Domain decomposition methods

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1 Theoretical study in 1D

Let $L \geq 0$ and $\Omega = (0, L)$. We consider the Poisson problem on Ω:

$$\begin{cases} -\frac{d^2u}{dx^2}(u) = f(x) \text{ in } (0, L) \\ u(0) = u(1) = 0 \end{cases}$$

Where $f \in L^2(\Omega)$ is given.

1.1 An overlapping domain decomposition method

Setting for $i = 1, 2$, $e_i^n = u_i^n - u$ we find the following equations:

$$\begin{cases} \frac{d^2}{dx^2} e_1^{n+1}(x) = 0 \text{ in } (0, L_1) \\ e_1^{n+1}(0) = 0 \\ e_1^{n+1}(L_1) = e_2^n(L_1) \end{cases}$$

$$\begin{cases} \frac{d^2}{dx^2} e_2^{n+1}(x) = 0 \text{ in } (l_1, L) \\ e_2^{n+1}(L) = 0 \\ e_2^{n+1}(l_2) = e_1^{n+1}(l_2) \end{cases}$$

Solving the first order derivative equation for $i = 2$ we find:

$$e_2^{n+1}(x) = \frac{e_1^{n+1}(l_2)}{l_2 - L} x + \frac{e^{n+1}(l_2)L}{L - l_2} \ \forall x \in [l_2, L]$$

and for $i = 1$:

$$e_1^{n+1}(x) = \frac{e_2^n(L_1)}{L_1} x \ \forall x \in [0, l_1]$$

finally we find:

$$e_2^{n+1}(x) = \frac{e_2^n(L_1) l_2}{L_1(l_2 - L)} x + \frac{e_2^n(L_1) l_2 L}{L_1(L - l_2)}$$

To finish with:

$$e_2^{n+1}(L_1) = \frac{(L - L_1) l_2}{(L - l_2) L_1} e_2^n(L_1)$$

We can conclude that $(e_2^n(L_1))$ is geometric with $q = \frac{(L-L_1)l_2}{(L-l_2)L_1}$ and since $L_1 < l_2$ we have $\frac{l_2}{L_1} < 1$ and $(L - L_1) < (L - l_2)$ to get $q < 1$. Then $\lim_{n \to +\infty} e_2^n(L_1) = 0$.
When $\delta \to 0$ we have $l_2 \to L_1$ so $\lim_{\delta \to 0^+} q = 1$. Thus convergence is slowed down.
Finally the algorithm converge if $\delta > 0$ and the convergence is exponential.

1.2 A non overlapping domain decomposition method

Setting for $i = 1, 2$, $e_i^n = v_i^n - u$ we find the following equations:

$$\begin{cases} \frac{d^2}{dx^2} e_1^{n+1}(x) = 0 \text{ in } (0, l) \\ e_1^{n+1}(0) = 0 \\ \frac{d}{dx} e_1^{n+1}(l) + \lambda e_1^{n+1}(l) = \frac{d}{dx} e_2^n(l) + \lambda e_2^n(l) = g_2^n \end{cases}$$

$$\begin{cases} \frac{d^2}{dx^2} e_2^{n+1}(x) = 0 \text{ in } (l, L) \\ e_2^{n+1}(L) = 0 \\ -\frac{d}{dx} e_2^{n+1}(l) + \lambda e_1^{n+1}(l) = -\frac{d}{dx} e_1^n(l) + \lambda e_1^n(l) = g_1^n \end{cases}$$

Solving the first order derivative equation for $i = 1$ we find:

$$e_1^{n+1}(x) = \frac{g_2^n}{1 + \lambda l} x$$

and the relation:

$$g_1^{n+1} = -\frac{d}{dx} e_1^{n+1}(l) + \lambda e_1^{n+1}(l) = \frac{g_2^n}{1 + \lambda l}(\lambda l - 1)$$

and for $i = 2$ we find:

$$e_2^{n+1}(x) = \frac{g_1^n}{\lambda(l - L) - 1} x + \frac{g_1^n L}{1 - \lambda(l - L)}$$

and the relation:

$$g_2^{n+1} = \frac{d}{dx} e_2^{n+1}(l) + \lambda e_2^{n+1}(l) = \frac{g_1^n}{\lambda(l - L) - 1}(1 + \lambda(l - L))$$

Finally we find the following relation:

$$\begin{pmatrix} g_1^{n+1} \\ g_2^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\lambda l - 1}{1 + \lambda l} \\ \frac{1 + \lambda(l - L)}{\lambda(l - L) - 1} & 0 \end{pmatrix} \begin{pmatrix} g_1^n \\ g_2^n \end{pmatrix}$$

And knowing that with $M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ we have:

$$M^k = \begin{cases} \begin{pmatrix} (ab)^{k-1} & 0 \\ 0 & (ab)^{k-1} \end{pmatrix} & \text{if } k \text{ is even} \\ \begin{pmatrix} 0 & a^{k-1} b^{k-2} \\ a^{k-2} b^{k-1} & 0 \end{pmatrix} & \text{if } k \text{ is odd} \end{cases}$$

Thus, the sufficient condition for which $\lim_{n\to+\infty}\begin{pmatrix} g_1^n \\ g_2^n \end{pmatrix} = 0$ is $\lim_{n\to+\infty}(\frac{\lambda l-1}{1+\lambda l}\frac{1+\lambda(l-L)}{\lambda(l-L)-1})^n = 0$.
Finally, we have: $\frac{\lambda l-1}{1+\lambda l}\frac{1+\lambda(l-L)}{\lambda(l-L)-1} = 1 + 2L\frac{\lambda}{\lambda^2 l(l-L)-\lambda L-1}$ for the convergence we need $1 + 2L\frac{\lambda}{\lambda^2 l(l-L)-\lambda L-1} < 1$.
Finding a minimum for $f(\lambda) = \frac{\lambda}{\lambda^2 l(l-L)-\lambda L-1}$ means that we have faster convergence. By derivation, $f'(\lambda) = \frac{\lambda^2 l(l-L)-\lambda(L+2l(l-L))+L}{(\lambda^2 l(l-L)-\lambda L-1)^2}$ the minimums are the solution of $\lambda^2 l(l-L) - \lambda(L+2l(l-L)) + L = 0$, and we also have to take the solution that follows the condition.

2 Numerical study in 2D

2.1 Introduction to the geometry and to the Poisson problem

We consider the Poisson equation:

$$\begin{cases} -\Delta u(x,y) = f(x,y) & \text{in } \Omega \\ u(x,y) = 0 & \text{on } \partial\Omega \end{cases}$$

where f is a given regular function and Ω is an open bounded domain defined as the union of a disk and a rectangle you can see in Figure 1.

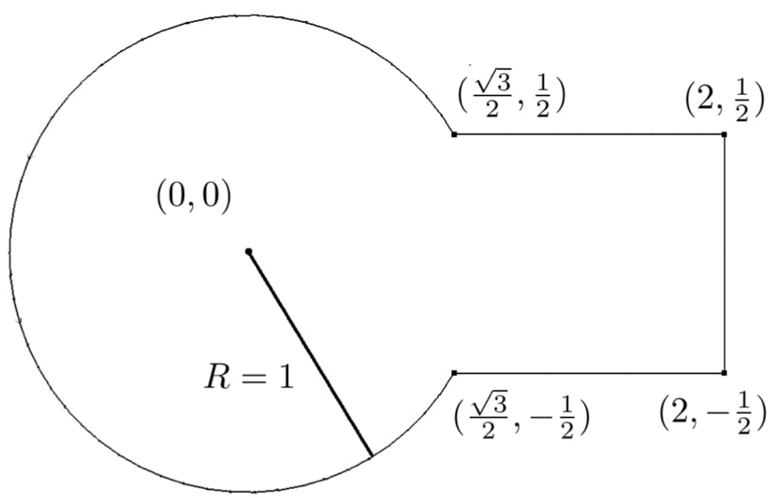

Figure 1: Domain Ω

We suppose the regularity of the solution u and we take a test function v regular too. Then we have, using the Green formula:

$$\int_\Omega fv\mathrm{d}x = -\int_\Omega v\Delta u\mathrm{d}x = \int_\Omega \nabla u.\nabla v\mathrm{d}x - \int_{\partial\Omega}\frac{\partial u}{\partial n}v\mathrm{d}s$$

But we know that $u = 0$ on $\partial\Omega$, so we choose $v \in V$ (with V an Hilbert space to define) such as $v = 0$ on $\partial\Omega$ too:

$$\underbrace{\int_\Omega f v \mathrm{d}x}_{v, f \in L^2(\Omega)} = \overbrace{\int_\Omega \nabla u . \nabla v \mathrm{d}x}^{\nabla u, \nabla v \in L^2(\Omega)}$$

Finally, taking $V = H_0^1(\Omega)$ we have the variational formulation:

$$\text{Find } u \in H_0^1(\Omega) \text{ such as } \int_\Omega f v \mathrm{d}x = \int_\Omega \nabla u . \nabla v \mathrm{d}x \ \forall v \in H_0^1(\Omega)$$

Following the discretization of Ω given, we can take:

$$V_h = \{v \in C(\bar{\Omega}) \text{ such as } v_{|K} \in \mathbb{P}_1 \ \forall K \in \mathcal{T} \text{ and } v = 0 \text{ on } \partial\Omega\}$$

The internal approximations gives the following formulation:

$$\text{Find } u_h \in V_h \text{ such as } \int_\Omega f v_h \mathrm{d}x = \int_\Omega \nabla u_h . \nabla v_h \mathrm{d}x \ \forall v \in V_h$$

Using Freefem $++$ we find a numerical solution of the equation with $f = 1$ you can see on Figure 2.

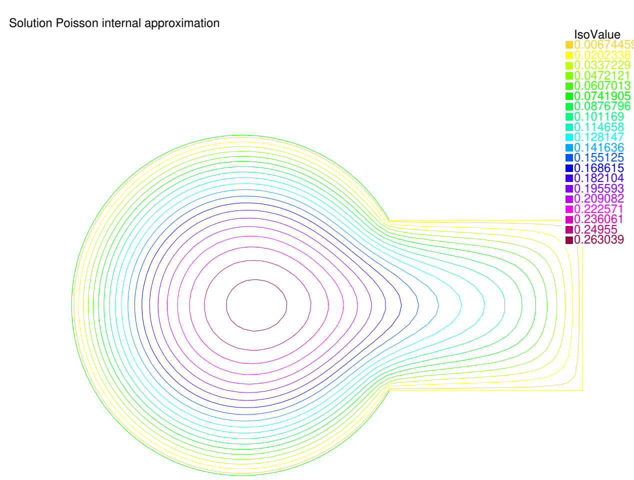

Figure 2: Poisson solution with internal approximation

2.2 An overlapping domain decomposition method

In this section we study the numerical solution with another decomposition of Ω, using an overlapped domain decomposition you can see in Figure 3.

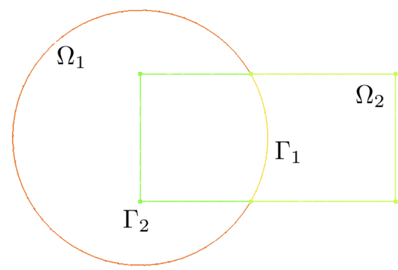

Figure 3: Decomposition of Ω

The alternating Schwarz method consists in building two sequences $(u_1^n)_n \in \mathbb{N}$ (defined in Ω_1) and $(u_2^n)_n \in \mathbb{N}$ (defined in Ω_2) such that $\forall n \geq 0$, $u_{n+1}^{1,2}$ are solution of:

$$\begin{cases} -\Delta u_1^{n+1} = f & \text{in } \Omega_1 \\ u_1^{n+1} = 0 & \text{on } \partial\Omega \cap \partial\bar{\Omega}_1 \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1 \end{cases}$$

$$\begin{cases} -\Delta u_2^{n+1} = f & \text{in } \Omega_2 \\ u_2^{n+1} = 0 & \text{on } \partial\Omega \cap \partial\bar{\Omega}_2 \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2 \end{cases}$$

Choosing $l = 0$ we get for $n \in \{1, ..., 4\}$ the figures you can see in Figure 4.

We are now going to study the convergence of the method comparing u_1^n and u_2^n with u numerically find in the previous section:

$$\mathcal{E}_{L^2} = ||u_1^n - u||_{L^2(\Omega_1)} + ||u_2^n - u||_{L^2(\Omega_2)}$$

$$\mathcal{E}_{H^1} = ||u_1^n - u||_{H^1(\Omega_1)} + ||u_2^n - u||_{H^1(\Omega_2)}$$

We find the following results:

n	$\mathcal{E}_{L^2}$	$\mathcal{E}_{H^1}$
1	0.0711983	0.229588
2	0.00348496	0.0208795
3	0.000520414	0.0159694
4	0.000595689	0.0159872

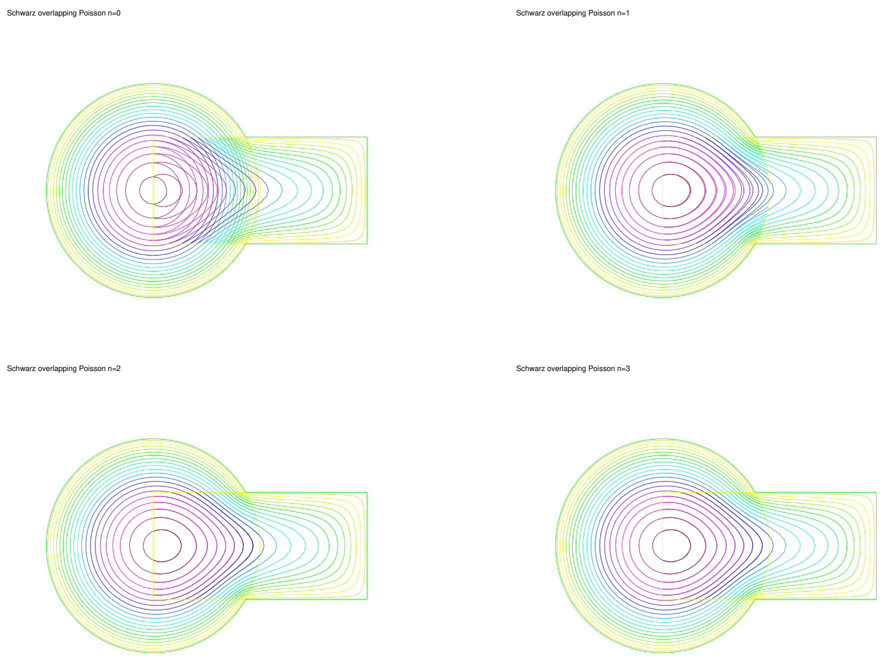

Figure 4: Numerical solution with an overlapping domain decomposition.

Those results show numerically that the method is converging, $5e-4$ for $||.||_{L^2}$ and $1.5e-2$ for $||.||_{H^1}$.

Supposing that the solution u is in $H^2(\Omega)$ we could compute: $\mathcal{E}_{L^2} = ||\Delta u_1^n - 1||_{L^2(\Omega_1)} + ||\Delta u_2^n - 1||_{L^2(\Omega_2)}$.

To study how the convergence is affected by the value l we can use the $\mathcal{E}_{L^1}$ criterion by looping on different values of l and see the different values for $\mathcal{E}_{L^2}$, we obtain:

n	0	0.1	0.3	0.5	0.8
1	0.0713915	0.0713915	0.0723178	0.0746788	0.0863955
2	0.00359688	0.00359688	0.00431534	0.00696119	0.0310829
3	0.000512441	0.000512441	0.000529725	0.000716562	0.0110686
4	0.000599739	0.000599739	0.00063672	0.000662095	0.00380915

We remark that the closer is l from 0 the better is the convergence. When we change $u_2^{n+1} = u_1^n$ on Γ_2 we see that the advantage of this method is that we can compute the two systems in parallel. //

2.3 A non overlapping domain decomposition method

We now consider another decomposition of the domain Ω you can see in Figure 5.

We now introduce a strictly positive parameter λ and two sequences $(v_1^n)_{n\in\mathbb{N}}$ (defined in Ω_1) and $(v_3^n)_{n\in\mathbb{N}}$ (defined in Ω_3) and the system:

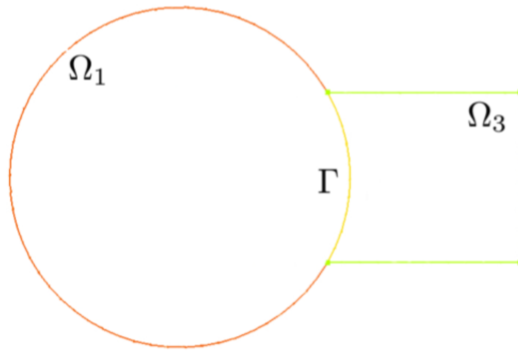

Figure 5: Domain Ω

$$\begin{cases} -\Delta v_1^{n+1} = f & \text{in } \Omega_1 \\ v_1^{n+1} = 0 & \text{on } \partial\Omega_1 \backslash \Gamma \\ \partial_{n_1} v_1^{n+1} + \lambda v_1^{n+1} = \partial_{n_1} v_3^n + \lambda v_3^n & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta v_3^{n+1} = f & \text{in } \Omega_3 \\ v_3^{n+1} = 0 & \text{on } \partial\Omega_3 \backslash \Gamma \\ \partial_{n_3} v_3^{n+1} + \lambda v_1^{n+1} = \partial_{n_3} v_1^n + \lambda v_1^n & \text{on } \Gamma \end{cases}$$

We have $\forall \phi \in H_0^1(\Omega_1 \backslash \Gamma)$:

$$\begin{aligned} \int_{\Omega_1} f\phi \mathrm{d}x = -\int_{\Omega_1} \Delta v_1^{n+1} \phi \mathrm{d}x &= \int_{\Omega_1} \nabla v_1^{n+1} . \nabla \phi \mathrm{d}x - \int_{\partial\Omega_1} \frac{\partial v_1^{n+1}}{\partial n_1} \phi \mathrm{d}s \\ &= \int_{\Omega_1} \nabla v_1^{n+1} . \nabla \phi \mathrm{d}x - \int_{\Gamma} \frac{\partial v_1^{n+1}}{\partial n_1} \phi \mathrm{d}s - \int_{\partial\Omega_1 \backslash \Gamma} \frac{\partial v_1^{n+1}}{\partial n_1} \phi \mathrm{d}s \\ &= \int_{\Omega_1} \nabla v_1^{n+1} . \nabla \phi \mathrm{d}x - \int_{\Gamma} \frac{\partial v_1^{n+1}}{\partial n_1} \phi \mathrm{d}s \\ &= \int_{\Omega_1} \nabla v_1^{n+1} . \nabla \phi \mathrm{d}x - \int_{\Gamma} (\frac{\partial v_3^n}{\partial n_1} + \lambda v_3^n) \phi \mathrm{d}s + \int_{\Gamma} \lambda v_1^{n+1} \phi \mathrm{d}s \end{aligned}$$

Finally, the variational formulation is:

$$\text{Find } v_1^{n+1} \in H_0^1(\Omega_1 \backslash \Gamma) \text{ such as}$$

$$\int_{\Omega_1} f\phi \mathrm{d}x = \int_{\Omega_1} \nabla v_1^{n+1} . \nabla \phi \mathrm{d}x - \int_{\Gamma} (\partial_{n1} v_3^n + \lambda v_3^n) \phi \mathrm{d}s + \int_{\Gamma} \lambda v_1^{n+1} \phi \mathrm{d}s \ \forall \phi \in H_0^1(\Omega_1 \backslash \Gamma)$$

After computing the equation we find plot for the first 4 iteration you can see in Figure 6.

We now compute $||v_1^n - u||_{L^2(\Omega_1)} + ||v_3^n - u||_{L^2(\Omega_3)}$ and we find the put the results in a table:

n	$\|\|v_1^n - u\|\|_{L^2(\Omega_1)} + \|\|v_3^n - u\|\|_{L^2(\Omega_3)}$
1	0.0148851
2	0.00552623
3	0.00392263
4	0.00358452

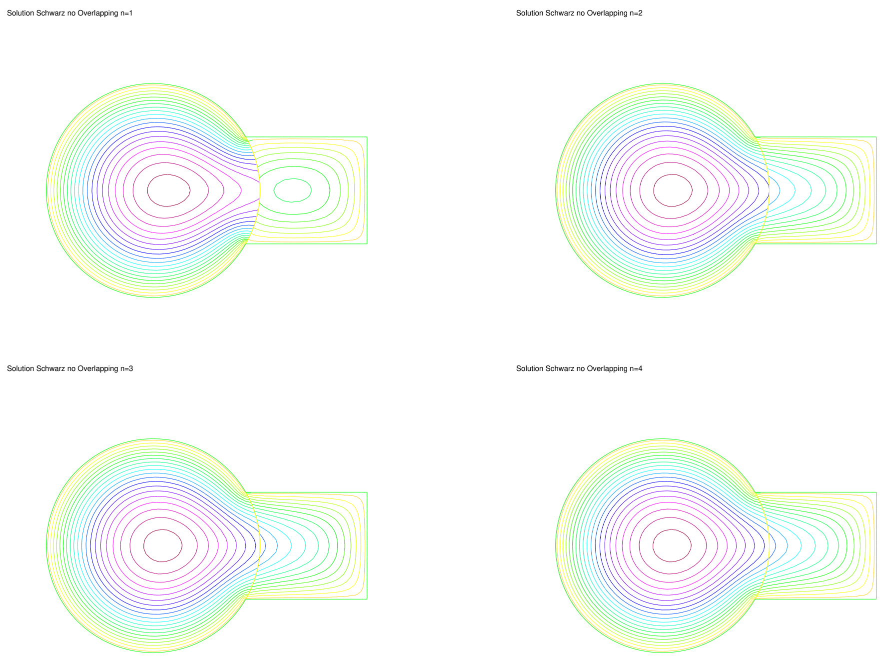

Figure 6: Numerical solution with an overlapping domain decomposition.

The method with no overlapping seems to converge faster to the solution contrary to the method with an overlapping.
To study the effect of the parameter λ we can print in a table the result of $||v_1^n - u||_{L^2(\Omega_1)} + ||v_3^n - u||_{L^2(\Omega_3)}$ for some values of λ:

n	0	0.5	1	1.5	2	2.5
1	0.0585339	0.0294172	0.0148851	0.00814685	0.00775998	0.0112978

It looks like that we have a minimum for $\lambda \in [2, 2.5]$, we write a python script so as to plot the result and see this minimum in Figure 7.

2.4 Conclusion

During this game, we saw two methods of decomposing Ω. The non-overlapping method seemed more efficient than the overlapping method. Moreover it is possible to study quite easily the λ effect on the error where we observe a minimum for a certain value (as predicted in dimension 1). A certain improvement of the algorithms would be to parallelize the calculations, which is made possible by the shape of the numerical sequence systems. Finally, for the method without overlapping, perhaps it would be judicious to modify λ at each iteration of the algorithm in order to always take the one that minimizes the error for a given u_i.

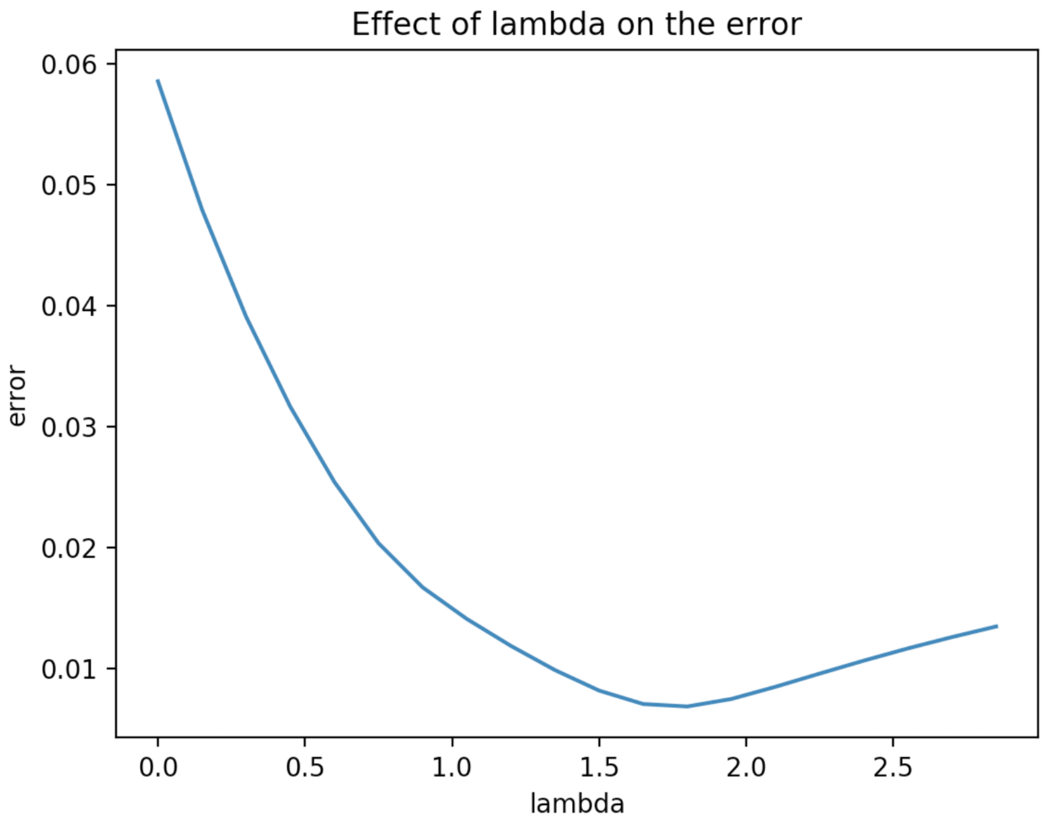

Figure 7: Effect of λ on the error

3 Wi-fi propagation in the oval office

Inspired by this tutorial on wifi propagation, we wanted to try to apply last sections methods to the Helmholtz equation.

$$\nabla^2 E + \frac{k^2}{n^2} E = 0$$

We kept last section shape that we saw as Donald J. Trump's office. We model the wifi hotspot as a little circle.
To create more realism, we make the office surrounded by very thick shielded walls that absorb and refract a given amount of the signal.

First of all we implemented the classic method in Helmholtz.edp. The variational formulation yields: Find $u \in H^1$ st for all $v \in H^1$

$$\int_\Omega (\nabla u \nabla v - \frac{k^2}{n^2} uv) - \int_{\partial\Omega} \partial_n uv = 0$$

As we assume that those very thick walls do not let pass by any amount of E, so we have $\partial_n u = 0$. Otherwise on the boundary of the hotspot we just impose a constant value on $u = E$.

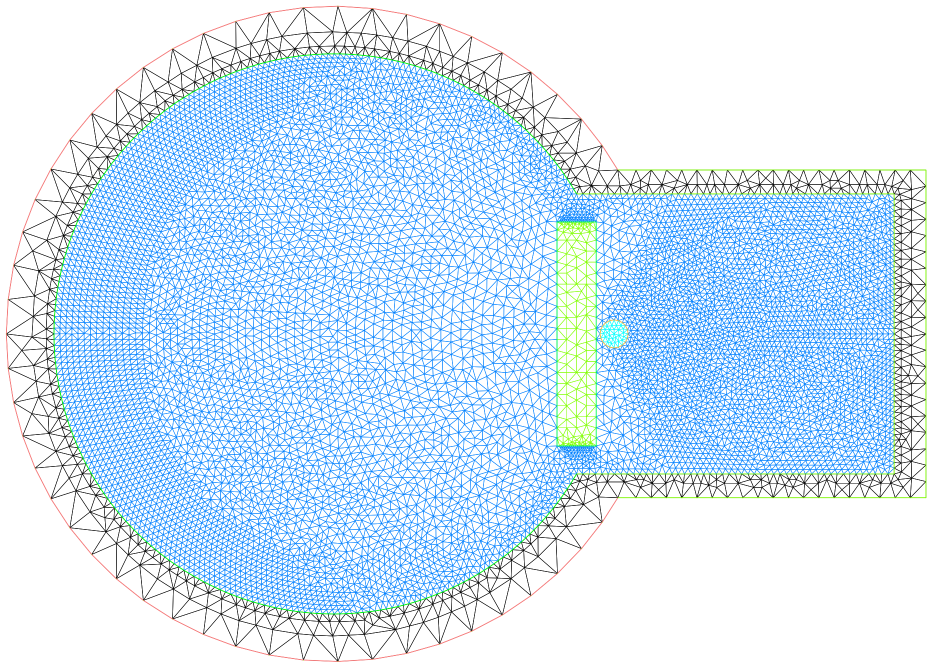

Figure 8: Our mesh for the oval office, in blue the inside with a green desk and in black the shielded walls

We also implemented 2 types of environment. One is air where $n = 1$ and the other are the walls/desk where $n = n_r + in_i$. The real part defines the reflexion of the wall (the amount of signal that doesn't pass) and the imaginary part defines the absorption of the wall (the amount that disappears). We ca see in Fig. **??** that behind the deskop the intensity of the field is lower.
We also did implement the non overlapping proposed by P.L. Lions in helmholtzNoOverlap.edp but there seems to be a problem. The solution seems normal after a few iterations, but after too many we find something all uniform. We can see them in 10.

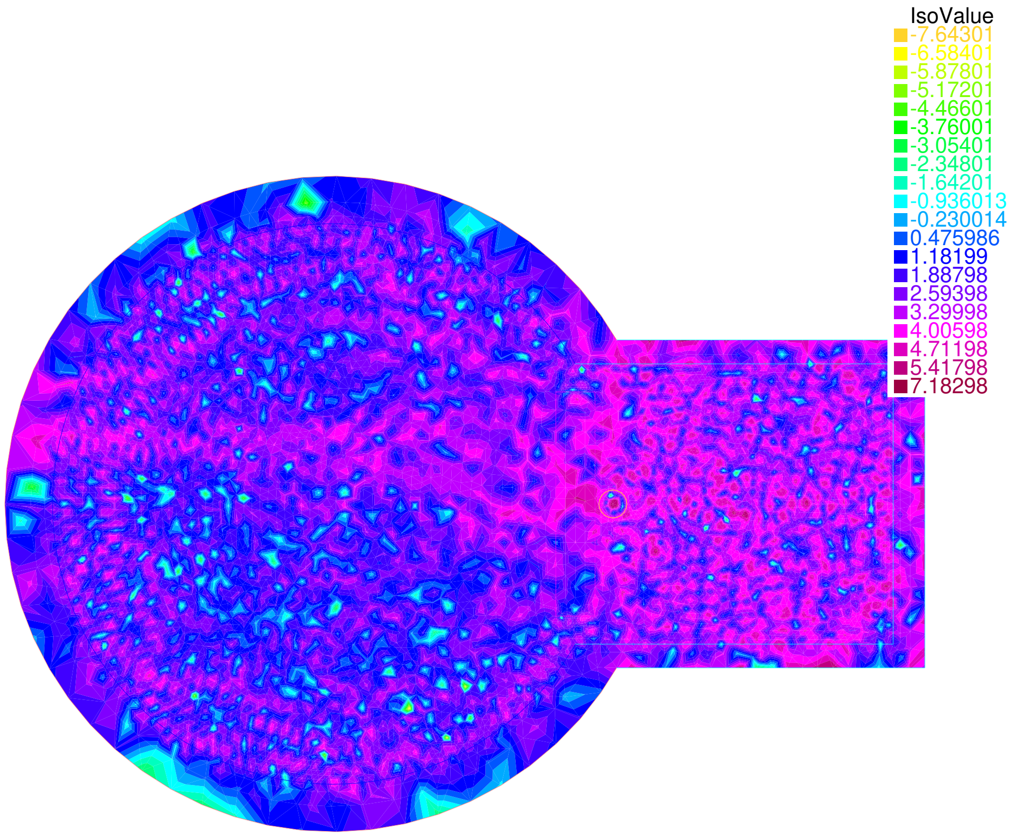

Figure 9: Log of the intensity of E in the office

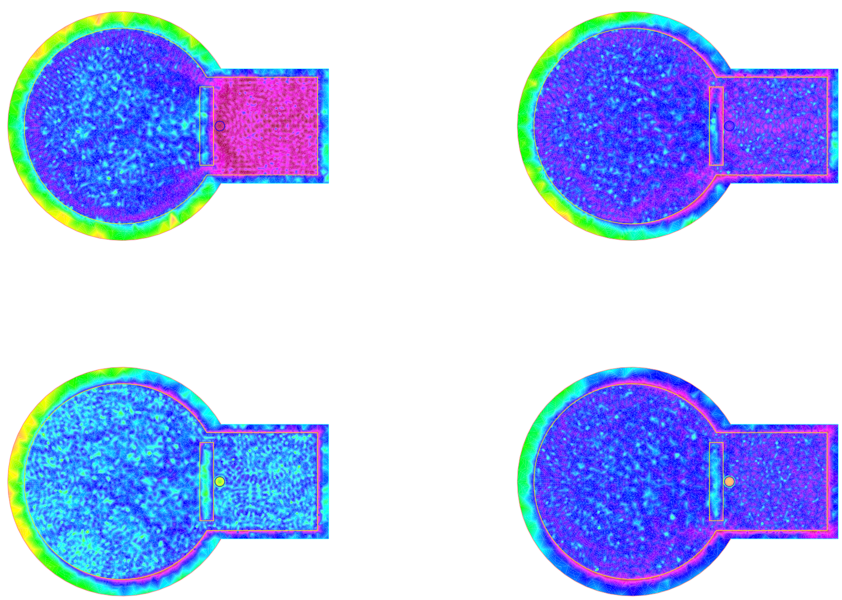

Figure 10: Some of the iterations of the non overlapping method