On the Properties of Certain Cosine Integrals

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The integral we wish to solve is

$$
\begin{aligned}
I &= I(\{a_j\},\{n_j\},N,m) \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{im\phi} \prod_{j=1}^{N} \frac{1}{(1+a_j\cos\phi)^{n_j}},
\end{aligned} \tag{1}
$$

where m is an integer, n_j are a positive integers, and a_j are real numbers. Note that if one of the a_j's is zero, then we can reduce the integral to one with $N-1$ sets of parameters.

Some things to note about this integral:

1. Since the range of integration is 2π, and all the functions in the integrand are periodic in that range, the range of integration can be shifted by an arbitrary amount.

2. When the variable of integration is changed from ϕ to $-\phi$, the limits of integration can be rewritten as

$$
\int_0^{2\pi} d\phi \to \int_0^{-2\pi} d(-\phi) = \int_{-2\pi}^{0} d\phi = \int_0^{2\pi} d\phi. \tag{2}
$$

   Here we have used item number **??**. Since $\cos(-z) = \cos(z)$, the only thing that changes in the integrand is the sign in the exponential. Hence, $I(m) = I(-m)$, or the integral is only a function of the magnitude of m.

3. The integral is real, since $I(m)^* = I(-m) = I(m)$.

4. We may write

$$
\frac{1}{(1+a\cos\phi)^n} = \prod_{j=2}^{n}\left[1+\frac{a}{j-1}\frac{d}{da}\right]\frac{1}{1+a\cos\phi} \tag{3}
$$

   so that

$$
\begin{aligned}
&\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \prod_{j=1}^{N} \frac{1}{(a_j+b_j\cos\phi)^{n_j}} \\
&\quad = \prod_{l=1}^{N} \frac{(-1)^{(n_l-1)}}{(n_l-1)!} \frac{d^{n_l-1}}{da_l^{n_l-1}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \prod_{j=1}^{N} \frac{1}{a_j+b_j\cos\phi}.
\end{aligned} \tag{4}
$$

Since $b_j \neq 0$, we can factor out a $\prod_{j=1}^{N}\frac{1}{b_j^{n_j}}$, and we relabel $\frac{a_j}{b_j} \to a_j$. Then, the non-trivial integral is

$$
\begin{aligned}
I(\{a_i\},m,N) &= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi|m|} \prod_{j=1}^{N} \frac{1}{a_j+\cos\phi} \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi|m|} \prod_{j=1}^{N} \frac{2e^{i\phi}}{e^{2i\phi}+2a_je^{i\phi}+1}.
\end{aligned} \tag{5}
$$

Now we let $z = e^{i\phi}$, so $d\phi e^{i\phi} = dz/i$, which allows the integral to be converted to a contour integral over the contour C, a unit circle in the complex plane. Then,

$$\begin{aligned} I(\{a_i\}, m, N) &= \int_C \frac{dz}{2\pi i} z^{|m|-1} \prod_{j=1}^{N} \frac{2z}{z^2 + 2a_j z + 1} \\ &= \int_C \frac{dz}{2\pi i} z^{|m|-1} \prod_{j=1}^{N} \frac{2z}{(z - z_{j+})(z - z_{j-})}, \end{aligned} \tag{6}$$

where

$$z_{j\pm} = \left\{ \begin{array}{ll} -a_j \pm \mathrm{sign}(a_j)\sqrt{a_j^2 - 1} & \text{if } |a_j| > 1 \\ -a_j \mp i\sqrt{1 - a_j^2} = -e^{\pm i\phi_i)} & \text{if } |a_j| \leq 1 \end{array} \right\}. \tag{7}$$

Here $\phi_i = \arccos\sqrt{1 - a_i^2}$.

We will only consider the case where $|a_j| > 1$. In this case, only the only poles that contribute are the ones at $z = z_{j+}$. The contour integral is done by inspection to obtain

$$I(\{a_i\}, m, N) = \sum_{k=1}^{N} \int_{z_{k+}} \frac{dz}{2\pi i} \frac{2z^{|m|}}{(z - z_{k+})(z - z_{k-})} \prod_{j=1, j\neq k}^{N} \frac{2z}{z^2 + 2a_j z + 1} \tag{8}$$

$$= \sum_{k=1}^{N} \frac{2z_{k+}^{|m|}}{z_{k+} - z_{k-}} \prod_{j=1, j\neq k}^{N} \frac{2z_{k+}}{z_{k+}^2 + 2a_j z_{k+} + 1}. \tag{9}$$

Note that

$$z_{k\pm}^2 + 2a_j z_{k\pm} + 1 = z_{k\pm}^2 + 2a_k z_{k\pm} + 1 + 2(a_j - a_k) z_{k\pm} = 2(a_j - a_k) z_{k\pm}, \tag{10}$$

$$z_{k+} - z_{k-} = 2\,\mathrm{sign}(a_k)\sqrt{a_k^2 - 1}. \tag{11}$$

We define c_k and d_k by

$$c_k \equiv \frac{z_{k+} - z_{k-}}{2}, \tag{12}$$

$$k_k \equiv z_{k+}^{|m|}. \tag{13}$$

With the appropriate replacements, we obtain

$$I(\{a_i\}, m, N) = \sum_{k=1}^{N} \frac{d_k}{c_k} \prod_{j=1, j\neq k}^{N} \frac{1}{a_j - a_k}. \tag{14}$$

Putting the b_j's back in using the $a_i \to \frac{a_i}{b_i}$, we get

$$c_k = \mathrm{sign}(a_k)\sqrt{a_k^2 - b_k^2}, \tag{15}$$

$$d_k = \left(\frac{c_k - a_k}{b_k}\right)^{|m|}, \tag{16}$$

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \prod_{j=1}^{N} \frac{1}{a_j + b_j \cos\phi} = \sum_{k=1}^{N} \frac{d_k}{c_k} \prod_{j=1, j\neq k}^{N} \frac{1}{a_j - a_k \frac{b_j}{b_k}}. \tag{17}$$

This expression is used in Eq. (??) get

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \prod_{j=1}^{N} \frac{1}{(a_j + b_j \cos\phi)^{n_j}}$$
$$= \prod_{l=1}^{N} \frac{(-1)^{(n_l-1)}}{(n_l-1)!} \frac{d^{n_l-1}}{da_l^{n_l-1}} \sum_{k=1}^{N} \frac{d_k}{c_k} \prod_{j=1, j\neq k}^{N} \frac{1}{a_j - a_k \frac{b_j}{b_k}}. \tag{18}$$

Specificly, for $N = 1$, we can drop the subscripts. For $n = 1$,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{a + b\cos\phi} = \frac{d}{c}, \tag{19}$$

and for $n = 2$,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{(a + b\cos\phi)^2} = -\frac{d}{da}\frac{d}{c} = f_2 \frac{d}{c} \tag{20}$$

$$f_2 = \frac{a + |m|\, c}{c^2}. \tag{21}$$

Since $I(a, b, m)$ is like an eigenfunction of $\frac{d}{da}$, (but not exactly, since the "eigenvalue" is a function of a), we an use recursion to find that for general $n > 2$,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{(a + b\cos\phi)^n} = f_n I(a, b, m), \tag{22}$$

$$f_n = \frac{1}{n-1}\left(f_2 f_{n-1} - \frac{d}{da} f_{n-1}\right), \tag{23}$$

Using this we get

$$f_1 = 1, \tag{24}$$

$$f_2 = \frac{a + |m|\, c}{c^2}, \tag{25}$$

$$f_3 = \frac{3a^2 + c^2(m^2 - 1) + 3a|m|c}{2c^4}. \tag{26}$$

The higher-order f_i's can easily be found from analytic iteration using a program like *Mathematica*, but apparently they do not have a nice analytic form. We note that in general, f_i is multiplied by a factor of $c^{-2(i-1)}$.

We now check the equation for situations that may be numerically unstable. First note that as $c \to 0$, the f's are singular, which is due to the form of the original integral. Since the singular part is multiplicative, it can easily be treated.

Instabilities appear when b is very small. in which case we have to take care of d carefully as it appears to be singular. Let $b = 2a\delta$, so we can write

$$d_k = \left(\frac{\sqrt{1 - 4\delta^2} - 1}{2\delta}\right)^{|m|} \approx \left[-\delta(1 + \delta^2(1 + \delta^2(2 + \delta^2(5 + \delta^2(14 + \delta^2 \ldots)))))\right]^{|m|}.$$

This expression demonstrates that d_k is not go as $b^{-|m|}$, but as $b^{+|m|}$ as $b \to 0$.

Now we analyze what happens where there is more than one term in the denominator. We note first that

$$\left[\left(\frac{d}{dx}\right)^n f(x)g(x)\right] = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\left[\left(\frac{d}{dx}\right)^m f(x)\right]\left[\left(\frac{d}{dx}\right)^{n-m} g(x)\right]. \tag{27}$$

This means that Eq. (**??**) can be written as

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi m}\prod_{j=1}^{N}\frac{1}{(a_j+b_j\cos\phi)^{n_j}} \\
&= \sum_{k=1}^{N}\left[\prod_{l=1}^{N}\sum_{i_l=0}^{n_l-1}\right]\left[\frac{(-1)^{(n_l-1-i_l)}}{(n_l-1-i_l)!}\frac{d^{n_l-1-i_l}}{da_l^{n_l-1-i_l}}\left\{\frac{d_k}{c_k}\right\}_{\text{on}}\right] \\
&\quad\times\left[\frac{(-1)^{(i_l)}}{(i_l)!}\frac{d^{i_l}}{da_l^{i_l}}\left\{\prod_{j=1,j\neq k}^{N}\frac{1}{a_j-a_k\frac{b_j}{b_k}}\right\}_{\text{on}}\right].
\end{aligned}\tag{28}$$

The product can be broken into two parts, one where $l=k$ and another where $l\neq k$.

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi m}\prod_{j=1}^{N}\frac{1}{(a_j+b_j\cos\phi)^{n_j}} \\
&= \sum_{k=1}^{N}\left[\prod_{l=1,l\neq k}^{N}\sum_{i_l=0}^{n_l-1}\right]\left[\frac{(-1)^{(n_l-1-i_l)}}{(n_l-1-i_l)!}\frac{d^{n_l-1-i_l}}{da_l^{n_l-1-i_l}}\left\{\frac{d_k}{c_k}\right\}_{\text{on}}\right] \\
&\quad\times\left[\frac{(-1)^{(i_l)}}{(i_l)!}\frac{d^{i_l}}{da_l^{i_l}}\left\{\prod_{j=1,j\neq k}^{N}\frac{1}{a_j-a_k\frac{b_j}{b_k}}\right\}_{\text{on}}\right] \\
&\times\left[\sum_{i_k=0}^{n_k-1}\right]\left[\frac{(-1)^{(n_k-1-i_k)}}{(n_k-1-i_k)!}\frac{d^{n_k-1-i_k}}{da_k^{n_k-1-i_k}}\left\{\frac{d_k}{c_k}\right\}_{\text{on}}\right] \\
&\quad\times\left[\frac{(-1)^{(i_k)}}{(i_k)!}\frac{d^{i_k}}{da_k^{i_k}}\left\{\prod_{j=1,j\neq k}^{N}\frac{1}{a_j-a_k\frac{b_j}{b_k}}\right\}_{\text{on}}\right].
\end{aligned}\tag{29}$$

Now we note that $\frac{d}{da_l}\frac{d_k}{c_k}=0$, and so on, to get

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi m}\prod_{j=1}^{N}\frac{1}{(a_j+b_j\cos\phi)^{n_j}} \\
&= \sum_{k=1}^{N}\prod_{l=1,l\neq k}^{N}\left[\frac{(-1)^{(n_l-1)}}{(n_l-1)!}\frac{d^{n_l-1}}{da_l^{n_l-1}}\left\{\prod_{j=1,j\neq k}^{N}\frac{1}{a_j-a_k\frac{b_j}{b_k}}\right\}_{\text{on}}\right] \\
&\quad\times\frac{d_k}{c_k}\sum_{i_k=0}^{n_k-1}f_{n_k-i_k}\left[\frac{(-1)^{i_k}}{i_k!}\frac{d^{i_k}}{da_k^{i_k}}\left\{\prod_{j=1,j\neq k}^{N}\frac{1}{a_j-a_k\frac{b_j}{b_k}}\right\}_{\text{on}}\right].
\end{aligned}\tag{30}$$

The first derivative term in Eq. (**??**) can be written as

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi m}\prod_{j=1}^{N}\frac{1}{(a_j+b_j\cos\phi)^{n_j}} \\
&= \sum_{k=1}^{N}\frac{d_k}{c_k}\sum_{i_k=0}^{n_k-1}f_{n_k-i_k}\frac{(-1)^{i_k}}{i_k!}\frac{d^{i_k}}{da_k^{i_k}}\prod_{j=1,j\neq k}\frac{1}{\left(a_j-a_k\frac{b_j}{b_k}\right)^{n_j}}.
\end{aligned}\tag{31}$$

Now, we consider the situation where all the b_i have the same value,b. In the case where $k = N$, the complicated part of Eq. (**??**) is

$$h_N(i_N, N) \equiv \frac{1}{i_N!}\frac{d^{i_N}}{da_N^{i_N}}\prod_{j=1}^{N-1}\frac{1}{(a_j - a_N)^{n_j}} \tag{32}$$

$$= \sum_{i_{N-1}=0}^{i_N} h_{N-1}(i_{N-1}, N-1)\frac{1}{(i_N - i_{N-1})!} \times \frac{d^{(i_N - i_{N-1})}}{da_N^{(i_N - i_{N-1})}}\frac{1}{(a_{N-1} - a_N)^{n_{N-1}}} \tag{33}$$

$$= \sum_{i_{N-1}=0}^{i_N} h_{N-1}(i_{N-1}, N-1)\frac{(i_N - i_{N-1} + n_{N-1} - 1)!}{(i_N - i_{N-1})!(n_{N-1} - 1)!} \times \frac{1}{(a_{N-1} - a_N)^{n_{N-1} + i_N - i_{N-1}}} \tag{34}$$

$$= \prod_{l=3}^{N}\left[\sum_{i_{l-1}=0}^{i_l}\frac{(i_l - i_{l-1} + n_{l-1} - 1)!}{(i_l - i_{l-1})!(n_{l-1} - 1)!}\frac{1}{(a_{l-1} - a_N)^{n_{l-1} + i_l - i_{l-1}}}\right] \tag{35}$$

$$\times \frac{(i_2 + n_1 - 1)!}{i_2!(n_1 - 1)!}\frac{1}{(a_1 - a_N)^{n_1 + i_2}}. \tag{36}$$

From this, we can cyclically permute (or shuffle in any order) the labels to get a similar expression for $k \neq N$.

For definiteness, we consider a few special cases. If $n_j = 1$ for all j except N, we find that

$$h_2(i, 1) = \frac{\delta_{i,0}}{(a_2 - a_1)^{n_2}}, \tag{37}$$

$$h_2(i, 2) = \frac{1}{(a_1 - a_2)^{i+1}}, \tag{38}$$

and

$$h_3(i, 1) = \frac{\delta_{i,0}}{(a_2 - a_1)(a_3 - a_1)^{n_3}}, \tag{39}$$

$$h_3(i, 2) = \frac{\delta_{i,0}}{(a_1 - a_2)(a_3 - a_2)^{n_3}}, \tag{40}$$

$$h_3(i, 3) = \sum_{j=0}^{i}\frac{1}{(a_1 - a_3)^{j+1}(a_2 - a_3)^{i-j+1}}. \tag{41}$$

Using these we can write

$$g_k = \sum_{i_k=0}^{n_k - 1}(-1)^{i_k} f_{n_k - i_k} h_N(i_k, k), \tag{42}$$

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi m}\prod_{j=1}^{N}\frac{1}{(a_j + b_j\cos\phi)^{n_j}} = \sum_{k=1}^{N}\frac{d_k}{c_k}g_k. \tag{43}$$

Approximations

For $N = 2$, consider what happens when $a_1 \approx a_2$. We can write $a_2 = a$, $a_1 = a - \delta$, so that

$$\begin{aligned} I(a_1, a_2, b, n_1, n_2, m) & \\ &= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{(a_2 + b\cos\phi - \delta)^{n_1}} \frac{1}{(a_2 + b\cos\phi)^{n_2}} && (44) \\ &= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{(a_2 + b\cos\phi)^{n_1+n_2}} \sum_{j=0}^{\infty} \frac{(n_1 - 1 + j)!}{(n_1 - 1)! j!} \frac{\delta^j}{(a + b\cos\phi)^j} && (45) \\ &= \left[\sum_{j=0}^{\infty} (a_2 - a_1)^j \frac{(n_1 - 1 + j)!}{(n_1 - 1)! j!} f_{n_1+n_2+j} \right] \frac{d_2}{c_2}. && (46) \end{aligned}$$

For $N = 3$, we could have $a_2 \approx a_3$ (or equivalently $a_1 \approx a_3$). In this case,

$$\begin{aligned} I(a_1, a_2, a_3, b, 1, 1, n, m) & \\ &= \sum_{j=0}^{\infty} (a_3 - a_2)^j \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \frac{1}{a_1 + b\cos\phi} \frac{1}{(a_3 + b\cos\phi)^{n+j+1}} && (47) \\ &= \sum_{j=0}^{\infty} (a_3 - a_2)^j I(a_1, a_3, b, 1, n + j + 1, m). && (48) \end{aligned}$$

When $a_1 \approx a_2$ but $a_1 \not\approx a_3$, we get

$$I(a_1, a_2, a_3, b, 1, 1, n, m) = \sum_{j=0}^{\infty} (a_2 - a_1)^j I(a_1, a_3, b, 2 + j, n, m). \quad (49)$$

Summary

We define

$$I(\{a_i\}, b, \{n_i\}, m) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi m} \prod_i \frac{1}{(a_i + b\cos\phi)^{n_i}}, \quad (50)$$

$$c_k = a_k \sqrt{1 - \left(\frac{b_k}{a_k}\right)^2}, \quad (51)$$

$$d_k = \left(\frac{c_k - a_k}{b_k}\right)^{|m|}. \quad (52)$$

We assume that $n_i > 0$ and $b_i \neq 0$.

For $N = 1$, the integral is

$$I(a, b, n, m) = f_n \frac{d}{c}, \quad (53)$$

and for $N = 2$, the integral is

$$I(a_1, a_2, b, n_1, n_2, m) = (-1)^{n_2} \left(\sum_{j=0}^{n_1-1} f_{n_1-j} \frac{(j + n_2 - 1)!}{j!(n_2 - 1)!} \frac{1}{(a_1 - a_2)^{j+n_2}} \right) \frac{d_1}{c_1}$$

$$+(-1)^{n_1}\left(\sum_{j=0}^{n_2-1} f_{n_2-j}\frac{(j+n_1-1)!}{j!(n_1-1)!}\frac{1}{(a_2-a_1)^{j+n_1}}\right)\frac{d_2}{c_2}. \qquad (54)$$

For $N=2$, and with $n_1=1$ the integral simplifies to

$$I(a_1,a_2,b,1,n,m) \;=\; \frac{1}{(a_2-a_1)^n}\frac{d_1}{c_1} - \left(\sum_{j=0}^{n-1} f_{n-j}\frac{1}{(a_2-a_1)^{j+1}}\right)\frac{d_2}{c_2}. \qquad (55)$$

When $N=3$ and $n_1=n_2=1$ the integral is

$$\begin{aligned} &I(a_1,a_2,a_3,b,1,1,n,m)\\ &\quad= \frac{1}{(a_2-a_1)(a_3-a_1)^n}\frac{d_1}{c_1} + \frac{1}{(a_1-a_2)(a_3-a_2)^n}\frac{d_2}{c_2}\\ &\qquad+\left[\sum_{i=0}^{n-1}(-1)^i f_{n-i}\sum_{j=0}^{i}\frac{1}{(a_1-a_3)^{j+1}(a_2-a_3)^{i-j+1}}\right]\frac{d_3}{c_3}. \end{aligned} \qquad (56)$$

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To do this integral, we pull out complex analysis, and use this corollary to the Cauchy-Gorsat theorem,

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\int_C \frac{ds\, f(s)}{(s-z)^{n+1}} \qquad n \in \{0,1,2,3,\ldots\}$$

where the s is contained in the contour, and f is analytic inside the contour. This can be rewritten as

$$\int_C \frac{dz}{(z-s)^n} f(z) \;=\; \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\, f(z)\big|_{z=s}$$

Using this,

$$\begin{aligned} I(a,b,l\geq 0,m,n) \;&=\; \frac{2^{m+n}}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{z^{l+m+n-1}}{(z-z_{a-})^m(z^2+2bz+1)^n}\right)_{z=z_{a+}} +\\ &\quad+\frac{2^{m+n}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z^{l+m+n-1}}{(z-z_{b-})^n(z^2+2az+1)^m}\right)_{z=z_{b+}} \end{aligned}$$

Specificly, for $m=1$ and $n=0$,

$$I(a,b,l\geq 0,m=1,n=0) = 2\frac{z_{a+}^l}{z_{a+}-z_{a-}} = \frac{\mathrm{sign}(a)}{\sqrt{a^2-1}}\left(-a+\mathrm{sign}(a)\sqrt{a^2-1}\right)^l$$

and for $m=2$, $n=0$,

$$\begin{aligned} &I(a,b,l\geq 0,m=2,n=0)\\ &= 4\frac{z_{a+}^{l+1}}{(z_{a+}-z_{a-})^2}\left(\frac{l+1}{z_{a+}}-\frac{2}{z_{a+}-z_{a-}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a^2-1}\left(-a+\mathrm{sign}(a)\sqrt{a^2-1}\right)^{l+1}\left(\frac{l+1}{-a+\mathrm{sign}(a)\sqrt{a^2-1}}-\frac{\mathrm{sign}(a)}{\sqrt{a^2-1}}\right)\\
&= \frac{1}{a^2-1}\left(-a+\mathrm{sign}(a)\sqrt{a^2-1}\right)^{l+1}\left(\frac{|a|+l\sqrt{a^2-1}}{(-a+\mathrm{sign}(a)\sqrt{a^2-1})\sqrt{a^2-1}}\right)\\
&= \frac{1}{(a^2-1)^{3/2}}\left(-a+\mathrm{sign}(a)\sqrt{a^2-1}\right)^{l}\left(|a|+l\sqrt{a^2-1}\right)\\
&= I(a,b,l\geq 0,m=1,n=0)\frac{a+l\,\mathrm{sign}(a)\sqrt{a^2-1}}{a^2-1}
\end{aligned}
$$

and for $m=1$, $n=1$, we first note that $z_{c\pm}^2+2dz_{c\pm}+1=z_{c\pm}^2+2cz_{c\pm}+1+2(d-c)z_{c\pm}=2(d-c)z_{c\pm}$. Also, it it happens that $a=b$, then we should use the formula above. Then,

$$
\begin{aligned}
&I(a,b,l\geq 0,m=1,n=1)\\
&= 4\frac{z_{a+}^{l+1}}{(z_{a+}-z_{a-})(z_{a+}^2-2bz_{a+}+1)}+4\frac{z_{b+}^{l+1}}{(z_{b+}-z_{b-})(z_{b+}^2-2az_{a+}+1)}\\
&= 4\frac{z_{a+}^{l+1}}{(2\,\mathrm{sign}(a)\sqrt{a^2-1})(2(b-a)z_{a+})}+4\frac{z_{b+}^{l+1}}{(2\,\mathrm{sign}(b)\sqrt{b^2-1})(2(a-b)z_{b+})}\\
&= \frac{1}{a-b}\left(\frac{\left(-b+\mathrm{sign}(b)\sqrt{b^2-1}\right)^l}{\mathrm{sign}(b)\sqrt{b^2-1}}-\frac{\left(-a+\mathrm{sign}(a)\sqrt{a^2-1}\right)^l}{\mathrm{sign}(a)\sqrt{a^2-1}}\right)
\end{aligned}
$$

Now we wish to return to our original integral, $I(a,b,l,m)=\frac{1}{b^m}I(a/b,l,m)$. Therefore, we get

$$
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{i\theta l}}{(a+c\cos\theta)} = \frac{\mathrm{sign}(a)}{\sqrt{a^2-c^2}}\left(\frac{-a+\mathrm{sign}(a)\sqrt{a^2-c^2}}{c}\right)^{|l|} \tag{57}
$$

if $a^2>c^2$.

And

$$
\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{i\theta l}}{(a+c\cos\theta)^2} = \left(\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{i\theta l}}{(a+c\cos\theta)}\right)\frac{a+l\,\mathrm{sign}(a)\sqrt{a^2-c^2}}{a^2-c^2} \tag{58}
$$

In particular, for the case when $a^2>b^2$, and $a<0$ and $b>0$,

$$
I(a,b,l,1) = -\frac{2\pi}{\sqrt{a^2-b^2}}\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)^{|l|} \tag{59}
$$

$$
\begin{aligned}
I(a,b,l,2) &= \frac{2\pi}{(a^2-b^2)^{3/2}}\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)^{|l|}\left(-a+|l|\sqrt{a^2-b^2}\right) \qquad (60)\\
&= I(a,b,l,1)\frac{a-|l|\sqrt{a^2-b^2}}{a^2-b^2}
\end{aligned}
$$

Note that the first equation is negative definite, while the second one is positive definite.