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Convex underestimation strategies for signomial functions

Andreas Lundell and Tapio Westerlund*

Process Design and Systems Engineering, Åbo Akademi University, Turku, Finland

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Different types of underestimation strategies are used in deterministic global optimization. In this paper, convexification and underestimation techniques applicable to problems containing signomial functions are studied. Especially, power transformation and exponential transformation (ET) will be considered in greater detail and some new theoretical results regarding the relation between the negative power transformation and the ET are given. The techniques are, furthermore, illustrated through examples and compared with other underestimating methods used in global optimization solvers such as αBB and BARON.

Keywords: signomial functions; convex underestimators; deterministic global optimization; MINLP; exponential transformation; power transformation

AMS Subject Classification: 90C26; 90C27; 90C30

1. Introduction

Optimization problems often include signomial functions. Convex envelopes, i.e. the tightest convex underestimators, are, however, only available for special cases of signomial functions like bi- and trilinear terms. Therefore, different transformation methods for convexifying and underestimating general signomial functions have been developed. In some techniques, the transformations depend on the signs of the signomial terms. The power transformations (PTs) [2,10] as well as the αBB underestimator [1] and the underestimation techniques in BARON [9] can be used for both positive and negative terms, whereas the exponential transformation (ET) [2,8] and the method proposed by Li *et al.* [4] are only applicable to positive terms. General results regarding some of the transformations can be found in [3].

This paper is structured as follows. In Section 2, the basic principles of a transformation technique for mixed integer nonlinear programming (MINLP) problems using single variable transformations are presented. The different transformations that can be used in this method are described, and some illustrative examples are provided for clarification. In Section 3, two other types of convex underestimation techniques are presented and in Section 4 new theoretical results regarding the relation between the negative power transformation (NPT) and the ET are given. The underestimation error of the different types of convex underestimators are compared numerically

*Corresponding author. Email: tapio.westerlund@abo.fi

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by applying the transformations to a few examples in Section 5, and finally, some conclusions regarding the results from the comparisons are drawn in Section 6.

2. Transformation of MINLP problems containing signomial terms

A signomial function is defined as the sum of signomial terms, where each term consists of products of power functions multiplied with a real constant, i.e.

$$\sigma_m(\boldsymbol{x}) = \sum_{j=1}^{J} c_j \prod_{i=1}^{I} x_i^{p_{ji}},$$

where $c_j,\ p_{ji} \in \mathbb{R}$. It is assumed that the integer- or real-valued variables x are positive. A signomial function where all coefficients c are positive is called a posynomial function. Thus, a signomial function can also be defined as the difference of two posynomial functions by grouping together the positive and the negative terms.

A MINLP problem with constraints containing signomial functions can be formulated as

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_I), \\ \text{subject to} \quad & \mathbf{A}\boldsymbol{x} = \boldsymbol{a}, \quad \mathbf{B}\boldsymbol{x} \leq \boldsymbol{b}, \\ & g_n(\boldsymbol{x}) \leq 0, \quad n = 1, 2, \ldots, J_n, \\ & q_m(\boldsymbol{x}) + \sigma_m(\boldsymbol{x}) \leq 0, \quad m = 1, 2, \ldots, J_m. \end{aligned}$$

If the functions f, g and q are convex, the whole problem can be convexified by the proposed technique. The nonconvex signomial terms in σ will be transformed to convex form by single variable transformations. Furthermore, by approximating the inverse transformations with piecewise linear functions, the transformed MINLP problem can be written in a convex relaxed form in such a way that its feasible region will overestimate the feasible region of the original problem [5,10]. In order to minimize the number of indices, from now on the index j corresponds to the jth nonconvex signomial term in the whole problem.

The transformation process described above consists of the following steps:

$$q_m(\boldsymbol{x}) + \sigma_m(\boldsymbol{x}) \leq 0 \xrightarrow{\text{(i)}} q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \boldsymbol{x}) \leq 0 \xrightarrow{\text{(ii)}} q_m(\boldsymbol{x}) + \sigma_m^C(\boldsymbol{x}, \hat{\boldsymbol{X}}) \leq 0.$$

In step (i), the signomial terms in the function σ are first convexified by applying single variable transformations of the type $x_i = T_{ji}(X_{ji})$ to the variables, and in step (ii), the signomial terms are underestimated by approximating the inverse transformations $X_{ji} = T_{ji}^{-1}(x_i)$ with the piecewise linear approximations $\hat{X}_{ji}$.

When applying the transformations $x_i = T_{ji}(X_{ji})$ to the nonconvex signomial terms in step (i), the MINLP problem is in principle not changed, that is, the feasible region with respect to the original variables is the same as long as the expressions describing the relation between the transformation variables X_{ji} and the original variables x_i are included in the problem. This is due to the fact that the nonconvex parts of the signomial functions have only been replaced by convex expressions in new variables and the relations between the new and original variables are now described by nonlinear equality constraints.

However, by approximating the inverse transformations $X_{ji} = T_{ji}^{-1}(x_i)$ with piecewise linear functions, the nonconvexities can be moved to the grid points in the piecewise linear functions. This results in a convex relaxation involving the original variables, the variables $\hat{X}_{ji}$ obtained from the approximations, as well as the variables used in the piecewise linear functions. The transformed problem is not only convex, but its feasible region will overestimate that of the original problem as long as the transformation has certain properties. This is described for different types of transformations in Sections 2.2 and 2.3 as well as in [5,10].

Since the transformed problem is now convex in the relaxed extended space consisting of the original variables and the additional variables used in the piecewise linear functions, it is solvable with a convex MINLP solver. By including more grid points the approximations are improved, however, in the process the complexity of the transformed MINLP problem is increased.

2.1 *Convexity of signomial terms*

The two main transformation types applicable to the individual variables in a signomial term are the PTs and ET. The PTs can be applied to variables in both negative and positive terms, whereas the ET can only be used on variables in positive terms.

The following two theorems, based on results from [6], state some conditions when a signomial term is convex. Note that for clarity the index j is omitted in the rest of the paper when only one signomial term is considered. The first theorem, proved in [6], is for positive signomial terms.

THEOREM 2.1 *A positive signomial or posynomial term* $c \cdot x_1^{p_1} x_2^{p_2} \cdots x_I^{p_I}$, *where* $c > 0$, $x_i \in \mathbb{R}_+$ *and* $p_i \in \mathbb{R}$, *is convex if*

(i) *all powers are negative, i.e.*

$$p_i \leq 0 \quad \forall i = 1, \ldots, I, \quad \textit{or}$$

(ii) *one power is positive, while the rest are negative, and the sum of the powers is greater than or equal to one, i.e.*

$$\exists k : p_k \geq 1 - \sum_{i \neq k} p_i \quad \textit{and} \quad p_i < 0 \;\; \forall i \neq k, i = 1, \ldots, I.$$

The corresponding result, also proved in [6], for negative signomial terms is as follows.

THEOREM 2.2 *A signomial term* $c \cdot x_1^{p_1} x_2^{p_2} \cdots x_I^{p_I}$, *where* $c < 0$, $x_i \in \mathbb{R}_+$ *and* $p_i \in \mathbb{R}$, *is convex if all powers are positive and the sum of the powers is less than or equal to one, i.e.*

$$p_i \geq 0 \;\; \forall i = 1, \ldots, I \quad \textit{and} \quad \sum_i p_i \leq 1.$$

From Theorem 2.1, two different types of PTs for positive signomial terms can be derived: the first one – the NPT – based on Theorem 2.1(i) and the other – the positive power transformation (PPT) – based on Theorem 2.1(ii). Similarly, the PT for negative signomial terms can be derived from Theorem 2.2.

The other type of transformation mentioned in the beginning of this section – the ET – is based on the following theorem (proved for example in [8]).

THEOREM 2.3 *The function*

$$f(\boldsymbol{x}) = c \cdot \mathrm{e}^{(p_1 x_1 + p_2 x_2 + \cdots + p_i x_i)} \cdot x_{i+1}^{p_{i+1}} x_{i+2}^{p_{i+2}} \cdots x_I^{p_I},$$

where $c > 0$, $p_1, \ldots, p_i > 0$ *and* $p_{i+1}, \ldots, p_I < 0$, *is convex on* $\mathbb{R}^n{}_+$.

The transformations described above have all been studied previously, for example, the PTs in [5,10] and the ET in [2,7,8]. They are also explained in more detail in Sections 2.2 and 2.3.

2.2 *Convexification and underestimation of positive signomial terms*

In Theorem 2.1, it was stated that a positive signomial term is convex if all powers in the term are negative or at most one is positive and the sum of all the powers is greater than or equal to one. It has previously been shown that, by applying one of the following two transformations to the term, it can be convexified as well as underestimated [10].

DEFINITION 2.4 (NPT) *A positive signomial term is convexified and underestimated when applying the transformation $x_i = X_i^{Q_i}$, where $Q_i < 0$, to all variables x_i with positive powers ($p_i > 0$) as long as the inverse transformation $X_i = x_i^{1/Q_i}$ is approximated by a piecewise linear function $\hat{X}_i$.*

DEFINITION 2.5 (PPT) *A positive signomial term is convexified and underestimated when applying the transformation $x_i = X_i^{Q_i}$ to all variables with positive powers, where the transformation powers $Q_i < 0$ for all indices i except for one ($i = k$) where $Q_k \geq 1$, as long as the condition*

$$\sum_{i:p_i>0} p_i Q_i + \sum_{i:p_i<0} p_i \geq 1$$

is fulfilled, and the inverse transformation $X_i = x_i^{1/Q_i}$ is approximated by a piecewise linear function $\hat{X}_i$.

So, by using the NPT or PPT, a positive signomial term can be convexified and underestimated as follows:

$$s(\boldsymbol{x}) = c \prod_i x_i^{p_i} = c \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} X_i^{p_i Q_i} = s^C(\boldsymbol{x}, \boldsymbol{X}) \geq s^C(\boldsymbol{x}, \hat{\boldsymbol{X}}).$$

Note that if $Q_k = 1$ the kth variable is not transformed and $X_k = x_k$.

In the PPT, the only requirement on the positive power Q_k for convexification is $Q_k > 0$. However, to guarantee that the original signomial term is also underestimated by the transformed signomial term when the inverse transformation is approximated by piecewise linear functions, an additional constraint, $Q_k \geq 1$, has to be added [10].

Another transformation applicable to positive signomial terms, based on the results from Theorem 2.3, is the ET.

DEFINITION 2.6 (ET) *A positive signomial term is convexified and underestimated when applying the transformation $x_i = \mathrm{e}^{X_i}$ to the individual variables with positive powers as long as the inverse transformation $X_i = \ln x_i$ is approximated by a piecewise linear function $\hat{X}_i$.*

By using the ET, a positive signomial term can then be convexified and underestimated as follows:

$$s(\boldsymbol{x}) = c \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} x_i^{p_i} = c \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} \mathrm{e}^{p_i X_i} = s^C(\boldsymbol{x}, \boldsymbol{X}) \geq s^C(\boldsymbol{x}, \hat{\boldsymbol{X}}).$$

2.3 *Convexification and underestimation of negative signomial terms*

For negative signomial terms a PT based on Theorem 2.2 can be used to convexify and underestimate the term [5,10].

DEFINITION 2.7 (PT) *A negative signomial term is convexified and underestimated when applying the transformation* $x_i = X_i^{Q_i}$, *where* $0 < Q_i \leq 1$ *for all variables with positive powers and* $Q_i < 0$ *for all variables with negative power, to the individual variables in the term. Furthermore, the condition*

$$0 < \sum_i p_i Q_i \leq 1, \tag{1}$$

must be fulfilled and the inverse transformation $X_i = x_i^{1/Q_i}$ *approximated by a piecewise linear function* $\hat{X}_i$.

For the PTs applied to a negative signomial term, the condition on the power Q_i for convexification is $Q_i > 0$. However, for the signomial term to also be underestimated by the transformed signomial term when the transformation variables are approximated by piecewise linear functions, the additional condition $Q_i \leq 1$ must also be fulfilled [10].

So, by using the PT, a negative signomial term can be convexified and underestimated as follows:

$$s(\boldsymbol{x}) = c \prod_i x_i^{p_i} = c \prod_i X_i^{p_i Q_i} = s^C(\boldsymbol{x}, \boldsymbol{X}) \geq s^C(\boldsymbol{x}, \hat{\boldsymbol{X}}).$$

Note that, whenever it is possible to choose $Q_i = 1$, while the condition (1) is still valid, $X_i = x_i$, i.e. the transformation variable X_i is equal to the original variable x_i and no transformation is needed for that variable.

2.4 *Some illustrative examples*

The transformations described above are now applied to two simple concave signomial functions to illustrate the transformation technique. In the first example, the signomial function consists of a positive signomial term and in the second a negative signomial term.

Example 2.8 The concave signomial function $f(x) = x^{0.5}$, $x > 0$ is convexified by using (a) a PT and (b) an ET:

(a) Applying the PT $x = X^Q$ to $f(x)$ gives the result

$$f(x) = x^{0.5} = (X^Q)^{0.5}, \quad \text{where } X(x) = x^{1/Q}.$$

If the NPT or PPT is used, the power Q must fulfill the conditions $Q < 0$ or $Q \geq 2$, respectively, for the transformed term to be convex. Furthermore, by replacing the inverse transformation $X(x)$ with a piecewise linear approximation $\hat{X}(x)$, the function f is underestimated. In this example, and in the rest of this paper, the piecewise linear approximations are made in one step only between the lower and upper bounds of the variables to simplify the expressions. The piecewise linear function in one step on the interval $[\underline{x}, \overline{x}]$ is then given by

$$\hat{X}(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})$$

and the convex underestimator of the function f becomes

$$f(T^{-1}(\hat{X}(x))) = \left(\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x}) \right)^{0.5Q}.$$

(b) Applying the ET $x = \mathrm{e}^X$ to the function f gives the result

$$f(x) = x^{0.5} = (\mathrm{e}^X)^{0.5}, \quad \text{where } X(x) = \ln x.$$

No additional constraints are needed for the transformed term to be convex. However, the inverse transformation $X(x) = \ln x$ must be approximated by a piecewise linear function $\hat{X}(x)$ for the convexified term to underestimate the original one. A piecewise linear approximation in one step on the interval $[\underline{x}, \overline{x}]$ is now given as

$$\hat{X}(x) = \ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x})$$

and the convex underestimator of the function f becomes

$$f(T^{-1}(\hat{X}(x))) = \mathrm{e}^{0.5(\ln \underline{x} + ((\ln \overline{x} - \ln \underline{x})/(\overline{x} - \underline{x}))(x - \underline{x}))}.$$

The underestimations obtained from the PTs ($Q = 2, -1, -2, -5$) as well as the ET are illustrated in Figure 1. From the figure, it can be seen that the PPT with $Q = 2$ actually in this case gives the convex envelope, and is thus, the best convex underestimator of the concave function. This is, however, not generally true for the PPT. Furthermore, it seems like the NPT gets closer to the ET as Q gets smaller. This is examined further in Section 4.

Example 2.9 The concave signomial function $f(x) = -x^2$, $x > 0$ is convexified by using a PT $x = X^Q$, where Q is such that $0 < Q \leq \frac{1}{2}$. Applying the PT results in

$$f(x) = -x^2 = -(X^Q)^2, \quad \text{where } X(x) = x^{1/Q}.$$

By replacing the inverse transformation $X(x)$ with a piecewise linear approximation $\hat{X}(x)$, the function f is underestimated. A piecewise linear approximation in one step on the interval $[\underline{x}, \overline{x}]$

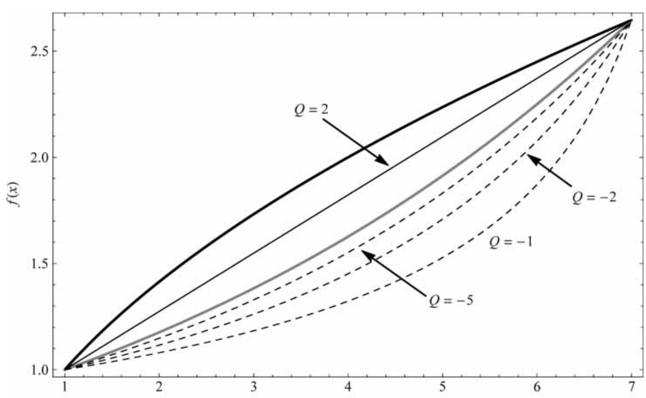


Figure 1. The signomial function $f(x) = x^{0.5}$ (thick, black) and the convex underestimators $f(T^{-1}(\hat{X}(x)))$. The transformations used are the exponential (thick, grey) and the PPT with $Q = 2$ and NPTs with $Q = -1, -2, -5$.

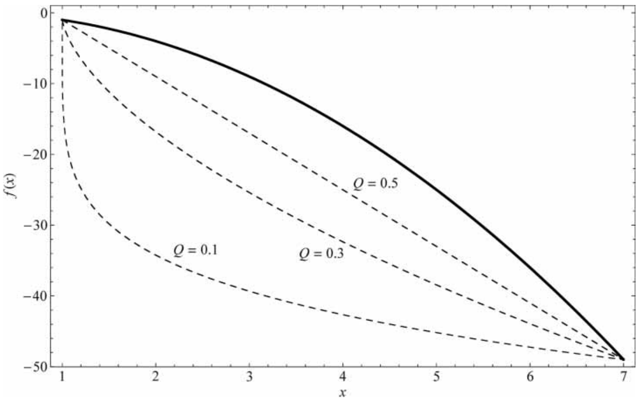


Figure 2. The signomial function $f(x) = -x^2$ (thick, black) and the convex underestimator $f(T^{-1}(\hat{X}(x)))$, where the transformation T is the PT with $Q = 0.5, 0.3, 0.1$.

is given as

$$\hat{X}(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})$$

and the convex underestimator of the function f then becomes

$$f(T^{-1}(\hat{X}(x))) = -\left(\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})\right)^{2Q}.$$

The underestimation obtained from the PTs ($Q = 0.5, 0.3, 0.1$) are illustrated in Figure 2. The PT with $Q = 0.5$ will in this case coincide with the convex envelope of f and is therefore the best convex underestimator of f.

3. Other convexification and underestimation techniques

There are also other techniques, which can be used to convexify and underestimate signomial terms. For example, the αBB type underestimator from [1] can be applied to any twice-differentiable function. This method is based on the fact that it is always possible to convexify and underestimate a twice-differentiable nonconvex function by adding a convex function fulfilling certain conditions. In the case of signomial functions, the method can be applied to either the individual terms or the signomial function as a whole.

THEOREM 3.1 *The function*

$$L(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_i \alpha(\underline{x}_i - x_i)(\overline{x}_i - x_i) \tag{2}$$

is a convex underestimator of the function $f \in C^2$ on the interval $[\underline{x}_i, \overline{x}_i]$, $x_i \in R$, if and only if, the parameters α fulfill

$$\alpha \geq \max\left\{0, -\frac{1}{2}\min_i \lambda_i\right\},$$

where the λ_i's are the eigenvalues of H_f, the Hessian matrix of the function $f(\boldsymbol{x})$ on the interval $[\underline{x}_i, \overline{x}_i]$.

It is also possible to use individual values of the parameters α_i *for each variable* x_i. *The Hessian matrix of Equation* (2) *is then given by*

$$H_f(\boldsymbol{x}) + 2\mathrm{diag}(\alpha_i), \tag{3}$$

where H_f *is the Hessian matrix of the function* f. *If the values of* α_i *are selected such that the matrix* (3) *is positive semidefinite for all* $x_i \in [\underline{x}_i, \overline{x}_i]$, *then* $L(\boldsymbol{x})$ (*with individual* α_i*-values*) *is a valid convex underestimator of the function* f *on the interval* $[\underline{x}_i, \overline{x}_i]$.

Different methods for obtaining valid values of the parameters α_i can be found in [1]. In this paper the scaled Gerschgorin method has been used.

Another convex underestimator for positive signomial terms is the method proposed by Li *et al.* [4]. This technique corresponds to the NPT with $Q = -1$ and is applied to variables having positive powers.

THEOREM 3.2 *The signomial function* $f(\boldsymbol{x}) = c \cdot x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, $c > 0$, $x_i \in [\underline{x}_i, \overline{x}_i]$, $\underline{x}_i > 0$ *can be underestimated by the function*

$$f(\boldsymbol{x}, \boldsymbol{X}) = c \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} X_i^{-p_i},$$

where the following is true for all indices i *corresponding to a positive power* p_i

$$\frac{x_i}{\overline{x}_i} + \underline{x}_i X_i - \frac{\underline{x}_i}{\overline{x}_i} \leq 1. \tag{4}$$

COROLLARY 3.3 *The expression* (4) *corresponds to a piecewise linear function in one step overestimating the inverse transformation* $X = x^{-1}$. *Thus*, *the expression can also be written as*

$$X(x) \leq \hat{X}(x), \quad \hat{X}(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x}), \quad \textit{where} \quad Q = -1.$$

4. Theoretical comparison of the ET and NPT

In Example 2.8, the ETs seemed to give tighter approximations than the NPTs regardless of the power Q used. In this section we prove that this is in fact a general result, i.e. the ET always gives a tighter convex underestimation of a positive signomial term than the NPT. To be able to prove this, the following result is needed.

LEMMA 4.1 *For a positive real variable* z *and a real constant* $\lambda \in [0, 1]$ *the following statement is true*

$$\lambda z \geq z^\lambda + \lambda - 1. \tag{5}$$

Proof The statement is equivalent to $f(z) = \lambda z - z^\lambda - \lambda + 1 \geq 0$. Differentiating $f(z)$ gives that $f'(z) = \lambda - \lambda z^{\lambda-1}$. Since $f'(z)$ has a root at $z = 1$ and $f''(1) \geq 0$ for $\lambda \in [0, 1]$, the function $f(z)$ obtains its minimum value 0 at $z = 1$ and the inequality (5) is valid. ■

The main theorem, regarding the relation between the NPT and ET, can now be proved.

THEOREM 4.2 *The ET $x = \mathrm{e}^X$ always gives a tighter convex underestimation than the NPT $x = X^Q$, $Q < 0$, when the inverse transformations $X = \ln x$ and $X = x^{1/Q}$ are approximated with piecewise linear functions.*

Proof A positive signomial term

$$s(\boldsymbol{x}) = \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} x_i^{p_i}, \quad x_i \in [\underline{x}, \overline{x}]$$

is, according to Definitions 2.4 and 2.6, convexified if one of the following two transformations is used on all variables x_i for which $p_i > 0$:

$$x_i = X_{i,\mathrm{P}}^{Q_i}, \quad Q_i < 0 \quad \text{or} \quad x_i = \mathrm{e}^{X_{i,\mathrm{E}}}.$$

As previously stated, the convexified term is underestimated when replacing the inverse transformations $X_{i,\mathrm{P}} = x_i^{1/Q_i}$ and $X_{i,\mathrm{E}} = \ln x_i$ with the piecewise linear approximations

$$\hat{X}_{i,\mathrm{P}} = \underline{x}_i^{1/Q_i} + \frac{\overline{x}_i^{1/Q_i} - \underline{x}_i^{1/Q_i}}{\overline{x}_i - \underline{x}_i}(x - \underline{x}_i) \quad \text{and} \quad \hat{X}_{i,\mathrm{E}} = \ln \underline{x}_i + \frac{\ln \overline{x}_i - \ln \underline{x}_i}{\overline{x}_i - \underline{x}_i}(x - \underline{x}_i),$$

respectively. The convexified and underestimated terms then become

$$\hat{s}_{\mathrm{P}}(\boldsymbol{x}) = \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} \hat{X}_{i,\mathrm{P}}^{p_i Q_i} \quad \text{and} \quad \hat{s}_{\mathrm{E}}(\boldsymbol{x}) = \prod_{i:p_i<0} x_i^{p_i} \cdot \prod_{i:p_i>0} \mathrm{e}^{p_i \hat{X}_{i,\mathrm{E}}}.$$

The claim is then that $\hat{s}_{\mathrm{P}}(\boldsymbol{x}) \leq \hat{s}_{\mathrm{E}}(\boldsymbol{x})$ is true for all values of the power $Q < 0$ in the NPT. This is the case if

$$\forall i : p_i > 0 : (\hat{X}_{i,\mathrm{P}}^{Q_i})^{p_i} \leq (\mathrm{e}^{\hat{X}_{i,\mathrm{E}}})^{p_i} \overset{p_i>0}{\Longrightarrow} \hat{X}_{i,\mathrm{P}}^{Q_i} \leq \mathrm{e}^{\hat{X}_{i,\mathrm{E}}}. \tag{6}$$

Thus, statement (6) is equivalent to (the indices i are excluded for clarity)

$$\left(\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})\right)^Q \leq \exp\left(\ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x})\right). \tag{7}$$

Since both the left- and right-hand sides of inequality (7) are greater than zero, this can be rewritten as

$$Q \cdot \ln\left[\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})\right] \leq \ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x}). \tag{8}$$

By further manipulation, the following equivalent form for expression (8) is received

$$\begin{aligned}
&Q \cdot \ln\left[\left(\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x})\right) \Big/ \underline{x}^{1/Q}\right] \leq \ln\left(\frac{\overline{x}}{\underline{x}}\right) \frac{x - \underline{x}}{\overline{x} - \underline{x}} \\
&\Longleftrightarrow \ln\left[\left(1 + \left(\left(\frac{\overline{x}}{\underline{x}}\right)^{1/Q} - 1\right) \frac{x - \underline{x}}{\overline{x} - \underline{x}}\right)^Q\right] \leq \ln\left[\left(\frac{\overline{x}}{\underline{x}}\right)^{(x-\underline{x})/(\overline{x}-\underline{x})}\right] \\
&\Longleftrightarrow \left(\left(\frac{\overline{x}}{\underline{x}}\right)^{1/Q} \frac{x - \underline{x}}{\overline{x} - \underline{x}} + \frac{\overline{x} - x}{\overline{x} - \underline{x}}\right)^Q \leq \left(\frac{\overline{x}}{\underline{x}}\right)^{(x-\underline{x})/(\overline{x}-\underline{x})}.
\end{aligned} \tag{9}$$

Finally, by setting

$$k = \frac{\overline{x}}{\underline{x}}, \quad \lambda = \frac{x - \underline{x}}{\overline{x} - \underline{x}} \quad \text{and} \quad 1 - \lambda = \frac{\overline{x} - x}{\overline{x} - \underline{x}},$$

the inequality (9) can be written as

$$\begin{aligned}(\lambda k^{1/Q} + (1-\lambda))^Q \leq k^\lambda &\overset{Q<0}{\iff} \lambda k^{1/Q} + (1-\lambda) \geq (k^{1/Q})^\lambda \\ &\iff \lambda z \geq z^\lambda + \lambda - 1,\end{aligned}$$

where in the last step $z = k^{1/Q}$ and $z \geq 0$, since $k \geq 1$ and $Q < 0$. The last inequality is true according to Lemma 4.1, and thus the proof is completed. ■

Although the previous theorem says that the ET always gives a better approximation than the NPT, the following result shows that the latter in fact gets arbitrary close to the ET when $Q \to -\infty$.

THEOREM 4.3 *For the piecewise linear approximations $\hat{X}_{\mathrm{P}}$ and $\hat{X}_{\mathrm{E}}$ of the NPT and ET the following statement is true*

$$\lim_{Q\to-\infty} \hat{X}_{\mathrm{P}}^Q = \mathrm{e}^{\hat{X}_{\mathrm{E}}}.$$

Proof

$$\begin{aligned}\lim_{Q\to-\infty} \hat{X}_{\mathrm{P}}^Q &= \lim_{Q\to-\infty} \left[\underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}} (x - \underline{x}) \right]^Q \\ &= \underline{x} \lim_{Q\to-\infty} \left[1 + \frac{x - \underline{x}}{\overline{x} - \underline{x}} \left(\left(\frac{\overline{x}}{\underline{x}} \right)^{1/Q} - 1 \right) \right]^Q. \qquad (10)\end{aligned}$$

By introducing the variable r, defined as

$$r = \left(\frac{\overline{x}}{\underline{x}} \right)^{1/Q} - 1 \Rightarrow Q = \frac{\ln(\overline{x}/\underline{x})}{\ln(r+1)},$$

where $r \to 0^-$ as $Q \to -\infty$, expression (10) can be rewritten as

$$\underline{x} \cdot \lim_{r\to 0^-} \left[1 + \frac{x - \underline{x}}{\overline{x} - \underline{x}} r \right]^{\ln(\overline{x}/\underline{x})/\ln(r+1)} = \underline{x} \cdot \lim_{r\to 0^-} \left(\left(\left[1 + \frac{x - \underline{x}}{\overline{x} - \underline{x}} r \right]^{1/r} \right)^{r/\ln(r+1)} \right)^{\ln(\overline{x}/\underline{x})}. \qquad (11)$$

Since $(1 + kr)^{1/r} \to \mathrm{e}^k$ and $r/\ln(r+1) \to 1$, whenever $r \to 0^-$ and $|k| < 1$, expression (11) is equal to

$$\underline{x} \cdot \exp\left[\frac{x - \underline{x}}{\overline{x} - \underline{x}} \ln\left(\frac{\overline{x}}{\underline{x}} \right) \right] = \exp\left[\ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}} (x - \underline{x}) \right] = \mathrm{e}^{\hat{X}_{\mathrm{E}}}.$$

So the error between the ET and the NPT goes to zero as Q goes to minus infinity. ■

COROLLARY 4.4 *The previous results hold also when including more grid points to the piecewise linear approximations as long as the same points are added to both approximations of the inverse transformations.*

5. Comparisons of the convex underestimation techniques

5.1 *Univariate functions*

To compare some of the convex underestimation techniques from Sections 2 and 3, they are here applied to a few examples. In the first example, the NPT and PPT as well as the ET is demonstrated on a signomial function of one variable only.

Example 5.1 The nonconvex function $f(x) = 0.05x^3 - 8x + 25x^{0.5}$, $0 < \underline{x} \le x \le \overline{x}$, consist of three signomial terms. The first two terms are convex and only the term $25x^{0.5}$ is nonconvex. Since the term is positive, either of the PPT, NPT or ET can be used to convexify it. Applying the PPT and NPT with the power Q will give the convexified and underestimated function

$$f_{\mathrm{P}}(x, \hat{X}_{\mathrm{P}}) = 0.05x^3 - 8x + 25\hat{X}_{\mathrm{P}}^{0.5Q},$$

where the piecewise linear approximation in one step of the inverse transformation $X_{\mathrm{P}} = x^{1/Q}$ is given as

$$\hat{X}_{\mathrm{P}}(x) = \underline{x}^{1/Q} + \frac{\overline{x}^{1/Q} - \underline{x}^{1/Q}}{\overline{x} - \underline{x}}(x - \underline{x}).$$

For the ET, the corresponding expressions are

$$f_{\mathrm{E}}(x, \hat{X}_{\mathrm{E}}) = 0.05x^3 - 8x + 25\mathrm{e}^{0.5\hat{X}_{\mathrm{E}}},$$

$$X_{\mathrm{E}}(x) = \ln x \quad \text{and} \quad \hat{X}_{\mathrm{E}}(x) = \ln \underline{x} + \frac{\ln \overline{x} - \ln \underline{x}}{\overline{x} - \underline{x}}(x - \underline{x}).$$

Graphs of the piecewise linear approximations are given in Figure 3 for $[\underline{x}, \overline{x}] = [1, 7]$. To illustrate how the approximations improve as the number of grid points are increased, figures when using only the interval endpoints as well as when adding an additional grid point at $x = 4$ are provided.

In Figure 4 plots of the function $f(x)$ and the convex underestimators $f_{\mathrm{P}}(x, \hat{X}_{\mathrm{P}})$ and $f_{\mathrm{E}}(x, \hat{X}_{\mathrm{E}})$ for the PPT ($Q = 2$), NPT ($Q = -1$) and ET are illustrated. For comparison, the αBB underestimator is also shown in the figure. The expression for the αBB underestimator on the interval

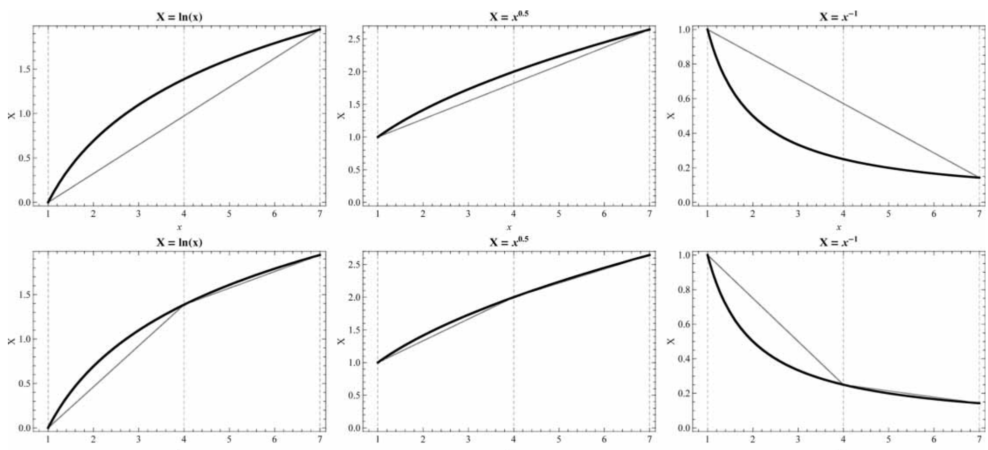


Figure 3. The inverse transformations $X = T^{-1}(x)$ (thick) and the piecewise linear approximations $\hat{X}$ (grey) of these in one step (above) and two steps (below). The transformations used are ET (left), PPT with $Q = 2$ (middle) and NPT with $Q = -1$ (right).

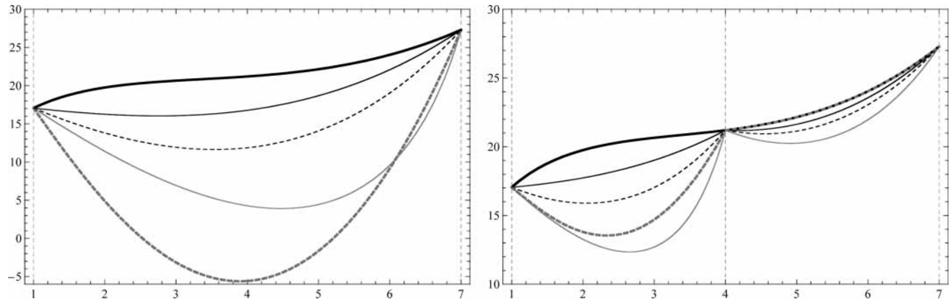


Figure 4. The signomial function $f(x) = 0.05x^3 - 8x + 25x^{0.5}$ (black, thick) underestimated by PPT with $Q = 2$ (black), ET (black, dashed), NPT with $Q = -1$ (grey) and αBB underestimator (grey, thick, dashed). In the left image, the piecewise approximations are made in one step, and in the right image in two steps. Note that the approximations are exact at the grid points.

[1,7] is given as

$$L^{[1,7]}(x) = 0.05x^3 - 8x + 25x^{0.5} + 2.975(1 - x)(7 - x),$$

i.e. the value of the parameter α in Theorem 3.1 is 2.975. The α-value was calculated with Mathematica using the scaled Gerschgorin method from [1].

If the domain is partitioned into two intervals [1,4] and [4,7] the αBB underestimators are given as

$$L^{[1,4]}(x) = 0.05x^3 - 8x + 25x^{0.5} + 2.975(1 - x)(4 - x),$$
$$L^{[4,7]}(x) = 0.05x^3 - 8x + 25x^{0.5}.$$

Since f is convex on the interval [4,7], the convex underestimator is equal to the function itself.

5.2 *Bivariate functions*

In Examples 5.2 and 5.3, some of the convex underestimation techniques are used to convexify and underestimate positive and negative signomial functions of two variables.

Example 5.2 The PPT and ET as well as the αBB underestimator is now applied to two positive signomial functions of two variables (a) $f_1(x, y) = x^{0.95}y^{0.9}$ and (b) $f_2(x, y) = x^{0.6}y^{0.5}$, defined on the interval $x, y \in [1, 7]$. The NPT is not included, since it is known from Section 4 that the ET always gives tighter underestimations.

(a) Piecewise convex underestimators of f_1 are given by the functions

$$f_{1,\mathrm{P}}(\hat{X}_\mathrm{P}, \hat{Y}_\mathrm{P}) = \hat{X}_\mathrm{P}^{0.95Q_x}\hat{Y}_\mathrm{P}^{0.9Q_y} \quad \text{and} \quad f_{1,\mathrm{E}}(\hat{X}_\mathrm{E}, \hat{Y}_\mathrm{E}) = \mathrm{e}^{0.95\hat{X}_\mathrm{E}+0.9\hat{Y}_\mathrm{E}},$$

when using the PPT and ET, respectively. Here $\hat{X}_\mathrm{P}$, $\hat{Y}_\mathrm{P}$ and $\hat{X}_\mathrm{E}$, $\hat{Y}_\mathrm{E}$ are the piecewise linear approximations of the inverse functions $X_\mathrm{P} = x^{1/Q_x}$, $Y_\mathrm{P} = y^{1/Q_y}$ and $X_\mathrm{E} = \ln x$, $Y_\mathrm{E} = \ln y$. The corresponding αBB underestimator for the function f_1 is

$$L(x, y) = x^{0.95}y^{0.9} + 0.564(1 - x)(7 - x) + 0.713(1 - y)(7 - y).$$

(b) Similarly, the functions

$$f_{2,\mathrm{P}}(\hat{X}_\mathrm{P}, \hat{Y}_\mathrm{P}) = \hat{X}_\mathrm{P}^{0.6Q_x} \hat{Y}_\mathrm{P}^{0.5Q_y} \quad \text{and} \quad f_{2,\mathrm{E}}(\hat{X}_\mathrm{E}, \hat{Y}_\mathrm{E}) = \mathrm{e}^{0.6\hat{X}_\mathrm{E}+0.5\hat{Y}_\mathrm{E}}$$

are piecewise convex underestimators of f_2 when using the PPT and ET, respectively. The αBB underestimator for the function f_2 is

$$L(x, y) = x^{0.6}y^{0.5} + 0.467(1 - x)(7 - x) + 0.552(1 - y)(7 - y).$$

Plots of the error of the different underestimators in Example 5.2 are provided in Figure 5. The values for the parameters Q in the PPTs are in case (a) $Q_x = 2$, $Q_y = -1$ and in case (b) $Q_x = 2.5$, $Q_y = -1$.

Example 5.3 The PT and αBB underestimator are in this example applied to the signomial function $f(x, y) = -x^{0.7}y$, where $x, y \in [1, 9]$. In the PTs, the powers Q can be chosen in many different ways; here two different pairs of Q-values are given, the first is $Q_x = 0.5/0.7$, $Q_y = 0.5$ and the second is $Q_x = 1$, $Q_y = 0.3$. To provide an example of how the approximations improve as the domain is partitioned, or in the case of the PTs, additional grid points are added, three different cases are considered: (a) with only the interval endpoints as grid points, (b) with extra grid points at $x = 5$ and $y = 5$ and (c) with extra grid points at $y = 3, 5, 7$ for the case where $Q_x = 1$ and $Q_y = 0.3$.

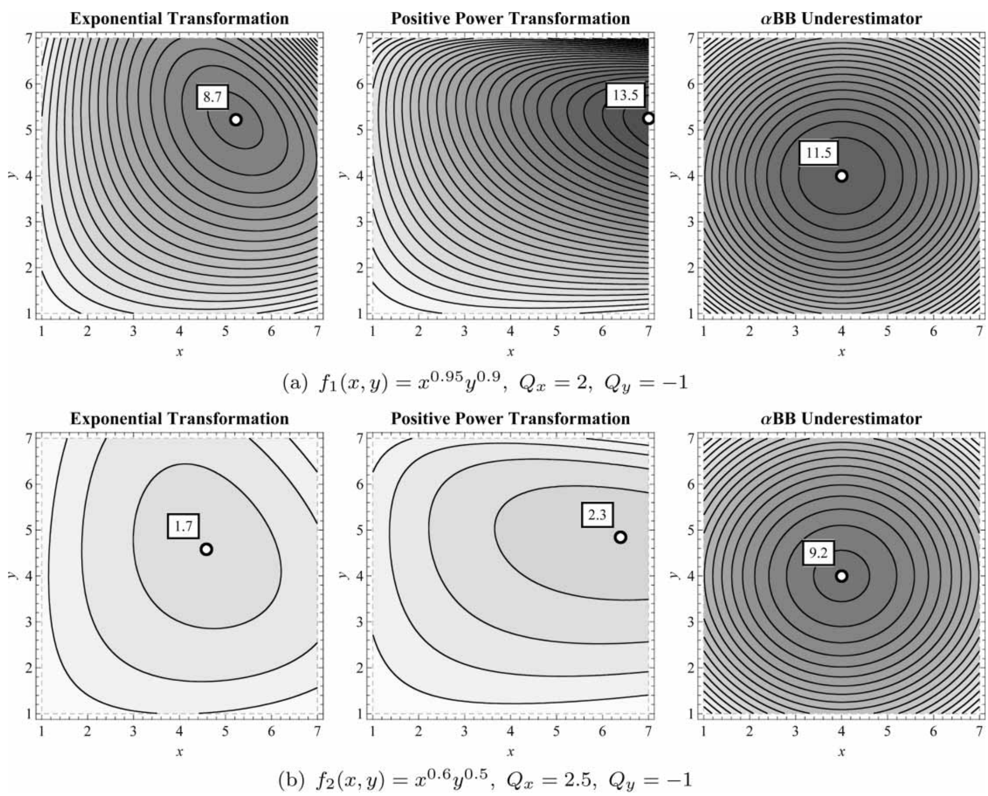


Figure 5. The absolute error of the convex underestimators in Example 5.2. Darker colours correspond to larger errors and the largest error is marked in the figure. The contours have a distance of 0.5. (a) $f_1(x, y) = x^{0.95}y^{0.9}$, $Q_x = 2$, $Q_y = -1$ and (b) $f_2(x, y) = x^{0.6}y^{0.5}$, $Q_x = 2.5$, $Q_y = -1$

(a) A piecewise convex underestimator of the function f is given by the function

$$f_{P_1}(\hat{X}_1, \hat{Y}_1) = -\hat{X}_1^{0.7\cdot 0.5/0.7}\hat{Y}_1^{1\cdot 0.5} = -\hat{X}_1^{0.5}\hat{Y}_1^{0.5},$$

when using the PT with the powers $Q_x = 0.5/0.7$, $Q_y = 0.5$, and the function

$$f_{P_2}(x, \hat{Y}_2) = -x^{0.7}\hat{Y}_2^{1\cdot 0.3} = -x^{0.7}\hat{Y}_2^{0.3},$$

when using the PT with the powers $Q_x = 1$ and $Q_y = 0.3$. Since the Q_x-value is equal to one, x is not transformed in this case. The corresponding αBB underestimator for the function f is

$$L(x, y) = -x^{0.7}y + 0.344(1-x)(9-x) + 0.350(1-y)(9-y).$$

(b) Exactly as in (a), piecewise convex underestimators for the function f are given by

$$f_{P_1}(\hat{X}_1, \hat{Y}_1) = -\hat{X}_1^{0.7\cdot 0.5/0.7}\hat{Y}_1^{1\cdot 0.5} = -\hat{X}_1^{0.5}\hat{Y}_1^{0.5} \quad \text{and}$$
$$f_{P_2}(x, \hat{Y})_2 = -x^{0.7}\hat{Y}_2^{1\cdot 0.3} = -x^{0.7}\hat{Y}_2^{0.3},$$

for values of Q according to $Q_x = 0.5/0.7$, $Q_y = 0.5$, and $Q_x = 1$, $Q_y = 0.3$, respectively. However, additional breakpoints are added to the piecewise linear approximations $\hat{X}_1$, $\hat{Y}_1$ and $\hat{Y}_2$. The αBB underestimators for the function f are different in the four partitions of the domain:

$$L^{[1,5]\times[1,5]}(x, y) = -x^{0.7}y + 0.337(1-x)(5-x) + 0.350(1-y)(5-y),$$
$$L^{[1,5]\times[5,9]}(x, y) = -x^{0.7}y + 0.285(1-x)(5-x) + 0.350(5-y)(9-y),$$
$$L^{[5,9]\times[1,5]}(x, y) = -x^{0.7}y + 0.210(5-x)(9-x) + 0.216(1-y)(5-y),$$
$$L^{[5,9]\times[5,9]}(x, y) = -x^{0.7}y + 0.186(5-x)(9-x) + 0.216(5-y)(9-y).$$

(c) In the case when $Q_x = 1$ and $Q_y = 0.3$, only the variable y is transformed and the function

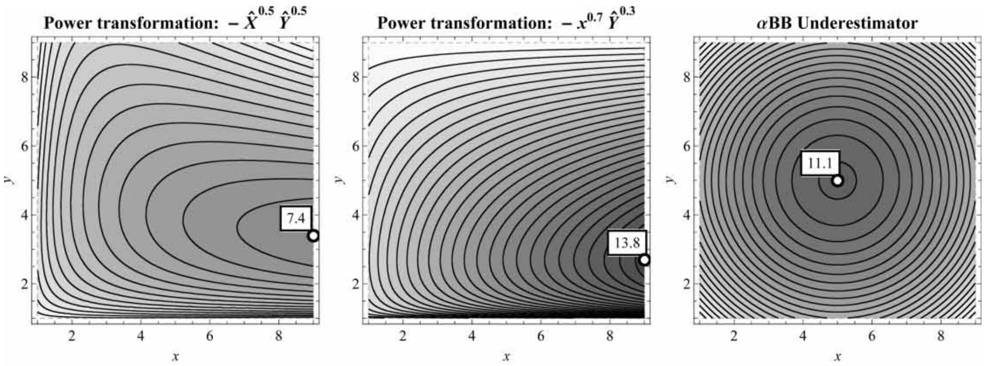
$$f_{P_2}(x, \hat{Y})_2 = -x^{0.7}\hat{Y}_2^{1\cdot 0.3} = -x^{0.7}\hat{Y}_2^{0.3}$$

is a piecewise convex underestimator of the function f. Since grid points can only be added to the piecewise linear approximation $\hat{Y}_2$, the domain is split into only two partitions when adding the grid point $y = 5$ as in case (b). To obtain a comparable combinatorial complexity as when transforming both variables, two additional grid points can be added; in this case $y = 3$ and $y = 5$. The result is a tighter convex underestimator than in any of the previous cases.

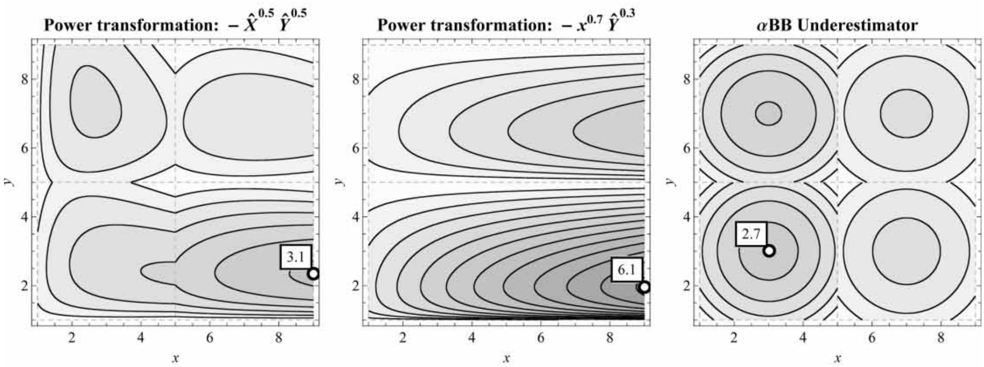
Plots of the absolute error of the different underestimators in Example 5.3 are provided in Figure 6.

5.3 *Multivariate functions*

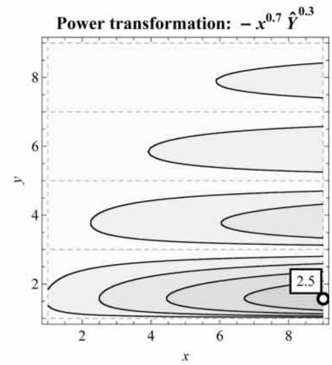
To be able to compare different types of convex underestimators for a function of arbitrarily many variables, the lower bound of the transformed function can, for example, be used.



(a) Only the interval endpoints as grid points.



(b) Additional grid points at $x = 5$ and $y = 5$.



(c) Additional grid points at $y = \{3, 5, 7\}$.

Figure 6. The absolute error of the convex underestimators in Example 5.3. Darker colors correspond to larger errors and the largest error is marked in the figure. The contours have an distance of 0.5. (a) Only the interval endpoints as grid points. (b) Additional grid points at $x = 5$ and $y = 5$. (c) Additional grid points at $y = 3, 5, 7$.

DEFINITION 5.4 *For a signomial function f the lower bound of the convexified and underestimated function $\hat{f}$ is given by the MINLP problem*

$$\begin{aligned} &\textit{minimize} \quad u, \\ &\textit{subject to} \quad \hat{f} - u \le 0. \end{aligned}$$

This method is used in the following two examples, originally from [4], to compare the convex underestimators. Since the domains of the variables are relatively large in these examples, the

Table 1. Comparison of convex underestimation techniques for Example 5.5.

Technique	ET	NPT	PPT	BARON	Li *et al.*	Optimal solution
Lower bound	−209.22	−215.73	−202.03	−224.44	−317.08	−202.00

ET, NPT and PPT are the exponential, negative power and positive power transformations. The result for BARON [9] is obtained at the root node, and the lower bound of Li *et al.*'s method is from [4].

αBB underestimator did not give a reasonable underestimator without partitioning the domains of the variables. It is, therefore, not included in the comparisons.

Example 5.5 In the signomial function $f(\boldsymbol{x}) = x_1x_2x_3x_4x_5 - x_2^{0.5}x_4^{0.5} - 3x_1 - x_5$, $1 \leq x_1, x_2, x_3, x_4, x_5 \leq 100$, only the first term is nonconvex and must be transformed. Transforming it using the ET gives the following convex underestimator

$$f_{\mathrm{E}}(\boldsymbol{x}, \hat{\boldsymbol{X}}_{\mathbf{E}}) = \mathrm{e}^{\hat{X}_{1,\mathrm{E}}+\hat{X}_{2,\mathrm{E}}+\hat{X}_{3,\mathrm{E}}+\hat{X}_{4,\mathrm{E}}+\hat{X}_{5,\mathrm{E}}} - x_2^{0.5}x_4^{0.5} - 3x_1 - x_5,$$

where $\hat{X}_{1,\mathrm{E}}, \ldots, \hat{X}_{5,\mathrm{E}}$ are the piecewise linear approximations of the inverse transformations $X_{i,\mathrm{E}} = \ln x_i$, i.e.,

$$\hat{X}_{i,\mathrm{E}} = \ln \underline{x}_i + \frac{\ln \overline{x}_i - \ln \underline{x}_i}{\overline{x}_i - \underline{x}_i}(x - \underline{x}_i).$$

Transforming $f(\boldsymbol{x})$ using any of the PTs gives the convex function

$$f_{\mathrm{P}}(\boldsymbol{x}, \hat{\boldsymbol{X}}_{\boldsymbol{P}}) = \hat{X}_{1,\mathrm{P}}^{Q_1}\hat{X}_{2,\mathrm{P}}^{Q_2}\hat{X}_{3,\mathrm{P}}^{Q_3}\hat{X}_{4,\mathrm{P}}^{Q_4}\hat{X}_{5,\mathrm{P}}^{Q_5} - x_2^{0.5}x_4^{0.5} - 3x_1 - x_5.$$

For the PPT the powers are here chosen such that $Q_1 = 5$ and $Q_2 = \cdots = Q_5 = -1$, and for the NPT the powers are chosen according to $Q_1 = \cdots = Q_5 = -10$. The function $f_{\mathrm{P}}(\boldsymbol{x}, \hat{\boldsymbol{X}}_{\boldsymbol{P}})$ will then be a convex underestimator of $f(\boldsymbol{x})$ if $\hat{X}_{1,\mathrm{P}}, \ldots, \hat{X}_{5,\mathrm{P}}$ are the piecewise linear approximations of the inverse transformations $X_{i,\mathrm{P}} = x_i^{1/Q}$, i.e.

$$\hat{X}_{i,\mathrm{P}} = \underline{x}_i^{1/Q_i} + \frac{\overline{x}_i^{1/Q_i} - \underline{x}_i^{1/Q_i}}{\overline{x}_i - \underline{x}_i}(x - \underline{x}_i).$$

The lower bounds of the different convex underestimators are provided in Table 1, and the impact of the power Q in the NPT is shown in Figure 7.

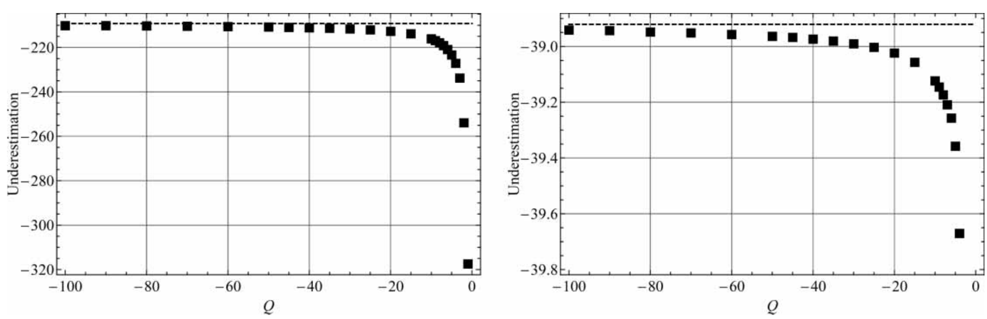


Figure 7. Impact of the values of the power Q in the NPT on the convex underestimations in Examples 5.5 (left) and 5.6 (right). The dashed line is the underestimation given by the ET.

Table 2. Comparison of convex underestimation techniques for Example 5.6.

Technique	ET	NPT	PPT	BARON	Li *et al.*	Optimal solution
Lower bound	−38.92	−39.12	−40.49	−42.31	−41.24	−38.08

ET, NPT and PPT are the exponential, negative power and positive power transformations. The result for BARON [9] is obtained at the root node, and the lower bound of Li *et al.*'s method is from [4].

Example 5.6 In the function $f(\boldsymbol{x}) = x_1^{-2}x_2^{-1.5}x_3^{1.2}x_4^3 - 3x_3^{0.5} + x_2 - 4x_4$, $1 \leq x_1, x_2, x_3, x_4 \leq 10$, only the first term is nonconvex and must be transformed. Transforming the function $f(\boldsymbol{x})$ using the ET gives the following expression for the convex underestimator

$$f_{\mathrm{E}}(\boldsymbol{x}, \hat{\boldsymbol{X}}_{\mathbf{E}}) = x_1^{-2}x_2^{-1.5}\mathrm{e}^{1.2\hat{X}_{3,\mathrm{E}}+3\hat{X}_{4,\mathrm{E}}} - 3x_3^{0.5} + x_2 - 4x_4,$$

where $\hat{X}_{1,\mathrm{E}}, \ldots, \hat{X}_{2,\mathrm{E}}$ are the piecewise linear approximations of the inverse transformations $X_{i,\mathrm{E}} = \ln x_i$. Transforming $f(\boldsymbol{x})$ using any of the PTs gives the following convex function

$$f_{\mathrm{P}}(\boldsymbol{x}, \hat{\boldsymbol{X}}_{\mathbf{P}}) = x_1^{-2}x_2^{-1.5}\hat{X}_{3,\mathrm{P}}^{1.2Q_3}\hat{X}_{4,\mathrm{P}}^{3Q_4} - 3x_3^{0.5} + x_2 - 4x_4,$$

which underestimates $f(\boldsymbol{x})$, if $\hat{X}_{3,\mathrm{P}}$ and $\hat{X}_{4,\mathrm{P}}$ are the piecewise linear approximations of the inverse transformations $X_{3,\mathrm{P}} = x_3^{1/Q_3}$ and $X_{4,\mathrm{P}} = x_4^{1/Q_4}$, and the powers Q are chosen so that the transformed term is convex. This can be done in many different ways. Here, the values on the powers Q_3 and Q_4 in the PPT were chosen as $Q_3 = -1$ and $Q_4 = 1.9$, and for the NPT as $Q_3 = Q_4 = -10$.

The lower bounds of the different convex underestimators are provided in Table 2, and the impact of the power Q in the NPT is presented in Figure 7.

5.4 *Some comments on the results*

In the case of the NPT, it was shown in this paper that the ET is in general a tighter underestimator. However, from Table 1, it may be noted that the PPT gives an even tighter underestimation of the function in Example 5.5 than the ET.

Table 2 indicates that the ET is the tightest of the underestimators examined in Example 5.6. However, the lower bound of the PPT can still in this case be improved if the powers Q_i would have been selected such that the lower bound is maximized. For example by choosing the powers as $Q_3 = 10.5$ and $Q_4 = -2$, the lower bound of the PPT would already increase to −38.73. The PPT is, thus, in fact tighter than ET in at least part of the considered domain. A more thorough analysis of the PPT will be given in some forthcoming papers.

6. Conclusions

The main result of this paper was Theorem 4.2 in Section 4, showing that the ET always gives tighter convex underestimations for positive signomial terms than the NPT. This is illustrated for Examples 5.5 and 5.6 in Figure 7. The PPT can, on the other hand, as Example 5.5 indicates, in some cases give better results than the ET. Also, when applying the PPT to a signomial term, it is sometimes possible to leave one of the variables with a positive power untransformed. This results in a simpler transformed MINLP problem than when using the ET, which may have a significant impact on the computational effort needed to solve the transformed problem.

Acknowledgments

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