Governing Equations (02/11/16)

Most of the following is taken from a couple of my jointly co-authored papers, but with some commentary added now in red and blue – this use of two colours is not very important, just to separate out the big block later in the text.

Within the domain Ω Maxwell's equations can be condensed to the condition

$$\nabla \cdot (\sigma \nabla \phi) = 0 \tag{1}$$

for the conductivity vector $\sigma \in [0, +\infty)^n$ with $\sigma = 1/\rho$ where ρ is the resistivity.

So the unknown to be estimated is σ within the domain.

The boundary of the domain ($\partial\Omega$) will comprise electrodes E_k ($k = 1, 2, \ldots, K$), and gaps where the boundary is insulating. Typically, $K = 8$. Current pattern $\underline{I} = \{I_1, I_2, \ldots, I_K\}$, with I_k the amplitude of the current injected through the electrode E_k, is applied to the electrodes.

In the case of the reference protocol, the $K - 1 = 7$ current patterns are:

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( 1, -1,  0,  0,  0,  0,  0,  0)
( 1,  0, -1,  0,  0,  0,  0,  0)
( 1,  0,  0, -1,  0,  0,  0,  0)
( 1,  0,  0,  0, -1,  0,  0,  0)
( 1,  0,  0,  0,  0, -1,  0,  0)
( 1,  0,  0,  0,  0,  0, -1,  0)
( 1,  0,  0,  0,  0,  0,  0, -1)
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Appropriate boundary conditions on the electrodes are

$$\left.\left(\phi + \zeta_k \sigma \frac{\partial \phi}{\partial \underline{n}}\right)\right|_{E_k} = U_k,$$

which defines the effect of the contact contact impedance and

$$\int_{E_k} \sigma \frac{\partial \phi}{\partial \underline{n}} dS = I_k, \qquad k = 1, 2, \ldots K,$$

which describes conservation of charge, whilst on the insulating boundaries between electrodes

$$\sigma \frac{\partial \phi}{\partial \underline{n}}\bigg|_{\partial\Omega \setminus \bigcup_{k=1}^{K} E_k} = 0, \tag{2}$$

where U_k denotes the potential on the k^{th} electrode E_k and $\underline{n}$ is the outward unit normal of the boundary.

In order to make the problem well posed and possess a unique solution, the following conditions must also be met

$$\sum_{k=1}^{K} I_k = 0$$

Which seems to be met by "physical" necessity and hence I am not sure if it is really needed and

$$\sum_{k=1}^{K} V_k = 0$$

which describe conservation of charge and specification of ground voltage respectively. First is clearly OK, but how does the second specify the ground voltage? The equations above describe the so-called complete electrode model of electrical tomography. The voltages as defined by equation (3) are used as the electrical tomography measurement data. This paper uses the 'reference measurement protocol' in which a single electrode is grounded and is always part of the drive and measurement circuits. To me this is saying $U_1 = 0$.

In all the papers I have looked at this morning (embarrassingly, mostly involving me as a co-author) the relationship between U and V is a bit unclear and so here is my current interpretation...

In truth I think that some of the equations follow from the statements, rather than being the mathematical equivalent.

So I believe that U is the electrical potential, but that V is a voltage, that is potential difference. So the voltage measured between Electrode 1, the reference electrode, and Electrode k is

$$V_{1,k}^{*} = U_k - U_1, \qquad k = 2, 3, \ldots, K.$$

However, if we ground Electrode 1, then I think we have $U_1 = 0$, and $V_k^* = V_{1,k}^* = U_k$ dropping the "$,1$" as it provides no useful information.

The dependence of this on the current pattern is also lost in the notation, but perhaps we can make it explicit in future notation.

We might define

$$V_{1,k}^{1,j} = U_k^{1,j} - U_1^{1,j}$$

where the superscript is defining the electrodes which are part of the drive circuit; perhaps simplify to $V_{j,k} = V_{1,k}^{1,j}$. Then the calculated voltage vector, that is noise-free data, is

$$\underline{V}^* = \{V_{j,k} : j = 2, 3, \ldots, K, k = 2, 3, \ldots, K\}.$$

The currents and voltages are given, the former are part of the experimental design and the latter are measured.

The contact impedances, ζ, are in truth unknown but estimates might come from some other experiment. I guess we will, at least initially, take these as given.

The data are related to the resistivities through a measurement model

$$\underline{V} = \underline{V}^*(\underline{\rho}, \underline{\zeta}) + \underline{\epsilon},$$

where $\underline{V}^*(\underline{\rho}, \underline{\zeta})$ are calculated from the forward model. This states that the measured voltages are equal to the theoretical values plus a, usually small, measurement error. It is assumed that the errors are independent and have common variance, which has been confirmed through extensive practical experience with the instrumentation. This leads to the likelihood: the conditional distribution of $\underline{V}$ given $\underline{\rho}$ and $\underline{\zeta}$, defined as $\underline{V}|\underline{\rho}, \underline{\zeta} \sim N(\underline{V}^*(\underline{\rho}, \underline{\zeta}), \sigma^2 I)$, with density function

$$\pi(\underline{V}|\underline{\rho}, \underline{\zeta}) = \frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left\{-\frac{1}{2\sigma^2}||\underline{V} - \underline{V}^*(\underline{\rho}, \underline{\zeta})||^2\right\}. \tag{3}$$