Math 131: Numerical Analysis
Homework Number 7
Due: April 27, 2024 5:00 PM

Special Instructions:

- If instructed to "write/implement a code" or "write/implement an algorithm", you should interpret this as meaning that you need to produce a code in Jupyter/python notebook. Unless otherwise stated, the codes should work for any generic function $f(x)$.
- Unless explicitly stated, you may not use any system/python packages or functions that implement one of the algorithms for this assignment. Standard mathematical functions (e.g. pow, abs, log, sin, exp, etc.) are allowed.
- You should include all the code source you implemented as one python notebook when turning in your assignment. Your code must run as is (and provide the correct results) to receive full credit.
- PDF files of your code, screenshots, etc. will not be graded.
- All code should be properly documented and include (**at a minimum**) 1) a summary of what the code is doing, 2) a description of all parameters that are used, and 3) a description of the output.
- The notebook itself should also be properly documented. Points will be **heavily** deducted for any notebook and/or code that is not properly documented.

1. Composite Quadrature (40 points)

This part will consider the following integral: $I = \int_0^2 e^{2x} \sin 3x dx$

(a) Theoretically determine the number of panels (r) required to approximate the integral to within 10^{-4} using

  (i) the Composite Trapezoidal rule,

  (ii) the Composite Simpson rule

(b) Implement a code for the Composite Simpson rule that takes as inputs a value of r, interval bounds a and b, and a function $f(x)$ and outputs an approximation to the integral $\int_a^b f(x)dx$. **Your code should work for any generic function.**

(c) Use your code to find the approximation to the integral above and verify that the value of r you found provides the correct approximation (**Hint:** you can compute the integral exactly using techniques from calculus and thereby compute the exact error of your approximation).

(d) Suppose we apply the Composite Trapezoid rule with r subintervals to approximate $\int_a^b f(x)dx$, for a general function $f(x)$. Derive a formula for the accumulated error (due to round-off) $e(h)$ and a bound for the error $e(h)$. Explain the result - specifically what conclusion can you make about Composite Trapezoid?

2. Initial Value Problems (30 points)

Consider the following initial value problem

$$\begin{aligned} y' &= -y + t + 1, \quad t \in [0,1] \\ y(0) &= 1. \end{aligned}$$

(a) Verify that the function $y(t) = e^{-t} + t$ is a solution to this problem.

(b) Formulate the Euler method approximation to the solution and write a code to solve this problem using the Euler method.

(c) Verify that Euler's method is first order by computing the error $e(h)$ for different time step sizes h and plotting $log(h)$ vs. $log(e(h))$. (**Hint:** note that the slope of this curve will be related to the order of a method since if a certain method is of order p we must have $e(h) = O(h) \approx Ch^p$ for some constant C. Then $\log e(h) = p \log(h) + C_1$.)

(d) What step size h do you need to use in order to approximate the solution $y(1)$ with accuracy 10^{-1}?

3. Initial Value Problems Higher-Order (30 points)

(a) Derive the Explicit Trapezoid Rule for solving an Initial Value Problem given by:

$$\begin{aligned} Y &= y_i + hf(t_i, y_i) \\ y_{i+1} &= y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, Y)) \end{aligned}$$

(b) Using the Explicit Trapezoid Rule, compute the first 3 steps, in other words y_1, y_2, and y_3, for the solution of the IVP given in (2) using $h = 0.2$. You may implement a python code or do this by hand.

4. Bonus: (20 points)

A colleague has come to you for advice on a method for solving

$$y' = y - t^2 + 1, \quad y(0) = 0.5, t \in [0, 2].$$

They tell you that high accuracy is needed for this problem. Given a choice between the two methods above (2) and 3) make a recommendation on which one to use. What questions might you ask to help you make a recommendation? Make any reasonable assumptions on the problem you like, but justify your assumptions and recommendations.