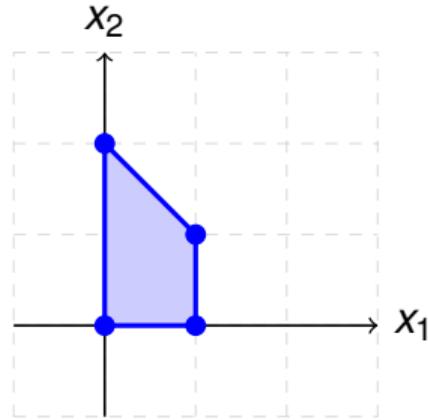


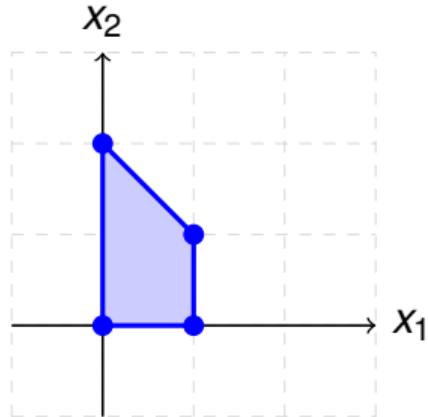
Figure: Illustration of  $PS_2((1, 1))$

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- the convex hull of finitely many vertices  $v$  in  $\mathbb{Z}^n$ , OR
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$$\begin{aligned}x_1 &\geq 0, \\x_2 &\geq 0, \\x_1 &\leq 1, \text{ and} \\x_1 + x_2 &\leq 1 + 1 = 2.\end{aligned}$$

Figure: Illustration of  $PS_2((1, 1))$

# Pitman-Stanley Polytope

## Definition

Given a vector  $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$ , the **Pitman-Stanley polytope** is

$$\text{PS}_n(\mathbf{a}) = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_{\geq 0}^n \mid \mathbf{x} \preceq \mathbf{a}\}.$$

## Definition

We say  $\mathbf{x}$  is less than  $\mathbf{a}$  in dominance order, denoted  $\mathbf{x} \preceq \mathbf{a}$ , if

$$x_1 \leq a_1$$

$$x_1 + x_2 \leq a_1 + a_2$$

⋮

$$x_1 + x_2 + \cdots + x_n \leq a_1 + a_2 + \cdots + a_n.$$

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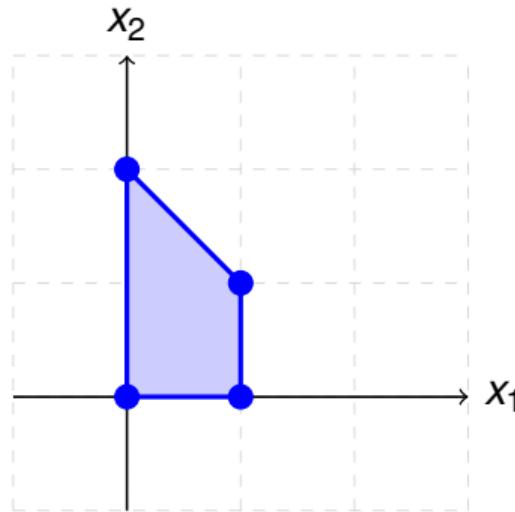
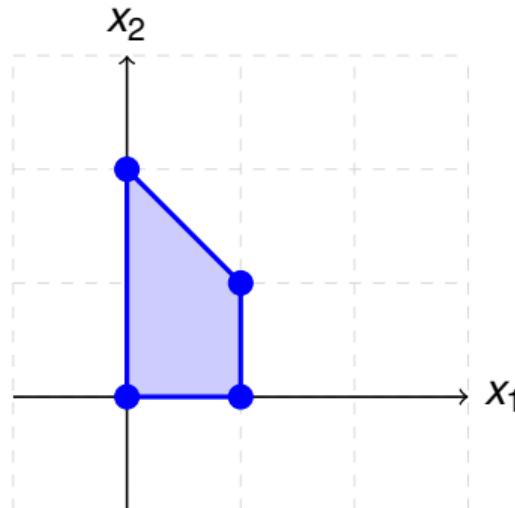


Figure: Illustration of  $PS_2((1, 1))$

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(here, yes boundary!)

- $d\text{vol } \text{PS}_2((1,1)) = 5$
- $\text{Evol } \text{PS}_2((1,1)) = \frac{3}{2}$
- $n\text{vol } \text{PS}_2((1,1)) = 2! \cdot \frac{3}{2} = 3$

Figure: Illustration of  $\text{PS}_2((1,1))$

# Volume of Pitman-Stanley polytope

Theorem (Stanley-Pitman, 2002)

With  $P_n(\mathbf{x})$  and  $\text{Evol PS}_n(\mathbf{x})$  as above, we have

$$P_n(\mathbf{x}) = \sum_{(a_1, a_2, \dots, a_n) \in PF_n} x_{a_1} x_{a_2} \cdots x_{a_n} = n! \cdot \text{Evol PS}_n(\mathbf{x}).$$



## Gap in the Literature

Adeniran et al. (2022) show that the number of parking completions is

$$|\text{PC}_n(\mathbf{t})| = \sum_{\ell \in L_n(\mathbf{t})} \binom{n-m}{\ell} \prod_{j=1}^{m+1} (\ell_j + 1)^{(\ell_j - 1)} \quad (1)$$

where  $L_n(\mathbf{t}) = \{\ell = (\ell_1, \ell_2, \dots, \ell_{m+1}) \in \mathbb{N}^{(m+1)} \text{ such that } \ell_1 + \ell_2 + \dots + \ell_j \geq t_j - j \text{ for all } j \in [m] \text{ and } \ell_1 + \ell_2 + \dots + \ell_{m+1} = n - m.$

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$$P_N(\mathbf{x}) = N! \sum_{\mathbf{k} \in K_N} \prod_{i=1}^N \frac{x_i^{k_i}}{k_i!} = \sum_{\mathbf{k} \in K_N} \binom{N}{\mathbf{k}} x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N} \quad (2)$$

where  $K_N = \{\mathbf{k} \in \mathbb{N}^N \text{ such that } \sum_{i=1}^j k_i \geq j \text{ for all } j \in [N-1] \text{ and } \sum_{i=1}^N k_i = N\}$ .

# Main Result

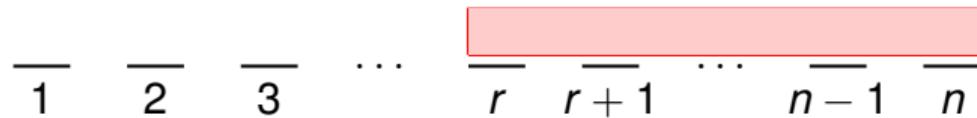


Figure: Parking Completion Street Diagram.

## Theorem (Hadaway, 2025+)

Fix  $n \in \mathbb{N}$ . If

- $\mathbf{t} = ()$ ,
- $\mathbf{t} = (1, 2, \dots, n)$ ,
- $\mathbf{t} = (n)$ , or
- $\mathbf{t} = (r, r+1, \dots, n)$  for  $r \in [n]$ ,

Then, the set of parking completions of  $\mathbf{t}$  is exactly the set of  $\mathbf{x}$ -parking functions.

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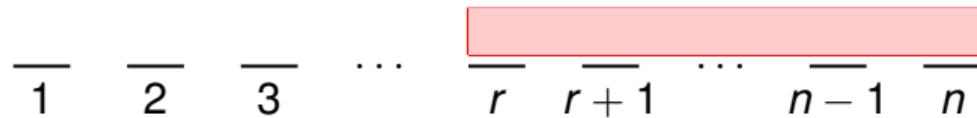


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## Sketch.

Recover the classical case! □

## Corollary

$$|\text{PC}_n(\mathbf{t})| = P_n(\mathbf{x}).$$

## Example

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

*The number of parking functions of length  $n$  is  $|PF_n| = (n + 1)^{n-1}$ .*

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Let  $n = 5$ . Let  $\mathbf{t} = (3, 4, 5)$ .

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## Example

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$$PC_5((3, 4, 5)) = 3^{(3-2)} = (2+1)^{2-1} = P_4((1, 1, 1)).$$

Thank you!!!



Email

kph3@iastate.edu

Website

<https://www.kimberlyphadaway.com/>

