Universal Hyperbolic Geometry:

Applying Archimedes function to help prove a theorem and fill a gap at the frontier of pure mathematics in geometry

(Discovering Exact Formulas for the Quadrea of a General Projective Quadrangle and the Quadreal of the dual Projective Quadrilateral)

Exact formulas for invariants of a general projective quadrangle and it's dual quadrilateral in Universal Hyperbolic Geometry

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Abstract

1. Concise Summary of what you wrote in the paper.
2. Selling Points (Why should someone invest the time to read the paper)
3. talk about the main key result. Invariant formula for quadrangle.
4. Descriptive but non-technical (people should get the just of what your paper is about quickly from reading the abstract)
5. What/Why/How structure for an abstract. What is it that you did in this paper, why did you do this, how did you do it, what sort of tools were you using to accomplish this result.

We are building on the results of Universal Hyperbolic Geometry I: Trigonometry[**?**, article]nd extending from triangle to quadrangles.

There is a complete duality in the theory between points and lines. Any result thus has a corresponding dual result, obtained by interchanging the roles of points and lines. We call this the duality principle, and use it often to eliminate repetition of statements and proofs of theorems

UHG is about formulas. The main formulas in the subject have been discovered for triangle projective geometry. We discovered exact formulas for the quadrea of a projective quadrangle and the corresponding dual formulas for the quadreal of the projective quadrilateral.

In this paper we establish and prove an exact formula for the quadrea of a projective quadrangle. Based on UHG[7]

1 Introduction

Why does this paper exist? Main takeaways. How to write a paper?

1. Introduction Part 1: Why is it important? Why should anyone care?

2. Introduction Part 2: What is missing? Long history of area of quadrilateral. Cite the existing studies?

3. Introduction Part 3? What is the value? What gaps are you filling? What is the value added?

1. State the problem

2. Say why it's and interesting problem

3. Say what your solution achieves

4. Say what follows from your solution

1. Summary

2. Background information / Context

3. The relative setting by which the paper exists, other established results in the literature. Catch the reader up on the results, theorems , definitions to which your paper is building on. Summary on this corner of research.

4. Motivation. Why were you doing this, what interested you about it.

5. Notation/ Jargon.

6. Outline of the paper. Description of the different major sections of the paper.

7. Make formulas part of sentences, via connections like .... so, thus, therefore, but, now we see that, observe, then define, etc

8. use careful diagrams.

A point in UHG is a proportion of 3 rational numbers [x:y:z] given in square brackets A line in UHG is a proportion of 3 rational numbers (l:m:n) given in round brackets. $q(a_1, a_2)$: the quadrance between points a_1, a_2 $S(L_1, L_2)$: the spread between lines L_1, L_2

A point **a** is just a proportion x to y to z enclosed in square brackets. **a** = [x:y:z]. A line **L** is a proportion l to m to n in round brackets. **L** = (l:m:n)

We have two projective points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ and we want to define a notion of separation between these two points. These are actually lines through the origin in the relativistic affine space. How are you going to assign a number that associates how far apart they are? We have the dot product but we want it to be homogeneous because the points are defined up to a scalar so we put the quadratic forms of the two individual points in the denominator.

The quadrance between points:

$$q(a_1, a_2) \equiv 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}$$

The spread between lines $S(L_1, L_2)$:

$$S(L_1, L_2) \equiv 1 - \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}$$

The cross ratio is the most important invariant in projective geometry[3]. Such a ratio is significant because projections distort most metric relationships (i.e., those involving the measured quantities of length and angle), while the study of projective geometry centres on finding those properties that remain invariant.

By the quadrance in coordinates theorem given in [4] the quadrance $q(a_1, a_2)$ [1] between points a_1 and a_2 is the cross ratio $R(a_1, b_2 : a_2, b_1) = \frac{\overrightarrow{a_1a_2}}{\overrightarrow{a_1b_1}} / \frac{\overrightarrow{b_2a_2}}{\overrightarrow{b_2b_1}}$ where b_1, b_2 are the intersection points of the dual lines of a_1 and a_2 and the line $L(a_1, a_2)$. In UHG quadrance q and spread S are dual concepts, so the spread between lines is also an invariant in projective geometry. Other metrical quantities that are invariant in UHG include the quadrea of a triangle and the quadreal of a quadrilateral which are functions of quadrances and spreads.

Quadrea of a triangle $\mathcal{A}(\overline{a_1a_2a_3}) = q_1q_2S_3 = q_2q_3S_1 = q_1q_3S_2$ by symmetry.

$$q_2q_3S_1 = -\frac{(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)(x_3^2 + y_3^2 - z_3^2)}$$

This turns out to be the most important triangle invariant[2]under scaling of any one of the coordinates a_1, a_2 or a_3. If you scale any of the coordinates in this equation then the numerator and denominator scale by the same amount and therefore the equation is invariant under scaling and it is a well defined expression. Standard Triangle 1: ST1

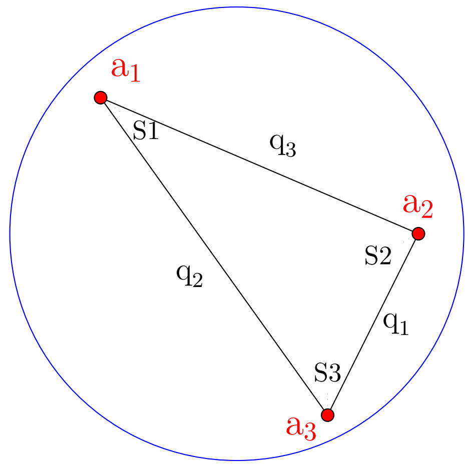

Figure 1: Standard Triangle 1

- a_1 = [-3:3:5]
- a_2 = [4:0:5]
- a_3 = [2:-4:5]
- $q_1 = quadrance(a_2, a_3) = \frac{-244}{45}$
- $q_2 = quadrance(a_3, a_1) = \frac{-1814}{35}$
- $q_3 = quadrance(a_1, a_2) = \frac{-1306}{63}$
- $L_1 = Join(a_2, a_3)$
- $L_2 = Join(a_3, a_1)$
- $L_3 = Join(a_1, a_2)$
- $S_1 = Spread(L_2, L_3) = \frac{50575}{592271}$
- $S_2 = Spread(L_3, L_1) = \frac{65025}{79666}$
- $S_3 = Spread(L_1, L_2) = \frac{36125}{110654}$
- $quadrea = q_1 q_2 S_3 = q_1 q_3 S_2 = q_2 q_3 S_1 = \frac{5780}{63}$

Def: A quadrangle $\overline{ABCD}$ is a list $[A, B, C, D]$ of distinct points with the convention that

$$\overline{ABCD} = \overline{BCDA} \qquad and \qquad \overline{ABCD} = \overline{ADBC}$$

$$Note : \overline{ABCD} \neq \overline{ABDC}$$

Def: The quadrangle $\overline{ABCD}$ has points A, B, C, D and lines AB, BC, CD, DA. It has diagonal lines AC and BD.

Standard Quadrangle 1: SQ1 Obtained by reflecting the orthocenter in the line L_2 = Join(a_1, a_3).

- a_1 = [-3:3:5]
- a_2 = [4:0:5]
- a_3 = [2:-4:5]
- a_4 = [4645765:12790075:-19645043]

Triangle geometry is a very much developed subject. There is some catching up to do in the quadrilateral world. Many beautiful things to be discovered about quadrilaterals. Important connections with both mathematics and physics.

Quadrea formula is a proper result of pure mathematics. All terms are precisely pinned down and we have a water tight (algebraic) proof.

Geometry is really about formulas. Algebraic expressions that capture relationships between geometrical concepts.

Formulas allow us to compute something. We are interested in making computations and answering explicit questions.

Formulas involve algebra. Build a library of useful formulas. We need formulas because our computers can deal with the formulas really efficiently.

If you want to make progress in understanding geometry you have to address the algebraic underpinnings of the subject. We need to be prepared for dealing with and appreciating more complicated formulas. Try to understand how these various formulas fit together. Try put some kind of coherence to these family of formulas that arise in geometry.

Rational numbers are at the heart of mathematics. Quadrea of quadrangle formula: Is a correct statement because it has ingredients that are all logically defined. The proof employs the quadratic compatibility theorem.

A nice equation gives you more out than you put in.

Metrical algebraic geometry: geometry looked at algebraically. With a metrical approach, we are interested in making measurements.

We need to add a quadratic form to get a metrical structure.

In projective geometry the metrical structure is introduced through the projective quadrance between projective points a,b .

We are interested in establishing fundamental formulas in this subject.

Henri Poincare quote "It is by logic that we prove but by intuition that we discover".

Archimedes was enraptured by pure mathematics and believed in studying mathematics for it's own sake. He produced formulas to calculate areas of regular shapes in his investigations of pure mathematics. He captured new shapes by using shapes he already understood. He also calculated volumes of solid objects.

Symmetries of tetrahedron is just the symmetric group of 4 letters. Any permutation of the 4 vertices is going to be a symmetry of the tetrahedron solid. That is another way of thinking of what the whole group of symmetries of the tetrahedron is. Aside from Symmetric Groups S_1 and S_2 all symmetric groups S_n are non-abelian for $n2$. The Symmetric Group S_n is the group of permutations on a set with n elements. There are $n!$ possible permutations so it is a finite group with $n!$ elements.

There are deep relationships between math and physics. At a deep level these structures are coming together between our model of reality and deep mathematical structures. There is this deep thing going on there that we only partially understand. This is a motivation to explore these deep connections.

The frontier is where the adventure of your life is. It is the place you go where there is magic, where there are things to discover. Where there is some thing that you don't have yet, some knowledge that you don't have yet.

Proper mathematics is built up from rational numbers. It is possible to create geometry over the rational numbers.

The cross ratio is the most important invariant in projective geometry. Pappus (300 A.D.). The quadrance between points a_1 and a_2 is the cross ratio $R(a_1, b_2 : a_2, b_1)$ where b_1, b_2 are the intersection points of the dual lines of a_1 and a_2 and the line $L(a_1, a_2)$.

Quadrea of Projective Quadrangle Definition: (X X is the quadrea of a projective quadrangle if and only if there exist a pair of compatible quadratic equations in X of the form (X-p1)^2 = $q1and(X-p2)^2 = q2wherep1, p2, q1andq2arefunctionsofthequadrancesandspreadsofthequadranglesothatXisthecomm$

Theorem: (Formal) (For all things in some domain, if you have a certain property e.g. compatible quadratic equations, then you get some other property.

For all pairs of quadratic equations in X with spreads and quadrances of the quadrangle, if the pair of quadratic equations are compatible then the common solution X is the quadrea of the quadrangle or its dual.

Proof:

1) Start with assumptions. (What do we know for sure, 4 distinct projective points forming a quadrangle, quadrances and spreads of quadrangle)

2) Apply the definitions. (quadrance and spread)

3) Use algebra, known theorems, logical inferences, facts that we proved previously. Perform manipulations.

4) Get to the conclusion,

Note: you can bring over at any point your formal definition, or formal statement of your theorem if you need to,

Suppose $a_1 = [x_1{:}y_1{:}z_1]$, $a_2 = [x_2{:}y_2{:}z_2]$, $a_3 = [x_3{:}y_3{:}z_3]$, and $a_4 = [x_4{:}y_4{:}z_4]$ are distinct projective points forming a projective quadrangle

2 Background Information

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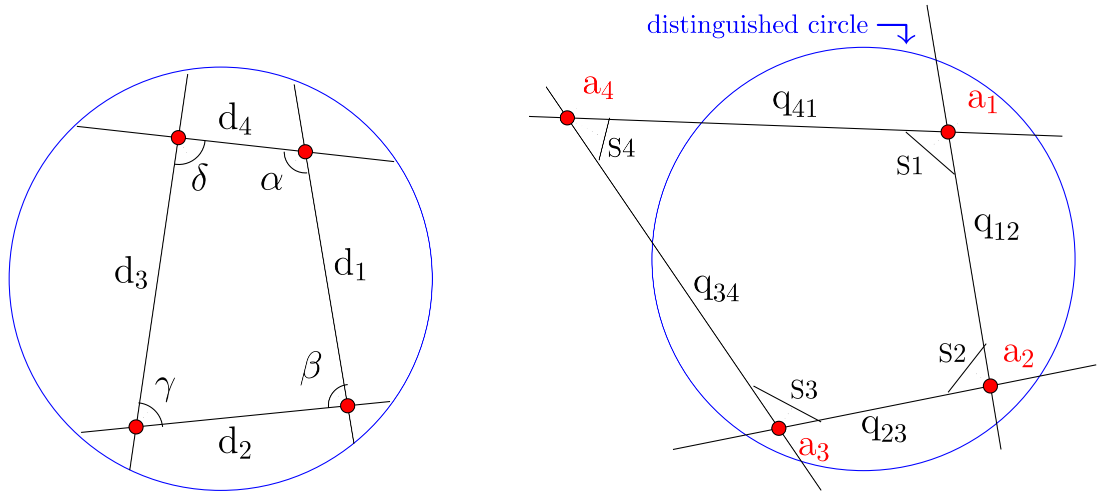
Universal Hyperbolic Geometry (UHG) is a mathematical framework that sits at the heart of mathematics. It is an approach to hyperbolic geometry that is completely algebraic, and replaces metrical notions of distance and angle with quadrance and spread. UHG models hyperbolic geometry in a similar manner to the Beltrami Klein projective model, which uses straight lines and takes place in the interior of the disk, but UHG replaces distance and angle with quadrance and spread and it works both inside and outside the disk. UHG adopts the point of view of Rational Trigonometry [6] and it combines projective geometry of the simplest kind with the simplest conic, the distinguished circle.

UHG works over a general field, in particular it works over rational numbers, it works over finite fields, and it works over complex numbers. The formulas extend outside the disk and the formulas unify the two standard non-Euclidean geometries, hyperbolic geometry and elliptic or spherical geometry. The exact same formulas work in both hyperbolic and elliptic geometry. These two geometries come together in this framework and the basic formulas are ultimately identical for



Beltrami Klein Projective Model

Universal Hyperbolic Geometry Projective Plane

Figure 2: Two Different Models of Hyperbolic Geometry

both subjects. Hyperbolic geometry in the UHG framework is a very beautiful subject and you can make exact calculations with the formulas in the subject rather than approximate calculations.

Formulas in the subject have been developed for hyperbolic triangle geometry and this paper presents new formulas for hyperbolic quadrangle geometry where the exact same formulas also work in both hyperbolic and elliptic geometries.

Geometry (after Felix Klein - 1870) was in terms of constructions instead of postulates by giving names to points in terms of coordinates and describing what happens geometrically in terms of some distance function.$d((x_1, y_1), (x_2, y_2))$

$$x^2 + y^2 < 1$$

$$d((x_1, y_1), (x_2, y_2)) = a\ cosh^{-1}\left[\frac{1 - x_1x_2 - y_1y_2}{\sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_1^2 - y_2^2}}\right]$$

Since this just boils down to algebra, then it is consistent if the algebra is consistent and the theorems about the real numbers are consistent, which nobody has been able to prove.

From the point of view of Rational Trigonometry (RT) and Universal Hyperbolic Geometry, geometry is really about formulas. The setting of RT is Affine geometry and the setting of UHG is projective geometry, both hyperbolic and elliptic projective geometry. The main laws in the two subjects are Pythagoras theorem, the Cross Law, the Spread Law, the Triple Quad Formula and the Triple Spread Formula and these main laws are shown in table 1. There is a metrical notion of quadrance and spread in both RT and UHG and the main laws are formulas involving quadrance

	Affine Geometry (Rational Trigonometry)	**Projective Geometry (Universal Hyperbolic Geometry)**
Pythagoras	$Q_1 + Q_2 = Q_3$	$q_3 = q_1 + q_2 - q_1 q_2$
Triple quad formula	$(Q_1+Q_2+Q_3)^2 = 2(Q_1^2+Q_2^2+Q_3^2)$	$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2) + 4q_1q_2q_3$
Spread law	$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$	$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}$
Cross Law	$(Q_1+Q_2-Q_3)^2 = 4Q_1Q_2(1-s_3)$	$(q_1q_2S_3 - q_1 - q_2 - q_3 + 2)^2 = 4(1-q_1)(1-q_2)(1-q_3)$
Triple spread formula	$(s_1+s_2+s_3)^2 = 2(s_1^2+s_2^2+s_3^2) + 4s_1s_2s_3$	not applicable

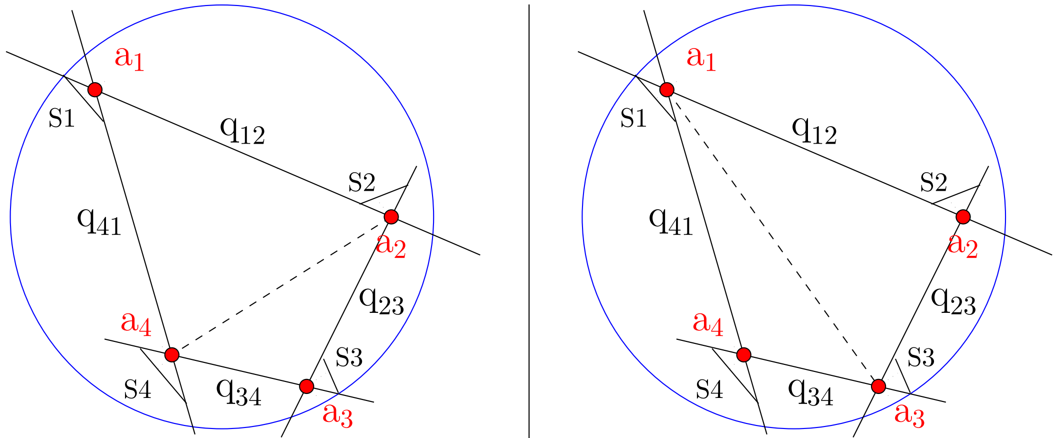
Table 1: Main Laws of Rational Trigonometry and Universal Hyperbolic Geometry

and spread. In affine geometry we use a capital 'Q' for quadrance and a lower case 's' for spread and in projective geometry it is the opposite, we use lower case 'q' for quadrance and capital 'S' for spread. That way you can identify which setting you are in when looking at a formula based on the capitalization of the letters for quadrance and spread.

3 The Problem

Formula	**Affine Geometry (Rational Trigonometry)**	**Projective Geometry (Universal Hyperbolic Geometry)**
Pythagoras	✓	✓
Triple quad formula	✓	✓
Spread law	✓	✓
Cross Law	✓	✓
Triple spread formula	✓	
Quadrance	✓	✓
Spread	✓	✓
Quadrea of Triangle	✓	✓
Quadreal of Trilateral		✓
Quadrea of Quadrilateral	✓	
Quadrea of Quadrangle		✗
Quadreal of Quadrilateral		✗

Table 2: Library of existing formulas



$$(\mathcal{B}-q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 = q_{12}q_{23}q_{34}q_{41}(1-S_2)(1-S_4)$$

$$(\mathcal{B}-q_{12}q_{23}S_2 - q_{34}q_{41}S_4)^2 = q_{12}q_{23}q_{34}q_{41}(1-S_1)(1-S_3)$$

Figure 3: Pair of compatible quadratic equations and associated triangulations for the quadrea $\mathcal{B}$ of a general projective quadrangle

4 My Idea

5 The Details

5.1 General Projective Quadrangle

We use the definitions from [7], which we include here for the reader's convenience:

Definition 5.1. *[7] 3-proportion*

*A **3-proportion** $x : y : z$ is an ordered triple of numbers x, y and z, not all zero, with the convention that for any non-zero number λ*

$$x : y : z \equiv x : \lambda y : \lambda\lambda z$$

.

Definition 5.2. *[7] (hyperbolic) point*

*A **(hyperbolic) point** is a 3-proportion $a \equiv [x : y : z]$ enclosed in square brackets.*

Definition 5.3. *[7] (hyperbolic) line*

*A **(hyperbolic) line** is a 3-proportion $L \equiv [x : y : z]$ enclosed in round brackets.*

Definition 5.4. *[7] triangle*

*A **triangle** $\overline{a_1 a_2 a_3}$ is a set $\{a_1, a_2, a_3\}$ of three non-collinear points.*

Definition 5.5. *[7] trilateral*

*A **trilateral** $\overline{L_1 L_2 L_3}$ is a set $\{L_1, L_2, L_3\}$ of three non-concurrent lines.*

Definition 5.6. *[7] (hyperbolic) quadrance*

*The **(hyperbolic) quadrance** between points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the number*

$$q(a_1, a_2) \equiv 1 - \frac{(x_1 x_2 + y_1 y_2 - z_1 z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}$$

.

Definition 5.7. *[7] (hyperbolic) spread*

*The **(hyperbolic) spread** between lines $L_1 \equiv [l_1 : m_1 : n_1]$ and $L_2 \equiv [l_2 : m_2 : n_2]$ is the number*

$$S(L_1, L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 - n_1 n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}$$

.

Theorem 5.1. *[7] Quadrea of Projective Triangle*

If $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$ are non-null points, then the quadrea of $\{a_1, a_2, a_3\}$ is the number

$$\mathcal{A} \equiv \mathcal{A}(a_1, a_2, a_3) \equiv -\frac{(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)(x_1^2 + y_1^2 - z_1^2)(x_3^2 + y_3^2 - z_3^2)}$$

.

Definition 5.8. *Projective Point*

*A projective point $a \equiv [x : y : z]$ is a proportion of 3 rational numbers x to y to z written in square brackets and given by a lower case letter like **a**.*

Definition 5.9. *Projective Line*

*A projective line $L \equiv [l : m : n]$ is a proportion of 3 rational numbers l to m to n written in round brackets and given by a upper case letter like **L**.*

We use the Join of Points and the Meet of Lines theorems which of theorems 1 and 2 of [7] which state

Join of Points: If $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are distinct points, then there is exactly one line L which passes through them both, namely

$$L \equiv a_1a_2 \equiv (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2$$

.

Meet of Lines: If $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are distinct lines, then there is exactly one point a which lies on them both, namely

$$a \equiv L_1L_2 \equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_2m_1 - l_1m_2]$$

.

Definition 5.10. *Quadrance*

*The quadrance **q** between projective points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ in the projective hyperbolic setting is*

$$q(a_1, a_2) \equiv 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}$$

and in the projective elliptic or spherical setting the quadrance is

$$q(a_1, a_2) \equiv 1 - \frac{(x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}$$

The quadrance is a well defined quantity, if you multiply $x_1, y_1 and z_1 in a_1$ by some number then the value of the quadrance $q(a_1, a_2)$ does not change.

Definition 5.11. *Spread*

*The spread **S** between projective lines $L_1 \equiv [l_1 : m_1 : n_1]$ and $L_2 \equiv [l_2 : m_2 : n_2]$ in the projective hyperbolic setting is*

$$S(L_1, L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 - n_1 n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}$$

and in the projective elliptic or spherical setting the spread is

$$S(L_1, L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 + n_1 n_2)^2}{(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2)}$$

Since points and lines are dual in projective geometry the definition of the spread is exactly the same as the quadrance but with points replaced by lines.

These are the main definitions and they are used in the four main laws in UHG, namely, the Pythagoras Theorem, the Triple Quad Formula, the Spread Law and the Cross Law and their corresponding Dual laws.

This is the basics of what is important in UHG and the proofs of these laws can be found in[7].

Theorem 5.2. *[6] Archimedes' formula*

$$\begin{aligned} A(x, y, z) &= (x + y + z)^2 - 2(x^2 + y^2 + z^2) \\ &= 4xy - (x + y - z)^2 \end{aligned} \tag{1}$$

Proof. See theorem 29 of [6] or slide 2 of [5] for the proof. □

Definition 5.12. $(x - p_1)^2 = q_1$ *and* $(x - p_2)^2 = q_2$

*in x are **compatible** precisely when they have a common solution.*

Definition 5.13. *Two quadratic equations*

$$(x - p_1)^2 = q_1 \qquad \textit{and} \qquad (x - p_2)^2 = q_2$$

*in x are **compatible** precisely when they have a common solution.*

Theorem 5.3. *Quadratic compatibility The quadratic equations*

$$(x - p_1)^2 = q_1 \qquad\qquad (x - p_2)^2 = q_2$$

are compatible precisely when

$$((p_1p_2)^2 - (q_1 + q_2))^2 = 4q_1q_2$$

In this case, if $p_1 \neq p_2$ then there is a unique common solution

$$x = \frac{p_1 + p_2}{2} - \frac{(q_1 - q_2)}{2(p_1 - p_2)}$$

Note that if $p_1 = p_2$ in the theorem, then the two quadratic equations are compatible precisely when they are identical, and so in this case there are two common solutions.

Definition 5.14. *Quadrangle*

A projective quadrangle $\square \equiv \overline{a_1a_2a_3a_4}$ is a list $[a_1, a_2, a_3, a_4]$ of distinct projective points with the convention that $\overline{a_1a_2a_3a_4} = \overline{a_2a_3a_4a_1}$ and $\overline{a_1a_2a_3a_4} = \overline{a_4a_3a_2a_1}$ Note $\overline{a_1a_2a_3a_4} \neq \overline{a_1a_2a_4a_3}$.

Definition 5.15. *Quadrilateral*

A projective quadrilateral $\lozenge \equiv \overline{L_1L_2L_3L_4}$ is a list $[L_1, L_2, L_3, L_4]$ of distinct projective lines with the convention that $\overline{L_1L_2L_3L_4} = \overline{L_2L_3L_4L_1}$ and $\overline{L_1L_2L_3L_4} = \overline{L_4L_3L_2L_1}$ Note $\overline{L_1L_2L_3L_4} \neq \overline{L_1L_2L_4L_3}$.

A quadrangle $\square \equiv \overline{a_1a_2a_3a_4}$ has an associated quadrilateral $\widetilde{\square} \equiv \overline{L_1L_2L_3L_4}$ consisting of the four lines of the quadrangle, namely

$$L_1 \equiv a_1a_2 \qquad L_2 \equiv a_2a_3 \qquad L_3 \equiv a_3a_4 \qquad L_4 \equiv a_4a_1$$

It also has a dual quadrilateral $\square^\perp \equiv \overline{a_1^\perp a_2^\perp a_3^\perp a_4^\perp}$ consisting of the four dual lines of the quadrangle, namely $a_1^\perp, a_2^\perp, a_3^\perp, a_4^\perp$.

A quadrilateral $\lozenge \equiv \overline{L_1L_2L_3L_4}$ has an associated quadrangle $\widetilde{\lozenge} \equiv \overline{a_1a_2a_3a_4}$ consisting of the four points of the quadrilateral, namely

$$a_1 \equiv L_4L_1 \qquad a_2 \equiv L_1L_2 \qquad a_3 \equiv L_2L_3 \qquad a_4 \equiv L_3L_4$$

It also has a dual quadrangle $\lozenge^\perp \equiv \overline{L_1^\perp L_2^\perp L_3^\perp L_4^\perp}$ consisting of the four dual points of the quadrilateral, namely $L_1^\perp, L_2^\perp, L_3^\perp, L_4^\perp$.

A quadrangle $\square \equiv \overline{a_1a_2a_3a_4}$ and its associated quadrilateral $\lozenge \equiv \overline{L_1L_2L_3L_4}$ both have sides a_1a_2, a_2a_3, a_3a_4 and a_4a_1 and vertices L_1L_2, L_2L_3, L_3L_4 and L_4L_1.

If $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$, $a_3 \equiv [x_3 : y_3 : z_3]$ and $a_4 \equiv [x_4 : y_5 : z_6]$ are non-null and

Definition 5.16. *Quadrea of Projective Quadrangle*

The quadrea B of a projective quadrangle $\square \equiv \overline{a_1a_2a_3a_4}$ is the common solution of two quadratic equations in B that are compatible.

$$(B - G_1(a_1, a_2, a_3, a_4))^2 = H_1(a_1, a_2, a_3, a_4) \tag{2}$$

$$(B - G_2(a_1, a_2, a_3, a_4))^2 = H_2(a_1, a_2, a_3, a_4) \tag{3}$$

Definition 5.17. *Quadreal of Projective Quadrilateral*

The quadreal $\boldsymbol{L}$ *of a projective quadrangle* $\overline{a_1a_2a_3a_4}$ *is the common solution of two quadratic equations in* L *that are compatible.*

$$(L - J_1(a_1, a_2, a_3, a_4))^2 = K_1(a_1, a_2, a_3, a_4) \tag{4}$$

$$(L - J_2(a_1, a_2, a_3, a_4))^2 = K_2(a_1, a_2, a_3, a_4) \tag{5}$$

Theorem 5.4 (Quadrea of a Projective Quadrangle)**.** *Suppose* a_1*,* a_2*,* a_3 *and* a_4 *are distinct points of a general projective quadrangle with quadrances* $q_{12} \equiv q(a_1, a_2)$*,* $q_{23} \equiv q(a_2, a_3)$*,* $q_{34} \equiv q(a_3, a_4)$ *and* $q_{41} \equiv q(a_4, a_1)$ *and spreads* $S_1 \equiv S(a_4a_1, a_1a_2)$*,* $S_2 \equiv S(a_1a_2, a_2a_3)$*,* $S_3 \equiv S(a_2a_3, a_3a_4)$ *and* $S_4 \equiv S(a_3a_4, a_4a_1)$ *and quadrea* $\mathcal{B}$*. Then the quadrea of the projective quadrangle is the common solution to the following pair of compatible quadratic equations*

$$\begin{aligned}(\mathcal{B} - q_{12}q_{23}S_2 - q_{34}q_{41}S_4)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_1)(1 - S_3)\\(\mathcal{B} - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_2)(1 - S_4)\end{aligned} \tag{6}$$

If $q_{12}q_{23}S_2 + q_{34}q_{41}S_4 - q_{41}q_{12}S_1 - q_{23}q_{34}S_3 \neq 0$ *then the quadrea is precisely*

$$\mathcal{B} = \frac{q_{12}q_{23}S_2 + q_{34}q_{41}S_4 + q_{41}q_{12}S_1 + q_{23}q_{34}S_3}{2} + \frac{q_{12}q_{23}q_{34}q_{41}((1 - S_1)(1 - S_3) - (1 - S_2)(1 - S_4))}{2(q_{12}q_{23}S_2 + q_{34}q_{41}S_4 - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)} \tag{7}$$

Proof. Since the quadrances q_{12}, q_{23}, q_{34} and q_{41} are defined, the points a_1, a_2, a_3 and a_4 are non-null. Also since the spreads S_1, S_2, S_3 and S_4 are defined, the lines a_1a_2, a_2a_3, a_3a_4 and a_4a_1 are non-null.

Using the quadratic compatibility theorem the following pair of quadratic equations

$$\begin{aligned}(\mathcal{B} - q_{12}q_{23}S_2 - q_{34}q_{41}S_4)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_1)(1 - S_3)\\(\mathcal{B} - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_2)(1 - S_4)\end{aligned} \tag{8}$$

are compatible precisely when

$$((q_{12}q_{23}S_2+q_{34}q_{41}S_4-q_{41}q_{12}S_1-q_{23}q_{34}S_3)^2-(q_{12}q_{23}q_{34}q_{41}(1-S_1)(1-S_3)+q_{12}q_{23}q_{34}q_{41}(1-S_2)(1-S_4))^2 = 4((q_{12}q_{23}q_{34}q_{41}(\tag{9}$$

Substitute the projective points
$a_1 = [x_1{:}y_1{:}z_1]$, $a_2 = [x_2{:}y_2{:}z_2]$, $a_3 = [x_3{:}y_3{:}z_3]$, and $a_4 = [x_4{:}y_4{:}z_4]$
into the definitions of the quadrance, line from join of points, and spread between lines.

$$q(a_1, a_2) \equiv 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}$$

.

$$L \equiv a_1a_2 \equiv (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2$$

.

$$S(L_1, L_2) \equiv 1 - \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}$$

After substituting into equation 9 we get a long expression in terms of $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3$ *and* z_4 for when the two quadratic equations are compatible. After expanding this all out and simplifying we find that the left hand side of the equation 9 is precisely equal to the right hand side.

Therefore, this proves that the two quadratic equations 12 are compatible and have a common solution $\mathcal{B}$ which is the quadrea of the projective quadrangle.

□

Theorem 5.5 (Quadreal of a Projective Quadrilateral)**.** *Suppose L_1, L_2, L_3 and L_4 are distinct lines with spreads $S_2 \equiv S(L_1, L_2)$, $S_3 \equiv S(L_2, L_3)$, $S_4 \equiv S(L_3, L_4)$ and $S_1 \equiv q(L_4, L_1)$ and quadrances $q_{12} \equiv q(L_4L_1, L_1L_2)$, $q_{23} \equiv q(L_1L_2, L_2L_3)$, $q_{34} \equiv q(L_2L_3, L_3L_4)$ and $q_{41} \equiv S(L_3L_4, L_4L_1)$ and quadrea $\mathcal{L}$. Then the quadreal of the projective quadrilateral is the common solution to the following pair of compatible quadratic equations*

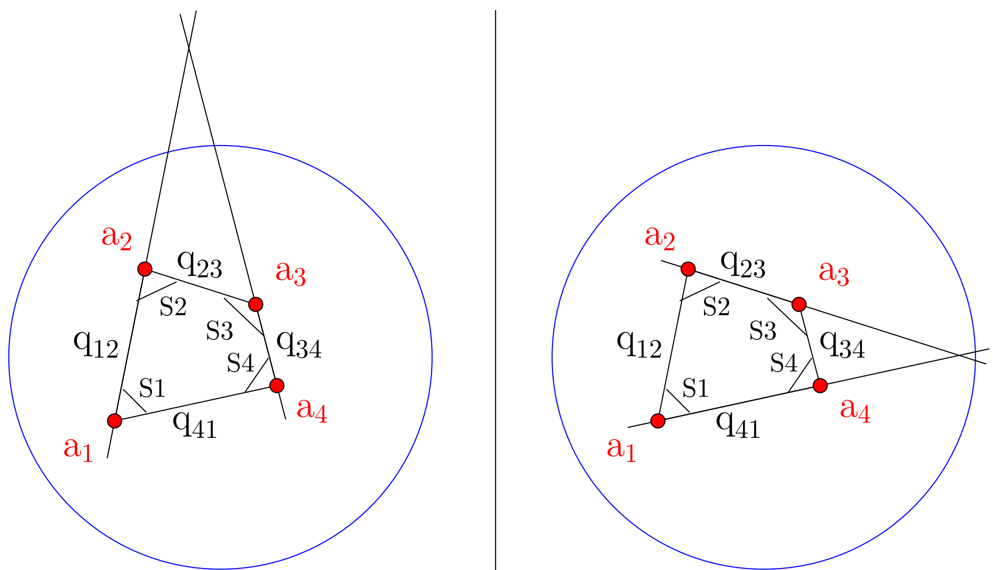
$$\begin{aligned} (\mathcal{L} - S_2S_3q_{23} - S_4S_1q_{41})^2 &= S_1S_2S_3S_4(1 - q_{12})(1 - q_{34}) \\ (\mathcal{L} - S_1S_2q_{12} - S_3S_4q_{34})^2 &= S_1S_2S_3S_4(1 - q_{23})(1 - q_{41}) \end{aligned} \tag{10}$$

If ${}_2S_3q_{23} + S_4S_1q_{41} - S_1S_2q_{12} - S_3S_4q_{34} \neq 0$ *then the quadreal is precisely*

$$\mathcal{L} = \frac{S_2S_3q_{23} + S_4S_1q_{41} + S_1S_2q_{12} + S_3S_4q_{34}}{2} + \frac{S_1S_2S_3S_4((1-q_{12})(1-q_{34}) - (1-q_{23})(1-q_{41}))}{2(S_2S_3q_{23} + S_4S_1q_{41} - S_1S_2q_{12} - S_3S_4q_{34})} \tag{11}$$

Proof. This is dual to the Quadrea of Projective Quadrangle theorem □

Here are the two triangulations associated with the two compatible quadratic equations for the quadreal of a general projective quadrilateral.



$$(\mathcal{L} - S_2S_3q_{23} - S_4S_1q_{41})^2 = S_1S_2S_3S_4(1-q_{12})(1-q_{34})$$

$$(\mathcal{L} - S_1S_2q_{12} - S_3S_4q_{34})^2 = S_1S_2S_3S_4(1-q_{23})(1-q_{41})$$

Figure 4: Pair of compatible quadratic equations and associated triangulations for the quadreal $\mathcal{L}$ of a general projective quadrilateral

Corollary 5.5.1 (Quadrea of Right Quadrangle)**.** *If one or more of the spreads in the quadrangle is equal to one then the right hand side of one or both of the quadratic equations 12 from Quadrea of a Projective Quadrilateral theorem is zero and the equation becomes linear. So the quadrea of the projective quadrilateral then becomes the sum of the quadreas of projective triangles on the left hand side of the equation.*

Proof. ☐

Corollary 5.5.2 (Reduction of Quadrangle Quadrea to Triangle Quadrea when two points coalesce)**.** *When two of the points of the projective quadrangle coalesce so that $a_i = a_j$ then $q_{ij} = 0$ and the projective quadrangle reduces to a projective triangle. In that case the quadrea B is equal to the quadrea of a projective triangle.*

Proof. Suppose a_4 and a_1 coalesce so that $a_4 = a_1$ and $q_{41} = 0$. Then the pair of quadratic equations for the projective quadrangle will change from

$$\begin{aligned}(\mathcal{B} - q_{12}q_{23}S_2 - q_{34}q_{41}S_4)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_1)(1 - S_3)\\(\mathcal{B} - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_2)(1 - S_4)\end{aligned} \tag{12}$$

to

$$\begin{aligned}\mathcal{B} - q_{12}q_{23}S_2 &= 0\\\mathcal{B} - q_{23}q_{31}S_3 &= 0\end{aligned} \tag{13}$$

Thus the resulting quadrea is $\mathcal{B} = q_{12}q_{23}S_2 = q_{23}q_{31}S_3$ which is the quadrea of the triangle $\overline{a_1a_2a_3}$ after the points a_1 and a_4 coalesce and the quadrangle reduces to a triangle.

☐

Corollary 5.5.3 (Projective Pythagoras Dual from Quadrangle Quadratic Equation Pair)**.** *If one of the spreads in the quadrangle is equal to zero and the adjacent line has quadrance equal to one, then the dual form of the projective Pythagoras formula emerges from the pair of quadrangle quadratic equations 12.*

Proof. Suppose we have a projective quadrangle with spread the spread $S_4 = 0$ and quadrance $q_{31} = 1$. Then the pair of quadratic equations will change from

$$\begin{aligned}(\mathcal{B} - q_{12}q_{23}S_2 - q_{34}q_{41}S_4)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_1)(1 - S_3)\\(\mathcal{B} - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_2)(1 - S_4)\end{aligned} \tag{14}$$

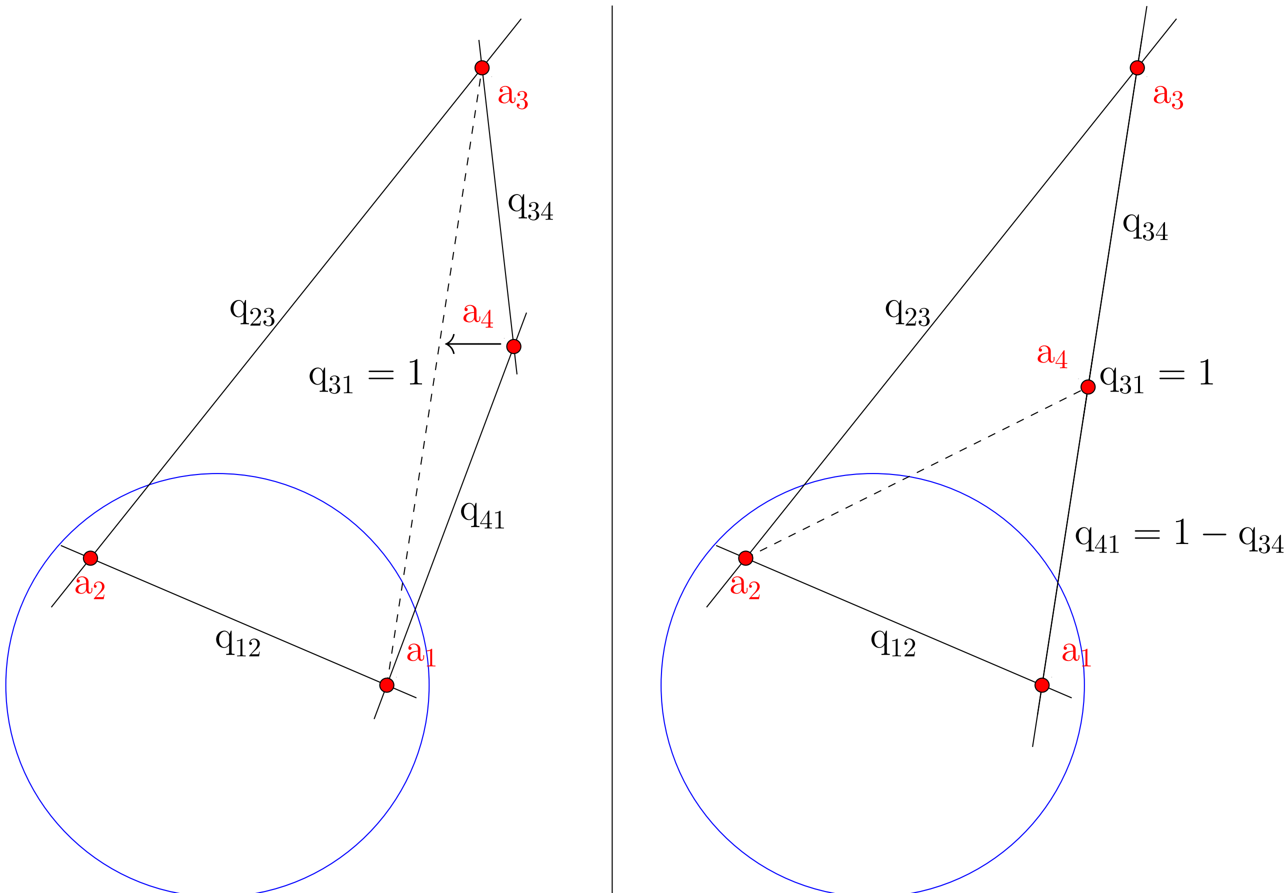
to

$$\begin{aligned}(\mathcal{B} - q_{12}q_{23}S_2)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_1)(1 - S_3)\\(\mathcal{B} - q_{41}q_{12}S_1 - q_{23}q_{34}S_3)^2 &= q_{12}q_{23}q_{34}q_{41}(1 - S_2)\end{aligned} \tag{15}$$

Since the spread $S_4 = 0$ then points $a_3, a_4, and a_1$ are collinear and form a quad triple. Also since $q_{13} = 1$ then $q_{41} = 1 - q34$. An alternate form of the quadrea of a triangle is base quadrance times altitude

$$(1 - S_2 = (1 - S_1)(1 - S_3) \tag{16}$$

where S_2 is opposite to $q_{13} = 1$ in triangle $\overline{a_1 a_2 a_3}$. Equation 16 is the dual form of the projective Pythagoras formula.



$q_{31} = 1$, Move a_4 onto line $\overline{a_1 a_3}$

$q_{31} = 1$ and a_4 is on line $\overline{a_3 a_1}$ so quadrea of $\overline{a_1 a_2 a_3}$ = quadrea of $\overline{a_1 a_2 a_4}$+ quadrea of $\overline{a_2 a_3 a_4}$

Figure 5: Dual projective Pythagoras formula emerges from quadrea of quadrangle equations for figure on right.

□

6 Conclusions and Further Work

1. Conclusion 1: Recapitulate your findings. This is what I found.

2. Conclusion 2: List your limitations

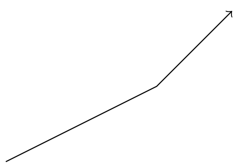
3. What are the implications? For future research. What are the next steps for future research.

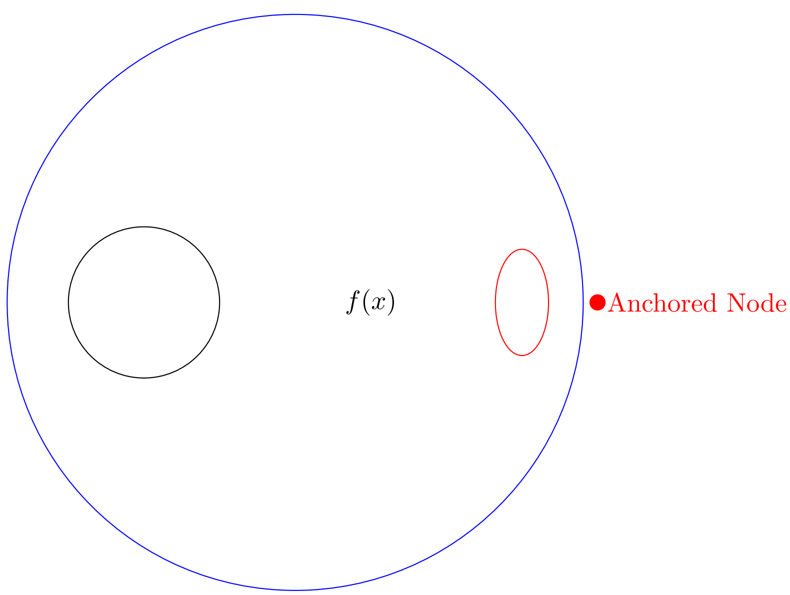
We have discovered exact equations for the quadrea of a projective quadrangle and its dual the quadreal of a projective quadrilateral in Universal Hyperbolic Geometry. These are scale invariant quantities which can be computed exactly. The exact same equations work in both hyperbolic and Elliptic or Spherical projective geometry. This is result in pure mathematics. If two of the projective points of a quadrangle coalesce then the quadrance between those points is zero then the shape becomes to a triangle and the equation of the quadrea of the quadrangle reduces to the quadrea of a triangle.

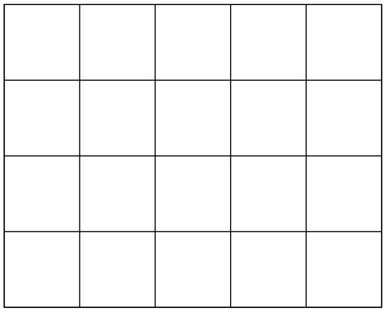
The limitations of these equations is that the quadrances and spreads used in the equations must exist. That is the quadrangle or quadrilateral does not contain null points or null lines for its vertices or sides (boundaries) of the quadrangle or dual quadrilateral.

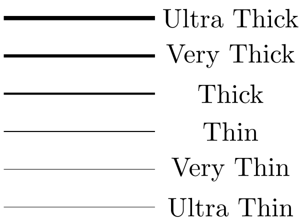
The implications for this result is that it opens up exploration of the projective quadrangle and quadrilateral in UHG. The projective triangle in UHG is well understood but the projective quadrangle in UHG can be explored and new discoveries made.

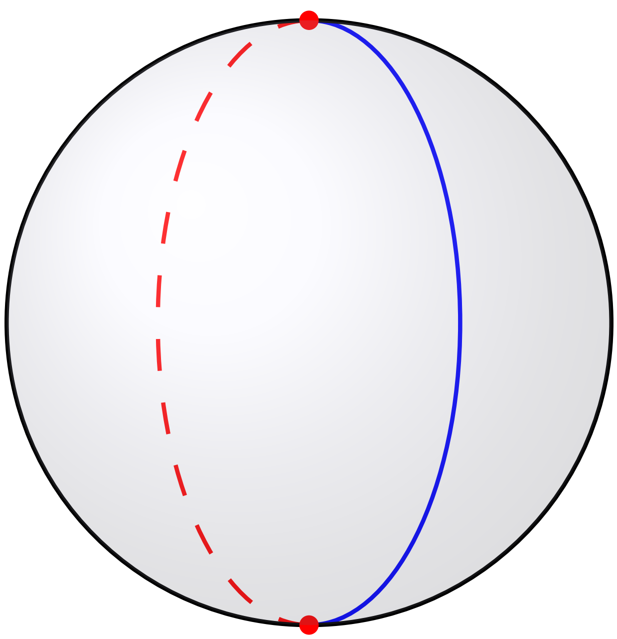
7 Tikz








Ultra Thick
Very Thick
Thick
Thin
Very Thin
Ultra Thin



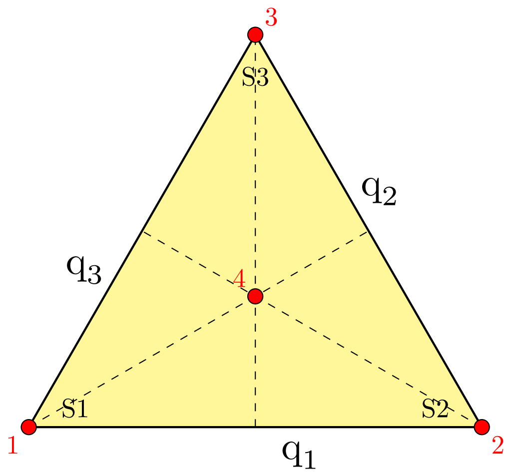
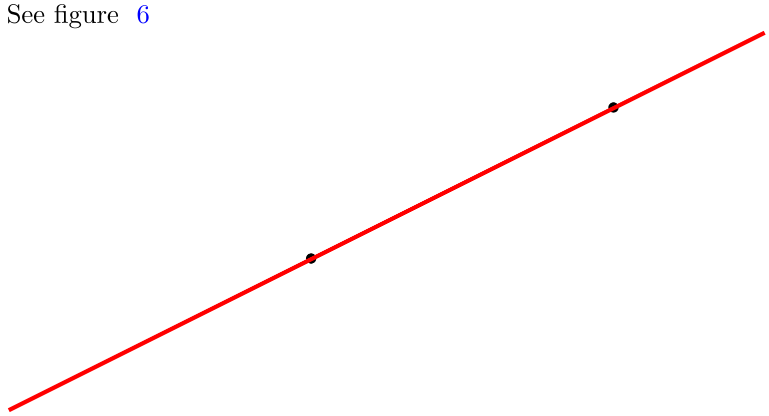


Figure 6: My sweet triangle.

7.1 Doubly Right Quadrangle

There is a red dot in the middle of this sentence.

See figure 6



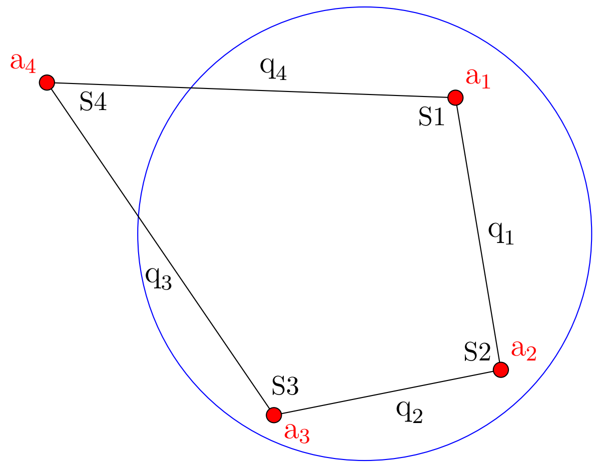


Figure 7: UHG Projective Quadrangle

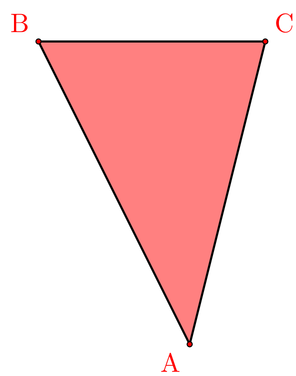


Figure 8: My sweet triangle2 .

A Appendix

$$\mathbf{ArchimedesFunction[p1\_, q1\_, p2\_, q2\_] := \left((p1 - p2)^2 - (q1 + q2)\right)^2 - 4q1 * q2}$$

$$\mathbf{compatibilityCheck = ArchimedesFunction\,[\boldsymbol{q_1 q_2 S_2 + q_3 q_4 S_4, 4q_1 q_2 q_3 q_4 (1 - S_1)(1 - S_3)},}$$
$$\mathbf{\boldsymbol{q_1 q_4 S_1 + q_2 q_3 S_3, 4q_1 q_2 q_3 q_4 (1 - S_2)(1 - S_4)}]}$$

$$-64q_1^2q_2^2q_3^2q_4^2(1 - S_1)(1 - S_2)(1 - S_3)(1 - S_4) + \big(-4q_1q_2q_3q_4(1 - S_1)(1 - S_3) - 4q_1q_2q_3q_4(1 - S_2)(1 - S_4) + (-q_1q_4S_1$$

$$\mathbf{compatible = Expand[compatibilityCheck]}$$

$$16q_1^2q_2^2q_3^2q_4^2S_1^2 - 16q_1^3q_2q_3q_4^3S_1^2 + 8q_1^3q_2q_3q_4^3S_1^3 + q_1^4q_4^4S_1^4 + 32q_1^3q_2^2q_3q_4^2S_1S_2 - 32q_1^2q_2^2q_3^2q_4^2S_1S_2 - 16q_1^3q_2^2q_3q_4^2S_1^2S_2 +$$

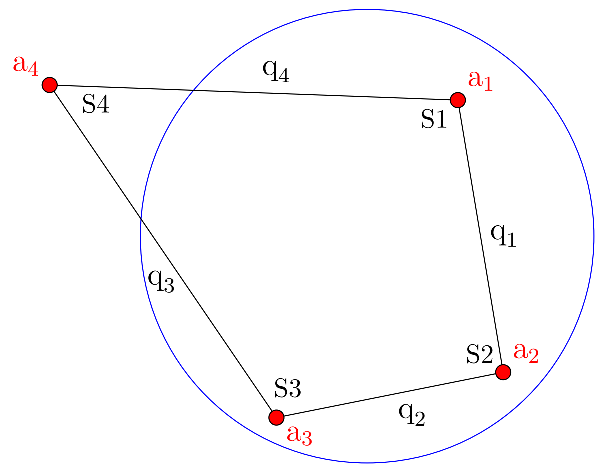


Figure 9: UHG Standard Triangle x

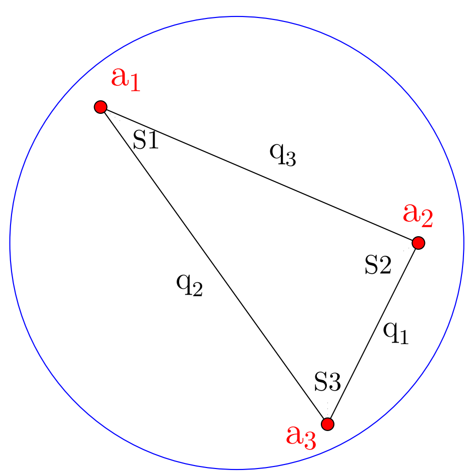


Figure 10: Standard Triangle 1

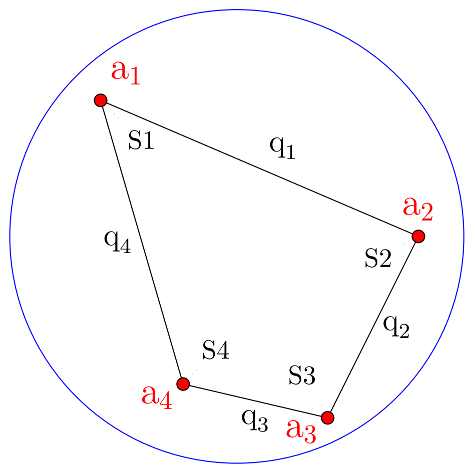


Figure 11: Standard Quadrangle 1

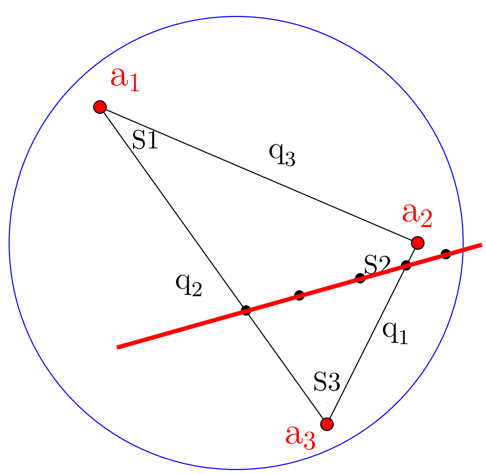


Figure 12: Standard Quadrangle 1 constructed from ST1 and ortho-axis

$8q_1^3q_2q_3q_4^3S_1^2S_2 - 4q_1^4q_2q_4^3S_1^3S_2 - 16q_1^3q_2^3q_3q_4S_2^2 + 16q_1^2q_2^2q_3^2q_4^2S_2^2 + 8q_1^3q_2^3q_3q_4S_1S_2^2 - 16q_1^3q_2^2q_3q_4^2S_1S_2^2 +$
$6q_1^4q_2^2q_4^2S_1^2S_2^2 + 8q_1^3q_2^3q_3q_4S_2^3 - 4q_1^4q_2^3q_4S_1S_2^3 + q_1^4q_2^4S_2^4 - 16q_1^2q_2^2q_3^2q_4^2S_1^2S_3 + 8q_1^3q_2q_3q_4^3S_1^2S_3 - 4q_1^3q_2q_3q_4^3S_1^3S_3 +$
$32q_1^2q_2^3q_3^2q_4S_2S_3 - 32q_1^2q_2^2q_3^2q_4^2S_2S_3 - 16q_1^2q_2^3q_3^2q_4S_1S_2S_3 - 16q_1^3q_2^2q_3q_4^2S_1S_2S_3 + 48q_1^2q_2^2q_3^2q_4^2S_1S_2S_3 +$
$4q_1^3q_2^2q_3q_4^2S_1^2S_2S_3 + 8q_1^3q_2^3q_3q_4S_2^2S_3 - 16q_1^2q_2^3q_3^2q_4S_2^2S_3 + 4q_1^3q_2^3q_3q_4S_1S_2^2S_3 - 4q_1^3q_2^4q_3S_2^3S_3 - 16q_1q_2^3q_3^3q_4S_3^2 +$
$16q_1^2q_2^2q_3^2q_4^2S_3^2 + 8q_1q_2^3q_3^3q_4S_1S_3^2 - 16q_1^2q_2^2q_3^2q_4^2S_1S_3^2 + 6q_1^2q_2^2q_3^2q_4^2S_1^2S_3^2 - 16q_1^2q_2^3q_3^2q_4S_2S_3^2 + 8q_1q_2^3q_3^3q_4S_2S_3^2 +$
$4q_1^2q_2^3q_3^2q_4S_1S_2S_3^2 + 6q_1^2q_2^4q_3^2S_2^2S_3^2 + 8q_1q_2^3q_3^3q_4S_3^3 - 4q_1q_2^3q_3^3q_4S_1S_3^3 - 4q_1q_2^4q_3^3S_2S_3^3 + q_2^4q_3^4S_3^4 - 32q_1^2q_2^2q_3^2q_4^2S_1S_4 +$
$32q_1^2q_2q_3^2q_4^3S_1S_4 + 8q_1^3q_2q_3q_4^3S_1^2S_4 - 16q_1^2q_2q_3^2q_4^3S_1^2S_4 - 4q_1^3q_3q_4^4S_1^3S_4 - 16q_1^3q_2^2q_3q_4^2S_1S_2S_4 + 48q_1^2q_2^2q_3^2q_4^2S_1S_2S_4 -$
$16q_1^2q_2q_3^2q_4^3S_1S_2S_4 + 4q_1^3q_2q_3q_4^3S_1^2S_2S_4 + 8q_1^3q_2^3q_3q_4S_2^2S_4 - 16q_1^2q_2^2q_3^2q_4^2S_2^2S_4 + 4q_1^3q_2^2q_3q_4^2S_1S_2^2S_4 -$
$4q_1^3q_2^3q_3q_4S_2^3S_4 - 32q_1^2q_2^2q_3^2q_4^2S_3S_4 + 32q_1q_2^2q_3^3q_4^2S_3S_4 + 48q_1^2q_2^2q_3^2q_4^2S_1S_3S_4 - 16q_1q_2^2q_3^3q_4^2S_1S_3S_4 - 16q_1^2q_2q_3^2q_4^3S_1S_3S_4 +$
$4q_1^2q_2q_3^2q_4^3S_1^2S_3S_4 - 16q_1^2q_2^3q_3^2q_4S_2S_3S_4 + 48q_1^2q_2^2q_3^2q_4^2S_2S_3S_4 - 16q_1q_2^2q_3^3q_4^2S_2S_3S_4 - 40q_1^2q_2^2q_3^2q_4^2S_1S_2S_3S_4 +$
$4q_1^2q_2^3q_3^2q_4S_2^2S_3S_4 + 8q_1q_2^3q_3^3q_4S_3^2S_4 - 16q_1q_2^2q_3^3q_4^2S_3^2S_4 + 4q_1q_2^2q_3^3q_4^2S_1S_3^2S_4 + 4q_1q_2^3q_3^3q_4S_2S_3^2S_4 - 4q_2^3q_3^4q_4S_3^3S_4 +$
$16q_1^2q_2^2q_3^2q_4^2S_4^2 - 16q_1q_2q_3^3q_4^3S_4^2 - 16q_1^2q_2q_3^2q_4^3S_1S_4^2 + 8q_1q_2q_3^3q_4^3S_1S_4^2 + 6q_1^2q_3^2q_4^4S_1^2S_4^2 - 16q_1^2q_2^2q_3^2q_4^2S_2S_4^2 +$
$8q_1q_2q_3^3q_4^3S_2S_4^2 + 4q_1^2q_2q_3^2q_4^3S_1S_2S_4^2 + 6q_1^2q_2^2q_3^2q_4^2S_2^2S_4^2 - 16q_1q_2^2q_3^3q_4^2S_3S_4^2 + 8q_1q_2q_3^3q_4^3S_3S_4^2 + 4q_1q_2q_3^3q_4^3S_1S_3S_4^2 +$
$4q_1q_2^2q_3^3q_4^2S_2S_3S_4^2 + 6q_2^2q_3^4q_4^2S_3^2S_4^2 + 8q_1q_2q_3^3q_4^3S_4^3 - 4q_1q_3^3q_4^4S_1S_4^3 - 4q_1q_2q_3^3q_4^3S_2S_4^3 - 4q_2q_3^4q_4^3S_3S_4^3 +$
$q_3^4q_4^4S_4^4$

$\boldsymbol{a_1 = \{x_1, y_1, z_1\}}$

$\boldsymbol{a_2 = \{x_2, y_2, z_2\}}$

$\boldsymbol{a_3 = \{x_3, y_3, z_3\}}$

$\boldsymbol{a_4 = \{x_4, y_4, z_4\}}$

$\{x_1, y_1, z_1\}$

$\{x_2, y_2, z_2\}$

$\{x_3, y_3, z_3\}$

$\{x_4, y_4, z_4\}$

J[a1_, a2_]:={a1[[2]]a2[[3]] − a1[[3]]a2[[2]], a1[[3]]a2[[1]] − a1[[1]]a2[[3]], a1[[2]]a2[[1]] − a1[[1]]a2[[2]]}

quadrance[a1_, a2_]:=

1 − (a1[[1]]a2[[1]] + a1[[2]]a2[[2]] − a1[[3]]a2[[3]])^2/

$((\mathrm{a1}[[1]]\mathrm{a1}[[1]] + \mathrm{a1}[[2]]\mathrm{a1}[[2]] - \mathrm{a1}[[3]]\mathrm{a1}[[3]])(\mathrm{a2}[[1]]\mathrm{a2}[[1]] + \mathrm{a2}[[2]]\mathrm{a2}[[2]] - \mathrm{a2}[[3]]\mathrm{a2}[[3]]))$

spread[l1_, l2_]:=

$1 - (\mathrm{l1}[[1]]\mathrm{l2}[[1]] + \mathrm{l1}[[2]]\mathrm{l2}[[2]] - \mathrm{l1}[[3]]\mathrm{l2}[[3]])^2/$

$((\mathrm{l1}[[1]]\mathrm{l1}[[1]] + \mathrm{l1}[[2]]\mathrm{l1}[[2]] - \mathrm{l1}[[3]]\mathrm{l1}[[3]])(\mathrm{l2}[[1]]\mathrm{l2}[[1]] + \mathrm{l2}[[2]]\mathrm{l2}[[2]] - \mathrm{l2}[[3]]\mathrm{l2}[[3]]))$

$q_1 = \text{quadrance}\,[a_1, a_2]$

$q_2 = \text{quadrance}\,[a_2, a_3]$

$q_3 = \text{quadrance}\,[a_3, a_4]$

$q_4 = \text{quadrance}\,[a_4, a_1]$

$1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)}$

$1 - \frac{(x_2x_3+y_2y_3-z_2z_3)^2}{\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}$

$1 - \frac{(x_3x_4+y_3y_4-z_3z_4)^2}{\left(x_3^2+y_3^2-z_3^2\right)\left(x_4^2+y_4^2-z_4^2\right)}$

$1 - \frac{(x_1x_4+y_1y_4-z_1z_4)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_4^2+y_4^2-z_4^2\right)}$

$L_1 = J\,[a_1, a_2]$

$L_2 = J\,[a_2, a_3]$

$L_3 = J\,[a_3, a_4]$

$L_4 = J\,[a_4, a_1]$

$\{-y_2z_1 + y_1z_2, x_2z_1 - x_1z_2, x_2y_1 - x_1y_2\}$

$\{-y_3z_2 + y_2z_3, x_3z_2 - x_2z_3, x_3y_2 - x_2y_3\}$

$\{-y_4z_3 + y_3z_4, x_4z_3 - x_3z_4, x_4y_3 - x_3y_4\}$

$\{y_4z_1 - y_1z_4, -x_4z_1 + x_1z_4, -x_4y_1 + x_1y_4\}$

$S_1 = \text{spread}\,[L_4, L_1]$

$S_2 = \text{spread}\,[L_1, L_2]$

$S_3 = \text{spread}\,[L_2, L_3]$

$S_4 = \text{spread}\,[L_3, L_4]$

$1 - \frac{(-((x_2y_1-x_1y_2)(-x_4y_1+x_1y_4))+(x_2z_1-x_1z_2)(-x_4z_1+x_1z_4)+(-y_2z_1+y_1z_2)(y_4z_1-y_1z_4))^2}{(-(x_2y_1-x_1y_2)^2+(x_2z_1-x_1z_2)^2+(-y_2z_1+y_1z_2)^2)(-(-x_4y_1+x_1y_4)^2+(-x_4z_1+x_1z_4)^2+(y_4z_1-y_1z_4)^2)}$

$$1-\frac{(-((x_2y_1-x_1y_2)(x_3y_2-x_2y_3))+(x_2z_1-x_1z_2)(x_3z_2-x_2z_3)+(-y_2z_1+y_1z_2)(-y_3z_2+y_2z_3))^2}{(-(x_2y_1-x_1y_2)^2+(x_2z_1-x_1z_2)^2+(-y_2z_1+y_1z_2)^2)(-(x_3y_2-x_2y_3)^2+(x_3z_2-x_2z_3)^2+(-y_3z_2+y_2z_3)^2)}$$

$$1-\frac{(-((x_3y_2-x_2y_3)(x_4y_3-x_3y_4))+(x_3z_2-x_2z_3)(x_4z_3-x_3z_4)+(-y_3z_2+y_2z_3)(-y_4z_3+y_3z_4))^2}{(-(x_3y_2-x_2y_3)^2+(x_3z_2-x_2z_3)^2+(-y_3z_2+y_2z_3)^2)(-(x_4y_3-x_3y_4)^2+(x_4z_3-x_3z_4)^2+(-y_4z_3+y_3z_4)^2)}$$

$$1-\frac{(-((-x_4y_1+x_1y_4)(x_4y_3-x_3y_4))+(-x_4z_1+x_1z_4)(x_4z_3-x_3z_4)+(y_4z_1-y_1z_4)(-y_4z_3+y_3z_4))^2}{(-(-x_4y_1+x_1y_4)^2+(-x_4z_1+x_1z_4)^2+(y_4z_1-y_1z_4)^2)(-(x_4y_3-x_3y_4)^2+(x_4z_3-x_3z_4)^2+(-y_4z_3+y_3z_4)^2)}$$

result = Factor[compatible]

0

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