Real-time GDA

ABSTRACT

1 SINGLE DAG QUERY

1.1 Model

Wide Area Network. Let $\mathcal{S}$ be the set of sites that hold data and run tasks, and $\mathcal{L}$ be the set of directed edges that represent inter-site links. For each inter-site WAN link $(i, j) \in \mathcal{L}$, let B_{ij} be the bandwidth from site i to site j. We assume that the bandwidths are stable within the time frame of doing real-time data analytics.

DAG of tasks. We define a stage in the DAG as a group of tasks that have the same input data dependencies. Let $\mathcal{T}$ be the set of stages and $\mathcal{D}$ be the directed set of edges that represent stage dependencies for the DAG, respectively. Each stage dependency $(k, l) \in \mathcal{D}$ also has a corresponding data amount D^{kl} that must be transfered from stage k to stage l.

Some of the stages do not have any incoming edges and so represent the raw input data. Let $\mathcal{R} := \{(k, i)\}$ be a set of stage and location pairs for the raw input data in which stage $k \in \mathcal{T}$ represents the input data and site $i \in \mathcal{S}$ is the location of this input data.

The DAG also has a start time T_0 and a finish time T_f for which the schedule of data transfers must respect.

Stage assignment. We first model each stage as a non-distributable group of tasks that must be assigned to one site. This constraint will be relaxed in the problem formulation section. The decision to place stage k at site i is represented by the binary decision variable x_i^k. This means that for any directed edge $(k, l) \in \mathcal{D}$, if $x_i^k = 1$ and $x_j^l = 1$, then D^{kl} of data will be transfered across link $(i, j) \in \mathcal{L}$. We can represent this data transfer amount by $x_i^k D^{kl} x_j^l$ and the duration of data transfer by $x_i^k (D^{kl}/B_{ij}) x_j^l$. Note that stages which represent raw data locations (no incoming edges) have a decision variable which is preset according to its location.

Stage scheduling. We assume that data transfers on a particular link are non-overlapping and non-interruptible. Let s_{ij}^{kl} be the scheduled start time for data transfer $(k, l) \in \mathcal{D}$ on link $(i, j) \in \mathcal{L}$. We also assume that a stage must have completely received all of its input data before it can process and start transferring data to another stage. For this reason we have the following scheduling

constraint to respect DAG dependencies:

$$\max_{k:(k,l)\in\mathcal{D}} \left\{ \max_{(i,j)\in\mathcal{L}} \left\{ s_{ij}^{kl} + x_i^k \frac{D^{kl}}{B_{ij}} x_j^l \right\} \right\} \leq \min_{(i,j)\in\mathcal{L}} \left\{ s_{ij}^{lm} \right\}$$
$$\forall (l, m) \in \mathcal{D} \quad (1)$$

Also, each data transfer must respect the start and finish time $[T_0, T_f]$ of the DAG:

$$\min_{(k,l)\in\mathcal{D}} \left\{ \min_{(i,j)\in\mathcal{L}} \left\{ s_{ij}^{kl} \right\} \right\} \geq T_0 \quad (2a)$$

$$\max_{(k,l)\in\mathcal{D}} \left\{ \max_{(i,j)\in\mathcal{L}} \left\{ s_{ij}^{kl} + x_i^k \frac{D^{kl}}{B_{ij}} x_j^l \right\} \right\} \leq T_f \quad (2b)$$

There are data transfers that don't have an ordering because the DAG does not contain a path that connects them. However, each link must have a specified ordering to schedule the data transfers. Let $z_{ij}^{kl|mn}$ be the binary decision on link $(i, j) \in \mathcal{L}$ that represents data transfer $(k, l) \in \mathcal{D}$ being scheduled to start and complete before data transfer $(m, n) \in \mathcal{D}$ starts. The following constraint defines this relation mathematically:

$$s_{ij}^{kl} + x_i^k \frac{D^{kl}}{B_{ij}} x_j^l \leq s_{ij}^{mn} + M(1 - z_{ij}^{kl|mn})$$
$$\forall \{(k, l), (m, n)\} \in \mathcal{D} : (k, l) \neq (m, n), \forall (i, j) \in \mathcal{L} \quad (3a)$$
$$z_{ij}^{kl|mn} + z_{ij}^{mn|kl} = 1$$
$$\forall \{(k, l), (m, n)\} \in \mathcal{D} : (k, l) \neq (m, n), \forall (i, j) \in \mathcal{L} \quad (3b)$$

where M is a large constant (e.g. $M := T_f - T_0$).

1.2 Problem

1.2.1 General DAG - Minimize WAN usage. Suppose that our objective is to minimize WAN usage by placing the stages from the DAG and scheduling the data transfers. We get the following optimization problem:

$$\min_{\mathbf{x}, \mathbf{z}, \mathbf{s}} \sum_{(k,l)\in\mathcal{D}} \sum_{(i,j)\in\mathcal{L}} x_i^k D^{kl} x_j^l \quad (4a)$$

$$\text{s.t.} \quad \sum_{i\in\mathcal{S}} x_i^k = 1 \quad \forall k \in \mathcal{T} \quad (4b)$$

$$x_i^k \in \{0, 1\} \quad \forall i \in \mathcal{S}, k \in \mathcal{T} \quad (4c)$$

$$x_i^k = 1 \quad \forall (k, i) \in \mathcal{R} \quad (4d)$$

$$z_{ij}^{kl|mn} \in \{0, 1\}$$
$$\forall \{(k, l), (m, n)\} \in \mathcal{D} : (k, l) \neq (m, n), \forall (i, j) \in \mathcal{L} \quad (4e)$$

$$(1)(2)(3) \quad (4f)$$

where (4a) is the total WAN usage, (4b) (4c) place each task at exactly one site, (4d) sets the locations of the raw data, (4e) (4f) are the scheduling constraints of the data transfers.

Notice that the objective function and first three constraints ((4a) (4b) (4c) (4d)) only depend on the stage placement decisions $\mathbf{x}$, while the following constraints ((4e) (4f)) depend on both the stage placement and scheduling ($\mathbf{z}, \mathbf{s}$) decisions.

1.2.2 MapReduce Relaxation. Suppose we relaxed the task placement decisions from being binary to nonnegative continuous, i.e. replacing (4c) with:

$$x_i^k \geq 0 \quad \forall i \in \mathcal{S}, k \in \mathcal{T}, \tag{5}$$

then the each stage fits the simplification used in a MapReduce model. Also, if (4d) can be set as fractions, then we can model a distributed data set based on its fraction at each location.

The MapReduce simplification works by allowing a reduce stage to be distributed across all sites where r_j^l represents the fraction of reduce tasks for stage l that are at site j. This model assumes that the amount of input data needed at a site and the amount of output data produced is proportional to the fraction of reduce tasks placed there. This means that if stage k precedes stage l in the DAG, stage l's reduce tasks at site j need a total of $D^{kl}r_j^l$ input data from all sites and the stage k's reduce tasks at site i produced the fraction r_i^k of that total. Therefore, $r_i^k D^{kl} r_j^l$ of data will be sent across link (i, j) and take the time duration $r_i^k (D^{kl}/B_{ij}^{kl}) r_j^l$ to transfer.

Replacing x_i^k with r_i^k and the MapReduce model in Problem (4) is exactly the same as relaxing the binary task placement decision variables (4c).

2 CLARINET

2.1 Given claims

The paper's introduction gives the following claims which were interrogated:

(1) Formulating an optimal solution for a multi-query, network-aware, joint query planning/placement/scheduling is computationally intractable.
  ✓ Cites two sources that even for a single query with the given scheduling assumptions is NP-hard.

(2) CLARINET does joint multi-query planning. It picks the best WAN-aware DAG/task placement/take scheduling per query and provides "hints" (location and start times of each task) to the query execution layer. It optimizes for minimal average query completion times. Task placement and task scheduling within a query are not done jointly.
  ✓ Joint multi-query planning although not optimal because of NP-hardness.

(3) They show how to (heuristically) compute the WAN-optimal DAG for a single query which includes task placement and task scheduling. The solution relies on reserving WAN links. This is an effective heuristic for the best single-query DAG, that decouples placement and scheduling.
  ✓ Task placement and scheduling are decoupled.
  × But not a jointly optimal decomposition which is NP-hard.
  × Task placement is minimizes time usage of WAN with a linear program, not completion time which would become a piecewise linear program.
  ✓ Given a task placement for a single query, the scheduling problem gives the optimal (non-interruptible) schedule to minimize that query's completion time.

(4) They allow for cross-query (heuristic) optimization of n queries. They order the queries by optimal DAG expected completion time. Then they choose the ith query's DAG considering the WAN impact of the $i - 1$ preceding queries.
  ✓ This mimics the heuristic rule Shortest Job First (SJF).

(5) They heuristically compact the schedules tightly in time by considering the above order and groups of $k \leq n$ queries to combat resource fragmentation.
  ✓ This is done at a cost to average completion time.

(6) They extend the above heuristic to accommodate (i) fair treatment of queries, (ii) minimize WAN bandwidth costs, and (iii) online query arrivals.
  ✓ This is done, but only heuristically.

2.2 Model

There are n queries and each query j has a set DAG-Set QS_j from which one DAG must be chosen. Also chosen are the task locations and task start times.

DAG assumptions:

(1) Each node in the DAG represents a stage which is a group of tasks.

(2) For the task placement decision, each stage is assumed to be either a set of Map tasks or a set of Reduce tasks. (Josh says: This seems pretty restrictive.)
  (i) Each Map stage is placed at the site holding its input data.
  (ii) Each Reduce stage can be distributed to all sites and is represented by a continuous fraction r_j, i.e. the fraction of reduce tasks for the stage at site j. (Josh says: I think this ability to geographically distribute tasks within a stage and the continuous decision simplification is the reason why they chose to model only Map and Reduce tasks.)

(3) Each stage in the DAG has a known amount of output data D_j at each site j.

Scheduling assumptions:

(1) Network transfers do not overlap. In fact, they have a theorem that proves that any optimal schedule has an equivalent non-overlapped schedule. This avoids having to decide concurrent flows.

(2) Obtain non-interruptible transfer schedules.

(3) The compute phase of a task can start after all its inputs are available to it at its site.

(4) Bandwidth B_{ij} between sites i and j is known and constant.

2.3 Single Query Problem

The goal is to minimize the query execution time. First, the tasks are placed across the sites. Then, the data transfers are scheduled.

2.3.1 Task placement within a (Reduce) stage. Given a DAG, each stage's task placement is decided one-by-one.

Within each stage (single node of the DAG), the decision of (Reduce) task placement is *decided* independently from other stages but *have* a dependency on each other through a reserved amount of time τ_{ij} for the link between sites i and j. This value is the length of time already reserved to use this link by other stages that do not have a DAG ordering with this stage.

The following linear program is given to decide the distribution of reduce tasks $\{r_j\}$ in a particular stage that respects the current reservation status of each link:

$$\min_r \quad \sum_j \sum_{i \neq j} \left(\frac{D_i r_j}{B_{ij}} + \tau_{ij} \right) \tag{6a}$$

$$\text{s.t.} \quad \sum_j r_j = 1 \tag{6b}$$

$$r_j \geq 0 \quad \forall j \tag{6c}$$

After $\{r_j\}$ is decided, each τ_{ij} is incremented by $\frac{D_i r_j}{B_{ij}}$.

(Josh says: This formulation is a bit strange. First, given the above formulation τ_{ij} is a constant, and so has no effect on the linear programming problem. Second, the summation is minimizing total time usage of the WAN and *not* stage execution time. To minimize stage execution time, it should be the max operator and not the summation operator. But then the order in which these problems are solved will matter. However, there is no discussion in order in which these problems are solved since it doesn't matter with the summation operator being used.)

After deciding the task placement for each stage, task scheduling for the DAG is decided.

2.3.2 Scheduling tasks in a DAG. Each DAG gives a partial ordering of the stages. For stages that compete over the same network links but do not have an ordering, an ordering must be decided.

The DAG is augmented in two ways. First, each data transfer is made into a node that is place between two the associated stages. Second, tasks from the same stage at the same site are coalesced into a sub-stage. From here on, only the start times s of the sub-stages will be scheduled.

For the explicitly stated order in the DAG and given two (data transfer) sub-stages u and v in a DAG, if u must finish before v then we have the following constraint:

$$s(v) \geq s(u) + d(u) \tag{7}$$

where $s(u)$ is the start time and $d(u)$ is the duration of stage u.

To decide the order between sub-stages that don't have an ordering but need to have a non-overlapping schedule. We add the following constraints to decide whether sub-stage u should be complete before v or vice versa:

$$s(v) \geq s(u) + d(u) - N(1 - z_{uv}) \tag{8a}$$
$$s(u) \geq s(v) + d(v) - N(z_{uv}) \tag{8b}$$

where z_{uv} is a binary decision variable that indicates if v is executed after u, and N is a large constant.

With the above scheduling constraints the i-th DAG, we measure the completion time of the query as the last finished task of the query:

$$\Phi^i := \max_{u \in \text{DAG}^i} s(u) + d(u) \tag{9a}$$
$$\text{s.t. } (7)(8) \tag{9b}$$

which is a binary integer linear program. (Josh says: This binary integer linear program may not scale well for larger DAGs. Can we do better?)

2.4 Multiple Query Problem

Essentially, multiple queries are handled as processing the Shortest Job First (SJF). However, CLARINET makes some modifications (see below).

2.4.1 Handling previously scheduled queries. Suppose $\text{low}(b)$ and $\text{high}(b)$ represents the start and end times of a reservation on a link for a previously scheduled query. Then we add the following two constraints to the scheduling problem (9) when scheduling other queries:

$$s(u) \geq \text{high}(b) - N(1 - x_{ub}) \tag{10a}$$
$$\text{low}(b) \geq s(u) + d(u) - N(x_{ub}) \tag{10b}$$

where x_{ub} is a binary indicator denoting that u is scheduled after interval b. The constraints work similar to (8).

2.4.2 Handling resource fragmentation. As queries get scheduled, resources (links) become fragmented from reservations happening without accounting for other queries. They present a heuristic that optimally packs subsets of queries of size $k \leq n$.

First, they calculate the execution time for each query as if the query was scheduled alone. Based on this, the queries are put in an increasing order.

The first k queries are then scheduled. While sweeping through time, each sub-stage is scheduled from this set when it and its link first become free. When a query has been completely scheduled, it leaves the set and the next shortest query left in original ordering replaces it. This continues until all queries have been scheduled.

This heuristic allows query link usages to become better packed. However, at high values of k, the average completion time may rise which means that there is a tradeoff between efficiently using the WAN and query completion time among multiple queries.

2.4.3 Fairness. The multi-query heuristics above favor shorter queries over longer queries. To combat this, they target that each query should finish within a specific multiple m of its stand alone completion time dur_j.

At time t when a query is to be added to the query set (of size k) that is being scheduled, each of the unscheduled queries is given the following score:

$$\text{Prox}_j(t) = 1 - \frac{n \times \text{dur}_j - t}{n \times \text{dur}_j} \tag{11}$$

and then sorted in descending order. From the first H of them in this order, the one with the shortest stand alone execution time is added to the query set being scheduled. This heuristic favors queries that are closer to their "fair" deadline.

2.4.4 WAN utilization. To limit WAN usage, they only select DAGs that have a WAN usage below a threshold β. This strictly limits each query's WAN usage.

2.4.5 Online Arrivals. When a new query arrives, update and recompute for all queries that have yet to have any tasks start to be executed.

2.5 Possible directions

(Josh says: In thinking about our own problem to solve, I think our main focus could be on:

(1) CLARINET assumes either a Map or Reduce structure for each stage of the DAG when deciding task placement. Can we extend this for a more general DAG stage type?
(2) CLARINET only changes the schedules for non-executed queries when a new query arrives. Can we also change partially executed queries to make the system more flexible and in return experience a switching cost?
(3) CLARINET focuses exclusively on minimizing the average query completion time. This may not be fair, and even with its fairness heuristic, it still favors shorter queries. Can we schedule queries' usage of the WAN to be more fair?
(4) CLARINET's task scheduling problem requires the use of a binary integer linear program. This may be intractable for large DAGs. Can we design a faster algorithm or formulate a more tractable problem?

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3 SINGLE MAPREDUCE QUERY

3.1 Model

Let $\mathcal{L}$ be the set of sites that holds data and runs tasks. For each inter-site WAN link, let B_{ij} be the bandwidth from site $i \in \mathcal{L}$ to site $j \in \mathcal{L}$. We assume that the bandwidths are stable within the time frame of doing real-time data analytics.

We perform the map tasks at the sites that contain the associated data and denote D_i as the output data from all of the map tasks at site i. The fraction of reduce tasks assigned to site j is denoted as r_j which is also the fraction of all other sites' data that must be transfered through the WAN to j. This means that the total WAN usage is for a given task distribution r is:

$$\sum_i \sum_{j \neq i} D_i r_j \qquad (12)$$

which can be rearranged to:

$$\sum_i \sum_{j \neq i} D_i r_j = \sum_i D_i \sum_j r_j - \sum_j D_j r_j = \sum_i D_i - \sum_j D_j r_j \quad (13)$$

$$= \sum_j D_j(1 - r_j) \qquad (14)$$

If we are given a start time s after the map steps are all completed and a data shuffle deadline t for the reduce tasks, then the completion time is bounded as such:

$$s + \max_{(i,j):j \neq i} \left\{ \frac{D_i r_j}{B_{ij}} \right\} \leq t \qquad (15)$$

which is caused by the heterogeneous WAN bandwidth and has a bottlenecking link(s). We assume that $t > s$.

3.2 Problem

Minimize WAN usage. When minimizing WAN usage for a MapReduce data shuffle we have the following optimization problem:

$$\min_r \quad \sum_j D_j(1 - r_j) \qquad (16a)$$

$$\text{s.t.} \quad \sum_j r_j = 1 \qquad (16b)$$

$$r_j \geq 0 \quad \forall j \qquad (16c)$$

Feasibility to meet deadline. We want to find a feasible r so that the data shuffle finishes at or before t:

$$\text{find } r \text{ s.t.} \quad \frac{D_i r_j}{B_{ij}} \leq t - s \quad \forall(i,j) : j \neq i \qquad (17a)$$

$$\sum_j r_j = 1 \qquad (17b)$$

$$r_j \geq 0 \quad \forall j \qquad (17c)$$

Minimize WAN usage given a shuffle deadline. When minimizing WAN usage for a MapReduce data shuffle for a given deadline t we have the following optimization problem:

$$\min_r \quad \sum_j D_j(1 - r_j) \qquad (18a)$$

$$\text{s.t.} \quad \frac{D_i r_j}{B_{ij}} \leq t - s \quad \forall(i,j) : i \neq j \qquad (18b)$$

$$\sum_j r_j = 1 \qquad (18c)$$

$$r_j \geq 0 \quad \forall j \qquad (18d)$$

3.3 Optima

Since Problem (18) is a convex optimization problem (linear program), we look at the Karush-Kuhn-Tucker (KKT) conditions for optimality using the following dual variables $(\mu_{ij}, \theta, \lambda_j) : \forall(i,j), i \neq j$:

$$-D_j + \sum_{i \neq j} \frac{D_i}{B_{ij}} \mu_{ij} + \theta - \lambda_j = 0 \quad \forall j \qquad (19a)$$

$$\mu_{ij} \left(\frac{D_i}{B_{ij}} r_j - (t - s) \right) = 0 \quad \forall(i,j) : i \neq j \qquad (19b)$$

$$r_j \lambda_j = 0 \quad \forall j \qquad (19c)$$

$$\mu_{ij} \geq 0 \quad \forall(i,j) : i \neq j \qquad (19d)$$

$$\lambda_j \geq 0 \quad \forall j \qquad (19e)$$

$$\frac{D_i}{B_{ij}} r_j - (t - s) \leq 0 \quad \forall(i,j) : i \neq j \qquad (19f)$$

$$\sum_j r_j - 1 = 0 \qquad (19g)$$

$$r_j \geq 0 \quad \forall j \qquad (19h)$$

where (19a) are the first stationary conditions, (19b) (19c) are the complimentary slackness conditions, (19d) (19e) are the dual feasibility conditions, and (19f) (19g) (19h) are the primal feasibility conditions.

Notice that for every site j, (19f) implies that there could be bottlenecking link if $\max_{i \neq j} \left\{ D_i/B_{ij} \right\} > t - s$. There would not be a bottlenecking link only if making $r_j = 1$ meets the deadline. Let us denote

$$U_j := \max_{i \neq j} \left\{ D_i/B_{ij} \right\} \qquad (20)$$

as the maximum upload time for site j if $r_j = 1$, and denote

$$b_j := \arg\max_{i \neq j} \left\{ D_i/B_{ij} \right\} \qquad (21)$$

as the set of bottlenecking data sources. If for any $i \notin b_j$, then (19f) is satisfied if it satisfied for b_j and also $D_i/B_{ij} < t - s$. This fact and (19b) implies that $\mu_{ij} = 0$ if $i \notin b_j$. Now the KKT conditions (19) have redundant conditions that can be eliminated to:

$$-D_j + U_j \sum_{i \in b_j} \mu_{ij} + \theta - \lambda_j = 0 \quad \forall j \qquad (22a)$$

$$\mu_{ij} \left(U_j r_j - (t - s) \right) = 0 \quad \forall(i,j) : i \in b_j \qquad (22b)$$

$$r_j \lambda_j = 0 \quad \forall j \qquad (22c)$$

$$\mu_{ij} \geq 0 \quad \forall(i,j) : i \in b_j \qquad (22d)$$

$$\lambda_j \geq 0 \quad \forall j \qquad (22e)$$

$$U_j r_j - (t - s) \leq 0 \quad \forall(i,j) : i \in b_j \qquad (22f)$$

$$\sum_j r_j - 1 = 0 \qquad (22g)$$

$$r_j \geq 0 \quad \forall j \qquad (22h)$$

We have three cases for each r_j:

(1) $r_j = (t - s)/U_j$. Since $t - s > 0$ and $U_j > 0$, then $r_j > 0$. Then (22c) results in $\lambda_j = 0$. In this case (22a) turns into:

$$\theta = D_j - U_j \sum_{i \in b_j} \mu_{ij} \qquad (23)$$

Since $\sum_{i \in b_j} \mu_{ij} \geq 0$ from (22d), then this means that $\theta \leq D_j$.

(2) $0 < r_j < (t-s)/U_j$. Then (22c) results in $\lambda_j = 0$ and (22b) results in $\mu_{ij} = 0 : \forall i \in b_j$. In this case (22a) turns into:

$$\theta = D_j \tag{24}$$

(3) $r_j = 0$. Then (22b) results in $\mu_{ij} = 0 : \forall i \in b_j$. In this case (22a) turns into:

$$\theta = D_j + \lambda_j \tag{25}$$

Since (22e), then $\theta \geq D_j$.

Since θ can only be a single value, the above cases give a natural ordering. For a particular θ: if $D_j > \theta$, then Case 1 applies; if $D_j < \theta$ then Case 3 applies; if $D_j = \theta$ then Cases 1,2, or 3 could apply. This also means that we can restrict θ to the set $\{D_j : \forall j\}$ without restricting the solution space of the task placements $r_j : \forall j$.

Also, if all sites are in Case 1 and we observe that from (22g) $\sum_j \frac{1}{U_j} < \frac{1}{t-s}$, then the deadline t is too soon and the problem is infeasible. Let us denote the lower bound on t as:

$$\underline{t} := s + \frac{1}{\sum_j \frac{1}{U_j}} \tag{26}$$

On the other hand, let $l := \arg\max_j\{D_j\}$ and if $U_l < t-s$, then $r_l = 1$ and $r_j = 0 : \forall j \neq l$. Let us denote the upper bound on t as:

$$\bar{t} := s + U_l \tag{27}$$

If $t \geq \bar{t}$, then $r_l = 1$, $r_j = 0 : \forall j \neq l$, and the WAN usage is $\sum_i D_i - D_l$.

If $t = \underline{t}$, then $r_j = (\underline{t}-s)/U_j : \forall j$ and the WAN usage is $\sum_i D_i - (\underline{t}-s)\sum_j(D_j/U_j)$.

Note that if either the maximum deadline range

$$\bar{t} - \underline{t} = U_l - \frac{1}{\sum_j \frac{1}{U_j}} \tag{28}$$

$$\tag{29}$$

or the maximum WAN savings

$$D_l - \frac{1}{\sum_j \frac{1}{U_j}} \sum_j \frac{D_j}{U_j} \tag{30}$$

has significant cost, then developing an optimization algorithm is worthwhile.

3.4 Algorithm

We have the following algorithm:

(1) **Initialize:**
   - Order D_j in descending order and relabel the indexes for this ordering.
   - Set $y := 1$, $k := 1$, and $r_j := 0 : \forall j$
   - For each site j, set:
     $U_j := \max_{i \neq j}\{D_i/B_{ij}\}$
     $b_j := \arg\max_{i \neq j}\{D_i/B_{ij}\}$.
(2) **Test for feasibility:**
   - If $t \leq s + \dfrac{1}{\sum_j \frac{1}{U_j}}$, then stop because the problem is infeasible.
(3) **Process:**
   - Replace $r_k := \min\{y, (t-s)/U_k\}$.
   - Update $y := y - r_k$ and $k := k + 1$.
   - If $y \leq 0$ or $k > |\mathcal{L}|$, then stop and output $r_j : \forall j$. Otherwise repeat Step (3).

APPENDIX