Mimetic Discretization of the Integration by Parts Formula

The divergence theorem states that

$$\int_{\Omega} (\nabla \cdot \mathbf{u})\, q \, dV = \int_{\partial\Omega} q\, (\mathbf{u} \cdot \mathbf{n})\, ds - \int_{\Omega} \mathbf{u} \cdot \nabla q \, dV \tag{1}$$

where the boundary integral represents the flux through $\partial\Omega$.
In one dimension, this reduces to the familiar integration by parts (IBP) formula:

$$\int_a^b u'(x)\, q(x)\, dx = \big[u(x)\, q(x)\big]_a^b - \int_a^b u(x)\, q'(x)\, dx \tag{2}$$

where the boundary term is $\big[u(x)\, q(x)\big]_a^b = u(b)q(b) - u(a)q(a)$. If the boundary term vanishes (e.g., homogeneous Dirichlet or periodic boundary conditions), we obtain the following

$$\int_a^b u'(x)\, q(x)\, dx = -\int_a^b u(x)\, q'(x)\, dx \tag{3}$$

and let

$$u_h \in \mathcal{F}_h, \qquad q_h \in \mathcal{C}_h$$

be discrete fields, where $\mathcal{F}_h$ denotes the discrete space associated with face-centered (vector) quantities, and $\mathcal{C}_h$ the space associated with cell-centered (scalar) quantities.
Define the one-dimensional mimetic operators *divergence* and *gradient*:

$$D : \mathcal{F}_h \to \mathcal{C}_h, \qquad G : \mathcal{C}_h \to \mathcal{F}_h$$

The weighted inner products on $\mathcal{F}_h$ and $\mathcal{C}_h$ are induced by diagonal, positive-definite matrices P and Q, respectively.
Then, the discrete analog of (3) is given by

$$\langle D u_h, q_h \rangle_Q = -\langle u_h, G q_h \rangle_P, \qquad \forall\, u_h, q_h \tag{4}$$

or, in matrix form,

$$(D u_h)^T Q\, q_h = -\, u_h^T P\, (G q_h) \tag{5}$$

The example `integration1D.m` illustrates how the weight matrix Q can be used to approximate the integral of a Ricker wavelet.