Quadratic Compatibility Theorem: Theorem 1 in the book Divine Proportions by Wildberger

The quadratic equations

$$(X-P1) = M$$

$$2(X-p2)=r2$$

are compatible precisely when

$$((p1-p2)^2-(m+r2))^2=4mr2$$

In this case, if p1 is not equal to p2 then there is a unique common solution

$$X = P1 + P2 - (r1 - r2)$$
  
2  $2(P1 - P2)$ 

Create a Mathematica function for unique common solution of a pair of compatible quadratic equations.

$$log(1) := X[p1_, r1_, p2_, r2_] := ((p1 + p2) / 2) - (r1 - r2) / (2 (p1 - p2))$$

Substitute values for p1, r1, p2 and r2 from quadratic equations (1) and (2) above to compute the quadreal of quadrilateral.

In[2]:= quadreal = 
$$X[S_1 S_2 q_{12} + S_3 S_4 q_{34}, 4 S_1 S_2 S_3 S_4 (1 - q_{23}) \times (1 - q_{41}),$$
  
 $S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1 - q_{12}) \times (1 - q_{34})]$ 

$$\begin{array}{l} \text{Out} [2] = \begin{array}{l} \displaystyle \frac{1}{2} & (q_{12} \; S_1 \; S_2 + q_{23} \; S_2 \; S_3 + q_{41} \; S_1 \; S_4 + q_{34} \; S_3 \; S_4) \; - \\ \\ \displaystyle \qquad \qquad \\ \displaystyle \frac{-4 \times \; (1 - q_{12}) \; \times \; (1 - q_{34}) \; \; S_1 \; S_2 \; S_3 \; S_4 + 4 \times \; (1 - q_{23}) \; \times \; (1 - q_{41}) \; \; S_1 \; S_2 \; S_3 \; S_4}{2 \; \; (q_{12} \; S_1 \; S_2 - q_{23} \; S_2 \; S_3 - q_{41} \; S_1 \; S_4 + q_{34} \; S_3 \; S_4)} \end{array}$$

The proof employs the quadratic compatibility theorem. We will show that equations (1) and (2) above are compatible since they meet the criteria for being compatible from the quadratic compatibility theorem.

Perform compatibility check:

$$ln[3]:=$$
 CompatibilityCheckLHS [p1\_, r1\_, p2\_, r2\_] :=  $((p1-p2)^2 - (r1+r2))^2$ 

$$ln[4]:=$$
 CompatibilityCheckRHS [p1\_, r1\_, p2\_, r2\_] :=  $4*r1*r2$ 

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ln[5]:= lhs = CompatibilityCheckLHS[S<sub>1</sub> S<sub>2</sub> q<sub>12</sub> + S<sub>3</sub> S<sub>4</sub> q<sub>34</sub>,
                                                                                                               4 S_1 S_2 S_3 S_4 (1 - q_{23}) \times (1 - q_{41}), S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1 - q_{12}) \times (1 - q_{34})
Out[5]= (-4 \times (1 - q_{12}) \times (1 - q_{34}) S_1 S_2 S_3 S_4 -
                                                                                                                               4 \times (1-q_{23}) \times (1-q_{41}) \, \, S_1 \, S_2 \, S_3 \, S_4 + \, \left(q_{12} \, S_1 \, S_2 - q_{23} \, S_2 \, S_3 - q_{41} \, S_1 \, S_4 + q_{34} \, S_3 \, S_4\right)^{\, 2}\right)^{\, 2}
         ln[6]:= rhs = CompatibilityCheckRHS[S<sub>1</sub> S<sub>2</sub> q<sub>12</sub> + S<sub>3</sub> S<sub>4</sub> q<sub>34</sub>,
                                                                                                               4 S_1 S_2 S_3 S_4 (1-q_{23}) \times (1-q_{41}), S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1-q_{12}) \times (1-q_{34})
Out[6]= 64 \times (1 - q_{12}) \times (1 - q_{23}) \times (1 - q_{34}) \times (1 - q_{41}) S_1^2 S_2^2 S_3^2 S_4^2
       ln[7]:= lhse = Expand[lhs]
 \text{Out} \text{[7]=} \quad q_{12}^4 \, \, S_1^4 \, \, S_2^4 \, - \, 4 \, \, q_{12}^3 \, \, q_{23}^3 \, \, S_1^3 \, \, S_2^4 \, S_3 \, + \, 6 \, \, q_{12}^2 \, \, q_{23}^2 \, \, S_1^2 \, \, S_2^4 \, \, S_3^2 \, - \, 4 \, \, q_{12} \, \, q_{23}^3 \, \, S_1 \, \, S_2^4 \, \, S_3^3 \, + \, q_{23}^4 \, \, S_2^4 \, \, S_3^4 \, - \, 4 \, \, q_{12}^3 \, \, q_{41}^4 \, \, S_1^4 \, \, S_2^3 \, \, S_4 \, - \, q_{12}^4 \, \, q_{12}^3 \, \, S_1^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, - \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^
                                                                                              8\ q_{12}^{2}\ q_{41}\ S_{1}^{3}\ S_{2}^{3}\ S_{3}\ S_{4}\ +\ 4\ q_{12}^{2}\ q_{23}\ q_{41}\ S_{1}^{3}\ S_{2}^{3}\ S_{3}\ S_{4}\ +\ 32\ q_{12}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{
                                                                                            16 q_{12} q_{23}^2 S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{12}^2 S_3^2 
                                                                                          4\,q_{12}\,q_{23}^2\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4 - 16\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{12}\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^3\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{23}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{23}^2\,S_1^2\,S_2^3\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^3\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_
                                                                                            4 q_{12} q_{23}^2 q_{34} S_1 S_2^3 S_3^3 S_4 + 8 q_{23}^2 q_{41} S_1 S_2^3 S_3^3 S_4 - 4 q_{23}^3 q_{41} S_1 S_2^3 S_3^3 S_4 - 4 q_{23}^3 q_{34} S_2^3 S_3^4 S_4 +
                                                                                          16 q_{12} q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 + 4 q_{12}^2 q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 - 16 q_{12} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 + 4 q_{12} q_{23} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 +
                                                                                              64\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,64\,q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,16\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,64\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_
                                                                                              16\ q_{23}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 64\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 64\ q_{12}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ S_2^2\ S_3^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ S_2^2\ S_3^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{24}\ S_2^2\ S_3^2\ S_3^2\ S_4^2\ +\ 32\ q_{24}\ S_2^2\ S_3^2\ S_3^2\ S_4^2\ +\ 32\ q_{24}\ S_2^2\ S_3^2\ S_
                                                                                              16 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12} q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12} q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 6 q_{12}^2 q_{12}^2 S_1^2 S_2^2 S_
                                                                                            64\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,64\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{23}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{23}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{13}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{13}\,S_2^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2
                                                                                              16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 +
                                                                                              24\,\,q_{12}\,\,q_{23}\,\,q_{34}\,\,q_{41}\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,-\,16\,\,q_{23}\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,6\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,S
                                                                                              4 q_{12} q_{23} q_{34}^2 S_1 S_2^2 S_3^3 S_4^2 - 16 q_{23} q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 + 4 q_{23}^2 q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 + 6 q_{23}^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{24}^2 S_3^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{24}^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{24}^2 S_3^2 S_4^2 + 6 q_{23}^2 S_4^2 S_4^2 + 6 q_{23
                                                                                            4 q_{12} q_{41}^3 S_1^4 S_2 S_4^3 - 16 q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{12} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{23} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{44} S_4^3 S_4^3 S_4^3 S_4^3 + 8 q_{44} S_4^3 S_4^3 S_4^3 + 8 q_{44} S_4^3 S_4^3 S_4^3 + 8 q_{44} S_4^3 S_4^3 S
                                                                                          4\,q_{12}\,q_{34}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3\,+\,8\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,-\,4\,q_{23}\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}\,q_{41}^3\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^2\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3^3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_2^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S
                                                                                              16\ q_{12}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{23}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{12}^2\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 - 16\ q_{12}^2\ q_{13}^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}
                                                                                            8 \, q_{23} \, q_{34}^2 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 4 \, q_{23} \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 4 \, q_{23} \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{23}^3 \, S_4^3 \, - \, 4 \, q_{23} \, q_{24}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_2^3 \, S_2^
                                                                                            4 \, q_{23} \, q_{34}^3 \, S_2 \, S_3^4 \, S_4^3 + q_{41}^4 \, S_1^4 \, S_4^4 - 4 \, q_{34} \, q_{41}^3 \, S_1^3 \, S_3 \, S_4^4 + 6 \, q_{34}^2 \, q_{41}^2 \, S_1^2 \, S_3^2 \, S_4^4 - 4 \, q_{34}^3 \, q_{41}^4 \, S_1 \, S_3^3 \, S_4^4 + q_{34}^4 \, S_3^4 \, S_3^4 \, S_4^4 + q_{34}^4 \, S_3^4 \, S_3^4 \, S_3^4 \, S_3^4 + q_{34}^4 \, S_3^4 \, S_3
         In[8]:= rhse = Expand[rhs]
Out[8]= 64S_1^2S_2^2S_3^2S_4^2 - 64q_{12}S_1^2S_2^2S_3^2S_4^2 - 64q_{23}S_1^2S_2^2S_3^2S_4^2 + 64q_{12}q_{23}S_1^2S_2^2S_3^2S_4^2 - 64q_{34}S_1^2S_2^2S_3^2S_4^2 + 64q_{12}q_{23}S_1^2S_2^2S_3^2S_4^2 - 64q_{34}S_1^2S_2^2S_3^2S_4^2 + 64q_{34}S_1^2S_2^2S_3^2S_3^2 + 64q_{34}S_3^2S_3^2S_3^2 + 64q_{34}S_3^2S_3^2 + 64q_{34}S_3^2S_3^2 + 64q_{34}S_3^2S_3^2 + 64q_{34}S_3^2 + 64q_{34}S_
                                                                                              64 q_{12} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 + 64 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 +
                                                                                            64 \, q_{12} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_
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 $64 q_{12} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 64 q_{12} q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2$ 

## In[9]:= ArchimedesFn = rhse - lhse Out[9]= $-q_{12}^4$ $S_1^4$ $S_2^4$ + 4 $q_{12}^3$ $q_{23}^3$ $S_1^3$ $S_2^4$ $S_3$ - 6 $q_{12}^2$ $q_{23}^2$ $S_1^2$ $S_2^4$ $S_3^2$ + 4 $q_{12}^3$ $q_{23}^3$ $S_1$ $S_2^4$ $S_3^3$ $- q_{23}^4$ $S_2^4$ $S_3^4$ + 4 $q_{12}^3$ $q_{41}^4$ $S_1^4$ $S_2^3$ $S_4$ + 4 $q_{12}^3$ $q_{23}^4$ $8q_{12}^2q_{41}S_1^3S_2^3S_3S_4 - 4q_{12}^2q_{23}q_{41}S_1^3S_2^3S_3S_4 - 32q_{12}q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_3S_4 + 16q_{12}^2q_{23}S_3S_3S_4 + 16q_{12}^2q_{23}S_3S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_5 + 16q_{12}^2q_{23}^2S_5 + 16q_{12}^2q_{23}^2S_5 + 16q_{1$ $4\,q_{12}\,q_{23}^2\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4 + 16\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{12}\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^3\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,S_1^3\,S_2^3\,S_2^3\,S_3^3\,S_2^3\,S$ $4\,q_{12}\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4 + 4\,q_{23}^3\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4 + 4\,q_{23}^3\,q_{34}\,S_2^3\,S_3^4\,S_4 - 6\,q_{12}^2\,q_{41}^2\,S_1^4\,S_2^2\,S_4^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^4\,S_2^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2 - 6\,q_$ $4q_{12}^2q_{34}q_{41}S_1^3S_2^2S_3S_4^2 + 16q_{12}q_{41}^2S_1^3S_2^2S_3S_4^2 - 4q_{12}q_{23}q_{41}^2S_1^3S_2^2S_3S_4^2 - 16q_{12}^2S_1^2S_2^2S_3^2S_4^2 +$ $32\ q_{12}\ q_{23}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{23}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 100\ q_{23}^2\ S_2^2\ S_3^2\ S_4^2\ +\ 100\ q_{23}^2\ S_2^2\ S_3^2\ S_3^2$ $48 \, q_{12} \, q_{23} \, q_{34} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 16 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12} \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 6 \, q_{12}^2 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 \, S_3^2$ $32 q_{12} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 48 q_{12} q_{23} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 48 q_{12} q_{23} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 48 q_{12} q_{23} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 60 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 30 q_{34} q_{41} S_2^2 S_3^2 S_4^2 + 30 q_{34} q_{41} S_4^2 S_4^2 S_4^2 + 30 q_{34} q_{41} S_4^2 S_4^2 + 30 q_{34} q_{41} S_4^2 S_4$ $48\,q_{12}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,48\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,Q_{12}^2\,Q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_2^2\,$ $16\ q_{23}\ q_{41}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 6\ q_{23}^2\ q_{41}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 32\ q_{23}\ q_{34}\ S_1\ S_2^2\ S_3^3\ S_4^2\ +\ 16\ q_{12}\ q_{23}\ q_{34}\ S_1\ S_2^2\ S_3^3\ S_4^2\ +$ $4\ q_{23}^2\ q_{34}\ q_{41}\ S_1\ S_2^2\ S_3^3\ S_4^2\ -\ 6\ q_{23}^2\ q_{34}^2\ S_2^2\ S_3^4\ S_4^2\ +\ 4\ q_{12}\ q_{41}^3\ S_1^4\ S_2\ S_3^3\ +\ 16\ q_{41}^2\ S_1^3\ S_2\ S_3\ S_4^3\ 8 \ q_{12} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 8 \ q_{23} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 8 \ q_{34} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 4 \ q_{12} \ q_{34} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3 \ S_3^3 \ S_3 \$ $8\ q_{41}^3\ S_1^3\ S_2\ S_3\ S_4^3\ +\ 4\ q_{23}\ q_{41}^3\ S_1^3\ S_2\ S_3\ S_4^3\ -\ 32\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ 16\ q_{12}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^3\ S_4^3\ +$ $16 q_{23} q_{34} q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41} S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^3 S_4^3 - 4 q_{12} q_{34}^2 q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{44}^2 q_{44}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{44}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{44}^2 S_1^2 S_2^2 S_2^2 S_4^2 + 16 q_{44}^2 S_1^2 S_2^2 S_2^2 S_4^2 S_4^2 S_4^2 + 16 q_{44}^2 S_2^2 S_4^2 S_4^2$ $4\ q_{23}\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3^2\ S_4^3 + 16\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8\ q_{12}\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8\ q_{23}\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8$ $8 q_{34}^3 S_1 S_2 S_3^3 S_4^3 + 4 q_{12} q_{34}^3 S_1 S_2 S_3^3 S_4^3 - 8 q_{34}^2 q_{41} S_1 S_2 S_3^3 S_4^3 - 4 q_{23} q_{34}^2 q_{41} S_1 S_2 S_3^3 S_4^3 +$ $4\,q_{23}\,q_{34}^3\,S_2\,S_3^4\,S_4^3-q_{41}^4\,S_1^4\,S_4^4+4\,q_{34}\,q_{41}^3\,S_1^3\,S_3\,S_4^4-6\,q_{34}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_4^4+4\,q_{34}^3\,q_{41}^4\,S_1\,S_3^3\,S_4^4-q_{34}^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_1^4\,$ $ln[10] = L_1 = \{l_1, m_1, n_1\}$ $L_2 = \{1_2, m_2, n_2\}$ $L_3 = \{l_3, m_3, n_3\}$ $L_4 = \{1_4, m_4, n_4\}$ Out[10]= $\{l_1, m_1, n_1\}$ Out[11]= $\{\,\mathbf{l}_{2}\,,\,m_{2}\,,\,n_{2}\,\}$ Out[12]= $\{1_3, m_3, n_3\}$ Out[13]= { $1_4$ , $m_4$ , $n_4$ } Meet of lines. In[14]:= **M[L1\_, L2\_] :=** $\{L1[2] \times L2[3] - L1[3] \times L2[2], L1[3] \times L2[1] - L1[1] \times L2[3], L1[1] \times L2[2] - L1[2] \times L2[1]\}$ $log(15) = quadrance[a1_, a2_] := 1 - (a1[1] \times a2[1] + a1[2] \times a2[2] + a1[3] \times a2[3])^2 / a1[3] = a1[3] + a1[$ $((a1[1] \times a1[1] + a1[2] \times a1[2] + a1[3] \times a1[3]) \times$ $(a2[1] \times a2[1] + a2[2] \times a2[2] + a2[3] \times a2[3])$

 $((11[1] \times 11[1] + 11[2] \times 11[2] + 11[3] \times 11[3]) \times$  $(12[1] \times 12[1] + 12[2] \times 12[2] + 12[3] \times 12[3])$ 

$$In[17]:=$$
  $S_1 = spread[L_4, L_1]$   
 $S_2 = spread[L_1, L_2]$ 

$$S_3 = spread[L_2, L_3]$$

$$S_4 = spread[L_3, L_4]$$

$$\text{Out[17]= } 1 - \frac{\left( 1_1 \ 1_4 + m_1 \ m_4 + n_1 \ n_4 \right)^2}{\left( 1_1^2 + m_1^2 + n_1^2 \right) \ \left( 1_4^2 + m_4^2 + n_4^2 \right)}$$

$$\text{Out[18]= } 1 - \frac{\left(1_{1} \; 1_{2} + \textit{m}_{1} \; \textit{m}_{2} + \textit{n}_{1} \; \textit{n}_{2}\right)^{\; 2}}{\left(1_{1}^{2} + \textit{m}_{1}^{2} + \textit{n}_{1}^{2}\right) \; \left(1_{2}^{2} + \textit{m}_{2}^{2} + \textit{n}_{2}^{2}\right)}$$

$$\text{Out[19]= } 1 - \frac{\left(1_2 \, 1_3 + m_2 \, m_3 + n_2 \, n_3\right)^2}{\left(1_2^2 + m_2^2 + n_2^2\right) \, \left(1_3^2 + m_3^2 + n_3^2\right)}$$

$$\text{Out}[\text{20}] = \ 1 - \frac{\left( \, 1_3 \, \, 1_4 \, + \, m_3 \, \, m_4 \, + \, n_3 \, \, n_4 \, \right) \, ^2}{\left( \, 1_3^2 \, + \, m_3^2 \, + \, n_3^2 \, \right) \, \, \left( \, 1_4^2 \, + \, m_4^2 \, + \, n_4^2 \, \right)}$$

$$In[21]:= a_1 = M[L_4, L_1]$$

$$a_2 = M[L_1, L_2]$$

$$a_3 = M[L_2, L_3]$$

$$a_4 = M[L_3, L_4]$$

Out[21]= 
$$\{ m_4 n_1 - m_1 n_4, -l_4 n_1 + l_1 n_4, l_4 m_1 - l_1 m_4 \}$$

Out[22]= 
$$\{-m_2 n_1 + m_1 n_2, l_2 n_1 - l_1 n_2, -l_2 m_1 + l_1 m_2\}$$

$$\text{Out} [\text{23}] = \; \left\{ \, -\, \text{m}_3 \; n_2 \, + \, \text{m}_2 \; n_3 \, , \; \, 1_3 \; n_2 \, - \, 1_2 \; n_3 \, , \; \, - \, 1_3 \; \text{m}_2 \, + \, 1_2 \; \text{m}_3 \, \right\}$$

Out[24]= 
$$\{-m_4 n_3 + m_3 n_4, 1_4 n_3 - 1_3 n_4, -1_4 m_3 + 1_3 m_4\}$$

```
ln[25] = q_{12} = quadrance[a_1, a_2]
                             q_{23} = quadrance[a_2, a_3]
                             q_{34} = quadrance[a_3, a_4]
                             q_{41} = quadrance[a_4, a_1]
 \text{Out} [25] = \ \ 1 - \left( \left( -1_2 \, \, \text{m}_1 + 1_1 \, \, \text{m}_2 \right) \, \left( 1_4 \, \, \text{m}_1 - 1_1 \, \, \text{m}_4 \right) \, + \, \left( 1_2 \, \, \text{n}_1 - 1_1 \, \, \text{n}_2 \right) \, \left( -1_4 \, \, \text{n}_1 + 1_1 \, \, \text{n}_4 \right) \, + \, \left( -m_2 \, \, \text{n}_1 + m_1 \, \, \text{n}_2 \right) \, \left( m_4 \, \, \text{n}_1 - m_1 \, \, \text{n}_4 \right) \, \right)^2 \, / \, 
                                           ((-1_2 m_1 + 1_1 m_2)^2 + (1_2 n_1 - 1_1 n_2)^2 + (-m_2 n_1 + m_1 n_2)^2)
                                                        \left(\;\left(\;1_{4}\;\mathsf{m_{1}}\;-\;1_{1}\;\mathsf{m_{4}}\;\right)^{\;2}\;+\;\left(\;-\;1_{4}\;\mathsf{n_{1}}\;+\;1_{1}\;\mathsf{n_{4}}\;\right)^{\;2}\;+\;\left(\;\mathsf{m_{4}}\;\mathsf{n_{1}}\;-\;\mathsf{m_{1}}\;\mathsf{n_{4}}\;\right)^{\;2}\;\right)\;\right)
 \text{Out} [26] = \ 1 - \left( \left( -l_2 \, m_1 + l_1 \, m_2 \right) \, \left( -l_3 \, m_2 + l_2 \, m_3 \right) \, + \, \left( l_2 \, n_1 - l_1 \, n_2 \right) \, \left( l_3 \, n_2 - l_2 \, n_3 \right) \, + \, \left( -m_2 \, n_1 + m_1 \, n_2 \right) \, \left( -m_3 \, n_2 + m_2 \, n_3 \right) \right)^2 \, / \, \left( -m_3 \, n_2 + l_2 \, m_3 \right) \, + \, \left( -m_3 \, n_2 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \, \left( -m_3 \, n_3 + l_3 \, m_3 \right) \, + \,
                                           \left( \left( \left( -1_2 m_1 + 1_1 m_2 \right)^2 + \left( 1_2 n_1 - 1_1 n_2 \right)^2 + \left( -m_2 n_1 + m_1 n_2 \right)^2 \right)
                                                        \left( \left( -1_3 \, m_2 + 1_2 \, m_3 \right)^2 + \left( 1_3 \, n_2 - 1_2 \, n_3 \right)^2 + \left( -m_3 \, n_2 + m_2 \, n_3 \right)^2 \right) \right)
(((-1_3 m_2 + 1_2 m_3)^2 + (1_3 n_2 - 1_2 n_3)^2 + (-m_3 n_2 + m_2 n_3)^2)
                                                        \left( \left( -1_4 m_3 + 1_3 m_4 \right)^2 + \left( 1_4 n_3 - 1_3 n_4 \right)^2 + \left( -m_4 n_3 + m_3 n_4 \right)^2 \right) \right)
 \text{Out}[28] = \ \ 1 - \left( \left( 1_4 \, \text{m}_1 - 1_1 \, \text{m}_4 \right) \, \left( -1_4 \, \text{m}_3 + 1_3 \, \text{m}_4 \right) \, + \left( -1_4 \, \text{n}_1 + 1_1 \, \text{n}_4 \right) \, \left( 1_4 \, \text{n}_3 - 1_3 \, \text{n}_4 \right) \, + \left( \text{m}_4 \, \text{n}_1 - \text{m}_1 \, \text{n}_4 \right) \, \left( -\text{m}_4 \, \text{n}_3 + \text{m}_3 \, \text{n}_4 \right) \right)^2 \, / \, 
                                           ((1_4 m_1 - 1_1 m_4)^2 + (-1_4 n_1 + 1_1 n_4)^2 + (m_4 n_1 - m_1 n_4)^2)
                                                        \left( \left( -1_4 m_3 + 1_3 m_4 \right)^2 + \left( 1_4 n_3 - 1_3 n_4 \right)^2 + \left( -m_4 n_3 + m_3 n_4 \right)^2 \right) \right)
```

The next cell may take a long time to execute since the q's and the S's in the "ArchimedesFn" equation are being substituted with the x's, y's and z's in the q's and S's expressions above and the whole thing is being factored. A result of zero indicates that the criteria for compatible equations in the quadratic compatibility theorem has been met.

```
In[29]:= result = Factor[ArchimedesFn]
Out[29]= 0
```

We have proved that the following two quadratic equations in B are compatible.