MASSACHUSETTS INSTITUTE OF TECHNOLOGY

ENTRANCE EXAMINATION, 1869-70

ALGEBRA

1. If e=8 find the numerical value of the following expresion:

$$e - \{\sqrt{e+1} + 2\} + (e - \sqrt[3]{e})\sqrt{e-4}$$

The solution becomes really easy when you notice that $8 = 2^3$,

$$8 - \{\sqrt{8+1} + 2\} + (8 - \sqrt[3]{2^3})\sqrt{8-4} =$$

$$= 8 - \{\sqrt{9} + 2\} + (8 - 2)\sqrt{4} =$$

$$= 8 - \{3 + 2\} + (6) \times 2 =$$

$$= 8 - 5 + 12 = 3 + 12 = 15$$

2. Simplify the following expression by removing the brackets and collecting like terms

$$3a - [b + (2a - b) - (a - b)]$$

Once again, the trick is working with no hurries,

$$3a - [b + (2a - b) - (a - b)] = 3a - [b + 2a - b - a + b] = 3a - [b + a] = 2a - b$$

3. Multiply $3a^2 + ab - b^2$ by $a^2 - 2ab - 3b^2$ and divide the product by $a + b$

Step 1: Multiply the polynomials

$(3a^2 + ab - b^2)(a^2 - 2ab - 3b^2)$

Distributing each term:

$$= 3a^2(a^2 - 2ab - 3b^2) + ab(a^2 - 2ab - 3b^2) - b^2(a^2 - 2ab - 3b^2)$$
$$= 3a^4 - 6a^3b - 9a^2b^2 + a^3b - 2a^2b^2 - 3ab^3 - a^2b^2 + 2ab^3 + 3b^4$$
$$= 3a^4 - 5a^3b - 12a^2b^2 - ab^3 + 3b^4$$

Step 2: Divide by $(a+b)$

We perform polynomial long division:

$$\begin{array}{r|l} a+b & 3a^4 - 5a^3b - 12a^2b^2 - ab^3 + 3b^4 \\ \hline & 3a^3 - 8a^2b - 4ab^2 + 3b^3 \end{array}$$

Therefore:

$$\boxed{3a^3 - 8a^2b - 4ab^2 + 3b^3}$$

Appendix: Detailed Long Division Process

We divide $3a^4 - 5a^3b - 12a^2b^2 - ab^3 + 3b^4$ by $a+b$.

Step-by-step process:

(a) **First term of quotient:** Divide the leading term of the dividend by the leading term of the divisor:

$$\frac{3a^4}{a} = 3a^3$$

Multiply $3a^3$ by $(a+b)$: $3a^3(a+b) = 3a^4 + 3a^3b$
Subtract from the dividend:

$$(3a^4 - 5a^3b - 12a^2b^2 - ab^3 + 3b^4) - (3a^4 + 3a^3b) = -8a^3b - 12a^2b^2 - ab^3 + 3b^4$$

(b) **Second term of quotient:** Divide the new leading term by a:

$$\frac{-8a^3b}{a} = -8a^2b$$

Multiply $-8a^2b$ by $(a+b)$: $-8a^2b(a+b) = -8a^3b - 8a^2b^2$
Subtract:

$$(-8a^3b - 12a^2b^2 - ab^3 + 3b^4) - (-8a^3b - 8a^2b^2) = -4a^2b^2 - ab^3 + 3b^4$$

(c) **Third term of quotient:** Divide the new leading term by a:

$$\frac{-4a^2b^2}{a} = -4ab^2$$

Multiply $-4ab^2$ by $(a+b)$: $-4ab^2(a+b) = -4a^2b^2 - 4ab^3$
Subtract:

$$(-4a^2b^2 - ab^3 + 3b^4) - (-4a^2b^2 - 4ab^3) = 3ab^3 + 3b^4$$

(d) **Fourth term of quotient:** Divide the new leading term by a:

$$\frac{3ab^3}{a} = 3b^3$$

Multiply $3b^3$ by $(a+b)$: $3b^3(a+b) = 3ab^3 + 3b^4$
Subtract:

$$(3ab^3 + 3b^4) - (3ab^3 + 3b^4) = 0$$

The remainder is zero, confirming that $(a+b)$ divides the product exactly.

Final quotient: $3a^3 - 8a^2b - 4ab^2 + 3b^3$

4. Reduce the following fraction to its lowest terms

$$\frac{x^6 + a^2x^3y}{x^6 - a^4y^2}$$

Solving this problem in no time requires an eagle eye to notice that

$$x^6 - a^4y^2 = (x^3 + a^2y)(x^3 - a^2y)$$

so

$$\frac{x^6 + a^2x^3y}{x^6 - a^4y^2} = \frac{x^3(x^3 + a^2y)}{(x^3 + a^2y)(x^3 - a^2y)} = \frac{x^3}{x^3 - a^2y}$$

5. Simplify

$$\left\{\frac{a+b}{a-b} + \frac{a-b}{a+b}\right\} \div \left\{\frac{a+b}{a-b} - \frac{a-b}{a+b}\right\}$$

I find that calculating the numerator and denominator separately is a very straightforward way to answer

$$\frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{(a+b)^2 + (a-b)^2}{a^2 - b^2} = \frac{a^2 + b^2}{a^2 - b^2}$$

$$\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)^2 - (a-b)^2}{a^2 - b^2} = \frac{2ab}{a^2 - b^2}$$

Now, division is just multiplying by the reciprocal so:

$$\left\{\frac{a+b}{a-b} + \frac{a-b}{a+b}\right\} \div \left\{\frac{a+b}{a-b} - \frac{a-b}{a+b}\right\} = \frac{a^2 + b^2}{a^2 - b^2} \times \frac{a^2 - b^2}{2ab} = \frac{a^2 + b^2}{2ab}$$

6. Solve
$$\frac{3x-4}{2}-\frac{6x-5}{8}=\frac{3x-1}{16}$$
This is an extremely simple exercise, it just takes adding the fractions first
$$\frac{3x-4}{2}-\frac{6x-5}{8}=\frac{8(3x-4)-2(6x-5)}{16}=\frac{12x-22}{16}$$
the equation is then
$$\frac{12x-22}{16}=\frac{3x-1}{16}$$
or
$$12x-22=3x-1$$
which is the same as
$$9x=21$$
so
$$x=\frac{21}{9}=\frac{7}{3}$$

7. Solve
$$\boxed{7x-5y=24\,,\quad 4x-3y=11}$$

We begin by writing the system in the usual forma:
$$7x-5y=24$$
$$4x-3y=11$$
Now it is easy to see that if we multiply the first eq. by 3 and the second by -5 I we get the equivalent system:
$$21x-15y=72$$
$$-20x+15y=-55$$
adding both equations yield
$$x=17$$
Substitution of this value in the first equation of the original system,
$$7(17)-5y=24\,,$$
or
$$-5y=24-119=-95$$
so
$$y=\frac{-95}{-5}=19$$
The complete solution of the system is thus
$$\boxed{(17,19)}$$

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