Quadratic Compatibility Theorem: Theorem 1 in the book Divine Proportions by Wildberger

The quadratic equations

$$(X-P1) = M$$

$$2(X-p2)=r2$$

are compatible precisely when

$$((p1-p2)^2-(m+r2))^2=4mr2$$

In this case, if p1 is not equal to p2 then there is a unique common solution

$$X = P1 + P2 - (r1 - r2)$$
  
2  $2(P1 - P2)$ 

Create a Mathematica function for unique common solution of a pair of compatible quadratic equations.

$$log(1) := X[p1_, r1_, p2_, r2_] := ((p1 + p2) / 2) - (r1 - r2) / (2 (p1 - p2))$$

Substitute values for p1, r1, p2 and r2 from quadratic equations (1) and (2) above to compute the quadreal of quadrilateral.

In[2]:= quadreal = 
$$X[S_1 S_2 q_{12} + S_3 S_4 q_{34}, 4 S_1 S_2 S_3 S_4 (1 - q_{23}) \times (1 - q_{41}),$$
  
 $S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1 - q_{12}) \times (1 - q_{34})]$ 

$$\begin{array}{l} \text{Out} [2] = \begin{array}{l} \displaystyle \frac{1}{2} & (q_{12} \; S_1 \; S_2 + q_{23} \; S_2 \; S_3 + q_{41} \; S_1 \; S_4 + q_{34} \; S_3 \; S_4) \; - \\ \\ \displaystyle \qquad \qquad \\ \displaystyle \frac{-4 \times \; (1 - q_{12}) \; \times \; (1 - q_{34}) \; \; S_1 \; S_2 \; S_3 \; S_4 + 4 \times \; (1 - q_{23}) \; \times \; (1 - q_{41}) \; \; S_1 \; S_2 \; S_3 \; S_4}{2 \; \; (q_{12} \; S_1 \; S_2 - q_{23} \; S_2 \; S_3 - q_{41} \; S_1 \; S_4 + q_{34} \; S_3 \; S_4)} \end{array}$$

The proof employs the quadratic compatibility theorem. We will show that equations (1) and (2) above are compatible since they meet the criteria for being compatible from the quadratic compatibility theorem.

Perform compatibility check:

$$ln[3]:=$$
 CompatibilityCheckLHS [p1\_, r1\_, p2\_, r2\_] :=  $((p1-p2)^2 - (r1+r2))^2$ 

$$ln[4]:=$$
 CompatibilityCheckRHS [p1\_, r1\_, p2\_, r2\_] :=  $4*r1*r2$ 

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ln[5]:= lhs = CompatibilityCheckLHS[S<sub>1</sub> S<sub>2</sub> q<sub>12</sub> + S<sub>3</sub> S<sub>4</sub> q<sub>34</sub>,
                                                                                                               4 S_1 S_2 S_3 S_4 (1 - q_{23}) \times (1 - q_{41}), S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1 - q_{12}) \times (1 - q_{34})
Out[5]= (-4 \times (1 - q_{12}) \times (1 - q_{34}) S_1 S_2 S_3 S_4 -
                                                                                                                                4 \times (1-q_{23}) \times (1-q_{41}) \, \, S_1 \, S_2 \, S_3 \, S_4 + \, \left(q_{12} \, S_1 \, S_2 - q_{23} \, S_2 \, S_3 - q_{41} \, S_1 \, S_4 + q_{34} \, S_3 \, S_4\right)^{\, 2}\right)^{\, 2}
         ln[6]:= rhs = CompatibilityCheckRHS[S<sub>1</sub> S<sub>2</sub> q<sub>12</sub> + S<sub>3</sub> S<sub>4</sub> q<sub>34</sub>,
                                                                                                               4 S_1 S_2 S_3 S_4 (1-q_{23}) \times (1-q_{41}), S_4 S_1 q_{41} + S_2 S_3 q_{23}, 4 S_1 S_2 S_3 S_4 (1-q_{12}) \times (1-q_{34})
Out[6]= 64 \times (1 - q_{12}) \times (1 - q_{23}) \times (1 - q_{34}) \times (1 - q_{41}) S_1^2 S_2^2 S_3^2 S_4^2
       ln[7]:= lhse = Expand[lhs]
 \text{Out} \text{[7]=} \quad q_{12}^4 \, \, S_1^4 \, \, S_2^4 \, - \, 4 \, \, q_{12}^3 \, \, q_{23}^3 \, \, S_1^3 \, \, S_2^4 \, S_3 \, + \, 6 \, \, q_{12}^2 \, \, q_{23}^2 \, \, S_1^2 \, \, S_2^4 \, \, S_3^2 \, - \, 4 \, \, q_{12} \, \, q_{23}^3 \, \, S_1 \, \, S_2^4 \, \, S_3^3 \, + \, q_{23}^4 \, \, S_2^4 \, \, S_3^4 \, - \, 4 \, \, q_{12}^3 \, \, q_{41}^4 \, \, S_1^4 \, \, S_2^3 \, \, S_4 \, - \, q_{12}^4 \, \, q_{12}^3 \, \, S_1^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, - \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^4 \, \, S_2^4 \, \, S_3^4 \, + \, q_{12}^
                                                                                              8\ q_{12}^{2}\ q_{41}\ S_{1}^{3}\ S_{2}^{3}\ S_{3}\ S_{4}\ +\ 4\ q_{12}^{2}\ q_{23}\ q_{41}\ S_{1}^{3}\ S_{2}^{3}\ S_{3}\ S_{4}\ +\ 32\ q_{12}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{3}^{2}\ S_{4}\ -\ 16\ q_{12}^{2}\ q_{23}\ S_{1}^{2}\ S_{2}^{3}\ S_{3}^{2}\ S_{3}^
                                                                                            16 q_{12} q_{23}^2 S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 16 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_3^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_3^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{23} q_{34} S_1^2 S_3^2 S_3^2 S_3^2 S_3^2 S_4 + 4 q_{12}^2 q_{12}^2 S_3^2 S_3^
                                                                                          4\,q_{12}\,q_{23}^2\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4 - 16\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{12}\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^3\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{23}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 + 8\,q_{23}^2\,q_{23}^2\,S_1^2\,S_2^3\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^3\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4 + 8\,q_{23}^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_
                                                                                            4 q_{12} q_{23}^2 q_{34} S_1 S_2^3 S_3^3 S_4 + 8 q_{23}^2 q_{41} S_1 S_2^3 S_3^3 S_4 - 4 q_{23}^3 q_{41} S_1 S_2^3 S_3^3 S_4 - 4 q_{23}^3 q_{34} S_2^3 S_3^4 S_4 +
                                                                                          16 q_{12} q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 + 4 q_{12}^2 q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 - 16 q_{12} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 + 4 q_{12} q_{23} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 +
                                                                                              64\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,64\,q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,16\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,64\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,Q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_3^2\,S_
                                                                                              16\ q_{23}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 64\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 64\ q_{12}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{23}\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{24}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{24}\ S_2^2\ S_3^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{24}\ S_2^2\ S_3^2\ S_
                                                                                              16\ q_{12}\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 16\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{34}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{12}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 6\ q_{12}^2\ q_{12}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{12}^2\ S_1^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{12}^2\ q_{12}^2\ S_1^2\ S_1^2\ S_1^2\ S_2^2\ S_3^2\ S_2^2\ S_3^2\ S_2^2\ S_3^2\ S_3
                                                                                            64\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{12}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,64\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{23}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{23}\,q_{23}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{12}\,q_{12}\,q_{12}\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,
                                                                                              16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{12} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 +
                                                                                              24\,\,q_{12}\,\,q_{23}\,\,q_{34}\,\,q_{41}\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,-\,16\,\,q_{23}\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,6\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{41}^2\,\,S_1^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_2^2\,\,S_3^2\,\,S_4^2\,+\,16\,\,q_{23}^2\,\,q_{23}^2\,\,S_2^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S_3^2\,\,S
                                                                                              4 q_{12} q_{23} q_{34}^2 S_1 S_2^2 S_3^3 S_4^2 - 16 q_{23} q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 + 4 q_{23}^2 q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 - 16 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{34}^2 S_2^2 S_3^2 S_3^2 S_4^2 + 6 q_{23}^2 q_{24}^2 S_3^2 S_4^2 + 6 q_{23}^2 S_4^2 + 6 q_{23}^2 S_4^2 + 6 q_{23}^2 S_4^2 S_4^2 
                                                                                            4 q_{12} q_{41}^3 S_1^4 S_2 S_4^3 - 16 q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{12} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{23} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 + 8 q_{44} S_1^3 S_2 S_2 S_3 S_4^3 + 8 q_{44} S_1^3 S_2 S_3 S_4^3 + 8 q_{44} S_1^3 S_2 S_3 S_4^3 + 8 q_{44} S_1^3 S_2 S_2 S_3 S_4^3 + 8 q_{44}
                                                                                          4\,q_{12}\,q_{34}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3\,+\,8\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,-\,4\,q_{23}\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}\,q_{41}^3\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^2\,S_2\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^2\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2\,S_3^3\,S_4^3\,+\,32\,q_{34}^3\,q_{41}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,q_{41}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_4^3\,-\,4\,q_{23}^3\,S_1^3\,S_2^3\,S_3^3\,S_2^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S_3^3\,S
                                                                                              16\ q_{12}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{23}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3 - 16\ q_{12}^2\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^2\ S_4^3 + 4\ q_{12}\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 - 16\ q_{12}^2\ q_{13}^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{12}^2\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{13}^2\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{12}\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_4^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3 + 4\ q_{14}^2\ S_1^2\ S_2\ S_2\ S_3^3 + 4\ q_{14}^2\ S_1^2
                                                                                            8 \, q_{23} \, q_{34}^2 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 4 \, q_{23} \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 4 \, q_{23} \, q_{34}^2 \, q_{41} \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, + \, 8 \, q_{23}^3 \, S_4^3 \, - \, 4 \, q_{23} \, q_{24}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{34}^3 \, S_1 \, S_2 \, S_3^3 \, S_4^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, S_2^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_1^3 \, S_2^3 \, - \, 4 \, q_{12} \, q_{12}^3 \, S_2^3 \, S_2^
                                                                                            4 \, q_{23} \, q_{34}^3 \, S_2 \, S_3^4 \, S_4^3 + q_{41}^4 \, S_1^4 \, S_4^4 - 4 \, q_{34} \, q_{41}^3 \, S_1^3 \, S_3 \, S_4^4 + 6 \, q_{34}^2 \, q_{41}^2 \, S_1^2 \, S_3^2 \, S_4^4 - 4 \, q_{34}^3 \, q_{41}^4 \, S_1 \, S_3^3 \, S_4^4 + q_{34}^4 \, S_3^4 \, S_3^4 \, S_4^4 + q_{34}^4 \, S_3^4 \, S_3^4 \, S_3^4 \, S_3^4 + q_{34}^4 \, S_3^4 \, S_3
         In[8]:= rhse = Expand[rhs]
Out[8]= 64S_1^2S_2^2S_3^2S_4^2 - 64q_{12}S_1^2S_2^2S_3^2S_4^2 - 64q_{23}S_1^2S_2^2S_3^2S_4^2 + 64q_{12}q_{23}S_1^2S_2^2S_3^2S_4^2 - 64q_{34}S_1^2S_2^2S_3^2S_4^2 + 64q_{12}q_{23}S_1^2S_2^2S_3^2S_4^2 - 64q_{34}S_1^2S_2^2S_3^2S_4^2 + 64q_{34}S_2^2S_2^2S_3^2S_4^2 + 64q_{34}S_2^2S_2^2S_3^2S_4^2 + 64q_{34}S_2^2S_2^2S_2^2S_2^2S_2^2 + 64q_{34}S_2^2S_2^2S_2^2S_2^2 + 64q_{34}S_2^2S_2^2S_2^2 + 64q_{34}S_2^2S_2^2S_2^2 + 64q_{34}S_2^2S_2^2 + 64q_{34}S_2^2S_2^2 + 64q_{34}S_2^2S_2^2 + 64q_{34}S_2^2 + 64q_{34}S_2^2 + 64q_{34}S_2^2 + 64q_{34}S_2^2 + 64q_{34}S_2^2 + 6
                                                                                              64 q_{12} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 + 64 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 +
                                                                                            64 \, q_{12} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{34} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 64 \, q_{12} \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{23} \, q_{41} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_3^2 \, S_2^2 \, S_3^2 \, S_4^2 + 64 \, q_{24} \, q_{24} \, S_1^2 \, S_2^2 \, S_
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 $64 q_{12} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 64 q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 64 q_{12} q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2$ 

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In[9]:= ArchimedesFn = rhse - lhse
         Out[9]= -q_{12}^4 S_1^4 S_2^4 + 4 q_{12}^3 q_{23}^3 S_1^3 S_2^4 S_3 - 6 q_{12}^2 q_{23}^2 S_1^2 S_2^4 S_3^2 + 4 q_{12}^3 q_{23}^3 S_1 S_2^4 S_3^3 - q_{23}^4 S_2^4 S_3^4 + 4 q_{12}^3 q_{41}^4 S_1^4 S_2^3 S_4 + 4 q_{12}^3 q_{23}^4 
                                                                                    8q_{12}^2q_{41}S_1^3S_2^3S_3S_4 - 4q_{12}^2q_{23}q_{41}S_1^3S_2^3S_3S_4 - 32q_{12}q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^3S_3S_4 + 16q_{12}^2q_{23}S_1^2S_2^2S_3S_4 + 16q_{12}^2q_{23}S_1^2S_3S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_4 + 16q_{12}^2q_{23}^2S_3S_5 + 16q_{12}^2q_{23}^2S_5 + 16q_{12}^
                                                                                    4\,q_{12}\,q_{23}^2\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4 + 16\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{12}\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^3\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{34}^2\,S_1^2\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,S_1^3\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,S_1^3\,S_2^3\,S_2^3\,S_3^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S_2^3\,S
                                                                                   4\,q_{12}\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4 - 8\,q_{23}^2\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4 + 4\,q_{23}^3\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4 + 4\,q_{23}^3\,q_{34}\,S_2^3\,S_3^4\,S_4 - 6\,q_{12}^2\,q_{41}^2\,S_1^4\,S_2^2\,S_4^2 - 6\,q_{12}^2\,q_{41}^2\,S_1^4\,S_2^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^4 - 6\,q_{12}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2\,S_3^2 - 6\,q_{12}^2\,S_3^2\,S_3^2 - 6\,q_{
                                                                                    4q_{12}^2q_{34}q_{41}S_1^3S_2^2S_3S_4^2 + 16q_{12}q_{41}^2S_1^3S_2^2S_3S_4^2 - 4q_{12}q_{23}q_{41}^2S_1^3S_2^2S_3S_4^2 - 16q_{12}^2S_1^2S_2^2S_3^2S_4^2 +
                                                                                   32\ q_{12}\ q_{23}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 16\ q_{23}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 16\ q_{12}^2\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ +\ 32\ q_{23}\ q_{34}\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 100\ q_{23}^2\ S_2^2\ S_3^2\ S_4^2\ +\ 100\ q_{23}^2\ S_2^2\ S_3^2\ S_3^2\ S_4^2\ +\ 100\ q_{23}^2\ S_2^2\ S_3^2\ 
                                                                                    48 \, q_{12} \, q_{23} \, q_{34} \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 16 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12} \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 - 6 \, q_{12}^2 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{34}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_1^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_2^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 \, S_3^2 \, S_4^2 + 16 \, q_{12}^2 \, q_{13}^2 \, S_3^2 
                                                                                    32\,q_{12}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,48\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,16\,q_{23}^2\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q_{41}\,S_2^2\,S_3^2\,S_4^2\,-\,32\,q
                                                                                    48\,q_{12}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,48\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,-\,16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,Q_{12}^2\,Q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_3^2\,S_4^2\,+\,40\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_2^2\,S_2^2\,S_2^2\,S_3^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,S_2^2\,
                                                                                   16\ q_{23}\ q_{41}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 6\ q_{23}^2\ q_{41}^2\ S_1^2\ S_2^2\ S_3^2\ S_4^2\ -\ 32\ q_{23}\ q_{34}\ S_1\ S_2^2\ S_3^3\ S_4^2\ +\ 16\ q_{12}\ q_{23}\ q_{34}\ S_1\ S_2^2\ S_3^3\ S_4^2\ +
                                                                                    4\ q_{23}^2\ q_{34}\ q_{41}\ S_1\ S_2^2\ S_3^3\ S_4^2\ -\ 6\ q_{23}^2\ q_{34}^2\ S_2^2\ S_3^4\ S_4^2\ +\ 4\ q_{12}\ q_{41}^3\ S_1^4\ S_2\ S_3^3\ +\ 16\ q_{41}^2\ S_1^3\ S_2\ S_3\ S_4^3\ -
                                                                                    8 \ q_{12} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 8 \ q_{23} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 8 \ q_{34} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - 4 \ q_{12} \ q_{34} \ q_{41}^2 \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_4^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 \ S_3 \ S_3^3 - q_{41} \ S_1^3 \ S_2 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3^3 \ S_3 \ S_3 \ S_3^3 \ S_3 \ S_3
                                                                                 8\ q_{41}^3\ S_1^3\ S_2\ S_3\ S_4^3\ +\ 4\ q_{23}\ q_{41}^3\ S_1^3\ S_2\ S_3\ S_4^3\ -\ 32\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ 16\ q_{12}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^3\ S_4^3\ +
                                                                                    16\ q_{23}\ q_{34}\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ 16\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ -\ 4\ q_{12}\ q_{34}^2\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ 16\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3^3\ S_4^3\ -\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ 16\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3^2\ S_4^3\ -\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ q_{41}\ S_1^2\ S_2\ S_3^2\ S_4^3\ +\ q_{41}\ S_1^2\ S_2\ S_3^3\ S_3^3\ +\ q_{41}\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3\ +\ q_{41}\ S_1^2\ S_2\ S_2\ S_3^3\ S_3^3\ +\ q_{41}\ S_1^2\ S_2\ S_2\ S_
                                                                                   4\ q_{23}\ q_{34}\ q_{41}^2\ S_1^2\ S_2\ S_3^2\ S_4^3 + 16\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8\ q_{12}\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8\ q_{23}\ q_{34}^2\ S_1\ S_2\ S_3^3\ S_4^3 - 8
                                                                                   8q_{34}^3S_1S_2S_3^3S_4^3 + 4q_{12}q_{34}^3S_1S_2S_3^3S_4^3 - 8q_{34}^2q_{41}S_1S_2S_3^3S_4^3 - 4q_{23}q_{34}^2q_{41}S_1S_2S_3^3S_4^3 +
                                                                                   4\,q_{23}\,q_{34}^3\,S_2\,S_3^4\,S_4^3-q_{41}^4\,S_1^4\,S_4^4+4\,q_{34}\,q_{41}^3\,S_1^3\,S_3\,S_4^4-6\,q_{34}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_4^4+4\,q_{34}^3\,q_{41}^4\,S_1\,S_3^3\,S_4^4-q_{34}^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_3^4\,S_4^4+q_{34}^2\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,S_1^4\,
       ln[10] = L_1 = \{l_1, m_1, n_1\}
                                                                   L_2 = \{1_2, m_2, n_2\}
                                                                   L_3 = \{l_3, m_3, n_3\}
                                                                   L_4 = \{1_4, m_4, n_4\}
Out[10]= \{l_1, m_1, n_1\}
Out[11]= \{\,\mathbf{l}_{2}\,,\,m_{2}\,,\,n_{2}\,\}
Out[12]= \{1_3, m_3, n_3\}
Out[13]= { 1_4, m_4, n_4}
                                                                   Meet of lines.
       In[14]:= M[L1_, L2_] :=
                                                                                    \{L1[2] \times L2[3] - L1[3] \times L2[2], L1[3] \times L2[1] - L1[1] \times L2[3], L1[2] \times L2[1] - L1[1] \times L2[2]\}
       log(15) = quadrance[a1_, a2_] := 1 - (a1[1] \times a2[1] + a1[2] \times a2[2] - a1[3] \times a2[3])^2 / a^2[3] + a^
                                                                                                                    ((a1[1] \times a1[1] + a1[2] \times a1[2] - a1[3] \times a1[3]) \times
                                                                                                                                                 (a2[1] \times a2[1] + a2[2] \times a2[2] - a2[3] \times a2[3])
       ln[16] = spread[11_, 12_] := 1 - (11[1] \times 12[1] + 11[2] \times 12[2] - 11[3] \times 12[3])^2/
```

 $((11[1] \times 11[1] + 11[2] \times 11[2] - 11[3] \times 11[3]) \times$  $(12[1] \times 12[1] + 12[2] \times 12[2] - 12[3] \times 12[3])$ 

$$ln[17] = S_1 = spread[L_4, L_1]$$

$$S_2 = spread[L_1, L_2]$$

$$S_3 = spread[L_2, L_3]$$

$$S_4 = spread[L_3, L_4]$$

$$\text{Out[17]= } 1 - \frac{\left( \, 1_1 \, 1_4 + m_1 \, m_4 - n_1 \, n_4 \, \right)^{\, 2}}{\left( \, 1_1^2 + m_1^2 - n_1^2 \, \right) \, \, \left( \, 1_4^2 + m_4^2 - n_4^2 \, \right)}$$

$$\text{Out[18]=} \ \ 1 - \frac{\left(\, 1_1 \,\, 1_2 \,+\, m_1 \,\, m_2 \,-\, n_1 \,\, n_2\,\right)^{\,2}}{\left(\, 1_1^2 \,+\, m_1^2 \,-\, n_1^2\,\right) \,\, \left(\, 1_2^2 \,+\, m_2^2 \,-\, n_2^2\,\right)}$$

$$\text{Out[19]= } 1 - \frac{\left(1_2 \, 1_3 + m_2 \, m_3 - n_2 \, n_3\right)^2}{\left(1_2^2 + m_2^2 - n_2^2\right) \, \left(1_3^2 + m_3^2 - n_3^2\right)}$$

$$\text{Out}[\text{20}] = \ 1 - \frac{\left( \, 1_3 \, \, 1_4 \, + \, m_3 \, \, m_4 \, - \, n_3 \, \, n_4 \, \right) \, ^2}{\left( \, 1_3^2 \, + \, m_3^2 \, - \, n_3^2 \, \right) \, \, \left( \, 1_4^2 \, + \, m_4^2 \, - \, n_4^2 \, \right)}$$

$$ln[21]:= a_1 = M[L_4, L_1]$$

$$a_2 = M[L_1, L_2]$$

$$a_3 = M[L_2, L_3]$$

$$a_4 = M[L_3, L_4]$$

Out[21]= 
$$\{ m_4 n_1 - m_1 n_4, -l_4 n_1 + l_1 n_4, -l_4 m_1 + l_1 m_4 \}$$

$$\text{Out} [\text{23}] = \ \big\{ -\,\text{m}_3\,\, n_2 \,+\, \text{m}_2\,\, n_3\, , \,\, 1_3\,\, n_2 \,-\, 1_2\,\, n_3\, , \,\, 1_3\,\, \text{m}_2 \,-\, 1_2\,\, \text{m}_3\, \big\}$$

$$\text{Out}[24] = \left\{ -\text{ } m_{4} \text{ } n_{3} + \text{ } m_{3} \text{ } n_{4} \text{, } 1_{4} \text{ } n_{3} - 1_{3} \text{ } n_{4} \text{, } 1_{4} \text{ } m_{3} - 1_{3} \text{ } m_{4} \right\}$$

```
ln[25] = q_{12} = quadrance[a_1, a_2]
                       q_{23} = quadrance[a_2, a_3]
                       q_{34} = quadrance[a_3, a_4]
                      q_{41} = quadrance[a_4, a_1]
 \text{Out}[25] = \ 1 - \left( - \left( \ \left( \ l_2 \ m_1 - l_1 \ m_2 \right) \ \left( - \ l_4 \ m_1 + l_1 \ m_4 \right) \ \right) \ + \ l_4 \ m_4 \right) \ + \ l_4 \ m_4 \ ) \ + \ l_4 \ l_4 \ m_4 \ ) \ + \ l_4 \ l_
                                                (l_2 n_1 - l_1 n_2) (-l_4 n_1 + l_1 n_4) + (-m_2 n_1 + m_1 n_2) (m_4 n_1 - m_1 n_4))^2
                                 ((-(1_2 m_1 - 1_1 m_2)^2 + (1_2 n_1 - 1_1 n_2)^2 + (-m_2 n_1 + m_1 n_2)^2)
                                          \left(-\left(-1_{4} \, m_{1} + 1_{1} \, m_{4}\right)^{2} + \left(-1_{4} \, n_{1} + 1_{1} \, n_{4}\right)^{2} + \left(m_{4} \, n_{1} - m_{1} \, n_{4}\right)^{2}\right)\right)
Out[26]= 1 -
                             ((-(1_2 m_1 - 1_1 m_2)^2 + (1_2 n_1 - 1_1 n_2)^2 + (-m_2 n_1 + m_1 n_2)^2)
                                          \left(-\left(1_{3} \, m_{2} - 1_{2} \, m_{3}\right)^{2} + \left(1_{3} \, n_{2} - 1_{2} \, n_{3}\right)^{2} + \left(-m_{3} \, n_{2} + m_{2} \, n_{3}\right)^{2}\right)
Out[27]= 1 -
                             \left( \left( - \left( \mathbf{1}_3 \, \, \mathbf{m}_2 - \mathbf{1}_2 \, \, \mathbf{m}_3 \right)^{\, 2} + \, \left( \mathbf{1}_3 \, \, \mathbf{n}_2 - \mathbf{1}_2 \, \, \mathbf{n}_3 \right)^{\, 2} + \, \left( - \, \mathbf{m}_3 \, \, \mathbf{n}_2 + \, \mathbf{m}_2 \, \, \mathbf{n}_3 \right)^{\, 2} \right)
                                          \left(-\left(1_{4}\,m_{3}-1_{3}\,m_{4}\right)^{2}+\left(1_{4}\,n_{3}-1_{3}\,n_{4}\right)^{2}+\left(-m_{4}\,n_{3}+m_{3}\,n_{4}\right)^{2}\right)
Out[28]= 1 - (-((-1_4 m_1 + 1_1 m_4) (1_4 m_3 - 1_3 m_4)) +
                                               (-1_4 n_1 + 1_1 n_4) (1_4 n_3 - 1_3 n_4) + (m_4 n_1 - m_1 n_4) (-m_4 n_3 + m_3 n_4))^2
                                 ((-(-1_4 m_1 + 1_1 m_4)^2 + (-1_4 n_1 + 1_1 n_4)^2 + (m_4 n_1 - m_1 n_4)^2)
                                          \left(-\left(1_{4}\,\mathsf{m}_{3}-1_{3}\,\mathsf{m}_{4}\right)^{2}+\left(1_{4}\,\mathsf{n}_{3}-1_{3}\,\mathsf{n}_{4}\right)^{2}+\left(-\,\mathsf{m}_{4}\,\mathsf{n}_{3}+\,\mathsf{m}_{3}\,\mathsf{n}_{4}\right)^{2}\right)\right)
```

The next cell may take a long time to execute since the q's and the S's in the "ArchimedesFn" equation are being substituted with the x's, y's and z's in the q's and S's expressions above and the whole thing is being factored. A result of zero indicates that the criteria for compatible equations in the quadratic compatibility theorem has been met.

```
In[29]:= result = Factor[ArchimedesFn]
Out[29]= 0
```

We have proved that the following two quadratic equations in B are compatible.