

Notes on Integration by Parts

The divergence theorem states that

$$\int_{\Omega} (\nabla \cdot \mathbf{u}) q \, dV = \int_{\partial\Omega} q (\mathbf{u} \cdot \mathbf{n}) \, ds - \int_{\Omega} \mathbf{u} \cdot \nabla q \, dV \quad (1)$$

where the boundary integral represents the flux through $\partial\Omega$. In one dimension, this reduces to the familiar IBP formula

$$\int_a^b u'(x) q(x) \, dx = [u(x)q(x)]_a^b - \int_a^b u(x) q'(x) \, dx \quad (2)$$

where the boundary term $[u(x)q(x)]_a^b = u(b)q(b) - u(a)q(a)$. If the boundary term vanishes (e.g., homogeneous Dirichlet or periodic BCs), we obtain the following

$$\int_a^b u'(x) q(x) \, dx = - \int_a^b u(x) q'(x) \, dx \quad (3)$$

Let

$$u_h \in \mathcal{F}_h, \quad q_h \in \mathcal{C}_h$$

be discrete fields. Define the mimetic operators

$$D : \mathcal{F}_h \rightarrow \mathcal{C}_h, \quad G : \mathcal{C}_h \rightarrow \mathcal{F}_h$$

The weighted inner products on the discrete spaces \mathcal{F}_h and \mathcal{C}_h are induced by the diagonal, positive definite matrices P and Q , respectively.

Then, the discrete analog of (3) is

$$\langle Du_h, q_h \rangle_Q = -\langle u_h, Gq_h \rangle_P, \quad \forall u_h, q_h \quad (4)$$

or, in matrix form,

$$(Du_h)^T Q q_h = -u_h^T P (Gq_h)$$

Example `integration1D.m` illustrates how the weight matrix Q can be used to approximate the integral of a Ricker wavelet.