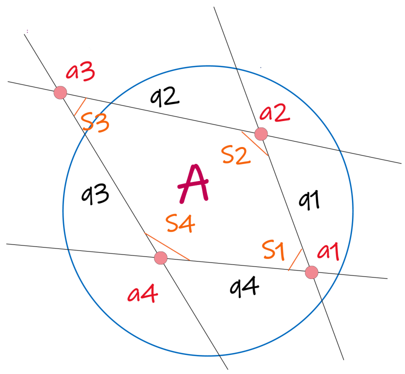
Universal Hyperbolic Geometry : Quadrea of Projective Quadrangle



$$(A - q1q2S2 - q3q4S4)^2 = 4q1q2q3q4(1-S1)(1-S3) \qquad (1)$$

$$(A - q1q4S1 - q2q3S3)^2 = 4q1q2q3q4(1-S2)(1-S4) \qquad (2)$$

$$A = \frac{(q1q2S2 + q3q4S4 + q1q4S1 + q2q3S3)}{2} - \frac{4(q1q2q3q4((1-S1)(1-S3)-(1-S2)(1-S4))}{2(q1q2S2 + q3q4S4 - q1q4S1 - q2q3S3)}$$

Quadratic Compatibility Theorem : Theorem 1 in the book Divine Proportions by Wildberger

The quadratic equations

$$(x - p1)^2 = r1$$

$$(x - p2)^2 = r2$$

are compatible precisely when

$$((p1 - p2)^2 - (r1 + r2))^2 = 4 r1 r2$$

In this case, if p1 is not equal to p2 then there is a unique common solution

$$x = \frac{p1 + p2}{2} - \frac{(r1 - r2)}{2(p1 - p2)}$$

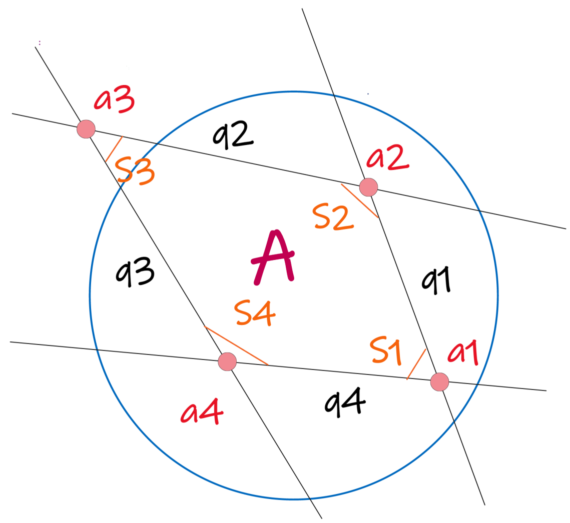
Create a Mathematica function for unique common solution of a pair of compatible quadratic equations.

In[∘]:=
```
X[p1_, r1_, p2_, r2_] := ((p1 + p2) / 2) - (r1 - r2) / (2 (p1 - p2))
```

Substitute values for p1, r1, p2 and r2 from quadratic equations (1) and (2) above to compute the qadrea A .

In[∘]:=
```
A = X[q1 q2 S2 + q3 q4 S4, 4 q1 q2 q3 q4 (1 - S1) × (1 - S3),
  q1 q4 S1 + q2 q3 S3, 4 q1 q2 q3 q4 (1 - S2) × (1 - S4)]
```

Out[∘]=
$$-\frac{4 q_1 q_2 q_3 q_4 (1 - S_1) \times (1 - S_3) - 4 q_1 q_2 q_3 q_4 (1 - S_2) \times (1 - S_4)}{2 (-q_1 q_4 S_1 + q_1 q_2 S_2 - q_2 q_3 S_3 + q_3 q_4 S_4)} + \frac{1}{2} (q_1 q_4 S_1 + q_1 q_2 S_2 + q_2 q_3 S_3 + q_3 q_4 S_4)$$



Quadrea of Projective Quadrangle Theorem

Suppose a1, a2, a3 and a4 are distinct projective points with quadrances q1 = q(a1, a2), q2 = q(a2, a3), q3 = q(a3, a4), q4 = q(a4, a1) and spreads S1 = S(a4a1, a1a2), S2 = S(a1a2, a2a3), S3 = S(a2a3, a3a4) and S4 = S(a3a4, a4a1) and quadrea A. Then the quadrea of the projective quadrangle is the common solution to the following pair of quadratic equations

$$(A - q1q2S2 - q3q4S4)^2 = 4q1q2q3q4(1-S1)(1-S3) \qquad (1)$$

$$(A - q1q4S1 - q2q3S3)^2 = 4q1q2q3q4(1-S2)(1-S4) \qquad (2)$$

The proof employs the quadratic compatibility theorem. We will show that equations (1) and (2) above are compatible since they meet the criteria for being compatible from the quadratic compatibility theorem.

Perform compatibility check:

```
In[◦]:= CompatibilityCheckLHS [p1_, r1_, p2_, r2_] := ((p1 - p2)^2 - (r1 + r2))^2
```

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In[◦]:= CompatibilityCheckRHS [p1_, r1_, p2_, r2_] := 4 * r1 * r2
```

In[∘]:= $$\texttt{lhs = CompatibilityCheckLHS}[q_1 q_2 S_2 + q_3 q_4 S_4,\ 4 q_1 q_2 q_3 q_4 (1 - S_1) \times (1 - S_3),\ q_1 q_4 S_1 + q_2 q_3 S_3,\ 4 q_1 q_2 q_3 q_4 (1 - S_2) \times (1 - S_4)]$$

Out[∘]= $$\left(-4 q_1 q_2 q_3 q_4 (1 - S_1) \times (1 - S_3) - 4 q_1 q_2 q_3 q_4 (1 - S_2) \times (1 - S_4) + (-q_1 q_4 S_1 + q_1 q_2 S_2 - q_2 q_3 S_3 + q_3 q_4 S_4)^2\right)^2$$

In[∘]:= $$\texttt{rhs = CompatibilityCheckRHS}[q_1 q_2 S_2 + q_3 q_4 S_4,\ 4 q_1 q_2 q_3 q_4 (1 - S_1) \times (1 - S_3),\ q_1 q_4 S_1 + q_2 q_3 S_3,\ 4 q_1 q_2 q_3 q_4 (1 - S_2) \times (1 - S_4)]$$

Out[∘]= $$64 q_1^2 q_2^2 q_3^2 q_4^2 (1 - S_1) \times (1 - S_2) \times (1 - S_3) \times (1 - S_4)$$

In[∘]:= $$\texttt{lhse = Expand[lhs]}$$

Out[∘]= $$\begin{aligned}
&64 q_1^2 q_2^2 q_3^2 q_4^2 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 + 16 q_1^2 q_2^2 q_3^2 q_4^2 S_1^2 - 16 q_1^3 q_2 q_3 q_4^3 S_1^2 + 8 q_1^3 q_2 q_3 q_4^3 S_1^3 + q_1^4 q_4^4 S_1^4 - \\
&64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 + 32 q_1^3 q_2^2 q_3 q_4^2 S_1 S_2 + 32 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 - 16 q_1^3 q_2^2 q_3 q_4^2 S_1^2 S_2 + 8 q_1^3 q_2 q_3 q_4^3 S_1^2 S_2 - \\
&4 q_1^4 q_2 q_4^3 S_1^3 S_2 - 16 q_1^3 q_2^3 q_3 q_4 S_2^2 + 16 q_1^2 q_2^2 q_3^2 q_4^2 S_2^2 + 8 q_1^3 q_2^3 q_3 q_4 S_1 S_2^2 - 16 q_1^3 q_2^2 q_3 q_4^2 S_1 S_2^2 + \\
&6 q_1^4 q_2^2 q_4^2 S_1^2 S_2^2 + 8 q_1^3 q_2^3 q_3 q_4 S_2^3 - 4 q_1^4 q_2^3 q_4 S_1 S_2^3 + q_1^4 q_2^4 S_2^4 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_3 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_3 - \\
&16 q_1^2 q_2^2 q_3^2 q_4^2 S_1^2 S_3 + 8 q_1^3 q_2 q_3 q_4^3 S_1^2 S_3 - 4 q_1^3 q_2 q_3 q_4^3 S_1^3 S_3 + 32 q_1^2 q_2^3 q_3^2 q_4 S_2 S_3 + 32 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_3 - \\
&16 q_1^2 q_2^3 q_3^2 q_4 S_1 S_2 S_3 - 16 q_1^3 q_2^2 q_3 q_4^2 S_1 S_2 S_3 - 16 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_3 + 4 q_1^3 q_2^2 q_3 q_4^2 S_1^2 S_2 S_3 + \\
&8 q_1^3 q_2^3 q_3 q_4 S_2^2 S_3 - 16 q_1^2 q_2^3 q_3^2 q_4 S_2^2 S_3 + 4 q_1^3 q_2^3 q_3 q_4 S_1 S_2^2 S_3 - 4 q_1^3 q_2^4 q_3 S_2^3 S_3 - 16 q_1 q_2^3 q_3^3 q_4 S_3^2 + \\
&16 q_1^2 q_2^2 q_3^2 q_4^2 S_3^2 + 8 q_1 q_2^3 q_3^3 q_4 S_1 S_3^2 - 16 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_3^2 + 6 q_1^2 q_2^2 q_3^2 q_4^2 S_1^2 S_3^2 - 16 q_1^2 q_2^3 q_3^2 q_4 S_2 S_3^2 + \\
&8 q_1 q_2^3 q_3^3 q_4 S_2 S_3^2 + 4 q_1^2 q_2^3 q_3^2 q_4 S_1 S_2 S_3^2 + 6 q_1^2 q_2^4 q_3^2 S_2^2 S_3^2 + 8 q_1 q_2^3 q_3^3 q_4 S_3^3 - 4 q_1 q_2^3 q_3^3 q_4 S_1 S_3^3 - \\
&4 q_1 q_2^4 q_3^3 S_2 S_3^3 + q_2^4 q_3^4 S_3^4 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_4 + 32 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_4 + 32 q_1^2 q_2 q_3^2 q_4^3 S_1 S_4 + \\
&8 q_1^3 q_2 q_3 q_4^3 S_1^2 S_4 - 16 q_1^2 q_2 q_3^2 q_4^3 S_1^2 S_4 - 4 q_1^3 q_3 q_4^4 S_1^3 S_4 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_4 - 16 q_1^3 q_2^2 q_3 q_4^2 S_1 S_2 S_4 - \\
&16 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_4 - 16 q_1^2 q_2 q_3^2 q_4^3 S_1 S_2 S_4 + 4 q_1^3 q_2 q_3 q_4^3 S_1^2 S_2 S_4 + 8 q_1^3 q_2^3 q_3 q_4 S_2^2 S_4 - \\
&16 q_1^2 q_2^2 q_3^2 q_4^2 S_2^2 S_4 + 4 q_1^3 q_2^2 q_3 q_4^2 S_1 S_2^2 S_4 - 4 q_1^3 q_2^3 q_3 q_4 S_2^3 S_4 + 32 q_1^2 q_2^2 q_3^2 q_4^2 S_3 S_4 + 32 q_1 q_2^2 q_3^3 q_4^2 S_3 S_4 - \\
&16 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_3 S_4 - 16 q_1 q_2^2 q_3^3 q_4^2 S_1 S_3 S_4 - 16 q_1^2 q_2 q_3^2 q_4^3 S_1 S_3 S_4 + 4 q_1^2 q_2 q_3^2 q_4^3 S_1^2 S_3 S_4 - \\
&16 q_1^2 q_2^3 q_3^2 q_4 S_2 S_3 S_4 - 16 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_3 S_4 - 16 q_1 q_2^2 q_3^3 q_4^2 S_2 S_3 S_4 + 24 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_3 S_4 + \\
&4 q_1^2 q_2^3 q_3^2 q_4 S_2^2 S_3 S_4 + 8 q_1 q_2^3 q_3^3 q_4 S_3^2 S_4 - 16 q_1 q_2^2 q_3^3 q_4^2 S_3^2 S_4 + 4 q_1 q_2^2 q_3^3 q_4^2 S_1 S_3^2 S_4 + \\
&4 q_1 q_2^3 q_3^3 q_4 S_2 S_3^2 S_4 - 4 q_2^3 q_3^4 q_4 S_3^3 S_4 + 16 q_1^2 q_2^2 q_3^2 q_4^2 S_4^2 - 16 q_1 q_2 q_3^3 q_4^3 S_4^2 - 16 q_1^2 q_2 q_3^2 q_4^3 S_1 S_4^2 + \\
&8 q_1 q_2 q_3^3 q_4^3 S_1 S_4^2 + 6 q_1^2 q_3^2 q_4^4 S_1^2 S_4^2 - 16 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_4^2 + 8 q_1 q_2 q_3^3 q_4^3 S_2 S_4^2 + 4 q_1^2 q_2 q_3^2 q_4^3 S_1 S_2 S_4^2 + \\
&6 q_1^2 q_2^2 q_3^2 q_4^2 S_2^2 S_4^2 - 16 q_1 q_2^2 q_3^3 q_4^2 S_3 S_4^2 + 8 q_1 q_2 q_3^3 q_4^3 S_3 S_4^2 + 4 q_1 q_2 q_3^3 q_4^3 S_1 S_3 S_4^2 + 4 q_1 q_2^2 q_3^3 q_4^2 S_2 S_3 S_4^2 + \\
&6 q_2^2 q_3^4 q_4^2 S_3^2 S_4^2 + 8 q_1 q_2 q_3^3 q_4^3 S_4^3 - 4 q_1 q_3^3 q_4^4 S_1 S_4^3 - 4 q_1 q_2 q_3^3 q_4^3 S_2 S_4^3 - 4 q_2 q_3^4 q_4^3 S_3 S_4^3 + q_3^4 q_4^4 S_4^4
\end{aligned}$$

In[∘]:= $$\texttt{rhse = Expand[rhs]}$$

Out[∘]= $$\begin{aligned}
&64 q_1^2 q_2^2 q_3^2 q_4^2 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 - \\
&64 q_1^2 q_2^2 q_3^2 q_4^2 S_3 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_3 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_3 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_3 - \\
&64 q_1^2 q_2^2 q_3^2 q_4^2 S_4 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_4 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_4 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_4 + \\
&64 q_1^2 q_2^2 q_3^2 q_4^2 S_3 S_4 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_3 S_4 - 64 q_1^2 q_2^2 q_3^2 q_4^2 S_2 S_3 S_4 + 64 q_1^2 q_2^2 q_3^2 q_4^2 S_1 S_2 S_3 S_4
\end{aligned}$$

In[∘]:= `res = lhse - rhse`

Out[∘]=

$$\begin{gathered}16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1^2 - 16\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^2 + 8\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^3 + q_1^4\, q_4^4\, S_1^4 + 32\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1\, S_2 - 32\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_2 - \\
16\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1^2\, S_2 + 8\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^2\, S_2 - 4\, q_1^4\, q_2\, q_4^3\, S_1^3\, S_2 - 16\, q_1^3\, q_2^3\, q_3\, q_4\, S_2^2 + 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2^2 + \\
8\, q_1^3\, q_2^3\, q_3\, q_4\, S_1\, S_2^2 - 16\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1\, S_2^2 + 6\, q_1^4\, q_2^2\, q_4^2\, S_1^2\, S_2^2 + 8\, q_1^3\, q_2^3\, q_3\, q_4\, S_2^3 - 4\, q_1^4\, q_2^3\, q_4\, S_1\, S_2^3 + \\
q_1^4\, q_2^4\, S_2^4 - 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1^2\, S_3 + 8\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^2\, S_3 - 4\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^3\, S_3 + 32\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_2\, S_3 - \\
32\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2\, S_3 - 16\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_1\, S_2\, S_3 - 16\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1\, S_2\, S_3 + 48\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_2\, S_3 + \\
4\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1^2\, S_2\, S_3 + 8\, q_1^3\, q_2^3\, q_3\, q_4\, S_2^2\, S_3 - 16\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_2^2\, S_3 + 4\, q_1^3\, q_2^3\, q_3\, q_4\, S_1\, S_2^2\, S_3 - 4\, q_1^3\, q_2^4\, q_3\, S_2^3\, S_3 - \\
16\, q_1\, q_2^3\, q_3^3\, q_4\, S_3^2 + 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_3^2 + 8\, q_1\, q_2^3\, q_3^3\, q_4\, S_1\, S_3^2 - 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_3^2 + 6\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1^2\, S_3^2 - \\
16\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_2\, S_3^2 + 8\, q_1\, q_2^3\, q_3^3\, q_4\, S_2\, S_3^2 + 4\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_1\, S_2\, S_3^2 + 6\, q_1^2\, q_2^4\, q_3^2\, S_2^2\, S_3^2 + 8\, q_1\, q_2^3\, q_3^3\, q_4\, S_3^3 - \\
4\, q_1\, q_2^3\, q_3^3\, q_4\, S_1\, S_3^3 - 4\, q_1\, q_2^4\, q_3^3\, S_2\, S_3^3 + q_2^4\, q_3^4\, S_3^4 - 32\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_4 + 32\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1\, S_4 + \\
8\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^2\, S_4 - 16\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1^2\, S_4 - 4\, q_1^3\, q_3\, q_4^4\, S_1^3\, S_4 - 16\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1\, S_2\, S_4 + 48\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_2\, S_4 - \\
16\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1\, S_2\, S_4 + 4\, q_1^3\, q_2\, q_3\, q_4^3\, S_1^2\, S_2\, S_4 + 8\, q_1^3\, q_2^3\, q_3\, q_4\, S_2^2\, S_4 - 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2^2\, S_4 + \\
4\, q_1^3\, q_2^2\, q_3\, q_4^2\, S_1\, S_2^2\, S_4 - 4\, q_1^3\, q_2^3\, q_3\, q_4\, S_2^3\, S_4 - 32\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_3\, S_4 + 32\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_3\, S_4 + \\
48\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_3\, S_4 - 16\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_1\, S_3\, S_4 - 16\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1\, S_3\, S_4 + 4\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1^2\, S_3\, S_4 - \\
16\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_2\, S_3\, S_4 + 48\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2\, S_3\, S_4 - 16\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_2\, S_3\, S_4 - 40\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_1\, S_2\, S_3\, S_4 + \\
4\, q_1^2\, q_2^3\, q_3^2\, q_4\, S_2^2\, S_3\, S_4 + 8\, q_1\, q_2^3\, q_3^3\, q_4\, S_3^2\, S_4 - 16\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_3^2\, S_4 + 4\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_1\, S_3^2\, S_4 + \\
4\, q_1\, q_2^3\, q_3^3\, q_4\, S_2\, S_3^2\, S_4 - 4\, q_2^3\, q_3^4\, q_4\, S_3^3\, S_4 + 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_4^2 - 16\, q_1\, q_2\, q_3^3\, q_4^3\, S_4^2 - 16\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1\, S_4^2 + \\
8\, q_1\, q_2\, q_3^3\, q_4^3\, S_1\, S_4^2 + 6\, q_1^2\, q_3^2\, q_4^4\, S_1^2\, S_4^2 - 16\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2\, S_4^2 + 8\, q_1\, q_2\, q_3^3\, q_4^3\, S_2\, S_4^2 + 4\, q_1^2\, q_2\, q_3^2\, q_4^3\, S_1\, S_2\, S_4^2 + \\
6\, q_1^2\, q_2^2\, q_3^2\, q_4^2\, S_2^2\, S_4^2 - 16\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_3\, S_4^2 + 8\, q_1\, q_2\, q_3^3\, q_4^3\, S_3\, S_4^2 + 4\, q_1\, q_2\, q_3^3\, q_4^3\, S_1\, S_3\, S_4^2 + 4\, q_1\, q_2^2\, q_3^3\, q_4^2\, S_2\, S_3\, S_4^2 + \\
6\, q_2^2\, q_3^4\, q_4^2\, S_3^2\, S_4^2 + 8\, q_1\, q_2\, q_3^3\, q_4^3\, S_4^3 - 4\, q_1\, q_3^3\, q_4^4\, S_1\, S_4^3 - 4\, q_1\, q_2\, q_3^3\, q_4^3\, S_2\, S_4^3 - 4\, q_2\, q_3^4\, q_4^3\, S_3\, S_4^3 + q_3^4\, q_4^4\, S_4^4\end{gathered}$$

In[∘]:=

$a_1 = \{x_1, y_1, z_1\}$
$a_2 = \{x_2, y_2, z_2\}$
$a_3 = \{x_3, y_3, z_3\}$
$a_4 = \{x_4, y_4, z_4\}$

Out[∘]= $\{x_1, y_1, z_1\}$

Out[∘]= $\{x_2, y_2, z_2\}$

Out[∘]= $\{x_3, y_3, z_3\}$

Out[∘]= $\{x_4, y_4, z_4\}$

In[∘]:=

```
J[a1_, a2_] :=
  {a1⟦2⟧ × a2⟦3⟧ - a1⟦3⟧ × a2⟦2⟧, a1⟦3⟧ × a2⟦1⟧ - a1⟦1⟧ × a2⟦3⟧, a1⟦2⟧ × a2⟦1⟧ - a1⟦1⟧ × a2⟦2⟧}
```

In[∘]:=

```
quadrance[a1_, a2_] := 1 - (a1⟦1⟧ × a2⟦1⟧ + a1⟦2⟧ × a2⟦2⟧ - a1⟦3⟧ × a2⟦3⟧)^2 /
    ((a1⟦1⟧ × a1⟦1⟧ + a1⟦2⟧ × a1⟦2⟧ - a1⟦3⟧ × a1⟦3⟧) ×
      (a2⟦1⟧ × a2⟦1⟧ + a2⟦2⟧ × a2⟦2⟧ - a2⟦3⟧ × a2⟦3⟧))
```

In[∘]:=

```
spread[l1_, l2_] := 1 - (l1⟦1⟧ × l2⟦1⟧ + l1⟦2⟧ × l2⟦2⟧ - l1⟦3⟧ × l2⟦3⟧)^2 /
    ((l1⟦1⟧ × l1⟦1⟧ + l1⟦2⟧ × l1⟦2⟧ - l1⟦3⟧ × l1⟦3⟧) ×
      (l2⟦1⟧ × l2⟦1⟧ + l2⟦2⟧ × l2⟦2⟧ - l2⟦3⟧ × l2⟦3⟧))
```

In[ ]:=
```
q1 = quadrance[a1, a2]
q2 = quadrance[a2, a3]
q3 = quadrance[a3, a4]
q4 = quadrance[a4, a1]
```

Out[ ]= $1-\frac{(x_1 x_2+y_1 y_2-z_1 z_2)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)}$

Out[ ]= $1-\frac{(x_2 x_3+y_2 y_3-z_2 z_3)^2}{\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}$

Out[ ]= $1-\frac{(x_3 x_4+y_3 y_4-z_3 z_4)^2}{\left(x_3^2+y_3^2-z_3^2\right)\left(x_4^2+y_4^2-z_4^2\right)}$

Out[ ]= $1-\frac{(x_1 x_4+y_1 y_4-z_1 z_4)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_4^2+y_4^2-z_4^2\right)}$

In[ ]:=
```
L1 = J[a1, a2]
L2 = J[a2, a3]
L3 = J[a3, a4]
L4 = J[a4, a1]
```

Out[ ]= $\{-y_2 z_1+y_1 z_2,\ x_2 z_1-x_1 z_2,\ x_2 y_1-x_1 y_2\}$

Out[ ]= $\{-y_3 z_2+y_2 z_3,\ x_3 z_2-x_2 z_3,\ x_3 y_2-x_2 y_3\}$

Out[ ]= $\{-y_4 z_3+y_3 z_4,\ x_4 z_3-x_3 z_4,\ x_4 y_3-x_3 y_4\}$

Out[ ]= $\{y_4 z_1-y_1 z_4,\ -x_4 z_1+x_1 z_4,\ -x_4 y_1+x_1 y_4\}$

In[∘]:=
```
S1 = spread[L4, L1]
S2 = spread[L1, L2]
S3 = spread[L2, L3]
S4 = spread[L3, L4]
```

Out[∘]= $1-\left(-\left(\left(x_2 y_1-x_1 y_2\right)\left(-x_4 y_1+x_1 y_4\right)\right)+\left(x_2 z_1-x_1 z_2\right)\left(-x_4 z_1+x_1 z_4\right)+\left(-y_2 z_1+y_1 z_2\right)\left(y_4 z_1-y_1 z_4\right)\right)^2 \Big/ \left(\left(-\left(x_2 y_1-x_1 y_2\right)^2+\left(x_2 z_1-x_1 z_2\right)^2+\left(-y_2 z_1+y_1 z_2\right)^2\right)\left(-\left(-x_4 y_1+x_1 y_4\right)^2+\left(-x_4 z_1+x_1 z_4\right)^2+\left(y_4 z_1-y_1 z_4\right)^2\right)\right)$

Out[∘]= $1-\left(-\left(\left(x_2 y_1-x_1 y_2\right)\left(x_3 y_2-x_2 y_3\right)\right)+\left(x_2 z_1-x_1 z_2\right)\left(x_3 z_2-x_2 z_3\right)+\left(-y_2 z_1+y_1 z_2\right)\left(-y_3 z_2+y_2 z_3\right)\right)^2 \Big/ \left(\left(-\left(x_2 y_1-x_1 y_2\right)^2+\left(x_2 z_1-x_1 z_2\right)^2+\left(-y_2 z_1+y_1 z_2\right)^2\right)\left(-\left(x_3 y_2-x_2 y_3\right)^2+\left(x_3 z_2-x_2 z_3\right)^2+\left(-y_3 z_2+y_2 z_3\right)^2\right)\right)$

Out[∘]= $1-\left(-\left(\left(x_3 y_2-x_2 y_3\right)\left(x_4 y_3-x_3 y_4\right)\right)+\left(x_3 z_2-x_2 z_3\right)\left(x_4 z_3-x_3 z_4\right)+\left(-y_3 z_2+y_2 z_3\right)\left(-y_4 z_3+y_3 z_4\right)\right)^2 \Big/ \left(\left(-\left(x_3 y_2-x_2 y_3\right)^2+\left(x_3 z_2-x_2 z_3\right)^2+\left(-y_3 z_2+y_2 z_3\right)^2\right)\left(-\left(x_4 y_3-x_3 y_4\right)^2+\left(x_4 z_3-x_3 z_4\right)^2+\left(-y_4 z_3+y_3 z_4\right)^2\right)\right)$

Out[∘]= $1-\left(-\left(\left(-x_4 y_1+x_1 y_4\right)\left(x_4 y_3-x_3 y_4\right)\right)+\left(-x_4 z_1+x_1 z_4\right)\left(x_4 z_3-x_3 z_4\right)+\left(y_4 z_1-y_1 z_4\right)\left(-y_4 z_3+y_3 z_4\right)\right)^2 \Big/ \left(\left(-\left(-x_4 y_1+x_1 y_4\right)^2+\left(-x_4 z_1+x_1 z_4\right)^2+\left(y_4 z_1-y_1 z_4\right)^2\right)\left(-\left(x_4 y_3-x_3 y_4\right)^2+\left(x_4 z_3-x_3 z_4\right)^2+\left(-y_4 z_3+y_3 z_4\right)^2\right)\right)$

The next cell may take a long time to execute since the q’s and the S’s in the “res” equation are being substituted with the x’s, y’s and z’s in the q’s and S’s expressions above and the whole thing is being factored. A result of zero indicates that the criteria for compatible equations in the quadratic compatibility theorem has been met.

In[∘]:=
```
result = Factor[res]
```

Out[∘]= 0

We have proved that the following two quadratic equations in A are compatible.

$$(A - q1q2S2 - q3q4S4)^2 = 4q1q2q3q4(1\text{-}S1)(1\text{-}S3) \qquad (1)$$

$$(A - q1q4S1 - q2q3S3)^2 = 4q1q2q3q4(1\text{-}S2)(1\text{-}S4) \qquad (2)$$