

**Lecture 2:** More on Graphs**Date:** January 26, 2026**Scribe:** Mark Johnson

# 1 Degree

## 1.1 Undirected Graphs

For an undirected graph  $G = (V, E)$ , for each  $u \in V$ ,  $\deg(u)$  is the number of edges adjacent to it. Formally,

$$\deg(u) := |\{e = \{v, w\} \in E : u \in e\}|.$$

Note the following terminology:

- If  $\deg(u) = 1$ ,  $u$  is a leaf.
- If  $\deg(u) = 0$ ,  $u$  is isolated.

## 1.2 Directed Graphs

For a directed graph, there are two different notions of degree. The in-degree (out-degree) of  $u \in V$  is the number of edges going into (out of)  $u$ . Formally,

$$\deg^-(u) := |\{e = (v, w) \in E : w = u\}|$$

and

$$\deg^+(u) := |\{e = (v, w) \in E : v = u\}|.$$

## 1.3 Examples

Consider the undirected graph in Figure 1. Then, we have

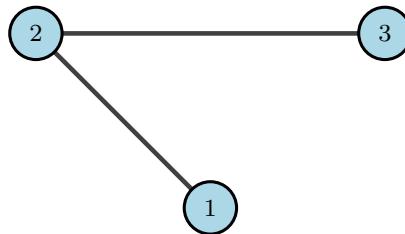


Figure 1: An undirected graph.

- $\deg(1) = 1$ ,
- $\deg(2) = 2$ , and
- $\deg(3) = 1$ .

Similarly, consider the directed graph in Figure 2. Then, we have

- $\deg^-(1) = 0$  and  $\deg^+(1) = 1$ ,
- $\deg^-(2) = 1$  and  $\deg^+(2) = 1$ , and
- $\deg^-(3) = 1$  and  $\deg^+(3) = 0$ .

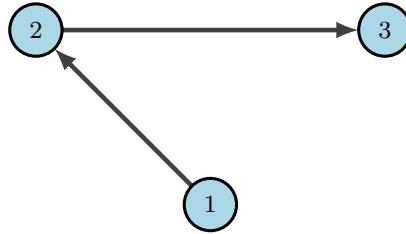


Figure 2: A directed graph.

## 2 Subgraphs

Let  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$ .  $H$  is a subgraph of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .

## 3 Union of Graphs

The union of  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  is a graph  $W = (V(W), E(W))$  such that

- $V(W) = V(G) \cup V(H)$  and
- $E(W) = E(G) \cup E(H)$ .

## 4 Paths

### 4.1 Undirected Graphs

A path is a graph  $P = (V(P), E(P))$  where  $V(P)$  can be totally ordered such that two nodes are adjacent in  $P$  only if they are adjacent in the order.

We can visualize this in Figure 3 and Figure 4.



Figure 3: A path.



Figure 4: Another path.

In the example, both graphs are paths. The graph in Figure 4 is a path: to see this algebraically, we can map each node to  $\mathbb{N}$  where  $a$  maps to 1,  $d$  maps to 2,  $w$  maps to 3, and so on. A path can be expressed as:

- A sequence of nodes, such as  $v_1, v_2, \dots, v_k$ .
- A sequence of edges, such as  $e_1, e_2, \dots, e_{k-1}$ .
- A sequence of nodes and edges, such as  $v_1, e_1, v_2, e_2, \dots, v_k, e_{k-1}$ .

Note that  $|E(P)| = |V(P)| - 1$ . If  $v_1 = s$  and  $v_k = t$ , we say  $P$  is an  $\{s, t\}$ -path. If  $P$  is an  $\{s, t\}$ -path  $P$ , then

- $\deg(s) = 1$ ,
- $\deg(t) = 1$ , and
- $\deg(u) = 2$  for all  $u \in V(P)$  with  $u \neq s, t$ .

## 4.2 Directed Graphs

Directed paths have the same formal definition with the additional requirement that

$$e_i = (v_i, v_{i+1})$$

for all  $i \in [k - 1]$ . In other words, the head of an edge is the same as the tail of the subsequent edge. Figure 5 shows a directed path.

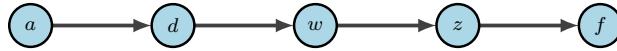


Figure 5: A directed path.

Note that again  $|E(P)| = |V(P)| - 1$ . If  $P$  is an  $(s, t)$ -path, then

- $\deg^-(s) = 0$  and  $\deg^+(s) = 1$ ,
- $\deg^-(t) = 1$  and  $\deg^+(t) = 0$ , and
- $\deg^-(u) = \deg^+(u) = 1$  for all  $u \in V(P)$  with  $u \neq s, t$ .

# 5 Cycles

## 5.1 Undirected Graphs

A cycle is a graph  $C = (V(C), E(C))$  such that its nodes can be placed around a circle on the plane with two nodes are adjacent on the circle if and only if they are adjacent in  $C$ . Figure 6 shows a cycle.

Alternatively, a cycle  $C$  is a path for which we connect its endpoints with an edge. Note that  $|E(C)| = |V(C)|$  and  $\deg(u) = 2$  for all  $u \in V(C)$ .

## 5.2 Directed Graphs

Directed cycles have the same formal definition with the additional requirement of respecting directionality.

# 6 Connectivity

## 6.1 Undirected Graphs

Let  $G = (V, E)$  be an undirected graph. Two (unordered) nodes  $u, v \in V$  are connected if  $G$  contains a  $\{u, v\}$ -path. The graph  $G$  is connected if all (unordered) pairs  $u, v \in V$  are connected. Figure 7 shows a connected graph with multiple  $\{u, v\}$ -paths.

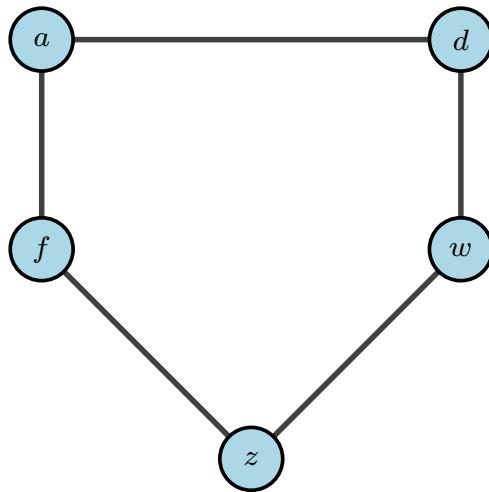


Figure 6: A cycle.

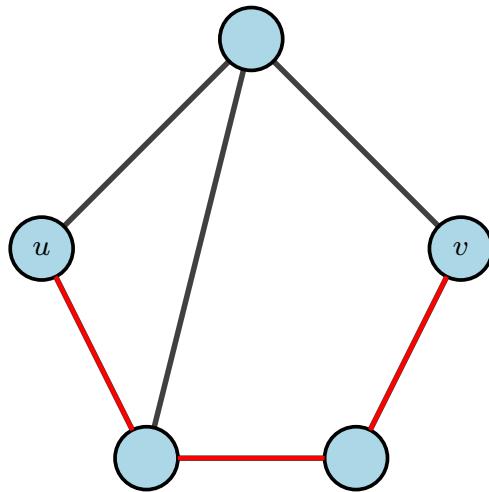


Figure 7: A connected graph.

## 6.2 Directed Graphs

For a directed graph  $G = (V, E)$ , there are two notions of connectivity:

- $G$  is weakly connected if its corresponding undirected graph is connected. For example, the graph in Figure 5 is weakly connected.
- $G$  is strongly connected if it contains a  $(u, v)$ -path for all ordered pairs  $u, v \in V$ .

## 7 Connected Components

### 7.1 Undirected Graphs

A connected component of graph  $G = (V, E)$  is an inclusion-wise maximal connected subgraph of  $G$ . Here, inclusion-wise maximal means there does not exist another other subgraph with the same property (i.e., connectivity) that strictly contains it. For example, Figure 8 shows a graph with three connected components, of sizes 4, 3, and 1 from left to right.

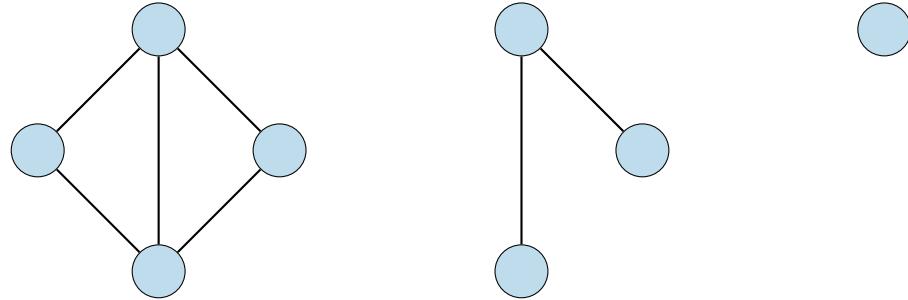


Figure 8: A graph with three connected components.

### 7.2 Directed Graphs

A weakly connected component of a directed graph  $G = (V, E)$  is an inclusion-wise maximal weakly connected subgraph of  $G$ . Similarly, a strongly connected component of  $G$  is an inclusion-wise maximal strongly connected subgraph of  $G$ .