Nonlinear Equations: Secant Method

Math 131: Numerical Analysis

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Secant Method (Quick Summary)

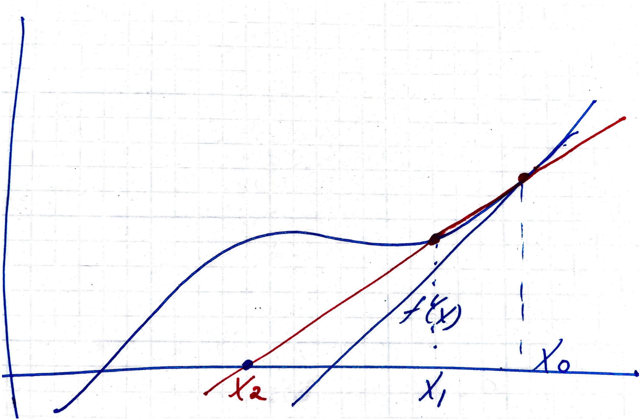
- Recall that Newton's method uses the derivative of the function whose roots we seek. That is both its power and its main disadvantage.
- In many real-world problems, the derivative may be difficult to compute. In other cases, it could be expensive. And in the worst case, it may not even be available.
- The secant method tries to address this disadvantage through an approximation to the derivative $f'(x)$ that uses two points close to each other, i.e. the secant. Using the secant, a new iterate is computed in a fashion similar to Newton's method.

Historical Note

- The secant method is one of the oldest methods for solving nonlinear equations
- Has an interesting history that can be traced back to the Rule of Double False Position described in the 18th-century BCE Egyptian Rhind Papyrus[@papakonstantinou2009].

Visually

Consider a line through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Let x_2 be the x intercept of this line.



Mathematically

Then it follows that

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

But notice that $f(x_2) = 0$, which leads to

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - 0}{x_1 - x_2}$$

General Form

Rearranging and solving for x_2 yields

$$x_2 = x_1 - \left[\frac{(x_1 - x_0)}{f(x_1) - f(x_0)} \right] f(x_1)$$

which is used as the next guess in our sequence.

This then yields the form for the general ***secant method***:

Secant Method

$$x_{k+1} = x_k - \left[\frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \right] f(x_k), \quad k = 0, 1, \ldots \tag{1}$$

Remark

- Another way to view this is to note that the term in the brackets

$$\frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

  approximates the derivative of a function (or rather in this case, the inverse).
- Therefore, one could interpret the secant method as just Newton's method with a finite difference approximation to the derivative.

Summary for Secant Method

Table 1: **Secant Method Summary**

Advantages	Disadvantages
Do not need to have derivatives	Need to provide 2 initial points.
Can have fast convergence (although not quadratic)	May not converge from all starting points
Generalizes to higher dimensions	Can be expensive in higher dimensions

Regula Falsi

- Given that both bisection and secant method require two points, it may not be surprising to learn that the two methods can be combined into a new method
- For example, where the updated points in the secant method are chosen in a manner similar to bisection.
- This method goes by several names including the ***method of false position*** and ***regula falsi*.**

Root Finding in Higher Dimensions

- Finding roots of nonlinear functions in dimensions higher than one has a long and rich history.
- Of the methods that we have discussed: 1) bisection, 2) Newton's, and 3) Secant, only Newton's method has an obvious path forward.
- This section gives a brief overview on how one proceeds in the case of Newton's method, and also provides a more general iterative procedure that is used in many applications.

Higher Dimensions (cont.)

Recall that Newton's method is based on approximating the next iterate in the sequence of approximations by using the following equation:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, ...$$

First, let's rewrite the equation as follows:

$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k), \quad k = 0, 1, ...$$

Higher Dimensions (cont.)

- Consider $F : \mathbb{R}^n \to \mathbb{R}^n$, where $n > 1$.
- We can still take the derivative of this function, following all the usual rules.
- In this case, it results in a matrix, which is called the ***Jacobian*** and is given by:

$$J(x_k) = F'(x_k) = \left[\frac{\partial f_i(x_k)}{\partial x_j} \right] \quad i, j = 1, \ldots, n.$$

Example

Let $F(x) : \mathbb{R}^2 \to \mathbb{R}^2, \quad x = (x_1, x_2)$

Consider

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} e^{x_1} - x_2 \\ x_1^2 - 2x_2 \end{bmatrix}$$

and

$$F'(x) = \begin{bmatrix} e^{x_1} & -1 \\ 2x_1 & -2 \end{bmatrix},$$

where $F'(x)$ is a 2×2 matrix.

Newton's Method in higher dimensions

Newton's method can then be written as:

$$x_{k+1} = x_k - J(x_k)^{-1} F(x_k), \quad k = 0, 1, ...$$

where the inverse is interpreted as *matrix inversion*.

or

$$J(x_k)(x_{k+1} - x_k) = -F(x_k), \quad k = 0, 1, ...$$

Remark

- It is a fundamental precept in numerical analysis that one rarely computes the inverse of a matrix.
- As such, the usual method for stating Newton's method in higher dimensions is as follows:

At each iteration k compute the step $s_k = (x_{k+1} - x_k)$ by solving the linear equation:

$$J(x_k)s_k = -F(x_k).$$

The new iterate is computed by:

$$x_{k+1} = x_k + s_k$$