# Math 131: Numerical Analysis
# Homework Number 5
# Due: April 10, 10:00 PM

### Special Instructions:

- This homework continues the work you did in the last assignment on polynomial interpolation and also includes some work on numerical differentiation.
- If instructed to "write/implement a code" or "write/implement an algorithm", you should interpret this as meaning that you need to produce a code in Jupyter/python notebook.
- Unless explicitly stated, you may not use any system/python packages or functions that implement one of the algorithms for this assignment. Standard mathematical functions (e.g. pow, abs, log, sin, exp, etc.) are allowed.
- You should include all the code source you implemented as one python notebook when turning in your assignment. Your code must run as is (and provide the correct results) to receive full credit.
- PDF files of your code, screenshots, etc. will not be graded.
- All code should be properly documented and include (**at a minimum**) 1) a summary of what the code is doing, 2) a description of all parameters that are used, and 3) a description of the output.
- The notebook itself should also be properly documented. Points will be **heavily** deducted for any notebook and/or code that is not properly documented.

## 1. Lagrange Polynomials: Chebyshev (20 points)

Using the codes you implemented in Homework 4 you will now modify them to construct a set of Chebyshev points instead of the uniform spaced nodes provided earlier.

(a) **Chebyshev points.** Modify the LagrInterpPolyError function from Homework 4 and replace the uniform spacing with Chebyshev points. Rerun the numerical experiments for each function with the following parameters:

1. $f(x) = \frac{e^{0.01x} \cdot \sin(17x^2)}{(1+25x^2)}, \quad 0 \leq x \leq 1$ and $N = 5, 10, 20$
2. $f(x) = \frac{1}{1+25x^2}, \quad -1 \leq x \leq 1$ and $N = 10, 20, 40$

(b) **Analysis.** Based on the plots you obtained compare and contrast your results from Homework 4 versus these results. Be specific, concise, and clear.

## 2. Numerical Differentiation (15 points)

In this part, you will compute an approximation to the derivatives of the function,

$$f(x) = \frac{e^{3x} \cdot \sin(3x^2)}{(1+3x^2)}.$$

(a) First compute a polynomial approximation to $f(x)$ using the interpolating polynomial code you constructed in Part 1(a)

(b) Now compute the first derivative using both the forward and centered difference formula at the points: $0.1, 0.2, \ldots, 0.9$ with $h = 10^{-6}$ using your polynomial approximation. In both cases, compute the absolute error. For this part you may take the analytical derivative to be given by:

$$f' = \frac{3e^{3x}\sin\left(3x^2\right)}{3x^2+1} - \frac{6e^{3x}x\sin\left(3x^2\right)}{\left(3x^2+1\right)^2} + \frac{6e^{3x}x\cos\left(3x^2\right)}{3x^2+1}$$

(c) Compute the second derivative at the points: $0.1, 0.2, \ldots, 0.9$ with $h = 10^{-6}$.

In case you're interested here is the second derivative:

$$\begin{aligned} f'' = &\left(e^{3x}\left(\frac{72x^2}{\left(3x^2+1\right)^3} - \frac{6}{\left(3x^2+1\right)^2}\right) + \frac{9e^{3x}}{3x^2+1} - \frac{36e^{3x}x}{\left(3x^2+1\right)^2}\right)\sin\left(3x^2\right) + \\ &12\left(\frac{3e^{3x}}{3x^2+1} - \frac{6e^{3x}x}{\left(3x^2+1\right)^2}\right)x\cos\left(3x^2\right) + \\ &\frac{e^{3x}\left(6\cos\left(3x^2\right) - 36x^2\sin\left(3x^2\right)\right)}{3x^2+1}. \end{aligned}$$

**However, you do not need to compute the absolute error.** This is just to highlight why we want to use numerical differentiation vs. the analytical derivative.

## 3. Numerical Differentiation - Optimal $h$ (15 points)

Suppose that a table of the values of a function $f(x_i)$ on some interval $[x_0, x_n]$ is computed where the values are rounded off to three decimal places and the inherent round-off error is $5 \times 10^{-4}$. Also assume that $|f^{(3)}(\xi)| \leq 1.5$ for $\xi \in [x_0, x_n]$.

(a) Find the optimal step size $h$ for the central difference formula:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

**Hint:** It might be helpful to derive expressions for the truncation and the roundoff errors for the formula.