Quadratic Compatibility Theorem : Theorem 1 in the book Divine Proportions by Wildberger

The quadratic equations

$$(X - p1)^2 = r1$$

$$(X - p2)^2 = r2$$

are compatible precisely when

$$((p1 - p2)^2 - (r1 + r2))^2 = 4r1r2$$

In this case, if p1 is not equal to p2 then there is a unique common solution

$$X = \frac{p1 + p2}{2} - \frac{(r1 - r2)}{2(p1 - p2)}$$

Create a Mathematica function for unique common solution of a pair of compatible quadratic equations.

In[1]:=
```
X[p1_, r1_, p2_, r2_] := ((p1 + p2) / 2) - (r1 - r2) / (2 (p1 - p2))
```

Substitute values for p1, r1, p2 and r2 from quadratic equations (1) and (2) above to compute the quadreal of quadrilateral.

In[2]:=
```
quadreal = X[S₁ S₂ q₁₂ + S₃ S₄ q₃₄, 4 S₁ S₂ S₃ S₄ (1 - q₂₃) × (1 - q₄₁),
  S₄ S₁ q₄₁ + S₂ S₃ q₂₃, 4 S₁ S₂ S₃ S₄ (1 - q₁₂) × (1 - q₃₄)]
```

Out[2]=

$$\frac{1}{2}\left(q_{12} S_1 S_2 + q_{23} S_2 S_3 + q_{41} S_1 S_4 + q_{34} S_3 S_4\right) - \frac{-4 \times (1 - q_{12}) \times (1 - q_{34})\, S_1 S_2 S_3 S_4 + 4 \times (1 - q_{23}) \times (1 - q_{41})\, S_1 S_2 S_3 S_4}{2\left(q_{12} S_1 S_2 - q_{23} S_2 S_3 - q_{41} S_1 S_4 + q_{34} S_3 S_4\right)}$$

The proof employs the quadratic compatibility theorem. We will show that equations (1) and (2) above are compatible since they meet the criteria for being compatible from the quadratic compatibility theorem.

Perform compatibility check:

In[3]:=
```
CompatibilityCheckLHS[p1_, r1_, p2_, r2_] := ((p1 - p2)^2 - (r1 + r2))^2
```

In[4]:=
```
CompatibilityCheckRHS[p1_, r1_, p2_, r2_] := 4 * r1 * r2
```

In[5]:=
```
lhs = CompatibilityCheckLHS[S1 S2 q12 + S3 S4 q34,
  4 S1 S2 S3 S4 (1 - q23) × (1 - q41), S4 S1 q41 + S2 S3 q23, 4 S1 S2 S3 S4 (1 - q12) × (1 - q34)]
```

Out[5]=
$$\begin{aligned}&\big(-4\times(1-q_{12})\times(1-q_{34})\,S_1\,S_2\,S_3\,S_4-\\&\quad 4\times(1-q_{23})\times(1-q_{41})\,S_1\,S_2\,S_3\,S_4+(q_{12}\,S_1\,S_2-q_{23}\,S_2\,S_3-q_{41}\,S_1\,S_4+q_{34}\,S_3\,S_4)^2\big)^2\end{aligned}$$

In[6]:=
```
rhs = CompatibilityCheckRHS[S1 S2 q12 + S3 S4 q34,
  4 S1 S2 S3 S4 (1 - q23) × (1 - q41), S4 S1 q41 + S2 S3 q23, 4 S1 S2 S3 S4 (1 - q12) × (1 - q34)]
```

Out[6]=
$$64\times(1-q_{12})\times(1-q_{23})\times(1-q_{34})\times(1-q_{41})\,S_1^2\,S_2^2\,S_3^2\,S_4^2$$

In[7]:=
```
lhse = Expand[lhs]
```

Out[7]=
$$\begin{aligned}&q_{12}^4\,S_1^4\,S_2^4-4\,q_{12}^3\,q_{23}\,S_1^3\,S_2^4\,S_3+6\,q_{12}^2\,q_{23}^2\,S_1^2\,S_2^4\,S_3^2-4\,q_{12}\,q_{23}^3\,S_1\,S_2^4\,S_3^3+q_{23}^4\,S_2^4\,S_3^4-4\,q_{12}^3\,q_{41}\,S_1^4\,S_2^3\,S_4-\\
&16\,q_{12}^2\,S_1^3\,S_2^3\,S_3\,S_4+8\,q_{12}^3\,S_1^3\,S_2^3\,S_3\,S_4+8\,q_{12}^2\,q_{23}\,S_1^3\,S_2^3\,S_3\,S_4+8\,q_{12}^2\,q_{34}\,S_1^3\,S_2^3\,S_3\,S_4-4\,q_{12}^3\,q_{34}\,S_1^3\,S_2^3\,S_3\,S_4+\\
&8\,q_{12}^2\,q_{41}\,S_1^3\,S_2^3\,S_3\,S_4+4\,q_{12}^2\,q_{23}\,q_{41}\,S_1^3\,S_2^3\,S_3\,S_4+32\,q_{12}\,q_{23}\,S_1^2\,S_2^3\,S_3^2\,S_4-16\,q_{12}^2\,q_{23}\,S_1^2\,S_2^3\,S_3^2\,S_4-\\
&16\,q_{12}\,q_{23}^2\,S_1^2\,S_2^3\,S_3^2\,S_4-16\,q_{12}\,q_{23}\,q_{34}\,S_1^2\,S_2^3\,S_3^2\,S_4+4\,q_{12}^2\,q_{23}\,q_{34}\,S_1^2\,S_2^3\,S_3^2\,S_4-16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4+\\
&4\,q_{12}\,q_{23}^2\,q_{41}\,S_1^2\,S_2^3\,S_3^2\,S_4-16\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4+8\,q_{12}\,q_{23}^2\,S_1\,S_2^3\,S_3^3\,S_4+8\,q_{23}^3\,S_1\,S_2^3\,S_3^3\,S_4+8\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4+\\
&4\,q_{12}\,q_{23}^2\,q_{34}\,S_1\,S_2^3\,S_3^3\,S_4+8\,q_{23}^2\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4-4\,q_{23}^3\,q_{41}\,S_1\,S_2^3\,S_3^3\,S_4-4\,q_{23}^3\,q_{34}\,S_2^3\,S_3^4\,S_4+\\
&6\,q_{12}^2\,q_{41}^2\,S_1^4\,S_2^2\,S_4^2+32\,q_{12}\,q_{41}\,S_1^3\,S_2^2\,S_3\,S_4^2-16\,q_{12}^2\,q_{41}\,S_1^3\,S_2^2\,S_3\,S_4^2-16\,q_{12}\,q_{23}\,q_{41}\,S_1^3\,S_2^2\,S_3\,S_4^2-\\
&16\,q_{12}\,q_{34}\,q_{41}\,S_1^3\,S_2^2\,S_3\,S_4^2+4\,q_{12}^2\,q_{34}\,q_{41}\,S_1^3\,S_2^2\,S_3\,S_4^2-16\,q_{12}\,q_{41}^2\,S_1^3\,S_2^2\,S_3\,S_4^2+4\,q_{12}\,q_{23}\,q_{41}^2\,S_1^3\,S_2^2\,S_3\,S_4^2+\\
&64\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+16\,q_{12}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+32\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+\\
&16\,q_{23}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{12}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{12}^2\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+32\,q_{23}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-\\
&16\,q_{12}\,q_{23}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+16\,q_{34}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{12}\,q_{34}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2+6\,q_{12}^2\,q_{34}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2-\\
&64\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+32\,q_{12}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-\\
&16\,q_{23}^2\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+32\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{12}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+\\
&24\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+16\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2-16\,q_{23}\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2+6\,q_{23}^2\,q_{41}^2\,S_1^2\,S_2^2\,S_3^2\,S_4^2+\\
&32\,q_{23}\,q_{34}\,S_1\,S_2^2\,S_3^3\,S_4^2-16\,q_{12}\,q_{23}\,q_{34}\,S_1\,S_2^2\,S_3^3\,S_4^2-16\,q_{23}^2\,q_{34}\,S_1\,S_2^2\,S_3^3\,S_4^2-16\,q_{23}\,q_{34}^2\,S_1\,S_2^2\,S_3^3\,S_4^2+\\
&4\,q_{12}\,q_{23}\,q_{34}^2\,S_1\,S_2^2\,S_3^3\,S_4^2-16\,q_{23}\,q_{34}\,q_{41}\,S_1\,S_2^2\,S_3^3\,S_4^2+4\,q_{23}^2\,q_{34}\,q_{41}\,S_1\,S_2^2\,S_3^3\,S_4^2+6\,q_{23}^2\,q_{34}^2\,S_2^2\,S_3^4\,S_4^2-\\
&4\,q_{12}\,q_{41}^3\,S_1^4\,S_2\,S_4^3-16\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3+8\,q_{12}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3+8\,q_{23}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3+8\,q_{34}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3+\\
&4\,q_{12}\,q_{34}\,q_{41}^2\,S_1^3\,S_2\,S_3\,S_4^3+8\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3-4\,q_{23}\,q_{41}^3\,S_1^3\,S_2\,S_3\,S_4^3+32\,q_{34}\,q_{41}\,S_1^2\,S_2\,S_3^2\,S_4^3-\\
&16\,q_{12}\,q_{34}\,q_{41}\,S_1^2\,S_2\,S_3^2\,S_4^3-16\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2\,S_3^2\,S_4^3-16\,q_{34}^2\,q_{41}\,S_1^2\,S_2\,S_3^2\,S_4^3+4\,q_{12}\,q_{34}^2\,q_{41}\,S_1^2\,S_2\,S_3^2\,S_4^3-\\
&16\,q_{34}\,q_{41}^2\,S_1^2\,S_2\,S_3^2\,S_4^3+4\,q_{23}\,q_{34}\,q_{41}^2\,S_1^2\,S_2\,S_3^2\,S_4^3-16\,q_{34}^2\,S_1\,S_2\,S_3^3\,S_4^3+8\,q_{12}\,q_{34}^2\,S_1\,S_2\,S_3^3\,S_4^3+\\
&8\,q_{23}\,q_{34}^2\,S_1\,S_2\,S_3^3\,S_4^3+8\,q_{34}^3\,S_1\,S_2\,S_3^3\,S_4^3-4\,q_{12}\,q_{34}^3\,S_1\,S_2\,S_3^3\,S_4^3+8\,q_{34}^2\,q_{41}\,S_1\,S_2\,S_3^3\,S_4^3+4\,q_{23}\,q_{34}^2\,q_{41}\,S_1\,S_2\,S_3^3\,S_4^3-\\
&4\,q_{23}\,q_{34}^3\,S_2\,S_3^4\,S_4^3+q_{41}^4\,S_1^4\,S_4^4-4\,q_{34}\,q_{41}^3\,S_1^3\,S_3\,S_4^4+6\,q_{34}^2\,q_{41}^2\,S_1^2\,S_3^2\,S_4^4-4\,q_{34}^3\,q_{41}\,S_1\,S_3^3\,S_4^4+q_{34}^4\,S_3^4\,S_4^4\end{aligned}$$

In[8]:=
```
rhse = Expand[rhs]
```

Out[8]=
$$\begin{aligned}&64\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{12}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{12}\,q_{23}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+\\
&64\,q_{12}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{23}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{12}\,q_{23}\,q_{34}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+\\
&64\,q_{12}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{12}\,q_{23}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-\\
&64\,q_{12}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2-64\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2+64\,q_{12}\,q_{23}\,q_{34}\,q_{41}\,S_1^2\,S_2^2\,S_3^2\,S_4^2\end{aligned}$$

In[9]:=
```
ArchimedesFn = rhse - lhse
```

Out[9]=

$$
\begin{aligned}
&-q_{12}^4 S_1^4 S_2^4 + 4 q_{12}^3 q_{23} S_1^3 S_2^4 S_3 - 6 q_{12}^2 q_{23}^2 S_1^2 S_2^4 S_3^2 + 4 q_{12} q_{23}^3 S_1 S_2^4 S_3^3 - q_{23}^4 S_2^4 S_3^4 + 4 q_{12}^3 q_{41} S_1^4 S_2^3 S_4 + \\
&16 q_{12}^2 S_1^3 S_2^3 S_3 S_4 - 8 q_{12}^3 S_1^3 S_2^3 S_3 S_4 - 8 q_{12}^2 q_{23} S_1^3 S_2^3 S_3 S_4 - 8 q_{12}^2 q_{34} S_1^3 S_2^3 S_3 S_4 + 4 q_{12}^3 q_{34} S_1^3 S_2^3 S_3 S_4 - \\
&8 q_{12}^2 q_{41} S_1^3 S_2^3 S_3 S_4 - 4 q_{12}^2 q_{23} q_{41} S_1^3 S_2^3 S_3 S_4 - 32 q_{12} q_{23} S_1^2 S_2^3 S_3^2 S_4 + 16 q_{12}^2 q_{23} S_1^2 S_2^3 S_3^2 S_4 + \\
&16 q_{12} q_{23}^2 S_1^2 S_2^3 S_3^2 S_4 + 16 q_{12} q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 - 4 q_{12}^2 q_{23} q_{34} S_1^2 S_2^3 S_3^2 S_4 + 16 q_{12} q_{23} q_{41} S_1^2 S_2^3 S_3^2 S_4 - \\
&4 q_{12} q_{23}^2 q_{41} S_1^2 S_2^3 S_3^2 S_4 + 16 q_{23}^2 S_1 S_2^3 S_3^3 S_4 - 8 q_{12} q_{23}^2 S_1 S_2^3 S_3^3 S_4 - 8 q_{23}^3 S_1 S_2^3 S_3^3 S_4 - 8 q_{23}^2 q_{34} S_1 S_2^3 S_3^3 S_4 - \\
&4 q_{12} q_{23}^2 q_{34} S_1 S_2^3 S_3^3 S_4 - 8 q_{23}^2 q_{41} S_1 S_2^3 S_3^3 S_4 + 4 q_{23}^3 q_{41} S_1 S_2^3 S_3^3 S_4 + 4 q_{23}^3 q_{34} S_2^3 S_3^4 S_4 - 6 q_{12}^2 q_{41}^2 S_1^4 S_2^2 S_4^2 - \\
&32 q_{12} q_{41} S_1^3 S_2^2 S_3 S_4^2 + 16 q_{12}^2 q_{41} S_1^3 S_2^2 S_3 S_4^2 + 16 q_{12} q_{23} q_{41} S_1^3 S_2^2 S_3 S_4^2 + 16 q_{12} q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 - \\
&4 q_{12}^2 q_{34} q_{41} S_1^3 S_2^2 S_3 S_4^2 + 16 q_{12} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 - 4 q_{12} q_{23} q_{41}^2 S_1^3 S_2^2 S_3 S_4^2 - 16 q_{12}^2 S_1^2 S_2^2 S_3^2 S_4^2 + \\
&32 q_{12} q_{23} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{23}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{12}^2 q_{34} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - \\
&48 q_{12} q_{23} q_{34} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{12} q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 6 q_{12}^2 q_{34}^2 S_1^2 S_2^2 S_3^2 S_4^2 + \\
&32 q_{12} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 48 q_{12} q_{23} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 16 q_{23}^2 q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 32 q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - \\
&48 q_{12} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 48 q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 + 40 q_{12} q_{23} q_{34} q_{41} S_1^2 S_2^2 S_3^2 S_4^2 - 16 q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^2 + \\
&16 q_{23} q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 6 q_{23}^2 q_{41}^2 S_1^2 S_2^2 S_3^2 S_4^2 - 32 q_{23} q_{34} S_1 S_2^2 S_3^3 S_4^2 + 16 q_{12} q_{23} q_{34} S_1 S_2^2 S_3^3 S_4^2 + \\
&16 q_{23}^2 q_{34} S_1 S_2^2 S_3^3 S_4^2 + 16 q_{23} q_{34}^2 S_1 S_2^2 S_3^3 S_4^2 - 4 q_{12} q_{23} q_{34}^2 S_1 S_2^2 S_3^3 S_4^2 + 16 q_{23} q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 - \\
&4 q_{23}^2 q_{34} q_{41} S_1 S_2^2 S_3^3 S_4^2 - 6 q_{23}^2 q_{34}^2 S_2^2 S_3^4 S_4^2 + 4 q_{12} q_{41}^3 S_1^4 S_2 S_4^3 + 16 q_{41}^2 S_1^3 S_2 S_3 S_4^3 - \\
&8 q_{12} q_{41}^2 S_1^3 S_2 S_3 S_4^3 - 8 q_{23} q_{41}^2 S_1^3 S_2 S_3 S_4^3 - 8 q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 - 4 q_{12} q_{34} q_{41}^2 S_1^3 S_2 S_3 S_4^3 - \\
&8 q_{41}^3 S_1^3 S_2 S_3 S_4^3 + 4 q_{23} q_{41}^3 S_1^3 S_2 S_3 S_4^3 - 32 q_{34} q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{12} q_{34} q_{41} S_1^2 S_2 S_3^2 S_4^3 + \\
&16 q_{23} q_{34} q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 q_{41} S_1^2 S_2 S_3^2 S_4^3 - 4 q_{12} q_{34}^2 q_{41} S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 - \\
&4 q_{23} q_{34} q_{41}^2 S_1^2 S_2 S_3^2 S_4^3 + 16 q_{34}^2 S_1 S_2 S_3^3 S_4^3 - 8 q_{12} q_{34}^2 S_1 S_2 S_3^3 S_4^3 - 8 q_{23} q_{34}^2 S_1 S_2 S_3^3 S_4^3 - \\
&8 q_{34}^3 S_1 S_2 S_3^3 S_4^3 + 4 q_{12} q_{34}^3 S_1 S_2 S_3^3 S_4^3 - 8 q_{34}^2 q_{41} S_1 S_2 S_3^3 S_4^3 - 4 q_{23} q_{34}^2 q_{41} S_1 S_2 S_3^3 S_4^3 + \\
&4 q_{23} q_{34}^3 S_2 S_3^4 S_4^3 - q_{41}^4 S_1^4 S_4^4 + 4 q_{34} q_{41}^3 S_1^3 S_3 S_4^4 - 6 q_{34}^2 q_{41}^2 S_1^2 S_3^2 S_4^4 + 4 q_{34}^3 q_{41} S_1 S_3^3 S_4^4 - q_{34}^4 S_3^4 S_4^4
\end{aligned}
$$

In[10]:=
```
L1 = {l1, m1, n1}
L2 = {l2, m2, n2}
L3 = {l3, m3, n3}
L4 = {l4, m4, n4}
```

Out[10]= $\{l_1, m_1, n_1\}$

Out[11]= $\{l_2, m_2, n_2\}$

Out[12]= $\{l_3, m_3, n_3\}$

Out[13]= $\{l_4, m_4, n_4\}$

Meet of lines.

In[14]:=
```
M[L1_, L2_] :=
 {L1⟦2⟧ × L2⟦3⟧ - L1⟦3⟧ × L2⟦2⟧, L1⟦3⟧ × L2⟦1⟧ - L1⟦1⟧ × L2⟦3⟧, L1⟦1⟧ × L2⟦2⟧ - L1⟦2⟧ × L2⟦1⟧}
```

In[15]:=
```
quadrance[a1_, a2_] := 1 - (a1⟦1⟧ × a2⟦1⟧ + a1⟦2⟧ × a2⟦2⟧ + a1⟦3⟧ × a2⟦3⟧)^2 /
   ((a1⟦1⟧ × a1⟦1⟧ + a1⟦2⟧ × a1⟦2⟧ + a1⟦3⟧ × a1⟦3⟧) ×
     (a2⟦1⟧ × a2⟦1⟧ + a2⟦2⟧ × a2⟦2⟧ + a2⟦3⟧ × a2⟦3⟧))
```

In[16]:=
```
spread[l1_, l2_] := 1 - (l1⟦1⟧ × l2⟦1⟧ + l1⟦2⟧ × l2⟦2⟧ + l1⟦3⟧ × l2⟦3⟧)^2 /
   ((l1⟦1⟧ × l1⟦1⟧ + l1⟦2⟧ × l1⟦2⟧ + l1⟦3⟧ × l1⟦3⟧) ×
     (l2⟦1⟧ × l2⟦1⟧ + l2⟦2⟧ × l2⟦2⟧ + l2⟦3⟧ × l2⟦3⟧))
```

In[17]:= S_1 = spread[L_4, L_1]
S_2 = spread[L_1, L_2]
S_3 = spread[L_2, L_3]
S_4 = spread[L_3, L_4]

Out[17]= $$1 - \frac{(l_1 l_4 + m_1 m_4 + n_1 n_4)^2}{\left(l_1^2 + m_1^2 + n_1^2\right)\left(l_4^2 + m_4^2 + n_4^2\right)}$$

Out[18]= $$1 - \frac{(l_1 l_2 + m_1 m_2 + n_1 n_2)^2}{\left(l_1^2 + m_1^2 + n_1^2\right)\left(l_2^2 + m_2^2 + n_2^2\right)}$$

Out[19]= $$1 - \frac{(l_2 l_3 + m_2 m_3 + n_2 n_3)^2}{\left(l_2^2 + m_2^2 + n_2^2\right)\left(l_3^2 + m_3^2 + n_3^2\right)}$$

Out[20]= $$1 - \frac{(l_3 l_4 + m_3 m_4 + n_3 n_4)^2}{\left(l_3^2 + m_3^2 + n_3^2\right)\left(l_4^2 + m_4^2 + n_4^2\right)}$$

In[21]:= a_1 = M[L_4, L_1]
a_2 = M[L_1, L_2]
a_3 = M[L_2, L_3]
a_4 = M[L_3, L_4]

Out[21]= $\{m_4 n_1 - m_1 n_4, -l_4 n_1 + l_1 n_4, l_4 m_1 - l_1 m_4\}$

Out[22]= $\{-m_2 n_1 + m_1 n_2, l_2 n_1 - l_1 n_2, -l_2 m_1 + l_1 m_2\}$

Out[23]= $\{-m_3 n_2 + m_2 n_3, l_3 n_2 - l_2 n_3, -l_3 m_2 + l_2 m_3\}$

Out[24]= $\{-m_4 n_3 + m_3 n_4, l_4 n_3 - l_3 n_4, -l_4 m_3 + l_3 m_4\}$

In[25]:=
```
q12 = quadrance[a1, a2]
q23 = quadrance[a2, a3]
q34 = quadrance[a3, a4]
q41 = quadrance[a4, a1]
```

Out[25]=
$$1 - ((-l_2 m_1 + l_1 m_2)(l_4 m_1 - l_1 m_4) + (l_2 n_1 - l_1 n_2)(-l_4 n_1 + l_1 n_4) + (-m_2 n_1 + m_1 n_2)(m_4 n_1 - m_1 n_4))^2 \Big/ \Big(\big((-l_2 m_1 + l_1 m_2)^2 + (l_2 n_1 - l_1 n_2)^2 + (-m_2 n_1 + m_1 n_2)^2\big) \big((l_4 m_1 - l_1 m_4)^2 + (-l_4 n_1 + l_1 n_4)^2 + (m_4 n_1 - m_1 n_4)^2\big)\Big)$$

Out[26]=
$$1 - ((-l_2 m_1 + l_1 m_2)(-l_3 m_2 + l_2 m_3) + (l_2 n_1 - l_1 n_2)(l_3 n_2 - l_2 n_3) + (-m_2 n_1 + m_1 n_2)(-m_3 n_2 + m_2 n_3))^2 \Big/ \Big(\big((-l_2 m_1 + l_1 m_2)^2 + (l_2 n_1 - l_1 n_2)^2 + (-m_2 n_1 + m_1 n_2)^2\big) \big((-l_3 m_2 + l_2 m_3)^2 + (l_3 n_2 - l_2 n_3)^2 + (-m_3 n_2 + m_2 n_3)^2\big)\Big)$$

Out[27]=
$$1 - ((-l_3 m_2 + l_2 m_3)(-l_4 m_3 + l_3 m_4) + (l_3 n_2 - l_2 n_3)(l_4 n_3 - l_3 n_4) + (-m_3 n_2 + m_2 n_3)(-m_4 n_3 + m_3 n_4))^2 \Big/ \Big(\big((-l_3 m_2 + l_2 m_3)^2 + (l_3 n_2 - l_2 n_3)^2 + (-m_3 n_2 + m_2 n_3)^2\big) \big((-l_4 m_3 + l_3 m_4)^2 + (l_4 n_3 - l_3 n_4)^2 + (-m_4 n_3 + m_3 n_4)^2\big)\Big)$$

Out[28]=
$$1 - ((l_4 m_1 - l_1 m_4)(-l_4 m_3 + l_3 m_4) + (-l_4 n_1 + l_1 n_4)(l_4 n_3 - l_3 n_4) + (m_4 n_1 - m_1 n_4)(-m_4 n_3 + m_3 n_4))^2 \Big/ \Big(\big((l_4 m_1 - l_1 m_4)^2 + (-l_4 n_1 + l_1 n_4)^2 + (m_4 n_1 - m_1 n_4)^2\big) \big((-l_4 m_3 + l_3 m_4)^2 + (l_4 n_3 - l_3 n_4)^2 + (-m_4 n_3 + m_3 n_4)^2\big)\Big)$$

The next cell may take a long time to execute since the q’s and the S’s in the “ArchimedesFn” equation are being substituted with the x’s, y’s and z’s in the q’s and S’s expressions above and the whole thing is being factored. A result of zero indicates that the criteria for compatible equations in the quadratic compatibility theorem has been met.

In[29]:=
```
result = Factor[ArchimedesFn]
```

Out[29]= 0

We have proved that the following two quadratic equations in B are compatible.