Quadratic Compatibility Theorem: Theorem 1 in the book Divine Proportions by Wildberger

The quadratic equations

$$(X-P1) = M$$

$$2(X-p2)=r2$$

are compatible precisely when

$$((p1-p2)^2-(m+r2))^2=4mr2$$

In this case, if p1 is not equal to p2 then there is a unique common solution

$$X = P1 + P2 - (r1 - r2)$$
  
2  $2(P1 - P2)$ 

Create a Mathematica function for unique common solution of a pair of compatible quadratic equations.

$$ln[1] = X[p1_, r1_, p2_, r2_] := ((p1 + p2) / 2) - (r1 - r2) / (2 (p1 - p2))$$

Substitute values for p1, r1, p2 and r2 from quadratic equations (1) and (2) above to compute the gadrea B.

The proof employs the quadratic compatibility theorem. We will show that equations (1) and (2) above are compatible since they meet the criteria for being compatible from the quadratic compatibility theorem.

Perform compatibility check:

$$ln[3]:=$$
 CompatibilityCheckLHS [p1\_, r1\_, p2\_, r2\_] :=  $((p1-p2)^2 - (r1+r2))^2$ 

$$ln[4]:=$$
 CompatibilityCheckRHS [p1\_, r1\_, p2\_, r2\_] :=  $4*r1*r2$ 

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ln[5]:= lhs = CompatibilityCheckLHS[q_{12} q_{23} S_2 + q_{34} q_{41} S_4,
                                                                                                                      4\ q_{12}\ q_{23}\ q_{34}\ q_{41}\ (\textbf{1}-\textbf{S}_{1})\ \times\ (\textbf{1}-\textbf{S}_{3})\ ,\ q_{12}\ q_{41}\ \textbf{S}_{1}+q_{23}\ q_{34}\ \textbf{S}_{3}\ ,\ 4\ q_{12}\ q_{23}\ q_{34}\ q_{41}\ (\textbf{1}-\textbf{S}_{2})\ \times\ (\textbf{1}-\textbf{S}_{4})\ ]
Out[5]= \left(-4\ q_{12}\ q_{23}\ q_{34}\ q_{41}\ (1-S_1)\ \times\ (1-S_3)\ -\right.
                                                                                                                                          4\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,\left(1-S_{2}\right)\,\times\,\left(1-S_{4}\right)\,+\,\left(-\,q_{12}\,q_{41}\,S_{1}\,+\,q_{12}\,q_{23}\,S_{2}\,-\,q_{23}\,q_{34}\,S_{3}\,+\,q_{34}\,q_{41}\,S_{4}\right)^{\,2}\right)^{\,2}
            ln[6]:= rhs = CompatibilityCheckRHS [q_{12} q_{23} S_2 + q_{34} q_{41} S_4,
                                                                                                                      4\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,\left(1-S_{1}\right)\times\left(1-S_{3}\right),\,q_{12}\,q_{41}\,S_{1}+q_{23}\,q_{34}\,S_{3},\,4\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,\left(1-S_{2}\right)\times\left(1-S_{4}\right)]
  Out[6]= 64 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 (1-S_1) \times (1-S_2) \times (1-S_3) \times (1-S_4)
            In[7]:= lhse = Expand[lhs]
  Out[7]= 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2 - 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1 + 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2 - 16\ q_{12}^3\ q_{23}\ q_{34}\ q_{41}^3\ S_1^2 +
                                                                                                     8\ q_{12}^{3}\ q_{23}\ q_{34}\ q_{41}^{3}\ S_{1}^{3}\ +\ q_{12}^{4}\ q_{41}^{4}\ S_{1}^{4}\ -\ 64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}\ +\ 32\ q_{12}^{3}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ -\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ -\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ -\ 32\ q_{12}^{2}\ q_{12}^{2}\ q_{12}^{2}\ q_{12}^{2}\ q_{12}^{2}\ S_{1}\ S_{2}\ -\ 32\ q_{12}^{2}\ q_{12}^{
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                                                                                                     16\ q_{12}^2\ q_{23}\ q_{34}^2\ q_{41}^3\ S_1\ S_4^2 + 8\ q_{12}\ q_{23}\ q_{34}^3\ q_{41}^3\ S_1\ S_4^2 + 6\ q_{12}^2\ q_{34}^2\ q_{41}^4\ S_1^2\ S_4^2 - 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_4^2 + 6\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^4\ S_1^2\ S_4^2 - 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_4^2 + 6\ q_{12}^2\ q_{23}^2\ q_{24}^2\ q_{24}^2\ S_2\ S_4^2 + 6\ q_{12}^2\ q_
                                                                                                   8 q_{12} q_{23} q_{34}^3 q_{41}^3 S_2 S_4^2 + 4 q_{12}^2 q_{23} q_{34}^2 q_{41}^3 S_1 S_2 S_4^2 + 6 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_2^2 S_4^2 - 16 q_{12} q_{23}^2 q_{34}^3 q_{41}^2 S_3 S_4^2 + 6 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_3 S_4^2 + 6 q_{12}^2 q_{23}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 S_4^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^2 q_{24}^2 q_{24}^2 + 6 q_{12}^2 q_{24}^
                                                                                                     8 q_{12} q_{23} q_{34}^3 q_{41}^3 S_3 S_4^2 + 4 q_{12} q_{23} q_{34}^3 q_{41}^3 S_1 S_3 S_4^2 + 4 q_{12} q_{23}^2 q_{34}^3 q_{41}^2 S_2 S_3 S_4^2 + 6 q_{23}^2 q_{34}^4 q_{41}^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^4 q_{41}^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 q_{41}^2 S_5^2 S_5^2 S_5^2 + 6 q_{23}^2 q_{24}^2 S_5^2 S_5^2 S_5^2 S_5^2 + 6 q_{23}^2 q_{24}^2 S_5^2 S_
                                                                                                8\ q_{12}\ q_{23}\ q_{34}^3\ q_{41}^3\ S_4^3\ -4\ q_{12}\ q_{34}^3\ q_{41}^4\ S_1\ S_4^3\ -4\ q_{12}\ q_{23}\ q_{34}^3\ q_{41}^3\ S_2\ S_4^3\ -4\ q_{23}\ q_{34}^4\ q_{41}^3\ S_3\ S_4^3\ +\ q_{34}^4\ q_{41}^4\ S_4^4
            In[8]:= rhse = Expand[rhs]
  Out[8]= 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2 - 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1 - 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2 + 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2 - 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ q_{24}^2\ S_1\ S_2 - 64\ q_{12}^2\ q_
                                                                                                     64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_3\ +\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_3\ +\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1\ S_2\ S_2\ S_2\ S_2\ S_3\ -\ 64\ q_{12}^2\ q_{1
                                                                                                   64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{4}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{4}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}\ S_{4}-64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ S_{4}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}\ S_{4}-64\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{2}\ S_{4}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{4}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}+64\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}+64\ q_{12}^{2}\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}+64\ q_{12}^{2}\ q_{12}^{2}\
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 $64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_3\ S_4\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_3\ S_4\ -\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_3\ S_4\ +\ 64\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ S_4$ 

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In[9]:= ArchimedesFn = rhse - lhse
Out[9]= -16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1^2 + 16 q_{12}^3 q_{23} q_{34} q_{41}^3 S_1^2 - 8 q_{12}^3 q_{23} q_{34} q_{41}^3 S_1^3 - q_{12}^4 q_{41}^4 S_1^4 - 32 q_{12}^3 q_{23}^3 q_{34} q_{41}^2 S_1 S_2 +
                                                                                              32\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ +\ 16\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ -\ 8\ q_{12}^3\ q_{23}\ q_{34}\ q_{41}^3\ S_1^2\ S_2\ +\ 4\ q_{12}^4\ q_{23}\ q_{41}^3\ S_1^3\ S_2\ +\ q_{41}^4\ q_{42}^2\ q_{42}^3\ q_{43}^3\ S_1^3\ S_2\ +\ q_{42}^4\ q_{42}^3\ q_{43}^3\ S_1^3\ S_2\ +\ q_{42}^4\ q_{42}^3\ q_{43}^3\ S_1^3\ S_2\ +\ q_{42}^4\ q_{42}^3\ q_{42}^3\ q_{42}^3\ S_1^3\ S_2\ +\ q_{42}^4\ q_{42}^3\ 
                                                                                              16\ q_{12}^3\ q_{23}^3\ q_{34}\ q_{41}\ S_2^2-16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2^2-8\ q_{12}^3\ q_{23}^3\ q_{34}\ q_{41}\ S_1\ S_2^2+16\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1\ S_2^2-100\ q_{12}^3\ q_{23}^2\ q_{23}^2\ q_{24}^2\ S_1^2\ S_2^2-100\ q_{12}^2\ q_{12}^2
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6\ q_{12}^4\ q_{23}^2\ q_{41}^2\ S_1^2\ S_2^2\ -\ 8\ q_{12}^3\ q_{23}^3\ q_{34}\ q_{41}\ S_2^3\ +\ 4\ q_{12}^4\ q_{23}^3\ q_{41}\ S_1\ S_2^3\ -\ q_{12}^4\ q_{23}^4\ S_2^4\ +\ 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2\ S_3\ -\ q_{12}^4\ q_{12}^4\ S_2^4\ S_2^4\ +\ 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2\ S_3\ -\ q_{12}^4\ q_{12}^4\ S_2^4\ S_2^4\ +\ 16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{21}^2\ S_3^2\ -\ q_{12}^2\ q_{23}^2\ q_{23}^2\ q_{24}^2\ S_3^2\ -\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_3^2\ +\ q_{12}^2\ S_3^2\ -\ q_{12}^2\ q_{12}^2\ q_{12}^2\ q_{12}^2\ q_{12}^2\ S_3^2\ -\ q_{12}^2\ q_{12}^2\ q_{12}^2\ q_{12}^2\ q_{12}^2\ S_3^2\ -\ q_{12}^2\ q_{
                                                                                                                 8\ q_{12}^{3}\ q_{23}\ q_{34}\ q_{41}^{3}\ S_{1}^{2}\ S_{3}\ +\ 4\ q_{12}^{3}\ q_{23}\ q_{34}\ q_{41}^{3}\ S_{1}^{3}\ S_{3}\ -\ 32\ q_{12}^{2}\ q_{23}^{3}\ q_{34}^{2}\ q_{41}\ S_{2}\ S_{3}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}\ S_{3}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{25}^{2}\ S_{3}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{25}^{2}\ S_{3}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{25}^{2}\ S_{3}\ +\ 32\ q_{12}^{2}\ q_{25}^{2}\ q_{25
                                                                                                                    16\ q_{12}^2\ q_{23}^3\ q_{34}^2\ q_{41}\ S_1\ S_2\ S_3\ +\ 16\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 48\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{23}^2\ q_{24}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{23}^2\ q_{23}^2\ q_{24}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{12}^2\ q_{23}^2\ q_{24}^2\ S_1^2\ S_2\ S_3\ -\ 4\ q_{12}^3\ q_{12}^2\ q_{12}^2\ q_{13}^2\ q_{12}^2\ q_{13}^2\ q_{12}^2\ q_{13}^2\ q
                                                                                                                    8\ q_{12}^{3}\ q_{23}^{3}\ q_{34}\ q_{41}\ S_{2}^{2}\ S_{3}\ +\ 16\ q_{12}^{2}\ q_{23}^{3}\ q_{34}^{2}\ q_{41}\ S_{2}^{2}\ S_{3}\ -\ 4\ q_{12}^{3}\ q_{23}^{3}\ q_{34}\ q_{41}\ S_{1}\ S_{2}^{2}\ S_{3}\ +\ 4\ q_{12}^{3}\ q_{23}^{4}\ q_{34}\ S_{2}^{3}\ S_{3}\ +
                                                                                                                    16 q_{12} q_{23}^3 q_{34}^3 q_{41} S_3^2 - 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_3^2 - 8 q_{12} q_{23}^3 q_{34}^3 q_{41} S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12} q_{23}^3 q_{34}^3 q_{41} S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12} q_{23}^3 q_{34}^3 q_{41} S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12} q_{23}^3 q_{34}^3 q_{41} S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12}^2 q_{23}^3 q_{34}^3 q_{41}^2 S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12}^2 q_{23}^3 q_{34}^2 q_{41}^2 S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_1 S_3^2 - 8 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_1 S_3^2 + 16 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_1 S_2^2 + 16 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_2^2 + 16 q_{12}^2 q_{24}^2 S_1 S_2^2 + 16 q_{12}^2 q_{23}^2 q_{24}^2 S_1 S_2^2 + 16 q_{12}^2 q_{12}^2 S_1 S_2^2 + 16 q_{12}^2 S_1 S_2^2 + 16 q_{12}^2 q_{12}^2 S_1 S_2^2 + 16 q_{12}^2 q_{12}^2 S_1 S_2^2 + 16 q_{12}^2 S_1 
                                                                                                                 6\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1^2\ S_3^2\ +\ 16\ q_{12}^2\ q_{23}^3\ q_{34}^2\ q_{41}\ S_2\ S_3^2\ -\ 8\ q_{12}\ q_{23}^3\ q_{34}^3\ q_{41}\ S_2\ S_3^2\ -\ 4\ q_{12}^2\ q_{23}^3\ q_{34}^2\ q_{41}\ S_1\ S_2\ S_3^2\ -\ q_{12}^2\ q_{12}^3\ q_{12}^2\ q_{13}^3\ q_{14}^2\ q_{14}\ S_1\ S_2\ S_3^2\ -\ q_{12}^2\ q_{12}^3\ q_{14}^2\ q_{15}^2\ q_{15}
                                                                                                                 6\ q_{12}^{2}\ q_{23}^{4}\ q_{34}^{2}\ S_{2}^{2}\ S_{3}^{2}-8\ q_{12}\ q_{23}^{3}\ q_{34}^{3}\ q_{41}\ S_{3}^{3}+4\ q_{12}\ q_{23}^{3}\ q_{34}^{3}\ q_{41}\ S_{1}\ S_{3}^{3}+4\ q_{12}\ q_{23}^{4}\ q_{34}^{3}\ S_{2}\ S_{3}^{3}-q_{23}^{4}\ q_{34}^{4}\ S_{3}^{4}+q_{12}^{2}\ q_{23}^{3}\ q_{34}^{3}\ q_{24}^{3}\ S_{2}^{3}+q_{24}^{3}\ q_{24}^{3}\ S_{3}^{3}+q_{24}^{3}\ S_{2}^{3}+q_{24}^{3}\ S_{2}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}+q_{24}^{3}
                                                                                                                    32\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_4 - 32\ q_{12}^2\ q_{23}\ q_{34}^2\ q_{41}^3\ S_1\ S_4 - 8\ q_{12}^3\ q_{23}\ q_{34}\ q_{41}^3\ S_1^2\ S_4 + 16\ q_{12}^2\ q_{23}\ q_{34}^2\ q_{34}^3\ S_1^2\ S_4 + 16\ q_{12}^2\ q_{12}^2\ q_{12}^2\ q_{13}^2\ S_1^2\ S_2 + 16\ q_{12}^2\ q_{13}^2\ q_{13}^2\ S_1^2\ S_2 + 16\ q_{13}^2\ q_{13}^2\ q_{13}^2\ S_1^2\ S_2 + 16\ q_{13}^2\ q_{13}^2\ q_{13}^2\ q_{13}^2\ S_1^2\ S_2 + 16\ q_{13}^2\ q_{13}^2\ q_{13}^2\ q_{13}^2\ S_1^2\ S_1^2
                                                                                                                    4\,q_{12}^3\,q_{34}\,q_{41}^4\,S_1^3\,S_4+16\,q_{12}^3\,q_{23}^2\,q_{34}\,q_{41}^2\,S_1\,S_2\,S_4-48\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^3\,S_1\,S_2\,S_4-48\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^3\,S_1\,S_2\,S_4-48\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{34}^2\,q_{41}^3\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_4+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{23}^2\,q_{24}^2\,S_1\,S_2\,S_2+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,S_1\,S_2+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,S_1+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,S_1+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,S_1+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,S_1+16\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12}^2\,q_{12
                                                                                                               4\ q_{12}^{3}\ q_{23}\ q_{34}\ q_{41}^{3}\ S_{1}^{2}\ S_{2}\ S_{4}-8\ q_{12}^{3}\ q_{23}^{3}\ q_{34}\ q_{41}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}^{2}\ S_{4}-4\ q_{12}^{3}\ q_{23}^{2}\ q_{34}\ q_{41}^{2}\ S_{1}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}^{2}\ S_{4}-4\ q_{12}^{3}\ q_{23}^{2}\ q_{34}\ q_{41}^{2}\ S_{1}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{2}^{2}\ S_{4}-4\ q_{12}^{3}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}^{2}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{24}^{2}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{24}^{2}\ S_{2}^{2}\ S_{4}-4\ q_{12}^{3}\ q_{24}^{2}\ q_{24}^{2}\ S_{1}^{2}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ q_{24}^{2}\ q_{24}^{2}\ q_{24}^{2}\ S_{1}^{2}\ S_{2}^{2}\ S_{4}+16\ q_{12}^{2}\ q_{24}^{2}\ 
                                                                                                                    4\ q_{12}^{3}\ q_{23}^{3}\ q_{34}\ q_{41}\ S_{2}^{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{3}\ S_{4}\ -\ 32\ q_{12}\ q_{23}^{2}\ q_{34}^{3}\ q_{41}^{2}\ S_{3}\ S_{4}\ -\ 48\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{41}^{2}\ S_{1}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{34}^{2}\ q_{24}^{2}\ S_{1}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{23}^{2}\ q_{24}^{2}\ S_{1}\ S_{2}\ S_{3}\ S_{4}\ +\ 32\ q_{12}^{2}\ q_{12}^{2}\
                                                                                                                    48\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_3\ S_4\ +\ 16\ q_{12}\ q_{23}^2\ q_{34}^3\ q_{41}^2\ S_2\ S_3\ S_4\ +\ 40\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_1\ S_2\ S_3\ S_4\ -
                                                                                                                 4\ q_{12}^{2}\ q_{23}^{3}\ q_{34}^{2}\ q_{41}\ S_{2}^{2}\ S_{3}\ S_{4}-8\ q_{12}\ q_{23}^{3}\ q_{34}^{3}\ q_{41}\ S_{3}^{2}\ S_{4}+16\ q_{12}\ q_{23}^{2}\ q_{34}^{3}\ q_{41}^{2}\ S_{3}^{2}\ S_{4}-4\ q_{12}\ q_{23}^{2}\ q_{34}^{3}\ q_{41}^{2}\ S_{1}\ S_{3}^{2}\ S_{4}-4
                                                                                                                    4 q_{12} q_{23}^3 q_{34}^3 q_{41} S_2 S_3^2 S_4 + 4 q_{23}^3 q_{34}^4 q_{41} S_3^3 S_4 - 16 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_4^2 + 16 q_{12} q_{23} q_{34}^3 q_{41}^3 S_4^2 + 16 q_{12} q_{23} q_{23}^3 q_{34}^3 q_{41}^3 S_4^2 + 16 q_{12} q_{23} q_{23}^3 q_{24}^3 q_{24}^3 S_4^2 + 16 q_{12} q_{23} q_{23}^3 q_{24}^3 q_{24}^3 S_4^2 + 16 q_{12} q_{23} q_{24}^3 q_{24}^3 S_4^2 + 16 q_{12} q_{23} q_{24}^3 q_{24}^3 S_4^2 + 16 q_{12} q_{23}^3 q_{24}^3 q_{24}^3 S_4^3 + 16 q_{12} q_{23}^3 q_{24}^3 q_{24}^3 S_4^3 + 16 q_{12} q_{23}^3 q_{24}^3 q_{24}^3 S_4^3 + 16 q_{12} q_{23}^3 q_{24}^3 q_{24}^3 + 16 q_{12}^3 q_{24}^3 + 16 q
                                                                                                                    16\ q_{12}^2\ q_{23}\ q_{34}^2\ q_{41}^3\ S_1\ S_4^2-8\ q_{12}\ q_{23}\ q_{34}^3\ q_{41}^3\ S_1\ S_4^2-6\ q_{12}^2\ q_{34}^2\ q_{41}^4\ S_1^2\ S_4^2+16\ q_{12}^2\ q_{23}^2\ q_{34}^2\ q_{41}^2\ S_2\ S_4^2-100\ q_{12}^2\ q_{13}^2\ q_{13}^2\ q_{14}^2\ S_2\ S_4^2-100\ q_{12}^2\ q_{13}^2\ q_{13}^2\ q_{14}^2\ S_2\ S_4^2+100\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_2\ S_4^2-100\ q_{14}^2\ S_2\ S_4^2+100\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_2\ S_4^2-100\ q_{14}^2\ S_2\ S_4^2+100\ q_{14}^2\ S_2\ S_3^2+100\ q_{14}^2\ S_3^2\ S_3^2+100\ q_{14}^2\ S_3^2+100\ 
                                                                                                                 8 q_{12} q_{23} q_{34}^3 q_{41}^3 S_2 S_4^2 - 4 q_{12}^2 q_{23} q_{34}^2 q_{41}^3 S_1 S_2 S_4^2 - 6 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_2^2 S_4^2 + 16 q_{12} q_{23}^2 q_{34}^3 q_{41}^2 S_3 S_4^2 - 6 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_2 S_4^2 + 16 q_{12} q_{23}^2 q_{34}^3 q_{41}^2 S_3 S_4^2 - 6 q_{12}^2 q_{23}^2 q_{34}^2 q_{41}^2 S_3 S_4^2 - 6 q_{12}^2 q_{12}^
                                                                                                                 8\ q_{12}\ q_{23}\ q_{34}^3\ q_{41}^3\ S_3\ S_4^2-4\ q_{12}\ q_{23}^3\ q_{34}^3\ q_{41}^3\ S_1\ S_3\ S_4^2-4\ q_{12}\ q_{23}^2\ q_{34}^3\ q_{41}^2\ S_2\ S_3\ q_{4}^2-6\ q_{23}^2\ q_{34}^4\ q_{41}^2\ S_3^2\ S_4^2-8\ q_{42}^2\ q_{42}^2\ S_3^2\ S_4^2-8\ q_{42}^2\ q_{42}^2\ S_3^2\ S_4^2-8\ q_{42}^2\ q_{42}^2\ S_3^2\ S_4^2-8\ q_{42}^2\ S_4^2\ S_4^2\
                                                                                                                 8 \ q_{12} \ q_{23} \ q_{34}^3 \ q_{41}^3 \ S_4^3 + 4 \ q_{12} \ q_{34}^3 \ q_{41}^4 \ S_1 \ S_4^3 + 4 \ q_{12} \ q_{23} \ q_{34}^3 \ q_{41}^3 \ S_2 \ S_4^3 + 4 \ q_{23} \ q_{44}^4 \ S_3 \ S_4^3 - q_{34}^4 \ q_{41}^4 \ S_4^4
          ln[10]:= a_1 = \{x_1, y_1, z_1\}
                                                                                             a_2 = \{x_2, y_2, z_2\}
                                                                                             a_3 = \{x_3, y_3, z_3\}
                                                                                             a_4 = \{x_4, y_4, z_4\}
Out[10]= \{X_1, Y_1, Z_1\}
Out[11]= \{x_2, y_2, z_2\}
Out[12]= \{X_3, Y_3, Z_3\}
Out[13]= \{x_4, y_4, z_4\}
       In[14]:= J[a1_, a2_] :=
                                                                                                                       \{a1[2] \times a2[3] - a1[3] \times a2[2], a1[3] \times a2[1] - a1[1] \times a2[3], a1[2] \times a2[1] - a1[1] \times a2[2] \}
          log_{15} = quadrance[a1_, a2_] := 1 - (a1[1] \times a2[1] + a1[2] \times a2[2] - a1[3] \times a2[3])^2 / a^2[3] + a
                                                                                                                                                                ((a1[1] \times a1[1] + a1[2] \times a1[2] - a1[3] \times a1[3]) \times
                                                                                                                                                                                                       (a2[1] \times a2[1] + a2[2] \times a2[2] - a2[3] \times a2[3])
       \ln[16] = \text{spread}[11\_, 12\_] := 1 - (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^{2} / (11[1] \times 12[2] \times 12[2] - 11[3] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2] - 11[2]
```

```
(\,(11[\![1]\!]\times11[\![1]\!]+11[\![2]\!]\times11[\![2]\!]-11[\![3]\!]\times11[\![3]\!])\,\times\\
    (12[1] \times 12[1] + 12[2] \times 12[2] - 12[3] \times 12[3])
```

$$ln[17]:= q_{12} = quadrance[a_1, a_2]$$

$$q_{23} = quadrance[a_2, a_3]$$

$$q_{34} = quadrance[a_3, a_4]$$

$$q_{41} = quadrance[a_4, a_1]$$

$$\text{Out[17]= } 1 - \frac{\left(x_1\,x_2 + y_1\,y_2 - z_1\,z_2\right)^2}{\left(x_1^2 + y_1^2 - z_1^2\right)\,\left(x_2^2 + y_2^2 - z_2^2\right)}$$

$$\text{Out[18]= } 1 - \frac{\left(x_2\,x_3 + y_2\,y_3 - z_2\,z_3\right)^2}{\left(x_2^2 + y_2^2 - z_2^2\right)\,\left(x_3^2 + y_3^2 - z_3^2\right)}$$

$$\text{Out[19]= } 1 - \frac{\left(x_3 \; x_4 + y_3 \; y_4 - z_3 \; z_4\right)^2}{\left(x_3^2 + y_3^2 - z_3^2\right) \; \left(x_4^2 + y_4^2 - z_4^2\right)}$$

$$\text{Out}[20] = \ 1 - \frac{\left(x_1 \, x_4 + y_1 \, y_4 - z_1 \, z_4\right)^2}{\left(x_1^2 + y_1^2 - z_1^2\right) \, \left(x_4^2 + y_4^2 - z_4^2\right)}$$

$$In[21]:= L_1 = J[a_1, a_2]$$

$$L_2 = J[a_2, a_3]$$

$$L_3 = J[a_3, a_4]$$

$$L_4 = J[a_4, a_1]$$

Out[21]= 
$$\{-y_2 z_1 + y_1 z_2, x_2 z_1 - x_1 z_2, x_2 y_1 - x_1 y_2\}$$

Out[22]= 
$$\{-y_3 z_2 + y_2 z_3, x_3 z_2 - x_2 z_3, x_3 y_2 - x_2 y_3\}$$

Out[23]= 
$$\{-y_4 z_3 + y_3 z_4, x_4 z_3 - x_3 z_4, x_4 y_3 - x_3 y_4\}$$

$$\text{Out} [\text{24}] = \hspace{0.1cm} \left\{ \hspace{0.1cm} y_4 \hspace{0.1cm} z_1 \hspace{0.1cm} - \hspace{0.1cm} y_1 \hspace{0.1cm} z_4 \hspace{0.1cm} , \hspace{0.1cm} - \hspace{0.1cm} x_4 \hspace{0.1cm} z_1 \hspace{0.1cm} + \hspace{0.1cm} x_1 \hspace{0.1cm} z_4 \hspace{0.1cm} , \hspace{0.1cm} - \hspace{0.1cm} x_4 \hspace{0.1cm} y_1 \hspace{0.1cm} + \hspace{0.1cm} x_1 \hspace{0.1cm} y_4 \hspace{0.1cm} \right\}$$

```
ln[25] = S_1 = spread[L_4, L_1]
              S_2 = spread[L_1, L_2]
              S_3 = spread[L_2, L_3]
              S_4 = spread[L_3, L_4]
Out[25]= 1 - (-((x_2 y_1 - x_1 y_2) (-x_4 y_1 + x_1 y_4)) +
                              (x_2 z_1 - x_1 z_2) (-x_4 z_1 + x_1 z_4) + (-y_2 z_1 + y_1 z_2) (y_4 z_1 - y_1 z_4))^2
                    \left( \, \left( \, - \, \left( \, x_{2} \, y_{1} - x_{1} \, y_{2} \, \right)^{\, 2} \, + \, \left( \, x_{2} \, z_{1} - x_{1} \, z_{2} \, \right)^{\, 2} \, + \, \left( \, - \, y_{2} \, z_{1} \, + \, y_{1} \, z_{2} \, \right)^{\, 2} \right)
                          \left(-\left(-X_{4} Y_{1}+X_{1} Y_{4}\right)^{2}+\left(-X_{4} Z_{1}+X_{1} Z_{4}\right)^{2}+\left(Y_{4} Z_{1}-Y_{1} Z_{4}\right)^{2}\right)
Out[26]= 1 -
                  \left(-\;\left(\;\left(x_{2}\,y_{1}-x_{1}\,y_{2}\right)\;\left(x_{3}\,y_{2}-x_{2}\,y_{3}\right)\;\right)\;+\;\left(x_{2}\,z_{1}-x_{1}\,z_{2}\right)\;\left(x_{3}\,z_{2}-x_{2}\,z_{3}\right)\;+\;\left(-\,y_{2}\,z_{1}+y_{1}\,z_{2}\right)\;\left(-\,y_{3}\,z_{2}+y_{2}\,z_{3}\right)\;\right)^{\,2}\;/
                    ((-(x_2 y_1 - x_1 y_2)^2 + (x_2 z_1 - x_1 z_2)^2 + (-y_2 z_1 + y_1 z_2)^2)
                          \left(-\left(x_{3}\,y_{2}-x_{2}\,y_{3}\right)^{2}+\left(x_{3}\,z_{2}-x_{2}\,z_{3}\right)^{2}+\left(-y_{3}\,z_{2}+y_{2}\,z_{3}\right)^{2}\right)
Out[27]= 1 -
                  \left(-\left(\left.\left(x_{3}\,y_{2}-x_{2}\,y_{3}\right)\,\left(x_{4}\,y_{3}-x_{3}\,y_{4}\right)\,\right)\right.\\ +\left.\left(x_{3}\,z_{2}-x_{2}\,z_{3}\right)\,\left(x_{4}\,z_{3}-x_{3}\,z_{4}\right)\right.\\ +\left.\left(-y_{3}\,z_{2}+y_{2}\,z_{3}\right)\,\left(-y_{4}\,z_{3}+y_{3}\,z_{4}\right)\right)^{2}\right/\left[-y_{4}\,z_{3}+y_{3}\,z_{4}\right]
                    ((-(x_3 y_2 - x_2 y_3)^2 + (x_3 z_2 - x_2 z_3)^2 + (-y_3 z_2 + y_2 z_3)^2)
                          \left(-\left(X_{4}\,y_{3}-X_{3}\,y_{4}\right)^{2}+\left(X_{4}\,Z_{3}-X_{3}\,Z_{4}\right)^{2}+\left(-y_{4}\,Z_{3}+y_{3}\,Z_{4}\right)^{2}\right)\right)
Out[28]= 1 - (-((-x_4 y_1 + x_1 y_4) (x_4 y_3 - x_3 y_4)) +
                             (-X_4 Z_1 + X_1 Z_4) (X_4 Z_3 - X_3 Z_4) + (y_4 Z_1 - y_1 Z_4) (-y_4 Z_3 + y_3 Z_4))^2
                    \left( \, \left( \, - \, \left( \, - \, x_{4} \, y_{1} + x_{1} \, y_{4} \, \right)^{\, 2} \, + \, \left( \, - \, x_{4} \, z_{1} + x_{1} \, z_{4} \, \right)^{\, 2} \, + \, \left( \, y_{4} \, z_{1} - y_{1} \, z_{4} \, \right)^{\, 2} \right)
                          \left(-\left(X_{4} Y_{3}-X_{3} Y_{4}\right)^{2}+\left(X_{4} Z_{3}-X_{3} Z_{4}\right)^{2}+\left(-Y_{4} Z_{3}+Y_{3} Z_{4}\right)^{2}\right)\right)
```

The next cell may take a long time to execute since the q's and the S's in the "ArchimedesFn" equation are being substituted with the x's, y's and z's in the q's and S's expressions above and the whole thing is being factored. A result of zero indicates that the criteria for compatible equations in the quadratic compatibility theorem has been met.

```
In[29]:= result = Factor[ArchimedesFn]
Out[29]= 0
```

We have proved that the following two quadratic equations in B are compatible.