Condensed Type Theory

by Johan Commelin

j/w Reid Barton

Menu

- Motivation
- Recap of type theory
- Condensed axioms
- Models
- Directed univalence
- Future ideas

Motivation: CAS for topology

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- If you squint a bit, then (cubical) HoTT can be viewed as CAS for homotopy theory.
- Our project provides some steps towards a CAS for general topology.

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- Condensed type theory provides some results of this flavour.

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  reorganize the proof, to require less human RAM.
- Condensed type theory rolled out of that,
  but might not be the final answer.

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  $\texttt{String.length} : \texttt{String} \to \texttt{Int}.$
- Compiler error: `String.length 3` is nonsense.

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- Examples of terms: `0` and π and `x ↦ (x^2 + 1)/2`.

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  `{x : X | P x}` is the *subtype* of terms of `X` satisfying `P`.
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- `Fin n`, for `n : ℕ`, is the subtype `{i : ℕ | i < n}`.
  It is the "canonical" type with `n` terms.

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- `CHaus : Type → Prop` and `ODisc : Type → Prop`.
- We abuse notation and define subuniverses corresponding to these predicates:

```
CHaus := {A : Type | CHaus A}
ODisc := {A : Type | ODisc A}
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- `A : CHaus` means that `A` is a compact Hausdorff space, and
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- The next slides present axioms on how these subuniverses interact.

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The subuniverses `CHaus` and `ODisc` are closed under:

- finite products, finite disjoint unions,
- quotients, formation of equalizers,
- isomorphisms,
- formation of Sigma types:

  if `A : CHaus` and `B : A → CHaus`,

  then `CHaus (Σ a, B a)`.

  And similarly for `ODisc`.

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- If `X : Type` and `Y : CHaus`, and if `f : X → Y` is surjective, then there exists a type `Y' : CHaus`, a surjective function `g : Y' → Y`, and a "section" `s : Y' → X` such that `f ∘ s = g`.

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$$\begin{tikzcd} & \mathtt{X} \arrow[d, "\mathtt{f}"] \\ \mathtt{Y'} \arrow[ur, "\mathtt{s}"] \arrow[r, "\mathtt{g}"'] & \mathtt{Y} \end{tikzcd}$$

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- The same statement holds for `ODisc` instead of `CHaus`.

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Condensed axioms: factorization

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- Let `X : CHaus` and `Y : ODisc`.

  Any function `f : X → Y` factors as

  `X → Fin n → Y`,

  for some `n :` $\mathbb{N}$.

- Slogan: “compact in discrete is finite”

Condensed axioms: Scott continuity

- Let `I, Y : ODisc` and `X : I → CHaus`.
  Let `f : (Π i, X i) → Y`.

Condensed axioms: Scott continuity

- Let `I, Y : ODisc` and `X : I → CHaus`.
  Let `f : (∏ i, X i) → Y`.

  Then `f` factors through a finite product of `X i`.

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Last but not least:

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We postulate `ODisc` $\mathbb{N}$.

Models: condensed sets

Almost Theorem (Barton–C). The internal type theory of the topos of condensed sets satisfies the condensed axioms.

We are almost done with the proof.

Provides a consistency check.

Models: computability

We are on the lookout for a computable model.

This would give CAS-like features to the type theory.

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{f : OProp → ODisc | f(⊥) = X ∧ f(⊤) = Y}
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Path(X, Y) ≅ (X → Y).
```

Formal proof in Lean.

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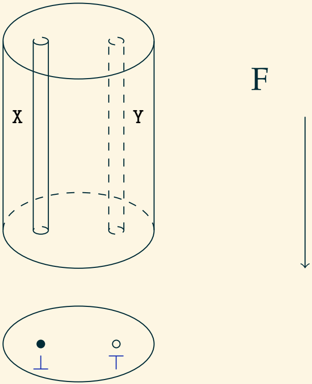
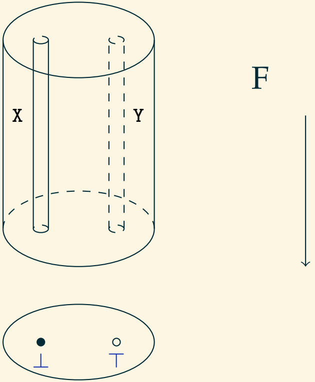
- Picture on next slide.

Directed univalence: picture



Directed univalence: picture



The fibration is a "local homeomorphism", so we get a map from the special fiber to the (nearby) generic fiber.

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This allows us to make the preceding picture precise. "□"

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Then $\mathrm{Et}^{\cong}$ is a stack of groupoids over CHaus, and Et is a stack of categories over CHaus.

Every functor of stacks $\mathrm{Et}^{\cong} \to \mathrm{Et}^{\cong}$ extends uniquely to a functor of stacks $\mathrm{Et} \to \mathrm{Et}$.

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- Explore the consequences of directed univalence.
- Work out the proof of LTE in condensed type theory.