Newton-Cotes

Math 131: Numerical Analysis

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Section 1

Introduction

Recall

Approximate

$$I(f) = \int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

General approach was to approximate the integral by:

1. Approximate $f(x)$ by an interpolating polynomial
2. Integrate the polynomial

Recall (cont.)

Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(x_0) + f(x_1)\right].$$

where $h = b - a, x_0 = a, x_1 = b$.

Simpson's rule

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]. \tag{1}$$

where $h = (b - a)/2, x_0 = 1, x_1 = (a + b)/2, x_2 = b$

Section 2

Exercise

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Compute the value of

$$\int_0^1 e^x dx$$

using the Trapezoid and Simpson's Rule for:

1. $a = 0$, $b = 1$
2. $a = 0.9$, $b = 1$

What is the error in each case?

Trapezoid Rule

Trapezoid Rule:

$$\int_a^b f(x)dx = \frac{(b-a)}{2}\Big[f(x_0) + f(x_1)\Big],$$

Simpson's Rule:

$$\int_a^b f(x)dx = \frac{(b-a)}{6}[f(x_0) + 4f(x_1) + f(x_2)]$$

Remarks

- Notice that both formulas have a rather large error when we compute the integral over a “large” interval,
- Whereas when we considered a smaller interval, the error was in fact quite small.
- How can we get better estimates?

Section 3

Higher-Order Methods

Newton-Cotes

- The basic quadrature rules derived so far are generally good, but what if we wanted to have formulas with greater accurarcy.
- The general approach we used still holds and leads to a family of quadrature formulas known as ***Newton-Cotes*** formulas.
- These are classified under either open or closed depending on whether the formulas include the end points or not.
  - ***Closed Newton-Cotes*** include the endpoints of closed interval $[a, b]$ as nodes.
  - ***Open Newton-Cotes*** do not include the endpoints.

In particular

To be specific, for a ***closed*** Newton-Cotes quadrature formula we would choose the node points x_i through the formula:

$$x_i = a + i\frac{b-a}{n-1}, \quad i = 0, 1, \dots, n-1. \tag{2}$$

For an ***open*** Newton-Cotes quadrature formula we would use the formula:

$$x_i = a + (i+1)\frac{b-a}{n+1}, \quad i = 0, 1, \dots, n-1. \tag{3}$$

Example

Suppose, we choose $n = 5$ on the interval [a,b] = [0,1].

Then Equation 2 (closed) would generate the points:

$$\begin{aligned} x_i &= a + i \ \cdot \frac{b-a}{n-1}, \\ &= 0 + i\frac{1}{4}, \\ &= \frac{i}{4}, \quad i = 0, 1, \ldots, 4, \end{aligned}$$

thereby yielding the set of nodes: $\{x\} = \{\ 0, .25, .5, .75, 1.0\ \}$.

Example

Similarly Equation 3 (open) would generate the points:

$$
\begin{aligned}
x_i &= a + (i+1) \cdot \frac{b-a}{n+1}, \\
&= 0 + (i+1)\frac{1}{6}, \\
&= \frac{i+1}{6}, \quad i = 0, 1, \dots, 4,
\end{aligned}
$$

which generates the set of nodes:

$$\{x\} = \{\ 1/6,\ 2/6,\ 3/6,\ 4/6,\ 5/6\ \}.$$

Some previous examples

- One example of an Open Newton-Cotes is the midpoint rule

$$\int_a^b f(x)dx = 2hf(x_0) + \frac{h^3}{3}f''(\xi)\ \xi \in (a,b)]$$

where x_0 is the midpoint between a and b.

- Likewise, both Trapezoidal and Simpson's rules can be categorized as Closed Newton-Cotes.

Other formulas

- There are many different formulas of both the Closed and Open variety all with corresponding error terms.
- All of them can be derived by the methods we've used for Trapezoid and Simpson's rule, so there is little to be gained by re-deriving them.
- Instead we will present them here because an interesting pattern arises that is worth knowing about:

Closed Newton-Cotes formulas:

$n = 2$ (Trapezoid)

$$I(f) = \frac{b-a}{2}[f(x_0) + f(x_1)] \tag{4}$$

$n = 3$ (Simpson's)

$$I(f) = \frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)] \tag{5}$$

$n = 4$ (Simpson's 3/8)

$$I(f) = \frac{b-a}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \tag{6}$$

$n = 5$ (Boole's rule)

$$I(f) = \frac{b-a}{90}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] \tag{7}$$

Trip-Hazard - Notation

- The formulas here are written using $b - a$ versus h to make them easier to compare.
- However, you will see these formulas written in terms of h in many other places.
- You should be careful in understanding exactly what h represents as it often is taken to mean $h = (b - a)/(n - 1), \; n \geq 1$, which is related to the number of node points used in the quadrature formula.

Higher-order formulas

- In theory, we could go as high as we wanted (and people have) in generating higher-order quadrature formulas, and of course with additional computational work.
- However, for large n the formulas can be shown to become numerically unstable $(n \geq 11.)$ One can actually prove that formulas do not converge for all integrands that are analytic.
- In practice, we tend to only use low-order formulas since they can still give us good accuracy (especially over small intervals (see Exercise 2.1 below).

Open Newton-Cotes formulas:

$n = 1$ (Midpoint)

$$I(f) = (b - a)f(x_0) \tag{8}$$

$n = 2$

$$I(f) = \frac{b-a}{2}[f(x_0) + f(x_1)] \tag{9}$$

$n = 3$

$$I(f) = \frac{b-a}{3}[2f(x_0) - f(x_1) + 2f(x_2)] \tag{10}$$

Similarly to the closed Newton-Cotes formulas, we could continue and derive higher-order formulas - with the same consequences.

Section 4

Error Estimates

Error Estimates

In both the closed and open Newton-Cotes cases, the formulas have error terms, which we have summarized in the table below, along with the precision of each:

Table 1: Summary of Error Terms for Newton-Cotes quadrature formulas

Name	N (npts)	Error	Precision
Trapezoid	2	$-\frac{(b-a)^3}{12}f^{(2)}(\xi)$	1
Simpson's	3	$-\frac{(b-a)^5}{2880}f^{(4)}(\xi)$	3
Simpson's 3/8	4	$-\frac{(b-a)^5}{6480}f^{(4)}(\xi)$	3
Boole	5	$-\frac{(b-a)^7}{1935360}f^{(6)}(\xi)$	5
Midpoint	1	$\frac{(b-a)^3}{24}f^{(2)}(\xi)$	1
	2	$\frac{(b-a)^3}{36}f^{(2)}(\xi)$	1
	3	$\frac{(b-a)^5}{23040}f^{(4)}(\xi)$	3

Remarks

Important

- An interesting feature of the quadrature formulas is that whenever N is odd then the precision of the formula $= N$.
- But when N is even then the precision is only $N - 1$.
- We lose one order in the precision whenever N is even! Or we could also say that we gain one order of precision for N odd.

Section 5

Summary

Summary

- A simple approach towards deriving basic quadrature rules is to replace the integrand with an interpolating polynomial on a chosen set of points and integrate the polynomial.
- Taylor's theorem yield error terms that provide us with estimates on how well the quadrature formula approximates the integral.
- The precision of a quadrature formula is the highest degree of the polynomial for which the formula is exact. When N is odd, the precision is also N; but when N is even, the precision is $N - 1$.
- Higher-order formulas yield greater accuracy, but at greater computational work as well as a fundamental assumption that the higher-order derivatives are nicely behaved (i.e. bounded).
- Basic (low-order) formulas can be accurate, but usually require a small interval. This observation will prove useful in the next sections.