

Global solution of optimization problems with signomial parts

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Abstract

In this paper a new approach for the global solution of nonconvex MINLP (Mixed Integer NonLinear Programming) problems that contain signomial (generalized geometric) expressions is proposed and illustrated. By applying different variable transformation techniques and a discretization scheme a lower bounding convex MINLP problem can be derived. The convexified MINLP problem can be solved with standard methods. The key element in this approach is that all transformations are applied termwise. In this way all convex parts of the problem are left unaffected by the transformations. The method is illustrated by four example problems.

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1. Introduction

Optimization problems that contain signomial expressions occur frequently in many areas of engineering and process synthesis. A *signomial* consists of a sum of terms of the form

$$ax_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}, \quad a, r_1, \dots, r_n \in \mathbf{R}.$$

The definition set is usually \mathbf{R}_+^n . A term with positive sign ($a > 0$) is called a posynomial term and a function that consists of a sum of positively signed terms is called a *posynomial*. Thus, by grouping together terms with identical sign a signomial function can be written as a difference between two posynomials. The type of optimization problem considered in this paper has the property that the objective function and all inequality constraints can be decomposed into a convex part and a signomial part. All equality constraints should be linear. Technically, this is not a stringent requirement since an equality constraint with signomial parts can be rewritten as two signomial inequality constraints. The focus in this paper is on the construction of transformations, which convexifies the original nonconvex problem without introducing any additional nonconvexity to the problem. Formally, the proposed transformations are of the

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form

$$(P) \begin{cases} \text{MIN} & f(z) + S_0(z) \\ \text{s.t.} & g_i(z) + S_i(z) \leq 0 \\ & Az = a \\ & l \leq z \leq u \\ & i \in \{1, 2, \dots, N\} \end{cases} \Rightarrow (P^{\text{conv}}) \begin{cases} \text{MIN} & f(z) + S_0^{\text{conv}}(z, \tilde{z}) \\ \text{s.t.} & g_i(z) + S_i^{\text{conv}}(z, \tilde{z}) \leq 0 \\ & Az = a \\ & T[z, \tilde{z}]^T = t \\ & l \leq z \leq u, \tilde{l} \leq \tilde{z} \leq \tilde{u} \\ & i \in \{1, 2, \dots, N\}, \end{cases}$$

where $S_i(z)$ represent the signomials. The functions $f(z)$, $g_i(z)$ are assumed to be once differentiable and convex on the hyper-rectangle defined by the bounds $l \leq z \leq u$. A and T are matrices and a and t column vectors of appropriate dimensions. The variables z can both be continuous and integer. If the functions $f(z)$ and $g_i(z)$ have no terms, the equalities $Az = a$ are not present and all functions $S_i(z)$ are posynomials, problem (P) corresponds to a classical geometric program. The additional variables \tilde{z} are linked to the original variables z through the linear discretization constraints $T[z, \tilde{z}]^T = t$. The transformations should be chosen so that all nonconvex functions in (P) will be convex in (P^{conv}) . The convexified signomials are called $S_i^{\text{conv}}(z, \tilde{z})$. Note that the convex and linear parts in (P) are left unaffected by the convexification, which is different from basic Geometric Programming (GP) techniques. For pure integer problems the convexification results in an equivalent convex problem. Some results for pure integer problems are given, for example, in [1–3]. In this paper we will concentrate on the convexification of continuous problems. In this case the convexification procedure will lead to an approximate and underestimating convex MINLP problem.

Optimization problems that contain only signomial expressions are usually called Generalized Geometric Programming (GGP) problems. GGP problems are a subclass of problem class (P) . GGPs was first studied by [4]. Unlike GP (Geometric Programming) problems, that contain posynomials only, GGP problems remain nonconvex both in their primal and dual representations. Many local optimization approaches have been developed for the solution of GGP problems. Some examples from the literature are [5–7]. A computational study of local GGP codes is reported in [8]. Specialized global optimization approaches for GGP problems are scarce. Falk [9] and Maranas and Floudas [10] proposed special purpose branch- and bound-based global optimization methods for the solution of GGP problems. Both methods use the exponential variable transformation and convex underestimation. The partition and branching scheme used in the methods differ. To the authors knowledge, there exists no global optimization method, which is specially designed to handle the structure of the important problem class (P) considered in this paper.

2. Transformation techniques

The basic idea in the convexification scheme is the construction of variable transformations that convexify signomial terms. First we need some results from convex analysis.

Property 1. The function $ae^{\{r_1x_1+r_2x_2+\dots+r_nx_n\}}$ is convex on \mathbf{R}_+^n if $a \geq 0$, $r_i \in \mathbf{R}$.

Property 2. The function $a \frac{e^{\{r_1x_1+r_2x_2+\dots+r_nx_n\}}}{x_1^{s_1}x_2^{s_2}\dots x_n^{s_n}}$ is convex on \mathbf{R}_+^n if $a, s_i \geq 0$, $r_i \in \mathbf{R}$.

Property 3. The function $ax_1^{r_1}x_2^{r_2}\dots x_n^{r_n}$ is convex on \mathbf{R}_+^n if $a \leq 0$, $r_i \geq 0$ and $R = \sum_{i=1}^n r_i \leq 1$.

The first property is elementary and the third was originally shown in [11] in a slightly different form but it is also found in [12,13]. The second is proved in Appendix A. The exponential term in Property 1 is a special case of the expression in Property 2. The idea is to transform a general signomial term

$$ax_1^{r_1}x_2^{r_2}\dots x_n^{r_n}, \quad a, r_1, \dots, r_n \in \mathbf{R}$$

into one of the types given above. Positively signed signomial terms are transformed into one of the first two types and signomial terms with negative sign are transformed into the third type. For simplicity we reorder the variables prior to the transformation so that variables with positive exponents appear first in the term and variables with negative exponents follows, i.e. $r_1, \dots, r_m > 0$ and $r_{m+1}, \dots, r_n < 0$.

Positively signed terms ($a > 0$).

New variables X_i are introduced according to $x_i = e^{X_i}$, $i = 1, 2, \dots, m$. The following equivalence is then established.

$$ax_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \Leftrightarrow \begin{cases} x_i = e^{X_i}, & i = 1, \dots, m \\ a \frac{e^{r_1 X_1 + \dots + r_m X_m}}{x_{m+1}^{|r_{m+1}|} \cdots x_n^{|r_n|}}. \end{cases} \quad (\text{ET})$$

The latter term on the right-hand side is now convex in the joint $X_1 \cdots X_m x_{m+1} \cdots x_n$ -space according to [Property 2](#), whereas the transformation constraint remains nonconvex. This transformation is called the *exponential transformation* (ET) and originates from GP. In GP all variables in a posynomial term are transformed, regardless of the sign of the exponents. Fewer transformations are needed if the convexity of the expression in [Property 2](#) is used. An alternative transformation that convexifies a posynomial term with positive sign is the *inverse transformation* (IT). New variables are in this case introduced through the equality $x_i = \frac{1}{X_i}$, $i = 1, 2, \dots, m$ and the resulting equivalence becomes

$$ax_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \Leftrightarrow \begin{cases} x_i = \frac{1}{X_i}, & i = 1, \dots, m \\ a \frac{1}{X_1^{r_1} \cdots X_m^{r_m} x_{m+1}^{|r_{m+1}|} \cdots x_n^{|r_n|}}. \end{cases} \quad (\text{IT})$$

Also in this case the convexified term is convex according to [Property 2](#) (a special case where the numerator is a constant).

Negatively signed terms ($a < 0$).

New variables are introduced by $x_i = X_i^{\frac{1}{R}}$, $i = 1, \dots, m$ and $x_i = X_i^{-\frac{1}{R}}$, $i = m+1, \dots, n$ where $R = \sum_{i=1}^n |r_i|$. The transformation results in the equivalence

$$ax_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \Leftrightarrow \begin{cases} x_i = X_i^{\frac{1}{R}}, & i = 1, \dots, m; & x_i = X_i^{-\frac{1}{R}}, & i = m+1, \dots, n \\ a X_1^{\frac{r_1}{R}} \cdots X_m^{\frac{r_m}{R}} X_{m+1}^{\frac{|r_{m+1}|}{R}} \cdots X_n^{\frac{|r_n|}{R}}. \end{cases} \quad (\text{PT})$$

If $R < 1$, only the variables with negative exponents are transformed according to the suggested scheme. The term on the right-hand side is now convex due to [Property 3](#), since all exponents are positive and their sum equals 1. This transformation is referred to as the *potential transformation* (PT). This transformation differs from the two given above, since it is dependent on the term that is convexified. Different terms have different R -values.

3. Discretization and convexification

If the transformations are used in this form in an optimization problem a dilemma becomes evident. The signomial term has been convexified, but the nonconvexity has moved to the transformation constraints instead. Nonlinear equality constraints are likely to cause multiple local optima in optimization problems. **To avoid this problem an approximation scheme is proposed.** The one-dimensional transformation constraints are approximated on closed intervals with piecewise linear functions. The piecewise linear approximation can then be modeled linearly by using 0–1 variables ([\[14\]](#), chapter 9, [\[15\]](#), chapter 7, [\[16\]](#) or [\[17\]](#) for instance). In the following, a well-known formulation of the piecewise linear function will be presented. The formulation may not always be the best one, but is widely used in the literature; for an alternative formulation, see [\[18\]](#), for instance. Suppose a piecewise linear function $pl(x)$ has break points p_1, p_2, \dots, p_k . If $x \in [p_j, p_{j+1}]$ there exists some j with $p_j \leq x \leq p_{j+1}$. Then for some real number $\lambda_j \in [0, 1]$ x can be written as $x = \lambda_j p_j + (1 - \lambda_j) p_{j+1}$. Then it holds that $pl(x) = \lambda_j pl(p_j) + (1 - \lambda_j) pl(p_{j+1})$. By associating a binary variable β_j with each interval $[p_j, p_{j+1}]$ the piecewise function can be represented linearly as

$$\begin{cases} pl(x) = \lambda_1 pl(p_1) + \lambda_2 pl(p_2) + \cdots + \lambda_k pl(p_k) \\ x = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_k p_k \\ \lambda_1 \leq \beta_1, \lambda_2 \leq \beta_1 + \beta_2, \dots, \lambda_{k-1} \leq \beta_{k-2} + \beta_{k-1}, \lambda_k \leq \beta_{k-1} \\ \beta_1 + \beta_2 + \cdots + \beta_{k-1} = 1 \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1 \\ \beta_j \in \{0, 1\}, \lambda_j \in [0, 1] \end{cases}$$

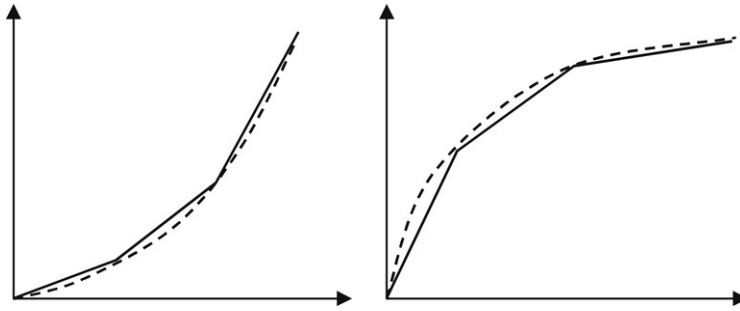


Fig. 1. Piecewise linear approximation of a convex and a concave function.

This set of variables and constraints represent a special ordered set of type 2 (SOS 2). In many modern MILP solvers ([19–21]) it is possible to declare special ordered sets explicitly. If the underlying solver supports explicit SOS declarations the piecewise linearization of the function $pl(x)$ can be rewritten in compact form as

$$\begin{cases} pl(x) = \lambda_1 pl(p_1) + \lambda_2 pl(p_2) + \cdots + \lambda_k pl(p_k) \\ x = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_k p_k \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1 \\ \lambda_j \in [0, 1] \\ \text{At most two adjacent } \lambda_i \text{ are nonzero.} \end{cases}$$

In the latter formulation the binary variables β_i can be omitted, since the logic is handled by the solver using SOS rules. A continuous one-dimensional function $f(x)$ can be approximated on a closed interval with a piecewise linear function which coincides with $f(x)$ at least at some given break points. Note that the formulation above is not optimal for performance, but is used for illustration purpose only. Any piecewise linear formulation will work fine and SOS 2 variables could preferably be used to expedite the calculations (see [22,23] for instance). A piecewise linear function will underestimate a concave function and overestimate a convex function. See Fig. 1. The nonlinear transformation constraints for the signomials are first rewritten as

(ET): $X_i = \ln(x_i)$ (concave)

(IT): $X_i = \frac{1}{x_i}$ (convex)

(PT): $X_i = x_i^R, X_i = \frac{1}{x_i^R}, R > 1$ (convex).

These constraints are then approximated with the piecewise linear expression given above. A separate set of constraints is needed for every variable in the signomial term. The variables introduced in the convexification procedure of a general signomial term can now be written as $\tilde{z} = (X_i, \lambda_{ij})$. The variables X_i and λ_{ij} are called *SOS2 variables* and the constraints introduced in the approximation are called *piecewise linearization constraints*. The ET and IT techniques are illustrated in the example below.

Example 1. This is a classical bilinear problem from [24].

$$\begin{aligned} \text{MIN} \quad & xy - x - y \\ \text{s.t.} \quad & -6x + 8y \leq 3 \\ & 3x - y \leq 3 \\ & 0 \leq x, y \leq 1.5, \quad x, y \in \mathbf{R}. \end{aligned}$$

Both variables participate in the positively signed posynomial xy . The break points are set to 0.0, 0.5, 1.0, 1.5 for both x and y . Together with a translation (+1), thus avoiding problems at $x = 0$ and $y = 0$, the exponential transformation is

$$xy \Leftrightarrow \begin{cases} x + 1 = e^X, & y + 1 = e^Y \\ e^{X+Y} - x - y - 1. \end{cases}$$

It is also possible to approximate zero with a small $\varepsilon > 0$. Following the guidelines for the piecewise linear approximation the following approximate convex underestimating MINLP problem is obtained

$$\begin{aligned}
 \text{MIN} \quad & e^{X+Y} - 2x - 2y - 1 \\
 \text{s.t.} \quad & -6x + 8y \leq 3, \quad 3x - y \leq 3 \\
 & x = 0.5\lambda_2^x + 1.0\lambda_3^x + 1.5\lambda_4^x, \quad X = \ln(1.5)\lambda_2^x + \ln(2.0)\lambda_3^x + \ln(2.5)\lambda_4^x \quad (\leq \ln(x)) \\
 & y = 0.5\lambda_2^y + 1.0\lambda_3^y + 1.5\lambda_4^y, \quad Y = \ln(1.5)\lambda_2^y + \ln(2.0)\lambda_3^y + \ln(2.5)\lambda_4^y \quad (\leq \ln(y)) \\
 & \lambda_1^x + \lambda_2^x + \lambda_3^x + \lambda_4^x = 1, \quad \lambda_1^y + \lambda_2^y + \lambda_3^y + \lambda_4^y = 1 \\
 & \text{At most two adjacent } \lambda_i^x \text{ are nonzero} \\
 & \text{At most two adjacent } \lambda_i^y \text{ are nonzero} \\
 & x, y \in [0, 1.5], \quad \lambda_j^x, \lambda_j^y \in [0, 1].
 \end{aligned}$$

The transformed objective is now convex in the joint $XYxy$ -space. The nonlinear transformation constraints that are approximated are $X = \ln(x)$ and $Y = \ln(y)$. The global optimal solution reported in [24] is $(x, y) = (7/6, 0.5)$ with optimal objective value -1.0833 . When the approximate convex MINLP is solved with the ECP method [25] the solution $(1.1143, 0.3429)$ with objective value -1.1351 is obtained. It is also worth noticing that one of the break points in the piecewise linearization coincides with the solution. If the solution values were found only at some break points (and not in between), the solution would be feasible and optimal in the original problem (P) as well. Now, since one solution value is found at the break point and one in between, the convexified approximate problem will underestimate the original nonconvex problem. If IT is used the objective and the fourth and sixth constraints are replaced by

$$\frac{1}{XY} - 2x - 2y + 1; \quad X = \frac{1}{1.5}\lambda_2^x + \frac{1}{2.0}\lambda_3^x + \frac{1}{2.5}\lambda_4^x; \quad Y = \frac{1}{1.5}\lambda_2^y + \frac{1}{2.0}\lambda_3^y + \frac{1}{2.5}\lambda_4^y.$$

In this case the solution to the approximate MINLP is $(1.2393, 0.7180)$ with objective value -1.1918 . Note that the solution to both approximate convex problems underestimates the solution to the original bilinear problem and that IT gave a looser underestimate than ET, which is, in fact, often the case. The error in both the transformations is derived in [Appendix B](#).

4. Properties of the approximate convex MINLP

Some important properties for the approximate problem are given in the theorem below. All the original variables are contained in the vector z and all the variables introduced in the convexification are called \tilde{z} . Each SOS2 variable is assumed to be linked to the original variable through a set of piecewise linearization constraints.

Theorem 1. *The following properties hold for the nonconvex problem (P) and the convexified problem (P^{conv}) (when (P^{conv}) includes the piecewise linear reformulations of the transformation constraints).*

(i) *Every convexified term underestimates the corresponding signomial term. That is*

$$\begin{aligned}
 (1) \quad a > 0 \quad & a \frac{e^{r_1 X_1 + r_2 X_2 + \dots + r_m X_m}}{x_{m+1}^{|r_{m+1}|} \dots x_n^{|r_n|}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \\
 (2) \quad a > 0 \quad & \frac{a}{X_1^{r_1} X_2^{r_2} \dots X_m^{r_m} x_{m+1}^{|r_{m+1}|} \dots x_n^{|r_n|}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \\
 (3) \quad a < 0 \quad & a X_1^{\frac{r_1}{R}} X_2^{\frac{r_2}{R}} \dots X_m^{\frac{r_m}{R}} X_{m+1}^{\frac{|r_{m+1}|}{R}} \dots X_n^{\frac{|r_n|}{R}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}.
 \end{aligned}$$

(ii) *The convexified functions underestimate the original functions. That is*

$$S_i^{\text{conv}}(z, \tilde{z}) \leq S_i(z), \quad i = 0, 1, \dots, N, \forall z, \tilde{z} \text{ with } l \leq z \leq u, \tilde{l} \leq \tilde{z} \leq \tilde{u}.$$

(iii) *The feasible region in (P) is a subset of the feasible region in (P^{conv}).*

(iv) *Let z^* be the global solution to (P) and $(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*)$ the solution to (P^{conv}). If z_{conv}^* is feasible in (P) then*

$$f(z_{\text{conv}}^*) + S_0^{\text{conv}}(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*) \leq f(z^*) + S_0(z^*) \leq f(z_{\text{conv}}^*) + S_0(z_{\text{conv}}^*).$$

If z_{conv}^ is infeasible in (P) then only the first inequality is valid.*

Proof. Case i. First we consider terms with $a > 0$. Case i.1. In the exponential transformation of a posynomial term only variables with positive exponents are transformed. Each factor $x_i^{r_i}$, $i = 1, \dots, m$ is replaced by the expression $e^{r_i X_i}$ in the convexified term. Due to the concavity of the ln-function the piecewise linear approximation X_i will underestimate $\ln(x_i)$. Since $r_i \geq 0$ we obtain $e^{r_i X_i} \leq e^{r_i \ln(x_i)} = x_i^{r_i}$. This holds for all factors $i = 1, \dots, m$. Since $a, r_i \geq 0$ we obtain

$$a \frac{e^{r_1 X_1 + \dots + r_m X_m}}{x_{m+1}^{|r_{m+1}|} x_{m+2}^{|r_{m+2}|} \dots x_n^{|r_n|}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}.$$

Hence, a term convexified with ET underestimates the posynomial term. Case i.2. In this case each factor $x_i^{r_i}$, $i = 1, \dots, m$ is replaced by the expression $\frac{1}{X_i^{r_i}}$. Due to the convexity of the inverse function the piecewise linear approximation X_i will overestimate $\frac{1}{x_i}$. Since $r_i \geq 0$ we obtain $X_i^{r_i} \geq \left(\frac{1}{x_i}\right)^{r_i} \Leftrightarrow \frac{1}{X_i^{r_i}} \leq x_i^{r_i}$. This holds for all factors $i = 1, \dots, m$. Since $a, r_i \geq 0$ we obtain the inequality

$$\frac{a}{x_1^{r_1} \dots x_m^{r_m} x_{m+1}^{|r_{m+1}|} x_{m+2}^{|r_{m+2}|} \dots x_n^{|r_n|}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}.$$

Hence, a term convexified with IT underestimates the posynomial term. Case i.3. Negatively signed terms are now considered ($a < 0$). In this case each factor $x_i^{r_i}$, $i = 1, \dots, m$ is replaced by the expression $X_i^{\frac{r_i}{R}}$ and each factor $x_i^{r_i}$, $i = m+1, \dots, n$ is replaced by $X_i^{-\frac{r_i}{R}}$. Since both the transformation constraints in PT are convex functions the piecewise linear approximation will be an overestimate. For $i = 1, \dots, m$ we have that $X_i^{\frac{r_i}{R}} \geq (x_i^R)^{\frac{r_i}{R}} = x_i^{r_i}$ and for $i = m+1, \dots, n$ we have that $X_i^{-\frac{r_i}{R}} = X_i^{\frac{|r_i|}{R}} \geq \left(\frac{1}{x_i^R}\right)^{\frac{|r_i|}{R}} = \frac{1}{x_i^{|r_i|}} = x_i^{r_i}$. Since this holds for all factors $i = 1, \dots, n$ we obtain the inequalities

$$X_1^{\frac{r_1}{R}} \dots X_m^{\frac{r_m}{R}} X_{m+1}^{\frac{|r_{m+1}|}{R}} \dots X_n^{\frac{|r_n|}{R}} \geq x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \Leftrightarrow a X_1^{\frac{r_1}{R}} \dots X_m^{\frac{r_m}{R}} X_{m+1}^{\frac{|r_{m+1}|}{R}} \dots X_n^{\frac{|r_n|}{R}} \leq a x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}.$$

Hence, a term convexified with PT underestimates a negatively signed signomial term. \square

Case ii. Since every convexified term underestimates the corresponding nonconvex signomial term it is clear that each convexified function $S_i^{\text{conv}}(z, \tilde{z})$ in (P^{conv}) underestimates the corresponding signomial function $S_i(z)$ in (P) . \square

Case iii. Take a feasible point z arbitrarily in (P) . For every choice of z the transformation variables \tilde{z} obtain unique values from the piecewise linearization constraints. According to case ii above we get for every $i = 1, \dots, N$

$$0 \geq g_i(z) + S_i(z) \geq g_i(z) + S_i^{\text{conv}}(z, \tilde{z}),$$

which implies that the feasible region in (P) is a subset of the feasible region in (P^{conv}) . \square

Case iv. Assume that z_{conv}^* is feasible in (P) . Since z^* is optimal in (P) we immediately get that

$$f(z^*) + S_0(z^*) \leq f(z_{\text{conv}}^*) + S_0(z^*, z_{\text{conv}}^*).$$

According to the underestimation property it follows that

$$f(z^*) + S_0^{\text{conv}}(z^*, \tilde{z}^*) \leq f(z^*) + S_0(z^*)$$

and since $(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*)$ was assumed optimal in (P^{conv}) we obtain

$$f(z_{\text{conv}}^*) + S_0(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*) \leq f(z^*) + S_0^{\text{conv}}(z^*, \tilde{z}^*).$$

On the other hand, if z_{conv}^* is infeasible in (P) then the right inequality may be invalid. \square

The theorem states that all convexified functions in (P^{conv}) are underestimators and the feasible region in (P^{conv}) is an overestimate. Point iv in Theorem 1 can be used to bound the global solution to the nonconvex problem (P) . The optimal objective value to the convex MINLP is a lower bound for the global solution to (P) . An upper bound can be computed as an evaluation of the objective in (P) at the solution to the underestimating convex MINLP problem.

After the solution of the first approximate problems in [Example 1](#) it can be concluded that the sought global optimal objective value to the bilinear problem lies in the interval $[-1.1351, -1.0751]$ if ET is used and in the interval $[-1.1918, -1.0675]$ if IT is used.

5. Description of the global optimization algorithm

A novel deterministic global optimization algorithm for problem class (P) can easily be constructed by solving a sequence of underestimating convex MINLP problems (P^{conv}) . The quality of the underestimation will primarily depend on the density of the grid used. If the gap between the upper bound and lower bound is considered unsatisfactory new grid points are sequentially added to the previous grid and the updated tighter underestimating convex problem is resolved. The steps of the algorithm are given below.

1. Pre-processing

Determine tight lower and upper bounds for all variables that participate in the signomial expressions. Let SI be the index set of all continuous variables that participate in the signomials and N_{conv} the index set of all pure convex constraints in (P) . Tight bounds on all variables can efficiently be obtained by solving a number of convex bounding problems of the form

$$\left\{ \begin{array}{ll} l_k := \text{MIN} & z_k \\ \text{s.t.} & g_i(z) \leq 0 \\ & Az = a \\ & l \leq z \leq u \\ & i \in N_{\text{conv}} \end{array} \right\} \quad \left\{ \begin{array}{ll} u_k := \text{MAX} & z_k \\ \text{s.t.} & g_i(z) \leq 0 \\ & Az = a \\ & l \leq z \leq u \\ & i \in N_{\text{conv}}, \end{array} \right.$$

for every $k \in SI$. This results in a tight hyper-rectangle that contains the feasible region defined by all linear and convex constraints. The same bounding technique is also frequently used in the pre-processing step of deterministic global optimization methods, for example in [\[13,27–30\]](#). Tight bounds can also be obtained by applying simple interval analysis of the constraints or by inspection.

2. Convexification

Define initial discretization grids for all variables that participate in the signomials. Convexify problem (P) . Let $\tilde{z} = (X_i, \lambda_{ij})$.

3. Solution

Solve the convexified problem (P^{conv}) by any MINLP method suitable for this purpose. Call the obtained solution $(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*)$ and the objective value Z_{conv} .

4. Bounding

Due to the underestimating property of (P^{conv}) Z_{conv} is a lower bound for the global solution to (P) . Set $LB = Z_{\text{conv}}$. An upper bound, UB , can be constructed in several ways.

- (i) Evaluate the objective in (P) at the solution to the convexified problem $(z_{\text{conv}}^*, \tilde{z}_{\text{conv}}^*)$. Call this value Z_{eval} and set $UB = Z_{\text{eval}}$.
- (ii) Apply a local NLP/MINLP method using z_{conv}^* as starting point. Call the result Z_{loc} . A valid upper bound can now be set to $UB = Z_{\text{loc}}$.

5. Termination

Terminate if $UB - LB < \text{tol}$ or if the solution is sufficiently close to an existing grid point.

6. Updating the grid

The discretization grid can be updated in several ways. Add the

- (1) solution to the convexified problem to the grid.
- (2) solution to the local search to the grid (if it is not already included).
- (3) midpoint of all intervals for which the solution to the convexified problem does not lie at a grid point.

New grid points are added for each variable that is not sufficiently close to an existing grid point. Go back to step 3.

To obtain a more detailed proof of convergence and the convergence properties of the method, the reader is referred to [\[25\]](#). In classical branch and bound methods for global optimization the partition of the space is done for only one variable at a time. In this method several new grid points are added in each iteration, i.e. the partition is done in several dimensions. If updating alternative 1 is used the grids are updated for all variables that do not lie exactly at an existing grid point at the solution to the convexified problem. Convergence for the method can generally be ensured by

periodically using updating alternative 3. This follows from the compactness and continuity assumption of problem (P). Once again we return to [Example 1](#) and the ET approach. If updating alternative 1 is used the point (1.1143, 0.3429) is added to the grid and the problem is resolved. The new constraints are

$$\begin{aligned} X &= \ln(1.5)\lambda_2^x + \ln(2.0)\lambda_3^x + \ln(2.1143)\lambda_4^x + \ln(2.5)\lambda_5^x \\ Y &= \ln(1.3429)\lambda_2^y + \ln(1.5)\lambda_3^y + \ln(2.0)\lambda_4^y + \ln(2.5)\lambda_5^y. \end{aligned}$$

The NLP-heuristic includes the point (7/6, 0.5) in the grid and alternative 3 (midpoint) the point (2.25, 1.25). The updated MINLP has four additional variables, two binary and two continuous, and two additional inequality constraints.

6. Test problems

In this chapter, three examples are given to illustrate the global optimization procedure. In all the examples, the Extended Cutting Plane algorithm [26] has been used to solve the convex underestimating MINLP problems.

Example 2. This example is taken from [31]. The problem consists of minimizing the total heat exchanger area-cost for a specific structure. Convergence to the global optimum is not guaranteed for traditional NLP solvers since the objective function consists of linear fractional terms. All constraints are linear.

$$\begin{aligned} \text{MIN} \quad & 270 \frac{q_1}{0.1 \cdot \Delta t_1} + 720 \frac{q_2}{0.1 \cdot \Delta t_2} + 240 \frac{q_3}{\Delta t_3} + 900 \frac{q_4}{\Delta t_4} \\ \text{s.t.} \quad & q_1 = 5.555 (t_1 - 395); \quad q_2 = 3.125 (t_2 - 398) \\ & q_3 = 4.545 (t_2 - 365); \quad q_4 = 5.555 (575 - t_1) \\ & q_4 = 3.571 (t_4 - 358); \quad q_4 = 3.125 (718 - t_2) \\ & q_1 + q_2 = 1000 \\ & \Delta t_1 = \frac{t_1 - 305}{2}; \quad \Delta t_2 = \frac{t_2 - 302}{2} \\ & \Delta t_3 = \frac{t_1 - t_3 + 210}{2}; \quad \Delta t_4 = \frac{t_2 - t_4 + 360}{2} \\ & 405 \leq t_1 \leq 575, \quad 405 \leq t_2 \leq 718, \\ & 365 \leq t_3, \quad 358 \leq t_4 \\ & q_i \geq 0, \quad \Delta t_i \geq 5, \quad i = 1, \dots, 4. \end{aligned}$$

If the variables q_i are transformed into e^{Q_i} for $i = 1, 2, 3, 4$, respectively, the objective function is convex according to [Property 2](#). The approximate convexified problem will be the original plus the constraints containing the piecewise approximation of $Q_i = \ln(q_i)$; $i = 1, 2, 3, 4$ and the objective function replaced by

$$\text{MIN} \quad 270 \frac{e^{Q_1}}{0.1 \cdot \Delta t_1} + 720 \frac{e^{Q_2}}{0.1 \cdot \Delta t_2} + 240 \frac{e^{Q_3}}{\Delta t_3} + 900 \frac{e^{Q_4}}{\Delta t_4}.$$

Piecewise linear functions in three steps (four grid points) were used to solve the first convexified subproblem. Each subproblem corresponds to a convex MINLP program. The solution from the convexified problems was subsequently added as new grid points to make the approximation more accurate (updating alternative 1). The upper bound was obtained by simply evaluating the nonconvex objective function at the solution to the convexified MINLP problem. The global optimum was found in the sixth subproblem with an objective value of 36163. The iteration path is illustrated in [Fig. 2](#).

If IT is used instead of ET in [Example 2](#) there is a need for solving significantly more convexified subproblems, since IT generally generates poorer lower bounds than ET. The iteration path for IT is illustrated in [Fig. 3](#).

An investigation of the underestimation quality of ET and IT is made in [Appendix B](#). Some general error estimates are derived in [32].

Example 3. This example is a nonconvex MINLP found in [33]. The nonconvexities are located in the two equality constraints. Each of these constraints is first rewritten as two inequalities and the resulting two concave constraints are finally convexified. The nonconvexities lie entirely in the continuous space.

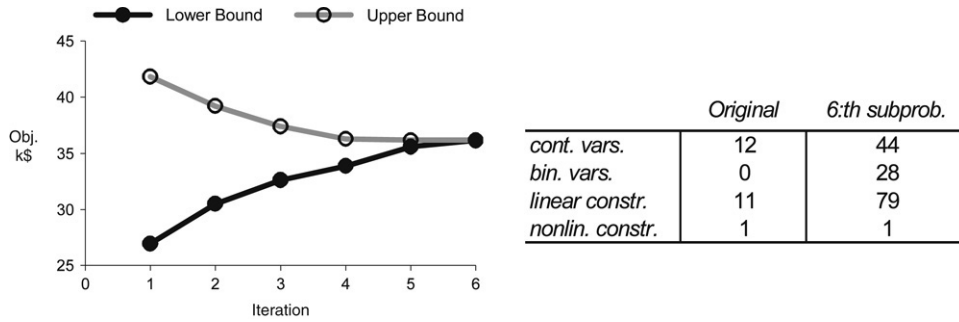


Fig. 2. Iteration path and a comparison between the sizes of the original problem and the last subproblem for Example 2.

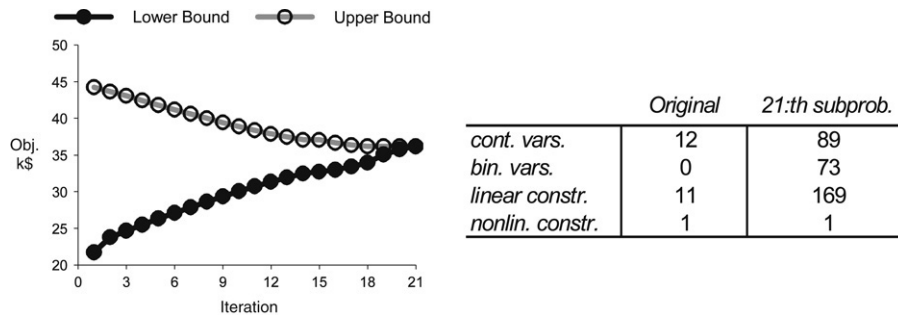


Fig. 3. Iteration path and a comparison between the sizes of the original problem and the last subproblem for Example 2 solved with IT.

$$\begin{array}{ll}
 \text{MIN} & 2x_1 + 3x_2 + 1.5y_1 + 2y_2 - 0.5y_3 \\
 \text{s.t.} & x_1^2 + y_1 = 1.25 \\
 & x_2^{1.5} + 1.5y_2 = 3 \\
 & x_1 + y_1 \leq 1.6 \\
 & 1.333x_2 + y_2 \leq 3 \\
 & -y_1 - y_2 + y_3 \leq 0 \\
 & x_1, x_2, y_3 \geq 0 \\
 & (y_1, y_2, y_3) \in \{0, 1\}
 \end{array}
 \quad
 \begin{array}{ll}
 \text{MIN} & 2x_1 + 3x_2 + 1.5y_1 + 2y_2 - 0.5y_3 \\
 \text{s.t.} & x_1^2 + y_1 \leq 1.25 \\
 & x_2^{1.5} + 1.5y_2 \leq 3 \\
 & -X_1 - y_1 \leq -1.25 \\
 & -X_2 - 1.5y_2 \leq -3 \\
 \Rightarrow & x_1 + y_1 \leq 1.6 \\
 & 1.333x_2 + y_2 \leq 3 \\
 & -y_1 - y_2 + y_3 \leq 0 \\
 & X_i = pl_i(x_i), \quad i = 1, 2 \\
 & x_1, x_2 \geq \varepsilon; \quad X_1 \geq \varepsilon^2; \quad X_2 \geq \varepsilon^{1.5} \\
 & (y_1, y_2, y_3) \in \{0, 1\}
 \end{array}
 \quad
 P^{\text{CONV}}$$

where $X_i = pl_i(x_i)$ represent the piecewise linearizations with some suitable grid points. The transformations used are $X_1 = x_1^2$, $X_2 = x_2^{1.5}$. In this problem, no upper bounds were calculated at all. On the other hand, the solution approach used consisted of subsequently solving the problem (P^{conv}) where the solution of the previous (P^{conv}) were added to the set of grid points in the piecewise linear approximations of the transformation constraints. The solution to the fifth convex subproblem introduced no new grid points (it was exactly at some old grid points), which implied that the solution was feasible in the original problem and, hence, optimal. The global optimal objective is 7.67.

Example 4. The last example is taken from [34] and includes bilinear equality and inequality constraints.

$$\begin{array}{ll}
 \text{MIN} & 6x_1 + 16x_2 - 9x_5 + 10x_6 - 15x_9 \\
 \text{s.t.} & x_1 + x_2 - x_3 - x_4 = 0; \quad x_3 - x_5 + x_7 = 0 \\
 & x_4 + x_8 - x_9 = 0; \quad -x_6 + x_7 + x_8 = 0 \\
 & -2.5x_5 + 2x_7 + x_3 \cdot x_{10} \leq 0 \\
 & 2x_8 - 1.5x_9 + x_4 \cdot x_{10} \leq 0 \\
 & 3x_1 + x_2 - x_3 \cdot x_{10} - x_4 \cdot x_{10} = 0 \\
 & x_i \geq 0, \quad i = 1, \dots, 9; \quad x_{10} \geq 1 \\
 & x_i \leq (300, 300, 100, 200, 100, 300, 100, 200, 3) .
 \end{array}$$

The bilinear equality constraint is relaxed into two inequalities prior to solution. If no upper bounding procedure was used, the global optimum (with an objective value of -400) was found after the solution of 11 convexified subproblems. After the 11th iteration no new grid points were added. If a NLP search is used as an upper bounding procedure after the first convexified subproblem the global optimum was, in fact, obtained as the upper bound. The local NLP solution was added as a new grid point, after which the lower bound solutions form the foundation of the new grid points in the following iterations. Having a good upper bound decreased the number of iterations from 11 to 9. More small and medium scale examples are solved in [32]. It is worth noticing that this method requires only one transformation for each variable that is found in a nonconvex constraint. This means that even if the number of nonconvex constraints is large, but the number of the variables in the constraints is small, the proposed method has good possibilities to solve the problem, for further details, see [25].

7. Discussion

In this paper a novel reformulation technique for NLP and MINLP problems, which contain signomial expressions, was presented. It was shown that, given any nonconvex optimization problem with an objective function and inequality constraints that can be decomposed into a convex and signomial part, it is possible to construct a corresponding convex underestimating MINLP problem. For this purpose, three different transformation techniques were derived and illustrated. The exponential transformation (ET) and the inverse transformation (IT) were applicable to positive terms and the potential transformation (PT) was used on negatively signed terms. The nonlinear transformation constraints were discretized in order to obtain a piecewise linear formulation. The quality of the underestimation produced by the ET and IT was also investigated. It was found that ET usually produces a tighter underestimate than IT. This conclusion is also supported by the bounding in the numerical examples. The reformulation could further be used for the construction of an iterative global optimization procedure. The initial discretization is subsequently updated in order to tighten the underestimating problem. The discretization could be updated in several ways. These updating rules correspond to different rectangular partition schemes used in branch and bound methods for global optimization. Finally, the method was used to successfully solve four example problems from the literature.

More work should be done to make the solution of the convex MINLP problems less expensive. At the moment, every MINLP problem is solved from scratch. It would be preferred to collect information after each subproblem and incorporate this in the solution scheme. For example, it would be possible to keep some important cuts (from the ECP method) derived in previous iterations and use them in subsequent ones. Another improvement would be to use information directly from different nodes in the branch and bound tree to reduce the combinatorial space in the following subproblems, i.e. to discard such regions from the initial feasible region, which cannot contain the optimal solution.

Appendix A

Proof of [Property 2](#).

Theorem. The function $a \frac{e^{\{r_1 x_1 + r_2 x_2 + \dots + r_n x_n\}}}{x_1^{s_1} x_2^{s_2} \dots x_n^{s_n}}$ is convex on \mathbf{R}_+^n if $a, s_i \geq 0$ and $r \in \mathbf{R}$.

Proof. The denominator can be rewritten as

$$\begin{aligned} a \frac{e^{\{r_1 x_1 + r_2 x_2 + \dots + r_n x_n\}}}{x_1^{s_1} x_2^{s_2} \dots x_n^{s_n}} &= a \frac{e^{\{r_1 x_1 + r_2 x_2 + \dots + r_n x_n\}}}{e^{\{s_1 \ln(x_1) + s_2 \ln(x_2) + \dots + s_n \ln(x_n)\}}} \\ &= a e^{\{r_1 x_1 + r_2 x_2 + \dots + r_n x_n\}} e^{-\{s_1 \ln(x_1) + s_2 \ln(x_2) + \dots + s_n \ln(x_n)\}} \\ &= a e^{\{r_1 x_1 + r_2 x_2 + \dots + r_n x_n - s_1 \ln(x_1) - s_2 \ln(x_2) - \dots - s_n \ln(x_n)\}} \end{aligned}$$

The \ln -function is concave on \mathbf{R}_+ . Then it follows that the function $r_1 x_1 + r_2 x_2 + \dots + r_n x_n - s_1 \ln(x_1) - s_2 \ln(x_2) - \dots - s_n \ln(x_n)$ is convex on \mathbf{R}_+^n since $s_i \geq 0$. Since $a \geq 0$ and the exp-function is convex and increasing on \mathbf{R} it follows from standard convex analysis [35] that the expression in [Property 2](#) is a convex function on \mathbf{R}_+^n . \square

x_1/x_2	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
1.0	0.0000	0.0428	0.0773	0.1032	0.1199	0.1270	0.1240	0.1104	0.0856	0.0490	0.0000
1.3	0.4539	0.5333	0.6032	0.6632	0.7127	0.7511	0.7779	0.7926	0.7943	0.7825	0.7565
1.6	0.8415	0.9540	1.0557	1.1459	1.2241	1.2896	1.3417	1.3799	1.4032	1.4110	1.4025
1.9	1.1529	1.2945	1.4237	1.5398	1.6420	1.7297	1.8020	1.8582	1.8974	1.9188	1.9214
2.2	1.3767	1.5429	1.6948	1.8316	1.9526	2.0567	2.1433	2.2114	2.2600	2.2880	2.2945
2.5	1.5000	1.6855	1.8546	2.0063	2.1398	2.2540	2.3481	2.4208	2.4712	2.4980	2.5000
2.8	1.5078	1.7066	1.8864	2.0464	2.1853	2.3022	2.3959	2.4651	2.5086	2.5251	2.5130
3.1	1.3830	1.5880	1.7713	1.9318	2.0681	2.1791	2.2635	2.3197	2.3464	2.3420	2.3049
3.4	1.1057	1.3091	1.4875	1.6396	1.7640	1.8593	1.9240	1.9564	1.9549	1.9177	1.8428
3.7	0.6534	0.8459	1.0097	1.1433	1.2451	1.3135	1.3467	1.3428	1.2999	1.2161	1.0890
4.0	0.0000	0.1711	0.3092	0.4126	0.4796	0.5081	0.4961	0.4417	0.3424	0.1960	0.0000

Fig. 4. ET underestimation errors for a bilinear term in the box $[1, 4] \times [3, 5]$.

x_1/x_2	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
1.0	0.0000	0.0750	0.1391	0.1909	0.2286	0.2500	0.2526	0.2333	0.1882	0.1125	0.0000
1.3	0.6568	0.7816	0.8947	0.9945	1.0790	1.1459	1.1926	1.2155	1.2105	1.1724	1.0946
1.6	1.2706	1.4435	1.6037	1.7493	1.8783	1.9882	2.0760	2.1380	2.1697	2.1653	2.1176
1.9	1.8290	2.0477	2.2524	2.4412	2.6117	2.7613	2.8866	2.9837	3.0474	3.0716	3.0484
2.2	2.3143	2.5757	2.8216	3.0499	3.2580	3.4429	3.6009	3.7276	3.8175	3.8636	3.8571
2.5	2.7000	3.0000	3.2826	3.5455	3.7857	4.0000	4.1842	4.3333	4.4412	4.5000	4.5000
2.8	2.9455	3.2782	3.5911	3.8817	4.1465	4.3818	4.5830	4.7442	4.8586	4.9173	4.9091
3.1	2.9842	3.3411	3.6750	3.9830	4.2612	4.5053	4.7098	4.8681	4.9721	5.0116	4.9737
3.4	2.7000	3.0675	3.4078	3.7173	3.9914	4.2250	4.4116	4.5433	4.6106	4.6013	4.5000
3.7	1.8692	2.2246	2.5466	2.8305	3.0710	3.2615	3.3943	3.4595	3.4453	3.3369	3.1154
4.0	0.0000	0.3000	0.5565	0.7636	0.9143	1.0000	1.0105	0.9333	0.7529	0.4500	0.0000

Fig. 5. IT underestimation errors for a bilinear term in the box $[1, 4] \times [3, 5]$.

Appendix B

Here we investigate the quality of the convex underestimation generated by ET and IT. We study a special case of a posynomial, namely the bilinear term x_1x_2 in the box $[a_1, b_1] \times [a_2, b_2]$. However, all results can be generalized to the case of a general posynomial term. The error, ε_i , for the piecewise linear approximation of the transformation constraints $X_i = \ln(x_i)$ is given by

$$\varepsilon_i(x_i) = \ln(x_i) - X_i = \ln(x_i) - (\alpha_i x_i + \beta_i); \quad \alpha_i = \frac{\ln(b_i) - \ln(a_i)}{b_i - a_i}; \quad \beta_i = \frac{b_i \ln(a_i) - a_i \ln(b_i)}{b_i - a_i}$$

for $i = 1, 2$. The error between the bilinear term and the corresponding convexified term at an arbitrarily point (x_1, x_2) in the grid box $[a_1, b_1] \times [a_2, b_2]$ is then given by the expression

$$\Delta^{ET}(x_1, x_2) = x_1x_2 - e^{X_1+X_2} = x_1x_2 - e^{\alpha_1x_1+\alpha_2x_2+\beta_1+\beta_2}.$$

By using similar arguments the error, $\bar{\varepsilon}_i$, between the piecewise linear approximation and the transformation constraint $X_i = \frac{1}{x_i}$ is

$$\bar{\varepsilon}_i = X_i - \frac{1}{x_i} = \bar{\alpha}_i x_i + \bar{\beta}_i - \frac{1}{x_i}; \quad \bar{\alpha}_i = -\frac{1}{a_i b_i}; \quad \bar{\beta}_i = \frac{1}{a_i} + \frac{1}{b_i}$$

and the error in the grid box is given by

$$\Delta^{IT}(x_1, x_2) = x_1x_2 - \frac{1}{X_1X_2} = x_1x_2 - \frac{1}{(\bar{\alpha}_1x_1 + \bar{\beta}_1)(\bar{\alpha}_2x_2 + \bar{\beta}_2)}.$$

As an illustration we assume that the box is $[1, 4] \times [3, 5]$. The error for ET and IT is calculated at 100 separate points in the given box. The errors are given in Figs. 4 and 5.

It is clear that the inverse transformation generates looser underestimation in this case. The maximal error in IT is about twice as large as in ET. The exact maximal error can be obtained by solving the corresponding maximization problems in the given grid box. This results in $\Delta_{\max}^{ET} = 2.5392$ and $\Delta_{\max}^{IT} = 5.0273$. A drawback with these transformations is clearly that they are exact only at the extreme points of the box. If we compare the quality of

x1/x2	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.3	0.0000	0.0600	0.1200	0.1800	0.2400	0.3000	0.3600	0.4200	0.4800	0.5400	0.0000
1.6	0.0000	0.1200	0.2400	0.3600	0.4800	0.6000	0.7200	0.8400	0.9600	0.4800	0.0000
1.9	0.0000	0.1800	0.3600	0.5400	0.7200	0.9000	1.0800	1.2600	0.8400	0.4200	0.0000
2.2	0.0000	0.2400	0.4800	0.7200	0.9600	1.2000	1.4400	1.0800	0.7200	0.3600	0.0000
2.5	0.0000	0.3000	0.6000	0.9000	1.2000	1.5000	1.2000	0.9000	0.6000	0.3000	0.0000
2.8	0.0000	0.3600	0.7200	1.0800	1.4400	1.2000	0.9600	0.7200	0.4800	0.2400	0.0000
3.1	0.0000	0.4200	0.8400	1.2600	1.0800	0.9000	0.7200	0.5400	0.3600	0.1800	0.0000
3.4	0.0000	0.4800	0.9600	0.8400	0.7200	0.6000	0.4800	0.3600	0.2400	0.1200	0.0000
3.7	0.0000	0.5400	0.4800	0.4200	0.3600	0.3000	0.2400	0.1800	0.1200	0.0600	0.0000
4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Fig. 6. Convex envelope underestimation errors for a bilinear term in the box $[1, 4] \times [3, 5]$.

ET and IT underestimations with the convex envelope [24] (tightest possible convex underestimator) this drawback can be clearly seen. The convex envelope coincides with the bilinear term at all edges of the box, which can be seen in Fig. 6. The maximal error for the convex envelope occurs exactly at the center of the box and is 1.5. However, a major advantage of the ET/IT approach is their applicability on general posynomial and fractional terms.

References

- [1] R. Pörn, I. Harjunkski, T. Westerlund, Convexification of different classes of nonconvex MINLP problems, *Computers and Chemical Engineering* 23 (1999) 439–448.
- [2] I. Harjunkski, T. Westerlund, R. Pörn, H. Skrifvars, Different transformations for solving nonconvex Trim-Loss problems by MINLP, *European Journal of Operational Research* 105 (1998) 594–603.
- [3] I. Harjunkski, Application of MINLP methods to a scheduling problem in the paper-converting industry, Ph.D. Thesis in Process Synthesis, Åbo Akademi University, 1997. ISBN 952-12-0091-X.
- [4] U. Passy, D.J. Wilde, Generalized polynomial optimization, *SIAM Journal of Applied Mathematics* 15 (1967) 1344.
- [5] G.E. Blau, D.J. Wilde, A Lagrangian algorithm for equality constrained generalized polynomial optimization, *AIChE* 17 (1971) 235.
- [6] R.J. Duffin, E.L. Peterson, Reversed geometric programming treated by harmonic means, *Indiana University Mathematics Journal* 22 (1972) 531.
- [7] M.J. Rijckaert, X.M. Martens, A bibliographical note on geometric programming, *JOTA* 2 (1978) 325.
- [8] R.S. Dembo, Current state of the art of algorithms and computer software for geometric programming, *JOTA* 26 (1978) 149.
- [9] J.E. Falk, Global solutions of signomial programs, Report, The George Washington University, Program in Logistics, 1973.
- [10] C.D. Maranas, C.A. Floudas, Global optimization in generalized geometric programming, *Computers and Chemical Engineering* 21 (1997) 351–369.
- [11] H. Minkowski, *Geometrie der zahlen*, Teubner, Leipzig, 1896.
- [12] M. Avriel, W.E. Diewert, S. Shaible, I. Zang, *Generalized Concavity*, Plenum Press, 1988.
- [13] C.D. Maranas, C.A. Floudas, Finding all solutions of nonlinearly constrained systems of equations, *Journal of Global Optimization* 7 (1995) 143–182.
- [14] W. Winston, *Operations Research—Applications and Algorithms*, 3rd edition, Duxbury Press, 1994.
- [15] C.A. Floudas, *Nonlinear and Mixed-Integer Optimization—Fundamentals and Applications*, Oxford University Press, 1995.
- [16] H.P. Williams, Experiments in the formulation of integer programming problems, *Mathematical Programming Study* 2 (1974) 180–197.
- [17] G.L. Nemhauser, L.A. Wolsey, *Integer and Combinatorial Optimization*, John Wiley and Sons, 1988.
- [18] M. Padberg, Approximating separable nonlinear functions via mixed zero-one programs, *Operations Research Letters* (2000) 1–5.
- [19] Cplex. <http://www.ilog.com/products/cplex/>, 2006.
- [20] XPress. http://www.dashoptimization.com/home/products/products_optimizer.html, 2006.
- [21] Lp-solve. <http://www.cs.sunysb.edu/~algorith/algorithm/implsolv/implsolv.shtml>, 2006.
- [22] E.M.L. Beale, Branch-and-bound methods for mathematical programming systems, *Annals of Discrete Mathematics* 5 (1979) 201–219.
- [23] H.P. Williams, *Model Solving in Mathematical Programming*, John Wiley & Sons Ltd., Chichester, 1993.
- [24] F.A. Al-Khayyal, J.E. Falk, Jointly constrained biconvex programming, *Mathematics of Operations Research* 8 (1983) 273–286.
- [25] T. Westerlund, Global optimization, in: L. Liberti, N. Maculan (Eds.), *Theory to Implementation*, Kluwer Academic Press, pp. 45–74 (Chapter 3).
- [26] T. Westerlund, F. Pettersson, An extended cutting plane method for solving convex MINLP problems, *Computers and Chemical Engineering* 19 (Suppl.) (1995) 131–136.
- [27] I. Quesada, I. Grossmann, A global optimization algorithm for linear fractional and bilinear programs, *Journal of Global Optimization* 6 (1995) 39–76.
- [28] H.S. Ryoo, N.V. Sahinidis, Global optimization of nonconvex NLPs and MINLPs with applications in process design, *Computers and Chemical Engineering* 19 (1995) 551–566.
- [29] C.S. Adjiman, I.P. Androulakis, C.A. Floudas, A global optimization method, aBB, for general twice-differentiable constrained NLPs – II. Implementation and computational results, *Computers and Chemical Engineering* 22 (1998) 1159–1179.

- [30] E.M.B. Smith, C.C. Pantelides, A symbolic reformulation/spatial branch and bound algorithm for the global solution of nonconvex MINLPs, *Computers and Chemical Engineering* 23 (1999) 457–478.
- [31] I. Quesada, I. Grossmann, Global optimization algorithm for heat exchanger networks, *Industrial Engineering and Chemical Resort* 32 (1993) 487–499.
- [32] R. Pörn, Mixed integer nonlinear programming: Convexification techniques and algorithm development, Ph.D. Thesis in Applied Mathematics, Åbo Akademi University, ISBN 952-12-0681-X, 2000.
- [33] G.R. Kocis, I. Grossmann, Global optimization of nonconvex mixed-integer nonlinear programming (MINLP) problems in process synthesis, *Industrial Engineering Chemical Resort* 27 (1988) 1407.
- [34] C.A. Haverly, Studies of the behavior of recursion for the pooling problem, *ACM SIGMAP Bulletin* 25 (1978) 19–28.
- [35] A.W. Roberts, D.E. Varberg, *Convex Functions*, Academic Press, 1973.