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Theory & Methods Section

Measures of Association for Cross Classifications, IV: Simplification of Asymptotic Variances

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The asymptotic sampling theory discussed in our 1963 article [3] for measures of association presented in earlier articles [1, 2] turns on the derivation of asymptotic variances that may be complex and tedious in specific cases. In the present article, we simplify and unify these derivations by exploiting the expression of measures of association as ratios. Comments on the use of asymptotic variances, and on a trap in their calculation, are also given.

1. INTRODUCTION AND SUMMARY

In our 1963 article [3], we discussed asymptotic sampling theory for some of the measures of association presented in our earlier articles [1, 2]. It would have been impractical to present in [3] asymptotic results for many measures under many sampling methods, so we gave results only for some of the more important cases, together with general methods so that others might more readily do their own asymptotics.

In the present article, we present a more unified way to derive asymptotic variances (which form the nub of the asymptotic theory) for the following two sampling methods:

a. Multinomial sampling over the entire two-way cross classification;
b. Independent multinomial sampling in the rows when total row proportions are known; that is, stratified sampling in the rows.

We then apply this more unified view to several measures of association and obtain fresh formulas for asymptotic variances. Further to illustrate the method, we rederive a few of the asymptotic variances in [3]. We also take this opportunity to correct (4.4.3) of [3]. Use of the asymptotic variances in practice is also discussed, and finally we give a cautionary note about a trap when working out asymptotic variances.

The basic notion of this article is to exploit the expression of most measures of association as ratios, and to do a portion of the manipulations towards asymptotic variances in advance.

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2. MULTINOMIAL SAMPLING OVER THE ENTIRE TWO-WAY CROSS CLASSIFICATION

2.1. The General Case

As before, let ρ_{ab} be the probability for the a, b cell of the cross classification, and let R_{ab} be the corresponding observed proportion. The ranges are $a=1, 2, \cdots, \alpha$ and $b=1, 2, \cdots, \beta$. The ρ_{ab} are unrestricted, except of course that they are non-negative and add to one.

Nearly all the measures of association we presented in our earlier articles were written as ratios. We exploit that structure and consider a generic measure in the form $\zeta=\nu/\delta$, where ν and δ are mnemonically chosen to stand for νumerator and δenominator; ζ is the generic measure of association, although it could be any ratio of two functions, ν and δ, of the ρ_{ab}'s. (We assume that $\delta\neq 0$ and that ν and δ are differentiable at the needed values of their arguments.)

In this section we assume a multinomial sample of size n over the entire $\alpha\times\beta$ cross classification, so that the R_{ab}'s are the maximum likelihood estimators of the ρ_{ab}'s. It is well known that the covariance between $\sqrt{n}(R_{ab}-\rho_{ab})$ and $\sqrt{n}(R_{a'b'}-\rho_{a'b'})$ is

$$\overset{K}{\delta}_{aa'}\overset{K}{\delta}_{bb'}\rho_{ab} - \rho_{ab}\rho_{a'b'}, \tag{2.1}$$

where the δ^K here is the Kronecker delta; the K superscript is used to avoid confusion with the denominator delta above.

The maximum likelihood estimators of ζ, ν, and δ are the same functions of the R_{ab}'s as ζ, ν, and δ are of the ρ_{ab}'s; we call these Z, N, and D, respectively, so that $Z=N/D$, and our concern is with the variance of the asymptotic normal distribution of $\sqrt{n}(Z-\zeta)$. (See Section 4 for comments on the possibility that $D=0$.) Let us define

$$\nu'_{ab} = \partial\nu/\partial\rho_{ab}, \qquad \delta'_{ab} = \partial\delta/\partial\rho_{ab}, \tag{2.2}$$

since these partial derivatives enter frequently. It is convenient also to define

$$\begin{aligned}\phi_{ab} &= \nu\delta'_{ab} - \delta\nu'_{ab},\\ \bar{\phi} &= \textstyle\sum_{a,b}\rho_{ab}\phi_{ab}.\end{aligned} \tag{2.3}$$

Note that $\bar{\phi}$ is a weighted average of the ϕ_{ab}'s.

Then by applying the delta method and the supplementary tools presented in [3], it is easily calculated that the asymptotic variance of $\sqrt{n}(Z-\zeta)$ is

$$\sigma^2 = \frac{1}{\delta^4} \sum_{a,b} \rho_{ab}(\phi_{ab} - \bar{\phi})^2 = \frac{1}{\delta^4}\left\{ \sum_{a,b} \rho_{ab}\phi_{a,b}^2 - \bar{\phi}^2 \right\}. \tag{2.4}$$

This is a simple general form for the asymptotic variance. We proceed to three examples, the first a reprise from [3] and the second two new.

2.2. The Measure of Association Gamma

In [1] we discussed a measure of association called γ that might be appropriate when both polytomies of the cross classification are ordered. For convenience we recapitulate the definition of γ; its interpretation is given in [1]. In the present setting, γ is most readily defined by a short chain of definitions,

$$\begin{aligned} \nu &= \Pi_s - \Pi_d, & \delta &= \Pi_s + \Pi_d, \\ \Pi_s &= 2\sum_{a,b} \rho_{ab}\Pi_{\mathrm{I};ab}, & \Pi_d &= 2\sum_{a,b} \rho_{ab}\Pi_{\mathrm{IV};ab}, \\ \Pi_{\mathrm{I};ab} &= \sum_{a'>a}\sum_{b'>b} \rho_{a'b'}, & \Pi_{\mathrm{IV};ab} &= \sum_{a'>a}\sum_{b'<b} \rho_{a'b'}. \end{aligned} \tag{2.5}$$

The Roman numeral subscripts are mnemonics for the "quadrants" relative to (a, b). (Quadrants here refer to the conventional Cartesian system with $a'-a$ corresponding to the horizontal axis and $b'-b$ to the vertical.)

To calculate the derivatives it helps to look at an example: what is the derivative of Π_s with respect to ρ_{22}? Think of Π_s as two times

$$\rho_{11}\Pi_{\mathrm{I};11} + \rho_{12}\Pi_{\mathrm{I};12} + \rho_{21}\Pi_{\mathrm{I};21} + \rho_{22}\Pi_{\mathrm{I};22} + \cdots.$$

Then one term of the derivative comes from the fourth summand above: $2\Pi_{\mathrm{I};22}$. Another term comes from the first summand: $2\rho_{11}$. No other summands contribute terms, and hence the desired derivative is $2(\Pi_{\mathrm{I};22}+\rho_{11})$. In general, the derivative of Π_s with respect to ρ_{ab} is $2(\Pi_{\mathrm{I};ab}+\Pi_{\mathrm{III};ab})$, where

$$\Pi_{\mathrm{III};ab} = \sum_{a'<a}\sum_{b'<b} \rho_{a'b'}$$

in accord with our mnemonic.[1] Similarly, the derivative of Π_d with respect to ρ_{ab} is $2(\Pi_{\mathrm{IV};ab}+\Pi_{\mathrm{II};ab})$, and hence

$$\nu'_{ab} = 2(\Pi_{\mathrm{I};ab} - \Pi_{\mathrm{II};ab} + \Pi_{\mathrm{III};ab} - \Pi_{\mathrm{IV};ab}),$$

$$\delta'_{ab} = 2(\Pi_{\mathrm{I};ab} + \Pi_{\mathrm{II};ab} + \Pi_{\mathrm{III};ab} + \Pi_{\mathrm{IV};ab}).$$

In terms of the notation of A7 of [3, p. 362],

$$\nu'_{ab} = 2(\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)}), \qquad \delta'_{ab} = 2(\mathcal{R}_{ab}^{(s)} + \mathcal{R}_{ab}^{(d)}).$$

Thus the general expression of (2.3) becomes

$$\begin{aligned} \phi_{ab} &= 2(\Pi_s - \Pi_d)(\mathcal{R}_{ab}^{(s)} + \mathcal{R}_{ab}^{(d)}) \\ &\quad - 2(\Pi_s + \Pi_d)(\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)}) \\ &= 4[\Pi_s\mathcal{R}_{ab}^{(d)} - \Pi_d\mathcal{R}_{ab}^{(s)}], \\ \bar{\phi} &= 4\sum_{a,b} \rho_{ab}[\Pi_s\mathcal{R}_{ab}^{(d)} - \Pi_d\mathcal{R}_{ab}^{(s)}] = 0 \end{aligned} \tag{2.6}$$

since $\sum \rho_{ab}\mathcal{R}_{ab}^{(d)} = \Pi_d$ and $\sum \rho_{ab}\mathcal{R}_{ab}^{(s)} = \Pi_s$.

Hence, if G is the sample analog of γ, the asymptotic variance of $\sqrt{n}(G-\gamma)$ is

$$\frac{16}{(\Pi_s + \Pi_d)^4} \sum_{a,b} \rho_{ab}[\Pi_s\mathcal{R}_{ab}^{(d)} - \Pi_d\mathcal{R}_{ab}^{(s)}]^2. \tag{2.7}$$

Except for minor notational differences, and the carrying out of the square, (2.7) is the same as (3.5.5) of [3]. (To see the equivalence, note that $\Pi_s+\Pi_d = 1-\Pi_t$, and refer to the manipulations at the bottom of p. 362 of [3].) For some purposes the present form may be simpler than the form in [3].

2.3. Somers' Asymmetrical Δ_{ba}

Somers [5] has introduced an asymmetrical modification[2] of γ,

$$\Delta_{ba} = \frac{\Pi_s - \Pi_d}{1 - \sum \rho_{a\cdot}^2}, \tag{2.8}$$

together with its mate Δ_{ab}, with the denominator replaced by $1-\sum \rho_{\cdot b}^2$. (Following our past convention, a dot replacing a subsubscript means summation over the replaced subscript, e.g., $\rho_{a\cdot} = \sum_b \rho_{ab}$). The denominator of Δ_{ba} is the probability that two independently chosen units from a population governed by the $\{\rho_{ab}\}$ cross classification probabilities do not lie in the same row (are not tied in a).

An interpretation of Δ_{ba} may be given in terms of two such independently chosen units; let us call their (random) row and column numbers $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{a}', \boldsymbol{b}')$, respectively, and say that the units are weakly concordant when $(\boldsymbol{a}-\boldsymbol{a}')(\boldsymbol{b}-\boldsymbol{b}')\geq 0$, i.e., when the order of the two columns is the same as that of the two rows or when there is a tie. Similarly we may define weak discordance as $(\boldsymbol{a}-\boldsymbol{a}')\cdot(\boldsymbol{b}-\boldsymbol{b}')\leq 0$. Then the conditional probability of weak concordance, given that there is a difference between the rows, is $[\Pi_s+\sum_b \rho_{\cdot b}^2 - \sum_{a,b} \rho_{ab}^2]/[1-\sum_a \rho_{a\cdot}^2]$, and the conditional probability of weak discordance, given $\boldsymbol{a}\neq\boldsymbol{a}'$, is $[\Pi_d+\sum_b \rho_{\cdot b}^2 - \sum_{a,b} \rho_{ab}^2]/[1-\sum_a \rho_{a\cdot}^2]$. Hence Δ_{ba} is the difference between these two conditional probabilities.

It is sometimes useful to write the denominator of Δ_{ba} in the form $\Pi_s+\Pi_d+\sum_{a,b} \rho_{ab}(\rho_{\cdot b}-\rho_{ab})$, where the third summand, which is equal to $\sum_b \rho_{\cdot b}^2 - \sum_{a,b} \rho_{ab}^2$, may be

[1] One might at first be concerned that these calculations of derivatives do not take into account that the sum of all the ρ_{ab} is one. That does not, however, introduce any difficulty so long as we use the correct covariance structure, which necessarily reflects linear restrictions like $\sum_{a,b} \rho_{ab}=1$, or its sample analogue $\sum_{a,b} R_{ab}=1$. There is a related possible difficulty, however, that is discussed in Section 6.

[2] Somers used the symbol d_{ba}. We have changed this to Δ_{ba} in order to use d_{ba} for the sample analogue.

thought of as the probability that two independent random units are tied in column but not in row.

Some important properties of Δ_{ba} are:

1. Δ_{ba} is indeterminate if and only if the population is concentrated in a single *row*;
2. Δ_{ba} is 1 if and only if both Π_d and $\sum_{a,b}\rho_{ab}(\rho_{\cdot b}-\rho_{ab})$ are zero. The second condition says that each column has at most one non-zero cell: hence, after removing all-zero columns, Δ_{ba} is 1 if and only if the non-zero cells descend in staircase fashion, perhaps with treads of unequal width. A similar interpretation holds for $\Delta_{ba}=-1$;
3. Δ_{ba} is 0 in the case of independence, but the converse need not hold except in the 2×2 case.

The numerator ν is the same for Δ_{ba} as for γ, and the denominator δ is $1-\sum \rho_{a\cdot}^2$. Hence ν'_{ab} is the same as for γ and, more simply, $\delta'_{ab}=-2\rho_{a\cdot}$. It follows that

$$\begin{aligned}\phi_{ab} &= -2\rho_{a\cdot}\nu - 2\delta(\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)}),\\ \bar{\phi} &= -2\nu\sum_a \rho_{a\cdot}^2 - 2\delta(\Pi_s - \Pi_d) = -2\nu,\end{aligned} \tag{2.9}$$

and that

$$\phi_{ab} - \bar{\phi} = 2\nu(1-\rho_{a\cdot}) - 2\delta(\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)}). \tag{2.10}$$

Letting d_{ba} denote the sample analogue to Δ_{ba}, the desired asymptotic variance for $\sqrt{n}(d_{ba}-\Delta_{ba})$ is, therefore,

$$\frac{4}{\delta^4}\sum_{a,b}\rho_{ab}\left[\nu(1-\rho_{a\cdot}) - \delta(\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)})\right]^2. \tag{2.11}$$

So far as we know, this result is newly published.

2.4. The Measure of Association τ_b

In [1] we presented an asymmetrical measure of association, τ_b, based on a notion—suggested to us by W. A. Wallis—of reconstructing as best one can the cross classification probabilities. Interpretative details are given in Section 9 of [1]. In the present setting, it is easier to work with $1-\tau_b$ (which will not affect the asymptotic variance) and to express $1-\tau_b$ in terms of its numerator and denominator,

$$\nu = 1 - \sum_{a,b}(\rho_{ab}^2/\rho_{a\cdot}), \qquad \delta = 1 - \sum_b \rho_{\cdot b}^2. \tag{2.12}$$

The maximum likelihood estimator of $1-\tau_b$ is

$$1 - t_b = \frac{1 - \sum_{a,b}(R_{ab}^2/R_{a\cdot})}{1 - \sum_b R_{\cdot b}^2},$$

and we want to find the asymptotic variance of $\sqrt{n}(t_b-\tau_b)$, which is the same as that of $\sqrt{n}[(1-t_b)-(1-\tau_b)]$. Following our general prescription, we find

$$\begin{aligned}\delta'_{ab} &= -2\rho_{\cdot b}\\ \nu'_{ab} &= -\sum_{a',b'}\frac{1}{\rho_{a'\cdot}^2}\left\{2\rho_{a\cdot}\rho_{ab}\delta^K_{aa'}\delta^K_{bb'} - \rho_{a'b'}^2\delta^K_{aa'}\right\}\\ &= -2\rho_{ab}/\rho_{a\cdot} + \sum_{b'}(\rho_{ab'}^2/\rho_{a\cdot}^2),\end{aligned}$$

where δ^K is the Kronecker delta. Thus

$$\begin{aligned}\phi_{ab} &= -2\nu\rho_{\cdot b} + 2\delta\rho_{ab}/\rho_{a\cdot} - \delta\sum_{b'}(\rho_{ab'}^2/\rho_{a\cdot}^2),\\ \bar{\phi} &= -2\nu\sum_b\rho_{\cdot b}^2 + \delta\sum_{a,b}(\rho_{ab}^2/\rho_{a\cdot})\\ &= -2\nu(1-\delta) + \delta(1-\nu)\\ &= -2\nu + \nu\delta + \delta\\ &= (1+\nu-2\tau_b)\delta\\ &= \left[\sum_{a,b}(\rho_{ab}^2/\rho_{a\cdot})\right]\left[1+\sum_b\rho_{\cdot b}^2\right] - 2\sum_b\rho_{\cdot b}^2.\end{aligned} \tag{2.13}$$

(We have expressed $\bar{\phi}$ in several different ways for convenience of reference.) Hence the asymptotic variance of $\sqrt{n}(t_b-\tau_b)$ under full $\alpha\times\beta$ multinomial sampling is

$$\frac{1}{\delta^4}\sum_{a,b}\rho_{ab}(\phi_{ab}-\bar{\phi})^2, \tag{2.14}$$

where δ, ϕ_{ab}, and $\bar{\phi}$ are defined above.

This result is new; we did not discuss the distribution of t_b under full multinomial sampling in [3].

3. INDEPENDENT MULTINOMIAL SAMPLING IN THE ROWS

3.1. The General Case

In Section 4 of [3] we dealt with a stratified sampling method that may sometimes arise. In this method there are separate, independent multinomial samples within each row of the cross classification; further, the sample size in row a is $n_{a\cdot}=n\omega_a$, where the ω_a's are supposed known, positive, and summing to one. (In practice, $n\omega_a$ will not in general be an integer, so one would take $n_{a\cdot}$ as that integer closest to $n\omega_a$. For purposes of asymptotic theory, we need only assume that the $n_{a\cdot}/(n\omega_a)$ have the limit one as n grows large.) We also assume that the $\rho_{a\cdot}$'s are known and positive.

To be specific, we treat separate sampling within rows; if there is separate sampling within columns, one need only interchange the roles of rows and columns. The sampling method under discussion may arise either from stratification of a single population, or in comparing several different populations.

If $\omega_a=\rho_{a\cdot}$, that is if $n_{a\cdot}$ is proportional to $\rho_{a\cdot}$, matters simplify a bit because then R_{ab} is still the maximum likelihood estimator of ρ_{ab}, just as in the case of full multinomial sampling. It was for this reason that we restricted ourselves in [3] to the proportional sampling rate case. The general case, however, is not essentially harder—it requires only carrying multiplicative factors along—and, to save space, we deal with it directly. (This remark qualifies a possibly misleading statement in the last paragraph of Section 4.4 of [3].)

It is convenient to deal with conditional row probabilities, and accordingly we write

$$\tilde{\rho}_{ab} = \rho_{ab}/\rho_{a\cdot}, \qquad \tilde{R}_{ab} = R_{ab}/R_{a\cdot}.$$

for all relevant values of a and b. (For asymptotic purposes, there is no problem about the positiveness of $R_{a\cdot}$, since we have assumed $\omega_a>0$ and $n_{a\cdot}$ is within 1 of $n\omega_a$, but there may be a problem with zero $R_{a\cdot}$ for small finite sample sizes.)

The maximum likelihood estimator of $\bar{\rho}_{ab}$ is clearly $\bar{R}_{ab}$; hence that of ρ_{ab} is $(\rho_{a\cdot}/R_{a\cdot})R_{ab}$, which is almost $(\rho_{a\cdot}/\omega_a)R_{ab}$. Further, the covariance between $\sqrt{n}(\bar{R}_{ab}-\bar{\rho}_{ab})$ and $\sqrt{n}(\bar{R}_{a'b'}-\bar{\rho}_{a'b'})$ is readily seen to be

$$\frac{1}{\omega_a}\overset{K}{\delta}_{aa'}[\overset{K}{\delta}_{bb'}\bar{\rho}_{ab}-\bar{\rho}_{ab}\bar{\rho}_{ab'}]. \tag{3.1}$$

Note the $1/\omega_a$ factor, and the first Kronecker delta outside the brackets, because of independence among the multinomials.

As before, we let $\zeta=\nu/\delta$ be a generic measure of association, but now we regard ζ, ν, and δ as functions of the $\bar{\rho}_{ab}$'s (and, of course, of the known $\rho_{a\cdot}$'s). Then Z, N, D $(Z=N/D)$ are the corresponding sample quantities, obtained by replacing $\bar{\rho}_{ab}$ with $\bar{R}_{ab}$ to obtain maximum likelihood estimators. The $\rho_{a\cdot}$'s stay unchanged since they are known constants.

Next, let ν^*_{ab} and δ^*_{ab} be the partial derivatives of ν and δ, respectively, with respect to $\bar{\rho}_{ab}$. We use asterisks instead of primes to avoid confusion about the argument of differentiation. (Nonetheless, we record the relationships $\nu^*_{ab}=\rho_{a\cdot}\nu'_{ab}$ and $\delta^*_{ab}=\rho_{a\cdot}\delta'_{ab}$.)

As in Section 2, we introduce

$$\overset{+}{\phi}_{ab}=\nu\overset{*}{\delta}_{ab}-\delta\overset{*}{\nu}_{ab}, \qquad \bar{\phi}^+_a=\textstyle\sum_b \bar{\rho}_{ab}\overset{+}{\phi}_{ab}, \tag{3.2}$$

where $\bar{\phi}^+_a$ is a weighted average of the ϕ^+_{ab}'s in row a only. (Some symbolic distinction from the notation of (2.3) is necessary, and the $+$ superscript is suggestive of fixed row marginals.) The methods of [3] then show easily that the asymptotic variance of $\sqrt{n}(Z-\zeta)$ is

$$\begin{aligned}&\frac{1}{\delta^4}\textstyle\sum_a\frac{1}{\omega_a}\sum_b\bar{\rho}_{ab}(\overset{+}{\phi}_{ab}-\overset{+}{\bar{\phi}}_a)^2\\ &\qquad=\frac{1}{\delta^4}\textstyle\sum_a\frac{1}{\omega_a}[\sum_b\bar{\rho}_{ab}\overset{+2}{\phi}_{ab}-\overset{+2}{\bar{\phi}}_a].\end{aligned} \tag{3.3}$$

In short, we obtain here a linear combination of the within-row variabilities of the ϕ^+_{ab}'s (weighted by the $\bar{\rho}_{ab}$'s), rather than the overall variability obtained in Section 2.

This is a simple general form for the asymptotic variance under independent sampling within rows. We illustrate it next with two examples, the first a generalization of Section 4.2 of [3], and the second a generalization and correction of formula (4.4.3) in [3].

3.2. The Measure of Association λ_b

In [1], we presented an asymmetrical measure of association, λ_b, based on the concept of optimal prediction. Interpretative details are given in Section 5.1 of [1]. In the present setting, it is easier to work with $1-\lambda_b$ (which will not affect the asymptotic variance) and to express $1-\lambda_b$ in terms of its numerator and denominator,

$$\nu=1-\textstyle\sum_a\rho_{am}, \qquad \delta=1-\rho_{\cdot m}, \tag{3.4}$$

where ρ_{am} is the maximum of $\rho_{a1},\cdots,\rho_{a\beta}$, and $\rho_{\cdot m}$ is the maximum of $\rho_{\cdot 1},\cdots,\rho_{\cdot\beta}$. We shall assume, as in [3] (see Sec. 3.1 there), that ρ_{am} equals exactly one of the ρ_{ab}, say $\rho_{ab(a)}$, and that $\rho_{\cdot m}$ equals exactly one of the $\rho_{\cdot b}$, say $\rho_{\cdot b(\cdot)}$ $(b=1,\cdots,\beta)$. (Note about symbolism: In [3] we used $\rho_{a(\cdot m)}$ for what is called $\rho_{ab(\cdot)}$ here, where $b_{(\cdot)}$ is the value of the column subscript index maximizing $\rho_{\cdot 1},\cdots,\rho_{\cdot\beta}$; and we did not in [3] need a symbol for what we now call $b_{(a)}$, the value of the column subscript index maximizing $\rho_{a1},\cdots,\rho_{a\beta}$.)

The maximum likelihood estimator of $1-\lambda_b$ is

$$1-L_b=\frac{1-\sum_a(\rho_{a\cdot}\bar{R}_{am})}{1-\operatorname*{Max}_b\sum_a(\rho_{a\cdot}\bar{R}_{ab})},$$

and we want to find the asymptotic variance of $\sqrt{n}(L_b-\lambda_b)$, which is the same as that of $\sqrt{n}[(1-L_b)-(1-\lambda_b)]$. To follow our general prescription,[3] first write

$$\nu=1-\textstyle\sum_a(\rho_{a\cdot}\bar{\rho}_{am}), \qquad \delta=1-\operatorname*{Max}_b\sum_a(\rho_{a\cdot}\bar{\rho}_{ab}). \tag{3.5}$$

Recalling that the $\rho_{a\cdot}$'s are fixed, we differentiate with respect to $\bar{\rho}_{ab}$ to find

$$\overset{*}{\nu}_{ab}=-\rho_{a\cdot}\overset{K}{\delta}_{bb(a)}, \qquad \overset{*}{\delta}_{ab}=-\rho_{a\cdot}\overset{K}{\delta}_{bb(\cdot)}.$$

A schematic sketch of the cross classification will aid in seeing why these are the derivatives. It follows that

$$\begin{aligned}\overset{+}{\phi}_{ab}&=-\nu\rho_{a\cdot}\overset{K}{\delta}_{bb(\cdot)}+\delta\rho_{a\cdot}\overset{K}{\delta}_{bb(a)},\\ \overset{+}{\bar{\phi}}_a&=\delta\rho_{ab(a)}-\nu\rho_{ab(\cdot)}.\end{aligned} \tag{3.6}$$

Hence the asymptotic variance of $\sqrt{n}(L_b-\lambda_b)$ under independent sampling within rows is, from the second form of (3.3),

$$\begin{aligned}&\frac{1}{\delta^4}\textstyle\sum_a\frac{1}{\omega_a}[\sum_b\bar{\rho}_{ab}\{\delta^2\rho^2_{a\cdot}\overset{K}{\delta}_{bb(a)}-2\delta\nu\rho^2_{a\cdot}\overset{K}{\delta}_{bb(a)}\overset{K}{\delta}_{bb(\cdot)}\\ &\qquad+\nu^2\rho^2_{a\cdot}\overset{K}{\delta}_{bb(\cdot)}\}-\overset{+2}{\bar{\phi}}_a]\\ &\quad=\frac{1}{\delta^4}\{\delta^2\textstyle\sum_a\theta_a\rho_{ab(a)}(1-\bar{\rho}_{ab(a)})\\ &\qquad-2\delta\nu[\textstyle\sum^r(\theta_a\rho_{ab(a)})-\sum_a\theta_a\rho_{ab(a)}\bar{\rho}_{ab(\cdot)}]\\ &\qquad+\nu^2\textstyle\sum_a\theta_a\rho_{ab(\cdot)}(1-\bar{\rho}_{ab(\cdot)})\},\end{aligned} \tag{3.7}$$

where $\theta_a=\rho_{a\cdot}/\omega_a$ and $\sum^r$ denotes summation over those values of a for which $b_{(a)}=b_{(\cdot)}$. (This summation usage had been used in [3].)

The quantity in curly brackets to the right of the equal-

[3] It might be feared that differentiation will lead to difficulty here because the maximum function is not everywhere differentiable. This problem is discussed in Sections A4 and A5 of [3]; there is no difficulty with the asymptotic theory under our assumptions.

ity sign in (3.7) is, when all $\theta_a=1$, exactly the same as (4.2.6) of [3], except for notation changes. Although these expressions appear rebarbative, they are often not difficult to use in specific cases; we illustrated the use of the sample analogue of (3.7) in Section 3.2 of [3].

3.3 The Measures of Association τ_b

We return to τ_b of Section 2.4, but now under independent sampling in the rows. It is convenient to work with $1-\tau_b$ and to express it as ν/δ, where

$$\nu = 1 - \textstyle\sum_{a,b} \rho_{a\cdot}\tilde{\rho}_{ab}^{2}, \qquad \delta = 1 - \textstyle\sum_b \left(\sum_a \rho_{a\cdot}\tilde{\rho}_{ab}\right)^{2}. \quad (3.8)$$

The maximum likelihood estimator of $1-\tau_b$ given in Section 2.4 should have $R_{a\cdot}$ replaced by $\rho_{a\cdot}$, since the $\rho_{a\cdot}$'s are known. Hence the estimator now is

$$[1 - \textstyle\sum_{a,b} \rho_{a\cdot}\tilde{R}_{ab}^{2}]/[1 - \sum_b \left(\sum_a \rho_{a\cdot}\tilde{R}_{ab}\right)^{2}].$$

Following our general prescription,

$$\overset{*}{\nu}_{ab} = -2\rho_{a\cdot}\tilde{\rho}_{ab} = -2\rho_{ab}, \qquad \overset{*}{\delta}_{ab} = -2\rho_{a\cdot}\rho_{\cdot b},$$

so that

$$\begin{aligned} \overset{+}{\phi}_{ab} &= 2\delta\rho_{ab} - 2\nu\rho_{a\cdot}\rho_{\cdot b} = 2\rho_{a\cdot}[\delta\tilde{\rho}_{ab} - \nu\rho_{\cdot b}], \\ \overset{+}{\bar{\phi}}_{a} &= 2\rho_{a\cdot}[\delta \textstyle\sum_b \tilde{\rho}_{ab}^{2} - \nu \sum_b \rho_{\cdot b}\tilde{\rho}_{ab}]. \end{aligned} \quad (3.9)$$

It is helpful to let $\psi_{ab}=\delta\tilde{\rho}_{ab}-\nu\rho_{\cdot b}$, so that

$$\overset{+}{\phi}_{ab} = 2\rho_{a\cdot}\overset{+}{\psi}_{ab}, \qquad \overset{+}{\bar{\phi}}_{a} = 2\rho_{a\cdot}\overset{+}{\bar{\psi}}_{a},$$

where $\overset{+}{\bar{\psi}}_{a} = \sum_b \tilde{\rho}_{ab}\overset{+}{\psi}_{ab}$. In these terms, the desired asymptotic variance for independent sampling in rows is

$$\frac{4}{\delta^4} \textstyle\sum_{a,b} \theta_a\rho_{ab}(\overset{+}{\psi}_{ab} - \overset{+}{\bar{\psi}}_{a})^{2}. \quad (3.10)$$

When the sample sizes by rows are proportional to the $\rho_{a\cdot}$'s, i.e., when $\omega_a=\rho_{a\cdot}$ so that all $\theta_a=1$, then (3.10) with "θ_a" deleted gives the asymptotic variance. Formula (4.4.3) of [3] purported to give that asymptotic variance, but in error; the second term of (4.4.3) of [3] is wrong.

4. USE OF THESE RESULTS IN PRACTICE

Probably the most common use of these results in practice (see Section 3.2 of [3]) is to treat $\sqrt{n}(Z-\zeta)/\hat{\sigma}$ as approximately unit-normal, where $\hat{\sigma}^2$ is a consistent estimator of the asymptotic variance σ^2. In the setting of our sequence of articles, $\hat{\sigma}^2$ is readily taken as the maximum likelihood estimator of σ^2, as follows.

Any σ^2 is a function of the ρ_{ab}'s, which may for convenience be written, perhaps in part, in terms of the $\tilde{\rho}_{ab}$'s. To find the maximum likelihood estimator of σ^2, make the replacements in the arguments of σ^2 as listed below. Recall that $R_{ab}=N_{ab}/n$, the proportion of all observations in the (a, b) cell, and that $\tilde{R}_{ab}=N_{ab}/n_{a\cdot}$, the proportion of observations in the (a, b) cell relative to row a.

Full multinomial sampling. (Sec. 2)

$$\rho_{ab} \to R_{ab}.$$

Independent sampling in rows. (Sec. 3)

$$\rho_{ab} \to R_{ab}(\rho_{a\cdot}/\omega_a) = R_{ab}\theta_a = \tilde{R}_{ab}\rho_{a\cdot}.$$

$$\tilde{\rho}_{ab} \to \tilde{R}_{ab}.$$

In practice, $\hat{\sigma}$ may be zero; we discuss this problem in [3], e.g., in connection with γ on p. 324 of [3]. Provided that $\sigma>0$, however, the probability that $\hat{\sigma}=0$ approaches zero as n grows, so for large enough samples the $\hat{\sigma}=0$ problem disappears. We have no analytic information about what "large enough" means, but we have encouraging evidence from the simulations reported in [3] and those of Rosenthal [4]. In the next section we consider the meaning of $\sigma=0$ for the measures of association described earlier.

It can also happen in practice that $D=0$, and then Z is undefined. This problem was discussed in [3, p. 320] for the case of λ_b. Since we assume throughout that $\delta\neq 0$, and since D converges to δ in probability, the $D=0$ problem also vanishes as n gets large.

5. WHEN DOES $\sigma=0$?

5.1. Full Multinomial Sampling

From (2.4), it is clear that, under full multinomial sampling, $\sigma=0$ if and only if $\rho_{ab}(\phi_{ab}-\bar{\phi})=0$ for all a, b. What does this mean for the examples of Section 2?

Gamma. Criteria for $\sigma=0$ in the case of G under full multinomial sampling were discussed in Section A7 of [3]. The basic condition given there may be rewritten, if $\Pi_d>0$, as follows:

> If two individuals, 1 and 2, are drawn independently at random from the population, then, whenever $\rho_{ab}>0$,
>
> $$\frac{\Pr\{2 \text{ is concordant with } 1 \mid 1 \text{ in } (a, b) \text{ cell}\}}{\Pr\{2 \text{ is discordant with } 1 \mid 1 \text{ in } (a, b) \text{ cell}\}}$$
>
> does not depend on the choice of (a, b), except that both numerator and denominator may be zero for some (a, b).

By interchanging numerator and denominator, a similar condition may be written under the assumption $\Pi_s>0$. By our general assumption, both Π_s and Π_d cannot be 0.

If either Π_s or Π_d is 0 (i.e., $\gamma=\pm 1$), then $\sigma=0$. If at least one corner cell has positive probability, this becomes an equivalence: $\sigma=0$ if and only if $\gamma=\pm 1$.

A family of cross classifications for which $\sigma=0$ is the balanced cruciform family: all the probability is in a single row and column (neither of them borders), and there is equal probability in the two horizontal arms of the "cross" as well as equal probability in the two vertical limbs. A specific numerical example is

	.1			
.2	.4	.1		.1
	.05			
	.05			

where cells without numbers have zero probabilities. In such a balanced cruciform case, $\Pi_s = \Pi_d$ so $\gamma = 0$, and $\mathcal{R}_{ab}^{(s)} = \mathcal{R}_{ab}^{(d)}$ for every cell with $\rho_{ab} > 0$.

There are, however, other cross classifications with $\sigma = 0$, but for which γ is not -1, 0, or 1. All appear to be very special. For example, consider the 4×4 case in which there is probability 0.25 in cells (1,2), (2,1), (3,4), and (4,3); other cells have, of course, zero probability. Here $\Pi_s = .5$, $\Pi_d = .25$ so that $\gamma = 1/3$, yet it is easy to check that $\sigma = 0$ because the concordance-discordance ratio is 2 for the four cells with positive probability.

The measure Δ_{ba}. A necessary and sufficient condition that $\sigma = 0$ here is that, for all cells with $\rho_{ab} > 0$,

$$\mathcal{R}_{ab}^{(s)} - \mathcal{R}_{ab}^{(d)} = \Delta_{ba}(1 - \rho_{a\cdot}).$$

We do not have a neat characterization of these cases.

The measure τ_b. This was not discussed in [3] for full multinomial sampling. The major finding here is that if all $\rho_{ab} > 0$, then $\sigma = 0$ if and only if independence holds, i.e., if and only if $\rho_{ab} = \rho_{a\cdot}\rho_{\cdot b}$ for all (a, b). (Note that this implies $\tau_b = 0$.)

To see this, assume that all $\rho_{ab} > 0$, so that to say $\sigma = 0$ is to say (see (2.13)) that for all a, b_1, b_2

$$0 = \phi_{ab_1} - \phi_{ab_2} = -2\nu(\rho_{\cdot b_1} - \rho_{\cdot b_2}) + 2\delta(\tilde{\rho}_{ab_1} - \tilde{\rho}_{ab_2}).$$

Hence, taking the difference of this quantity between rows a_1 and a_2,

$$\tilde{\rho}_{a_1 b_1} - \tilde{\rho}_{a_1 b_2} = \tilde{\rho}_{a_2 b_1} - \tilde{\rho}_{a_2 b_2}.$$

Finally, add over b_2 and recall that $\tilde{\rho}_{a\cdot} = 1$. It follows that $\tilde{\rho}_{a_1 b_1} = \tilde{\rho}_{a_2 b_1}$, or that $\rho_{a_1 b_1} = \rho_{a_1\cdot}\tilde{\rho}_{a_2 b_1}$. Now average both sides over a_2 to obtain $\rho_{a_1 b_1}$ as a product of a factor depending only on a_1 and another depending only on b_1. This shows independence. Conversely, if $\rho_{ab} = \rho_{a\cdot}\rho_{\cdot b}$ for all a, b, substitution shows immediately that $\delta = \nu$ and that all $\phi_{ab} = 0$.

If some ρ_{ab}'s are 0, we do not know a nice way to characterize $\sigma = 0$.

5.2. Independent Sampling in Rows

Here $\sigma = 0$ if and only if $\tilde{\rho}_{ab}(\phi_{ab}^{+} - \bar{\phi}_a^{+}) = 0$ for all a, b. What does this mean for the examples of Section 3?

The measure λ_b. In [3, p. 315] we asserted that for λ_b, $\sigma = 0$ if and only if $\lambda_b = 0$ or 1, but we did not there give a proof. In our current notation and approach, a proof may be given relatively easily. We assume, without loss of generality, that all $\rho_{a\cdot} > 0$; if $\rho_{a\cdot} = 0$, just delete that row.

Recall first that

$$\lambda_b = 0 \quad \text{means } \rho_{ab(a)} = \rho_{ab(\cdot)}, \quad \text{for all } a, \text{ or, equivalently, } \textstyle\sum^r \rho_{a\cdot} = \sum \rho_{a\cdot},$$

$$\lambda_b = 1 \quad \text{means } \textstyle\sum \rho_{am} = \sum \rho_{ab(a)} = 1.$$

Now suppose that $\sigma = 0$, i.e., that $\tilde{\rho}_{ab}(\phi_{ab}^{+} - \bar{\phi}_a^{+}) = 0$ for all a, b. In row a, look at the a, $b_{(a)}$ cell, for which $\tilde{\rho}_{ab(a)}$ must be positive. Thus, for all a,

$$\overset{+}{\phi}_{ab(a)} - \overset{+}{\bar{\phi}}_a = \delta(\rho_{a\cdot} - \rho_{ab(a)}) - \nu(\rho_{a\cdot}\overset{K}{\delta}_{b(a)b(\cdot)} - \rho_{ab(\cdot)}) = 0.$$

Next, add over a, to obtain $\delta\nu - (\sum^r \rho_{a\cdot} - \rho_{\cdot m})\nu = 0$. Hence, either $\nu = 0$ (whence $1 = \sum \rho_{am}$ and $\lambda_b = 1$) or else

$$1 - \rho_{\cdot m} - \textstyle\sum^r \rho_{a\cdot} + \rho_{\cdot m} = 1 - \sum^r \rho_{a\cdot} = 0.$$

If $1 = \sum^r \rho_{a\cdot}$, then $\sum^r \rho_{a\cdot} = \sum \rho_{a\cdot}$, $\rho_{ab(a)} = \rho_{ab(\cdot)}$ for all a, and $\lambda_b = 0$.

Conversely, if $\lambda_b = 1$, $1 - \lambda_b = 0$, $\nu = 0$, and $\rho_{a\cdot} - \rho_{ab(a)} = 0$. Hence $\phi_{ab(a)}^{+} - \bar{\phi}_a^{+} = 0$. Similarly, if $\lambda_b = 0$, $1 - \lambda_b = 1$, $\delta = \nu$, and $\delta_{b(a)b(\cdot)}^{K} = 1$ for all a. Hence

$$\overset{+}{\phi}_{ab(a)} - \overset{+}{\bar{\phi}}_a = \delta[\rho_{a\cdot} - \rho_{ab(a)} - \rho_{a\cdot} + \rho_{ab(\cdot)}] = 0.$$

This completes the proof.

The measure τ_b. As in Section 5.1, the result here is that if all $\rho_{ab} > 0$, $\sigma = 0$ if and only if independence holds. The argument from independence to $\sigma = 0$ is immediate; in the other direction, if all $\rho_{ab} > 0$ and $\sigma = 0$, then

$$\tfrac{1}{2}(\overset{+}{\phi}_{ab_1} - \overset{+}{\phi}_{ab_2}) = \delta(\tilde{\rho}_{ab_1} - \tilde{\rho}_{ab_2}) - \nu(\rho_{\cdot b_1} - \rho_{\cdot b_2}) = 0$$

for all a, b_1, b_2. The corresponding demonstration in Section 5.1 then applies.

6. CAUTIONARY NOTE ABOUT ASYMPTOTIC VARIANCES

In working out asymptotic variances of the above kind, there is a trap that stems from the singularity of the distributions, i.e., from relationships like $\sum_{a,b} \rho_{ab} = 1$ or $\sum_b \rho_{ab} = \rho_{a\cdot}$. (See Footnote 1.) Because of these relationships, a given function of the ρ_{ab}'s may be expressed in a variety of ways, and sometimes one way is more convenient than another. Which way an expression is written makes no difference (except for convenience of computation) in the final asymptotic variance, *provided that* the same symbolic functional form is used throughout in finding derivatives. If not, incorrect results may be obtained.

We illustrate with a very simple case. Suppose that (X_{1n}, X_{2n}) form a sequence of pairs of random variables $(n = 1, 2, 3, \cdots)$ such that $X_{1n} + X_{2n} = 0$, and such that the pair $(\sqrt{n}(X_{1n} - 2), \sqrt{n}(X_{2n} + 2))$ has in the limit as n becomes large the (singular) bivariate normal distribution with means zero, variances 1, and covariance -1.

Note that we are treating the singularity consistently: first, $2 + (-2) = 0$; second, the asymptotic variance of $\sqrt{n}[(X_{1n} - 2) + (X_{2n} + 2)]$, which should be zero, is indeed $1 - 2 + 1 = 0$.

Now let the function of interest be $Y_n = X_{1n}^2$. Its derivative with respect to X_{1n} (evaluated at $X_{1n} = 2$) is $2 \times 2 = 4$; the corresponding derivative with respect to X_{2n} is zero. Hence the asymptotic variance of $\sqrt{n}(Y_n - 4)$ is $16 (= (4)^2 \times 1)$.

But the function might just as well have been written $Y_n = X_{2n}^2$. The evaluated derivatives with respect to X_{1n}, X_{2n}, respectively, are 0 and -4. Hence the asymptotic variance of $\sqrt{n}(Y_n - 4)$ is again 16.

A more interesting way of writing the function for illustrative purposes is $Y_n = \frac{1}{3}X_{1n}^2 + \frac{2}{3}X_{2n}^2$. The evaluated derivatives now are $(\frac{2}{3})(2) = 4/3$ and $(\frac{4}{3})(-2) = -8/3$, respectively. Hence the asymptotic variance is

$$\left(\frac{4}{3}\right)^2 - 2\left(\frac{4}{3}\right)\left(-\frac{8}{3}\right) + \left(-\frac{8}{3}\right)^2 = 16$$

as before. Thus, no matter how we choose to write the function, we get the same asymptotic variance, provided we remain faithful to the same symbolic form during the differentiation process.

If, however, we do not remain with one symbolic form, incorrect results may occur. In the above example, suppose that we write the function as X_{2n}^2 before getting the X_{1n} derivative, and as X_{1n}^2 before getting the X_{2n} derivative. Both evaluated derivatives will then be zero, and we will obtain the grossly wrong asymptotic variance of 0, instead of the foursquare correct value of 16.

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