# Math 131: Numerical Analysis
# Homework Number 2
# Due: February 8, 2024, 5:00 PM

**Special Instructions:**

- If instructed to "write/implement a code" or "write/implement an algorithm", you should interpret this as meaning that you need to produce a code in Jupyter/python notebook.
- You should include all the code source you implemented when turning in your assignment. Your code must run as is (and provide the correct results) to receive full credit.
- All code should be properly documented and include (**at a minimum**) 1) a summary of what the code is doing, 2) a description of all parameters that are used, and 3) a description of the output.
- The notebook itself should also be properly documented. Points will be **heavily** deducted for any notebook and/or code that is not properly documented.

## 1. Roundoff Errors (10 pts)

Use three-digit (i) rounding and (ii) chopping arithmetic to perform the following calculations. Compute the absolute and relative error with the exact value determined to at least five digits.

1. $3\pi - 4e + \sqrt{2.1}$
2. $\frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}-\sqrt{7}}$

## 2. Roundoff Error (10 pts)

(a) Show that

$$\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$$

(b) Which of the two formulas is more suitable for numerical computation? Explain your rationale and provide a numerical example (complete with code) that highlights the differences in accuracy.

(c) What (if any) precautions should you take to ensure that you have a well-defined and robust algorithm?

## 3. Roundoff Error (20 pts)

Given the following 2 expressions:

$$E_1(x) = \frac{1 - \cos x}{\sin^2 x} \quad \text{and} \quad E_2(x) = \frac{1}{1 + \cos x}$$

(a) Show that $E_1$ is mathematically equivalent to $E_2$.

(b) Write a code to compute and print out both expressions using the values $x = 1.0, 10^{-1}, 10^{-2}, 10^{-3}, \ldots, 10^{-15}$, and explain your results.

(c) What (if any) precautions should you take to ensure that you have a well-defined and robust algorithm?

## 4. Efficiency/Accuracy (20 pts)

In statistics, we often need to compute the mean $\hat{x}$, and standard deviation, $s$. One approach is to use the following 2 formulas:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where $x_1, x_2, x_3, \ldots, x_n$ are given data. It is also easy to show that the formula for $s^2$ can be written as

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2.$$

Assume that $n$ is large, say $> 100{,}000$.

(a) Compare the two methods for computing $s^2$ in terms of computational cost? You may assume that $\bar{x}$ has already been computed.

(b) Which method do you expect will produce the more accurate results for $x^2$ in general?

(c) Provide a small example, using a decimal system with 2 digits of precision and numbers of your own choosing to back up your responses in (a) and (b).

## 5. Stability (40 pts)

Suppose we are asked to compute

$$y_n = \int_0^1 \frac{x^n}{x+10} dx, \; n = 1, 2, \ldots 30$$

(a) Show that

$$y_n + 10y_{n-1} = \frac{1}{n}$$

(b) Write a code to numerically compute a value for $y_0$.

(c) Using (a) and (b) propose an algorithm and write a code that computes $y_n, n = 1, 2, \ldots, 30$. You may not use any other software except for what you write yourself.

(d) The true values for the integrals are given in Table 1. For each of the $y_n$, compute the actual and relative errors (where applicable). Analyze and explain your results in terms of what we discussed in class on the stability of algorithms.

(e) Modify your code so that it can compute more accurate values for the integral (without resorting to numerical integration or any other software). Justify your algorithm, demonstrate why it is more accurate, and provide all code.

Table 1: Values for $y_n, n = 0, 1, \ldots, 30$

| | |
|---|---|
| 0 | 0.09531024 |
| 1 | 0.0468976 |
| 2 | 0.031024 |
| 3 | 0.02316271 |
| 4 | 0.01847704 |
| 5 | 0.01536839 |
| 6 | 0.0131563 |
| 7 | 0.01150236 |
| 8 | 0.01021934 |
| 9 | 0.0091953 |
| 10 | 0.00835922 |
| 11 | 0.00766384 |
| 12 | 0.00707652 |
| 13 | 0.00657401 |
| 14 | 0.00613926 |
| 15 | 0.0057595 |
| 16 | 0.00542499 |
| 17 | 0.00512816 |
| 18 | 0.00486303 |
| 19 | 0.00462483 |
| 20 | 0.00440971 |
| 21 | 0.00421449 |
| 22 | 0.00403658 |
| 23 | 0.00387381 |
| 24 | 0.00372434 |
| 25 | 0.00358665 |
| 26 | 0.00345942 |
| 27 | 0.00334152 |
| 28 | 0.00323199 |
| 29 | 0.00312998 |
| 30 | 0. |