Planar MIP

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1 Introduction

FIGURE 1 – Segways.

2 Planar approximation

2.1 Derivation of State Space Equations

2.1.1 Notations

Figure 2 presents the notations used in this document.

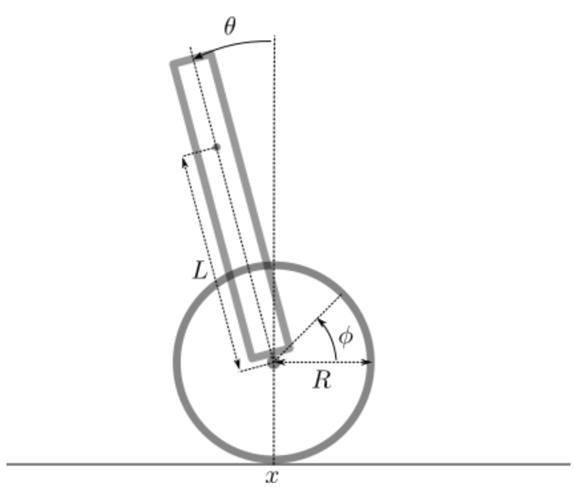

FIGURE 2 – Planar Mobile Inverted Pendulum.

- m_w, m_b, I_w, I_b : respectively wheel and body masses and inertias
- L : distance between wheel axis and body center of mass

— R : wheel radius
— θ : angle between vertical and body axis
— ϕ : angle between horizontal and the wheel
— x : horizontal coordinate of the wheel axis

2.1.2 Kinematics

Position of the body center of mass :

$$\underline{r} = \begin{pmatrix} x - L\sin\theta \\ R + L\cos\theta \end{pmatrix} \tag{1}$$

Velocity of the body center of mass :

$$\underline{\dot{r}} = \begin{pmatrix} \dot{x} - L\dot{\theta}\cos\theta \\ -L\dot{\theta}\sin\theta \end{pmatrix} \tag{2}$$

Acceleration of the body center of mass :

$$\underline{\ddot{r}} = \begin{pmatrix} \ddot{x} - L\ddot{\theta}\cos\theta + L\dot{\theta}^2\sin\theta \\ -L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta \end{pmatrix} \tag{3}$$

The *no slip* hypothesis also brings :

$$x = -R\phi \tag{4}$$

2.1.3 Dynamics

1. Angular acceleration of the wheel

$$I_w\ddot{\phi} = \tau - FR \tag{5}$$

2. Linear acceleration of the wheel

$$m_w \begin{pmatrix} \ddot{x} \\ 0 \end{pmatrix} = \begin{pmatrix} -P_x - F \\ N - P_y - m_w g \end{pmatrix} \tag{6}$$

3. Angular acceleration of the body

$$I_b\ddot{\theta} = -\tau + P_y L\sin\theta + P_x L\cos\theta \tag{7}$$

4. Linear acceleration of the body in $\underline{i}$ direction :

$$m_b(\ddot{x} - \dot{\theta}L\cos\theta + \dot{theta}^2 L\sin\theta) = P_x \tag{8}$$

2.2 State Space Equation

$$(m_b RL\cos\theta)\ddot{\phi} + (I_b + m_b L^2)\ddot{\theta} = m_b g L\sin\theta - \tau \tag{9}$$

$$\left(I_w + (m_b + m_w)R^2\right)\ddot{\phi} + (m_b RL\cos\theta)\ddot{\theta} = m_b RL\dot{\theta}^2\sin\theta + \tau \tag{10}$$

or matricially

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}\begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{11}$$

with

$$a = m_b RL\cos\theta \quad b = I_b + m_b L^2 \quad c = I_w + (m_b + m_w)R^2 \tag{12}$$

and

$$d = m_b g L \sin\theta - \tau \quad e = m_b R L \dot{\theta}^2 \sin\theta + \tau \tag{13}$$

When the system has full rank ($a^2 - bc \neq 0$), equations can be separated, leading to

$$\begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \end{pmatrix} = \frac{1}{a^2 - bc} \begin{pmatrix} a & -b \\ -c & a \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix} \tag{14}$$

or

$$\begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \end{pmatrix} = \frac{1}{a^2 - bc} \begin{pmatrix} ad - be \\ -cd + ae \end{pmatrix} \tag{15}$$