Homework 2

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1 Github

The repository for this homework can be found in: https://github.com/jcmendev/HW2

It contains 10 routines described in the sections of this document

1. Main_Chebyshev.m: Chebyshev main file for running the codes
2. residual_function.m: Chebyshev residual function
3. plot_policy_functions.m: Chebyshev policy functions
4. Main_Perturbation.m: Perturbation main file to run
5. Perturbation_Dynare.mod: Perturbation main dynare file
6. Main_FEM.m: Finite Elements main file to run
7. residual_function_FEM.m: Finite Elements residual function
8. plot_policy_functions_FEM.m: Finite Elements policy functions
9. Main_Labor_DL.ipynb: Main file for Deep Learning
10. MC_Tauchen.m: Tauchen discretization

2 Model

Consider the representative household with preferences over private consumption, c_t, and labor, l_t,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - \frac{l_t^{1+\psi}}{1+\psi} \right)$$

the household consumes, sabes and works with a budget constraint given by,

$$c_t + i_t = w_t l_t + r_t k_t$$

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there is a production function,

$$y_t = e^{z_t} k_t^{\alpha} l_t^{1-\alpha}$$

such that the aggregate resource constraint is,

$$c_t + i_t = y_t$$

and a law of motion for capital

$$k_{t+1} = (1-\delta) k_t + i_t$$

Technology level z_t follos an AR(1) process,

$$z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t$$

where $\varepsilon_t \sim \mathcal{N}(0,1)$.

Let a social planner solve the following sequential problem,

$$\begin{aligned} \max_{\{c_t, l_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - \frac{l_t^{1+\psi}}{1+\psi} \right) \\ \text{s.t} \\ k_{t+1} = (1-\delta) k_t + e^{z_t} k_t^{\alpha} l_t^{1-\alpha} - c_t \\ z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t \end{aligned}$$

given some initial conditions $k_0$and z_0.First Order Conditions (FOC) of this problem are given by,

$$\begin{aligned} [l_t] : l_t^{\psi} &= (1-\alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} \frac{1}{c_t} \\ [k_{t+1}] : \frac{1}{c_t} &= \beta \mathbb{E} \left[\left(\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \frac{1}{c_{t+1}} \right] \end{aligned}$$

We can solve the social planner's problem instead of the competitive equilibrium since welfare theorems hold in this economy.

The recursive formulation for this planner, is analogously given by

$$\begin{aligned} V(k_t, z_t) = \max_{c_t, l_t} \left(\log c_t - \frac{l_t^{1+\psi}}{1+\psi} \right) + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1}) \\ \text{s.t} \\ k_{t+1} = (1-\delta) k_t + e^{z_t} k_t^{\alpha} l_t^{1-\alpha} - c_t \\ z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t \end{aligned}$$

For the different solution methods in this document, we are going to use Tauchen's procedure to discretize z_t into a Markov chain. $\{z_1, ..., z_{n_z}\}$, with probability transition matrix, $\Pi_{z,z'}$, and

$n_z = 3$. We use an standard coverage of ± 3 unconditional standard deviations of z_t. Then, let the planner's recursive problem be rewritten as

$$V(k_t, z_t) = \max_{c_t, l_t} \left(\log c_t - \frac{l_t^{1+\psi}}{1+\psi} \right) + \beta \sum_z \Pi_{z,z'} V(k_{t+1}, z_{t+1})$$

$$\text{s.t}$$

$$k_{t+1} = (1-\delta) k_t + e^{z_t} k_t^\alpha l_t^{1-\alpha} - c_t$$

$$z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t$$

with first order conditions given by

$$[l_t] : l_t^\psi = (1-\alpha) e^{z_t} k_t^\alpha l_t^{-\alpha} \frac{1}{c_t}$$

and envelope condition given by,

$$V_k(k_t, z_t) = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} (1-\delta) \frac{1}{c_t}$$

and the Euler equation is then given by

$$\frac{1}{c_t} = \beta V_k(k_{t+1}, z_{t+1})$$

Projection

Chebyshev

To solve the model with Chebyshev polynomials on capital, let the approximation for next period capital use six Chebyshev polynomials for $j = 1, ..., 5$, as

$$k^j \left(k_t | \theta^{k,j} \right) = \sum_{i=0}^{5} \theta_i^{k,j} \Psi_i(k_t). \tag{1}$$

Using first order optimality condition that relates marginal utility of consumption, marginal disutility for labor and marginal productivity of labor we can obtain labor as function of consumption

$$l_t(c_t) = \left((1-\alpha) e^{z_t} k_t^\alpha \frac{1}{c_t} \right)^{\frac{1}{\psi+\alpha}}$$

then, from resource constraint, and using the approximation for next-period capital, we can solve for consumption $c^j(k_t)$ by solving the following non-linear equation,

$$e^{z_t} k_t^\alpha \left(l_t \left(c^j(k_t) \right) \right)^{1-\alpha} - c^j(k_t) = k^j \left(k_t | \theta^{k,j} \right) - (1-\delta) k_t$$

and then labor decision rule follows:

$$l^j(k_t) = \left((1-\alpha) e^{z_t} k_t^\alpha \frac{1}{c^j(k_t)} \right)^{\frac{1}{\psi+\alpha}}$$

To solve for the unknown coefficients θ^k, we plug functions $k^j\left(k_t|\theta^{k,j}\right), c^j\left(k_t\right), l^j\left(k_t\right)$ into the Bellman equation to get

$$V\left(k_t,z_t\right)=\left(\log c^j\left(k_t\right)-\frac{\left(l^j\left(k_t\right)\right)^{1+\psi}}{1+\psi}\right)+\beta\sum_{m=1}^{5}\pi_{jm}V\left(k^j\left(k_t|\theta^{k,j}\right),z_{t+1}\right)$$

where π_{jm} is the entry (j,m) of $\Pi_{z,z'}$. And the Euler equation,

$$\frac{1}{c^j\left(k_t\right)}=\beta\sum_{m=1}^{5}\pi_{jm}\left[\left(\alpha e^{z_{t+1}}\left(k^j\left(k_t|\theta^{k,j}\right)\right)^{\alpha-1}\left(l^l\left(k^j\left(k_t|\theta^{k,j}\right)\right)\right)^{1-\alpha}+(1-\delta)\right)\frac{1}{c^j\left(k^j\left(k_t|\theta^{k,j}\right)\right)}\right]$$

where $c^j\left(k^j\left(k_t|\theta^{k,j}\right)\right)$ is found solving the corresponding non-linear equation. Then, the residual function is given by

$$\begin{aligned}R\left(k_t,z_t|\theta^k\right)=&\frac{1}{c^j\left(k_t\right)}-\\&\beta\sum_{m=1}^{5}\pi_{jm}\left[\left(\alpha e^{z_{t+1}}\left(k^j\left(k_t|\theta^{k,j}\right)\right)^{\alpha-1}\left(l^l\left(k^j\left(k_t|\theta^{k,j}\right)\right)\right)^{1-\alpha}+(1-\delta)\right)\frac{1}{c^j\left(k^j\left(k_t|\theta^{k,j}\right)\right)}\right]\end{aligned}$$

where θ^k has 18 elements, given the six Chebyshev polynomials for next period capital for each of the 3 levels of productivity z. If we evaluate the residual function at each of the 6 zeros of the Chebyshev for capital and the 3 levels of productivity, we will have 18 equations required for the 18 elements in θ^k. As we mentioned along this discussion, we use a collocation weight function given by an orthogonal collocation, that is, we define the collocation points to be the zeros of the $6th$ order Chebyshev polynomial in each dimension of the state variable. Then we use the closed form solution for the zeros of the chebyshev polynomials given by

$$x_k=\cos\left(\frac{2k-1}{2n}\pi\right),k=1,...,n$$

Note that we can also choose to approximate the decision rule on consumption instead of solving the non-linear equation, by defining an approximation with six Chebyshev polynomials as,

$$c^j\left(k_t|\theta^{c,j}\right)=\sum_{i=0}^{5}\theta_i^{c,j}\Psi_i\left(k_t\right).$$

and including the additional residual equations, $\tilde{R}\left(k_t,z_t|\theta^k\right)$, given by the resource constraint

$$\tilde{R}\left(k_t,z_t|\theta\right)=e^{z_t}k_t^{\alpha}\left(l_t\left(c^j\left(k_t|\theta^{c,j}\right)\right)\right)^{1-\alpha}-c^j\left(k_t|\theta^{c,j}\right)=k^j\left(k_t|\theta^{k,j}\right)-(1-\delta)\,k_t$$

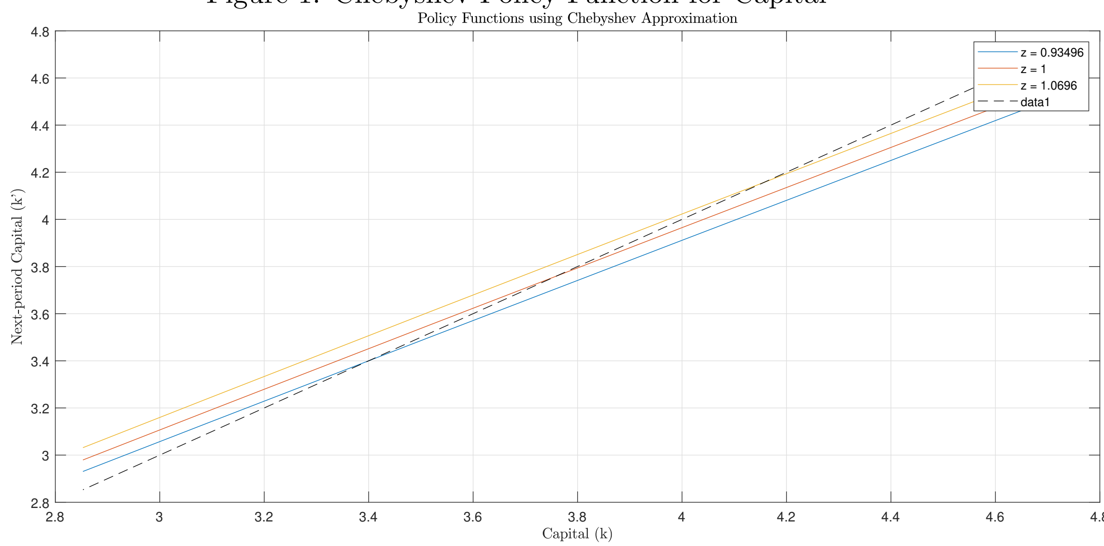
and then the residual equation

$$\hat{R}\left(k_t,z_t|\theta\right)=\left[R\left(k_t,z_t|\theta\right);\tilde{R}\left(k_t,z_t|\theta\right)\right]$$

where now θ stacks θ^k and θ^c and it is composed of 36 elements: the six Chebyshev polinomials for next period capital, six for consumption, for each of the 3 levels of productivity z, $(36 = 6 \times 2 \times 3)$. The first approach involves using a non-linear solver to find $c^j(k_t)$ and $c^j\left(k^j\left(k_t|\theta^{k,j}\right)\right)$, while the second approach, tackles directly the residual function by solving for coefficients θ^c.

The computed policy function for capital is given by,

Figure 1: Chebyshev Policy Function for Capital



Finite Elements

For the finite elements algorithm we bound the domain Ω,of the state variable $0 < k_t < mk^{ss}$, where $m > 1$, and k^{ss} is the steady state capital. Then we partition the domain Ω into eight equal size elements. Finally, we use a standard basis for the policy functions in each elements. Then, for the nodes of the partition of Ω,$\{k_0, ..., .k_7\}$ define the tent functions

$$\Psi_0(k) = \begin{cases} \frac{k_0 - k}{k_1 - k_0}, & \text{if } k \in [k_0, k_1] \\ 0 & \text{elsewhere} \end{cases}$$

for $i \in \{1, 6\}$ be given by

$$\Psi_i(k) = \begin{cases} \frac{k - k_{i-1}}{k_i - k_{i-1}}, & \text{if } k \in [k_{i-1}, k_i] \\ \frac{k_{i+1} - k_i}{k_{i+1} - k_i}, & \text{if } k \in [k_i, k_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

and for the last one

$$\Psi_7(k) = \begin{cases} \frac{k - k_6}{k_7 - k_6}, & \text{if } k \in [k_6, k_7] \\ 0 & \text{elsewhere} \end{cases}$$

then, as in Chebyshev, define

$$k^j\left(k_t|\theta^k\right) = \sum_{i=0}^{j} \theta_i^k \Psi_i\left(k_t\right).$$

Finally, we choose weighting function to follow a Galerkin scheme, where

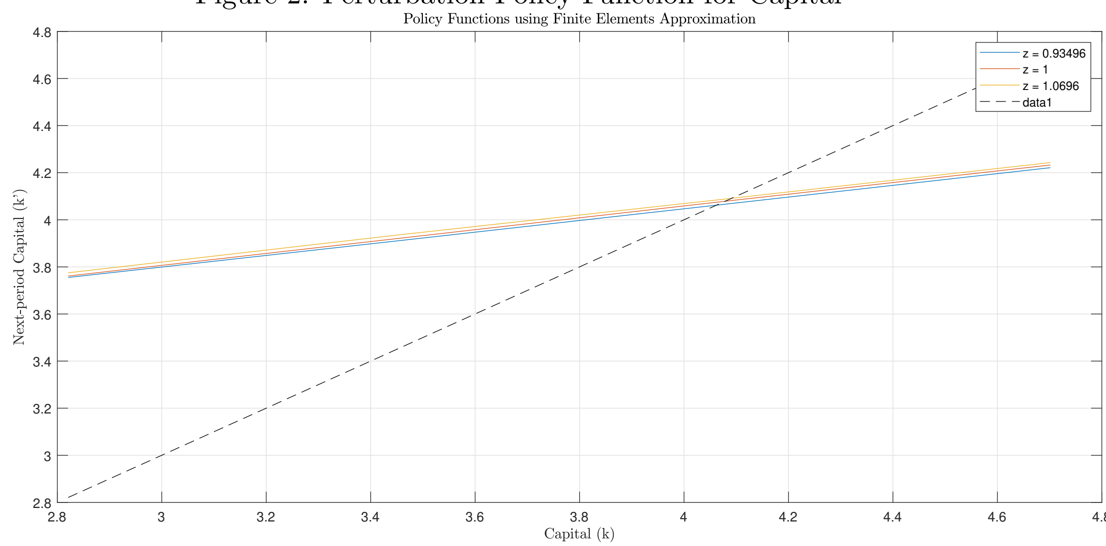
$$\phi_i\left(x\right) = \Psi_{i-1}\left(x\right)$$

and thus the weighting scheme

$$\int_\Omega \Phi_i\left(x\right) R\left(.|\theta\right) dx = 0, i = 1, ..., j+1$$

The computed policy function for capital is given by,

Figure 2: Perturbation Policy Function for Capital



Perturbation

We implement the third-order perturbation in Dynare. As in projection, perturbation methods solve the functional equation problem

$$\mathcal{H}\left(d\right) = 0$$

by specifying a Taylor series to the unknown function d. In particular, we specify a third-order expansion around the steady state for capital, consumption, and labor policies. Particularly, as we have mentioned throughout this document, the states are defined by capital and productivity, (k, z),

Let: k_t and z_t be the deviations of capital and productivity from their respective steady states k_{ss} and z_{ss}.

And define the state deviation vector:

$$\mathbf{s}_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}.$$

The capital policy function k_{t+1} is approximated around the steady state using a Taylor expansion up to the third order:

$$k_{t+1} = k_{ss} + \underbrace{\mathbf{gh}_x \cdot \mathbf{s}_t}_{\text{First-order term}} + \underbrace{\frac{1}{2}\mathbf{s}_t^\top \cdot \mathbf{gh}_{xx} \cdot \mathbf{s}_t}_{\text{Second-order term}} + \underbrace{\frac{1}{6}\sum_{i=1}^{n_s} \mathbf{s}_t^\top \cdot \mathbf{gh}_{xxx}^i \cdot \mathbf{s}_t}_{\text{Third-order term}}.$$

where the components of this expansion are given by

1. **Steady-State Term:**

$$k_{ss} : \text{Capital at steady state.}$$

2. **First-Order Term:**

$$\mathbf{gh}_x \cdot \mathbf{s}_t,$$

where $\mathbf{gh}_x$ is the vector of first-order coefficients.

3. **Second-Order Term:**

$$\frac{1}{2}\mathbf{s}_t^\top \cdot \mathbf{gh}_{xx} \cdot \mathbf{s}_t,$$

where $\mathbf{gh}_{xx}$ is the reshaped second-order coefficient matrix.

4. **Third-Order Term:**

$$\frac{1}{6}\sum_{i=1}^{n_s} \mathbf{s}_t^\top \cdot \mathbf{gh}_{xxx}^i \cdot \mathbf{s}_t,$$

where $\mathbf{gh}_{xxx}^i$ represents slices of the third-order coefficient tensor $\mathbf{gh}_{xxx}$.

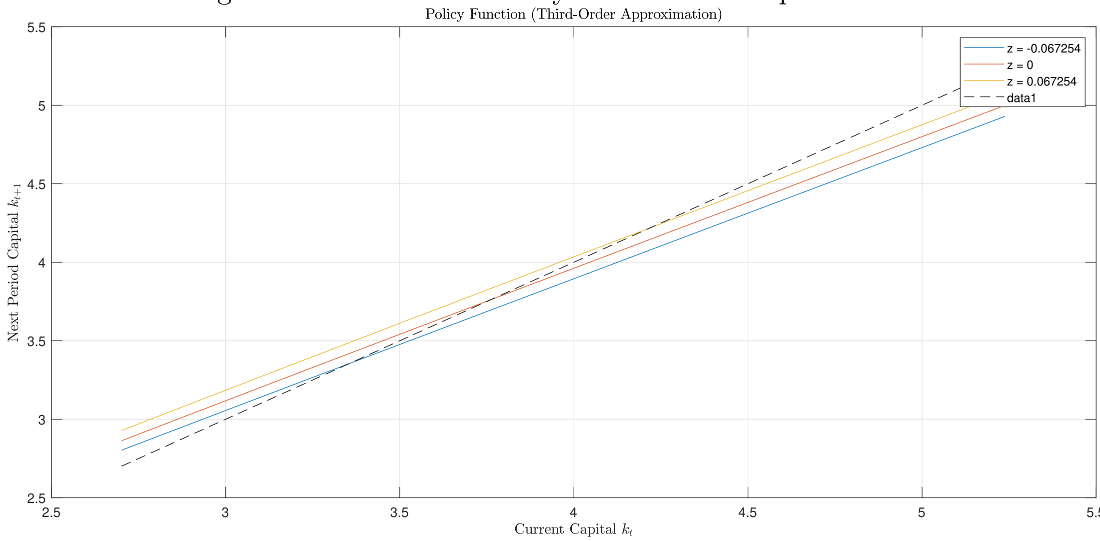
Compact Form

Combining all terms, the capital policy function can be written as:

$$k_{t+1} = k_{ss} + \mathbf{gh}_x \cdot \mathbf{s}_t + \frac{1}{2}\mathbf{s}_t^\top \cdot \mathbf{gh}_{xx} \cdot \mathbf{s}_t + \frac{1}{6}\sum_{i=1}^{n_s} \mathbf{s}_t^\top \cdot \mathbf{gh}_{xxx}^i \cdot \mathbf{s}_t.$$

The computed policy function for capital is given by,

Figure 3: Perturbation Policy Function for Capital



Deep Learning

Finally, for deep learning. We train a neural network with three layers and 25 nodes in each layer. We follow the approach of simulating the model jointly solving for the Euler equation, intratemporal labor condition, initial condition for capital, and budget constraint. The codes for this algorithm, follows those from Mahdi Ebrahimi Kahou, and can be found in the following link: https://colab.research.google.com/drive/1BFY-EqjCGAPDMI9nLI-jju0grdg2Pv87?usp=sharing

We compare the total computation time and Euler Error of the methods, we obtain

Table 1: Comparison in performance and Euler Error

	Chebyshev	**FEM**	**Perturbation**	**DL**
Time Running sec.	10.076	16	0.075	120
Euler Error	1e-16	1e-12	1e-9	1e-10

Figure 4: DL simulation Function for Capital and consumption

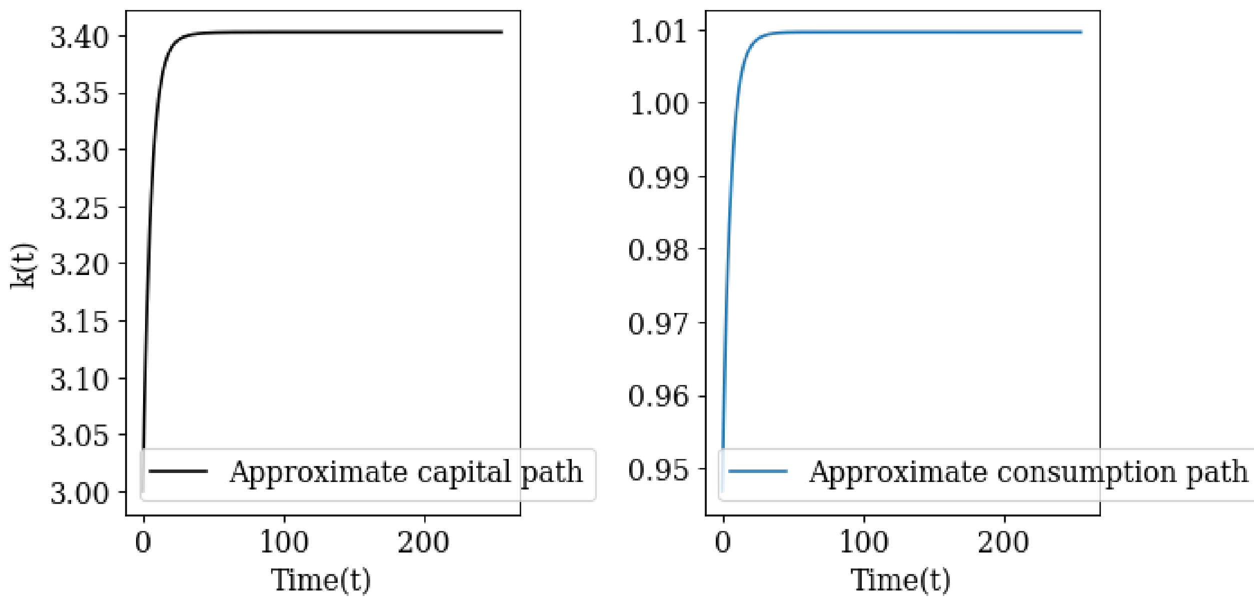


Figure 5: DL simulation Function for labor