Proof 12-10-2020

1 Proof for minimum function

Note: The word monotonically increasing and decreasing are referring to weakly increasing and decreasing respectively. I.e. not strictly increasing or decreasing.

1.1 Theorem. *Suppose f is continuous and monotonically decreasing on $[a, b]$. This implies that $h(x) = \min_{t\in[a,x]} f(t)$ is continuous on $[a, b]$.*

Proof. By definition of $f(x)$ being monotonically decreasing on $[a, b]$,

$$f(x) \leq f(y)\ \forall x, y \in [a, b] \text{ s.t. } y \leq x$$

This is fits the definition for a minimum of $f(x)$ over the interval $[a, x]$. This minimum is $f(x)$. Thus,

$$h(x) = \min_{t\in[a,x]} f(t) = f(x)\ \{\forall x \in [a, b]\}$$

Therefore, $h(x) = f(x)$ on $[a, b]$ and since $f(x)$ is continuous on $[a, b]$, $h(x)$ is continuous on $[a, b]$. □

1.2 Theorem. *If f is C^1 (in a piecewise fashion) on $[a, b]$, then $h(x) = \min_{t\in[a,x]} f(t)$ is continuous on $[a, b]$.*

Proof. f piecewise $C^1 \implies \exists x_0, x_1, \ldots, x_n$ s.t. $a = x_0 < x_1 < \ldots < x_n = b$ such that f is monotonic on $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

$\forall k \in \{1, \ldots, n\}$:

- Case 1: f is monotonically increasing on $[x_{k-1}, x_k]$:
  $h(x) = h(x_{k-1}) \implies h(x)$ continuous on $[x_{k-1}, x_k]$.
- Case 2: f is monotonically decreasing on $[x_{k-1}, x_k]$:
  By definition:

$$\begin{aligned} h(x) &:= \min_{t\in[x_0,x]} f(t) = \min_{t\in[x_0,x_{k-1}]\cup[x_{k-1},x]} f(t) \\ &= \min\left(\min_{t\in[x_0,x_{k-1}]} f(t), \min_{t\in[x_{k-1},x]} f(t)\right) \\ &= \min\left(h(x_{k-1}), \min_{t\in[x_{k-1},x]} f(t)\right) \end{aligned}$$

  By Theorem 1.1, $\min_{t\in[x_{k-1},x]} f(t)$ is continuous on $[x_{k-1}, x]$ since f is continuous (more specifically, on $[x_{k-1}, x]$). Since the minimum of two continuous functions is continuous and $h(x_{k-1})$ is constant (and therefore continuous on $[x_{k-1}, x_x]$), $h(x)$ is continuous on $[x_{k-1}, x_k]$.

Therefore, $h(x)$ is continuous on $[x_0, x_n]$. □

2 Proof for maximum function

2.1 Theorem. *Suppose f is continuous and monotonically increasing on $[a, b]$. This implies that $h(x) = \max_{t \in [a,x]} f(t)$ is continuous on $[a, b]$.*

Proof. By definition of $f(x)$ being monotonically increasing on $[a, b]$,

$$f(x) \geq f(y) \ \forall x, y \in [a, b] \text{ s.t. } y \leq x$$

This is fits the definition for a maximum of $f(x)$ over the interval $[a, x]$. This maximum is $f(x)$. Thus,

$$h(x) = \max_{t \in [a,x]} f(t) = f(x) \ \{\forall x \in [a, b]\}$$

Therefore, $h(x) = f(x)$ on $[a, b]$ and since $f(x)$ is continuous on $[a, b]$, $h(x)$ is continuous on $[a, b]$. □

2.2 Theorem. *If f is C^1 (in a piecewise fashion) on $[a, b]$, then $h(x) = \max_{t \in [a,x]} f(t)$ is continuous on $[a, b]$.*

Proof. f piecewise $C^1 \implies \exists x_0, x_1, \ldots, x_n$ s.t. $a = x_0 < x_1 < \ldots < x_n = b$ such that f is monotonic on $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

$\forall k \in \{1, \ldots, n\}$:

- Case 1: f is monotonically increasing on $[x_{k-1}, x_k]$:
  By definition:

$$\begin{aligned} h(x) &:= \max_{t \in [x_0,x]} f(t) = \max_{t \in [x_0,x_{k-1}] \cup [x_{k-1},x]} f(t) \\ &= \max\left(\max_{t \in [x_0,x_{k-1}]} f(t), \max_{t \in [x_{k-1},x]} f(t) \right) \\ &= \max\left(h(x_{k-1}), \max_{t \in [x_{k-1},x]} f(t) \right) \end{aligned}$$

  By Theorem 2.1, $\max_{t \in [x_{k-1},x]} f(t)$ is continuous on $[x_{k-1}, x]$ since f is continuous (more specifically, on $[x_{k-1}, x]$). Since the maximum of two continuous functions is continuous and $h(x_{k-1})$ is constant (and therefore continuous on $[x_{k-1}, x_x]$), $h(x)$ is continuous on $[x_{k-1}, x_k]$.

- Case 2: f is monotonically decreasing on $[x_{k-1}, x_k]$:
  $h(x) = h(x_{k-1}) \implies h(x)$ continuous on $[x_{k-1}, x_k]$.

Therefore, $h(x)$ is continuous on $[x_0, x_n]$. □