1 Foundations

> *If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.*
>
> *John von Neumann*

The material we will be studying in numerical analysis assumes a good foundation in calculus. In particular, we will have a need to use basic facts of continuity, differentiability, integration, and power series. In addition, we will present many of the ideas in the form of numerical algorithms, so a good working knowledge of some programming language will be needed. Our suggestions include Matlab, python, or R.

1.1 Frequently Used Theorems from Calculus

Numerical analysis relies on several fundamental theorems or analysis. We will refer to several of these repeatedly and have use of the following 4 in particular.

Theorem 1.1 (Mean Value Theorem) Suppose that *(1)* f is continuous on the closed finite interval $[a, b]$ and *(2)* $f'(x)$ exists for every x in the open interval (a, b). Then there exists a point c such that

$$a < c < b$$

and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

We will be looking at numerous cases of continuous functions over closed and bounded intervals. The following theorem will prove useful in our analyses.

Theorem 1.2 (Intermediate Value Theorem) Suppose that *(1)* f is continuous on the closed finite interval $[a, b]$ and *(2)* $f(a) < c < f(b)$. Then there exists some point $x \in [a, b]$ such that $f(x) = c$.

Remark: One way to interpret the IVT says is that if a continuous function on an interval takes on any 2 values, it takes on every value in between. This will be particularly useful in our analysis of root finding methods.

Similar to the MVT above, there is a variation that applies to integrals. It is well worth noting that in this case, there is an important assumption without which the theorem does not apply, so care must be taken when applying it to certain problems.

Theorem 1.3 (Weighted Mean Value Theorem for Integrals) Suppose that $f \in C[a, b]$, the Riemann integral of g exists on $[a, b]$, and $g(x)$ does not change sign of $[a, b]$. Then there exists a number $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

1.2 Computer Programming

In terms of programming, we suggest one of 3 possible languages: Matlab, python, or R. Python and R have the advantage of being open-source and most of our examples will be in R and sometimes in python. In addition, both python and R can be easily installed on most computer platforma and both have powerful programming environments similar to Matlab. For python, one can use JupyterLab (jupyter); for R, one can use Rstudio/Posit (Posit).

We will also note that in real-world applications, most scientific codes will use other languages such as Fortran or C++. For the purposes of this introductory course, any high-level language will suffice.

1.3 Other Useful References

You should be able to find references to all of the material here in standard introductory courses on calculus. A good online reference for some of the material above can be found at openstax.org.

- Mean Value Theorem
- Intermediate Value Theorem
- Mean Value Theorem for Integrals
- Taylor polynomials and Taylor's Theorem

Another good set of resources are the one-pagers provided by the UCM Math Center. You can check them all out at: UCM The Math Center and in particular the Math 23 refresher one-pager might prove useful: Math 23 refresher.