Math 202B: Topology and Analysis II Lecture Notes

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Contents

Chapter 1

Preliminaries

1.1 Linear Transformations

Let V, W be normed vector spaces, with $T : V \to W$ a linear transformation.

Def: T is **bounded** if there exists $c > 0$ such that $\|T(v)\|_W \leq c\,\|v\|_V$ for all $v \in V$.

Theorem: The following are equivalent:

(i) T is continuous.

(ii) T is continuous at 0.

(iii) T is bounded.

(iv) T is Lipschitz (and thus uniformly continuous)

Proof: Note that (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2) is obvious.
(3) $\Longrightarrow$ (4): $\|T(v_1) - T(v_2)\| = \|T(v_1 - v_2)\| \leq c\,\|v_1 - v_2\|$.

(2) $\Longrightarrow$ (3): Consider $B_1(0_W) \subseteq W$. Then $T^{-1}(B_1(0_W))$ contains a neighborhood of 0_V. That is, there is a $\delta > 0$ such that $T(B_\delta(0_V)) \subseteq B_1(T(0_W)) = B_1(0_W)$. For any nonzero $v \in V$,

$$\frac{\|T(v)\|}{\frac{2}{\delta}\,\|v\|} = \left\|T\Big(\frac{\delta v}{2\,\|v\|}\Big)\right\| \leq 1,$$

since $\frac{\delta v}{2\|v\|} \in B_\delta(0_V)$. Therefore $\|T(v)\| \leq \frac{2}{\delta} \cdot \|v\|$. ■

Denote $B(V, W) = \{T : V \to W \mid T \text{ is bounded \& linear}\}$. We can define a norm on $\{T : V \to W \mid T \text{ is bounded \& linear}\}$, by

$$\|T\| = \sup\left\{\frac{\|Tv\|_W}{\|v\|_V} : v \in V\right\} = \sup\{\|Tv\|_w \,:\, \|v\|_V \leq 1\}.$$

This is called the **operator norm** of T.

It is easily seen that $B(V,W)$ is a normed vector space with the operator norm. Indeed, for $S,T \in B(V,W)$ and $v \in V$, we have $\|(S+T)v\| = \|Sv + Tv\| \leq (\|S\| + \|T\|)\|v\|$. Therefore, $\|S+T\| \leq \|S\| + \|T\|$. Furthermore, for any scalar α, we have $\|\alpha T\| = |\alpha| \cdot \|T\|$.

If $T \in B(V,W)$, then in fact T is uniformly continuous. If W is complete, then T extends to the completion of V.

Proposition: Let $T \in B(V,W)$. Furthermore, suppose $(X, \mathcal{S}, \mu)$ is a measure space. The following propositions are left as exercises:

(i) If (f_n) is a sequence of ISFs converging to f pointwise, then (Tf_n) converges to Tf.

(ii) If (f_n) is a sequence of SMFs that is Cauchy in mean, then (Tf_n) is also Cauchy in mean.

(iii) If f is integrable, then Tf is integrable. In fact, $\int Tf\, d\mu = T\Big(\int f\, d\mu\Big)$.

Note that the map $\Phi_T : \mathcal{L}^1(X, \mathcal{S}, \mu, V) \to \mathcal{L}^1(X, \mathcal{S}, \mu, W)$ defined by $\Phi_T(f) = Tf$, then we can see that $\|\Phi_T\| \leq \|T\|$, so that $\Phi_T \in B(\mathcal{L}^1(X, \mathcal{S}, \mu, V), \mathcal{L}^1(X, \mathcal{S}, \mu, W))$.

Def: If V is a normed vector space, then the **dual space of V** is the vector space of all continuous linear functionals on V, denoted V'.

Let h be a bounded measurable function, $\|h\|_\infty < \infty$. Define φ_h on V' by

$$\varphi_h(f) = \int h(x) f(x)\, d\mu(x).$$

Then note that

$$|\varphi_h(f)| \leq \int |h(x)||f(x)|\, d\mu(x) \leq \|h\|_\infty \|f\|_1$$

so that $\|\varphi_h\| \leq \|h\|_\infty$.

Def: For $g : X \to V$ measurable, we say that g is **essentially bounded** if there is $c > 0$ such that $\mu(\{x : \|g(x)\| > c\}) = 0$.

Set $\|g\|_\infty = \inf\{c > 0 \ : \ \mu(\{x : \|g(x)\| > c\}) = 0\}$, for g an essentially bounded function.

Let $\mathcal{L}^\infty(X, \mathcal{S}, \mu, V)$ denote the vector space of all essentially bounded functions. Then we can check that $\|\cdot\|_\infty$ is a seminorm on $\mathcal{L}^\infty(X, \mathcal{S}, \mu, V)$.

Let $L^\infty(X, \mathcal{S}, \mu, V)$ denote the equivalence class of functions in $\mathcal{L}^\infty$ equal a.e.

Chapter 2

L^p Spaces

Def: Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $0 < p < \infty$. Let B be a Banach space, and let $f : X \to B$ be measurable. Then f is ***p*-integrable**, denoted $f \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$, if the map $x \mapsto \|f(x)\|^p$ is in $\mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$.

Let L^p denote the equivalence class of functions in $\mathcal{L}^p$ equal a.e. For now, we write $f \in L^p$ as any function in $\mathcal{L}^p$ equal a.e. to f.

Proposition: $L^p(X, \mathcal{S}, \mu, B)$ is a vector space for pointwise operations.
Proof: Let $f, g \in L^p$. Then

$$\|f(x) + g(x)\|^p \leq (2 \cdot \max\{\|f(x)\|, \|g(x)\|\})^p \leq 2^p(\|f(x)\|^p + \|g(x)\|^p) < \infty,$$

so $f + g \in L^p$. It is obvious that for $c \in \mathbb{R}$, $cf \in L^p$. $\blacksquare$

For $f \in L^p$, for now let us write $\|f\|_p = \left(\int \|f(x)\|^p \ d\mu(x)\right)^{1/p}$. $\|\cdot\|_p$ is not in general a norm. However, we will show that it is a norm for $p \geq 1$.

Lemma: If $p, q \in (1, \infty)$, and $p^{-1} + q^{-1} = 1$, and if $r > 0,\ s > 0$, then $rs \leq \frac{r^p}{p} + \frac{s^q}{q}$.
Proof: Since log is concave, we have

$$\log\left(\frac{1}{p} \cdot r^p + \frac{1}{q} \cdot s^q\right) \geq \frac{1}{p}\log(r^p) + \frac{1}{q}\log(s^q) = \log(r) + \log(s) = \log(rs).$$

$\blacksquare$

Holder's Inequality: If $p, q \in (1, \infty)$ and $p^{-1} + q^{-1} = 1$, and if $f \in L^p,\ g \in L^q$, then $\int \|fg\| \ d\mu \leq \|f\|_p \|g\|_q$.
Proof: From the lemma, we have

$$\frac{\|f(x)\|}{\|f\|_p} \cdot \frac{\|g(x)\|}{\|g\|_q} \leq \frac{\|f(x)\|^p}{p\|f\|_p^p} + \frac{\|g(x)\|^q}{q\|g\|_q^q}.$$

Integrating, we get

$$\int \frac{\|f(x)\|}{\|f\|_p} \cdot \frac{\|g(x)\|}{\|g\|_q} \, d\mu(x) \leq \frac{1}{p\,\|f\|_p^p} \int \|f\|^p \, d\mu + \frac{1}{q\,\|g\|_q^q} \int \|g\|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

■

For $g \in L^q$, we can define φ_g by $\varphi_g(f) = \int f(x)g(x)\, d\mu(x)$ for $f \in L^p$. Then φ_g is a linear functional, so its operator norm is $\|\varphi_g\| \leq \|g\|_q$.

Proposition: For any $f \in L^p$, there is $g \in L^q$, $\|g\|_q = 1$, such that $\varphi_g(f) = \|f\|_p$.
Proof: Observe that $|f(x)|^p = (|f(x)|^{p/q})^q$, so the map $x \mapsto |f(x)|^{p/q} \in L^q$. It suffices to assume $\|f\|_p = 1$.
Let

$$u(x) = \begin{cases} 0, & \text{if } f(x) = 0, \\ \frac{f(x)}{|f(x)|}, & \text{if } f(x) \neq 0 \end{cases}$$

Then

$$|u(x)| = \begin{cases} 0, & \text{if } f(x) = 0, \\ 1, & \text{if } f(x) \neq 0 \end{cases}$$

Let $g(x) = \overline{u(x)}|f(x)|^{p/q}$ so that $g \in L^q$. Then

$$\|g\|_q = \left(\int (|f(x)^{p/q})^q\right)^{1/q} = \|f\|_p^{p/q} = 1,$$

and

$$\varphi_g(f) = \int \overline{u(x)}|f(x)|^{p/q} f(x)\, d\mu(x) = \int \overline{u(x)}|f(x)|^{p/q} u(x)|f(x)|\, d\mu(x) =$$

$$= \int |f(x)|^{1+p/q}\, d\mu(x) = 1 = \|f\|_p^p.$$

■

Minkowski's Inequality: For $f, g \in L^p$, we have $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.
Proof:

$$\|f+g\|_p^p = \int |f(x) + g(x)|^p \, d\mu(x).$$

By the previous proposition, there is $h \in L^q$, where $h \geq 0$, $\|h\|_q = 1$, and

$$\|f+g\|_p = \int |f+g| h\, d\mu \leq \int \big(|f| + |g|\big) h\, d\mu = \int |f| h\, d\mu + \int |g| h\, d\mu \leq$$

$$\leq \|f\|_p \|h\|_q + \|g\|_p \|h\|_q = \|f\|_p + \|g\|_p.$$

■

Theorem: $L^p(X, \mathcal{S}, \mu, B)$ is complete.
Proof: Let $\{f_n\}$ be a Cauchy sequence for the p-norm. We claim that $\{f_n\}$ is Cauchy in measure. For $m, n \in \mathbb{N}$, and $\epsilon > 0$, define

$$E^{\epsilon}_{m,n} := \{x : \|f_m(x) - f_n(x)\| \geq \epsilon\}.$$

For any $\epsilon > 0$, we have $\frac{1}{\epsilon}\chi_{E^{\epsilon}_{m,n}}(x) \leq \|f_m(x) - f_n(x)\|$, so

$$\frac{1}{\epsilon^p}\mu(E^{\epsilon}_{m,n}) = \int \left(\chi_{E^{\epsilon}_{m,n}}\right)^p \, d\mu \leq \int \|f_m(x) - f_n(x)\|^p \, d\mu(x) = \|f_m - f_n\|_p^p \to 0$$

as $m, n \to \infty$. By the Riesz-Weyl Theorem, there is a subsequence that converges to some function f a.e. Thus, we can assume without loss of generality that $f_n \to f$ a.e.
For fixed m, we have $|f_n(x) - f_m(x)|^p \to |f(x) - f_m(x)|^p$ as $n \to \infty$, and by Fatou's Lemma, we have

$$\int |f - f_m|^p \, d\mu \leq \liminf_{n\to\infty} \int |f_n - f_m|^p \, d\mu.$$

For fixed m, given $\epsilon > 0$, there is N such that for $m, n \geq N$,

$$\|f_m - f_n\|_p < \epsilon^{1/p}, \text{ i.e., } \int |f_n - f_m|^p \, d\mu < \epsilon.$$

Therefore $\|f - f_n\|_p^p < \epsilon$, and thus $f_n \to f$ in the p-norm. $\blacksquare$

Chapter 3

Normed Vector Spaces

3.1 The Hahn-Banach Theorem

We begin our study of Banach spaces with the Hahn-Banach theorems concerning continuous linear functionals.

Def: Let V be a vector space over $\mathbb{R}$. A **Minkowski gauge** is a function $p : V \to \mathbb{R}$ such that

(1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$

(2) If $r \geq 0$, then $p(rv) = r \cdot p(v)$.

The first theorem we will prove is the Hahn-Banach Extension theorem. Given a vector space V, it is conceivable that linear functionals may not exist, or may be too complicated to be useful at all. This theorem will show us that there are lots of useful linear functionals on V given a simple gauge, i.e., a norm.

Lemma: Let W be a subspace of V, and let $p : V \to \mathbb{R}$ be a gauge on V. Let φ be a linear functional on W such that $\varphi(w) \leq p(w)$ for all $w \in W$. For any $v_0 \in V \setminus W$, let $Z = W + \mathbb{R}v_0$. Then there is a linear extension $\tilde{\varphi}$ of φ to Z such that $\tilde{\varphi}(z) \leq p(z)$ for all $z \in Z$.
Proof: We will find $\alpha \in \mathbb{R}$ such that if $\tilde{\varphi}(w + rv_0) = \varphi(w) + r\alpha$, then $\tilde{\varphi}$ is the desired extension.
Since we need $\tilde{\varphi}(w + v_0) \leq p(w + v_0)$, it follows that we need $\varphi(w) + \alpha \leq p(w + v_0)$. In other words, we have $\alpha \leq p(w + v_0) - \varphi(w)$ for all $w \in W$.
For $r > 0$, we need $\tilde{\varphi}(w - rv_0) \leq p(w - rv_0)$. In particular, we have

$$\tilde{\varphi}(w - v_0) \leq p(w - v_0) \implies \varphi(w) - \alpha \leq p(w - v_0) \implies \varphi(w) - p(w - v_0) \leq \alpha.$$

Note that for any $w, w' \in W$,

$$\varphi(w) + \varphi(w') = \varphi(w + w') \leq p(w + w') \leq p(w + v_0) + p(w' - v_0), \text{ so}$$
$$\varphi(w') - p(w' - v_0) \leq p(w + v_0) - \varphi(w).$$

Thus, for any fixed w', we have $\varphi(w') - p(w' - v_0) \leq \inf\{p(w + v_0) - \varphi(w) \ : \ w \in W\}$. Similarly, it follows that

$$\sup\{\varphi(w') - p(w' - v_0) \ : \ w' \in W\} \leq \inf\{p(w + v_0) - \varphi(w) \ : \ w \in W\}.$$

Then any α lying between these two values will suffice. ■

Hahn-Banach Extension Theorem: Let W be a subspace of V. With the same set up as in the lemma above, there is an extension $\tilde{\varphi}$ of φ to V such that $\tilde{\varphi}(v) \leq p(v)$ for all $v \in V$.
Proof: We can use the lemma above with Zorn's lemma to prove this.
Let S be the set of all pairs (U, φ_U), where $W \subseteq U \subseteq V$, U is a subspace, and φ_U is an extension of φ to U satisfying $\varphi(u) \leq p(u)$ for all $u \in U$.
Order S by inclusion, i.e., we say $(U, \varphi_U) \leq (U', \varphi_{U'})$ if $U \subseteq U'$ and $\varphi_{U'}$ is an extension of φ_U.
Let $\mathcal{T}$ be a totally-ordered subset of S, and let $U_{\mathcal{T}} = \bigcup_{(U,\varphi_U)\in\mathcal{T}} U$.
Since $\mathcal{T}$ is totally-ordered, it is easily seen that $U_{\mathcal{T}}$ is a subspace containing W. Similarly, let $\varphi_{\mathcal{T}} := \bigcup_{(U,\varphi_U)\in\mathcal{T}} \varphi_U$ be a linear functional on $U_{\mathcal{T}}$. Note that $\varphi_{\mathcal{T}}(u) \leq p(u)$ for all $u \in U_{\mathcal{T}}$.
Clearly, $(U_{\mathcal{T}}, \varphi_{\mathcal{T}}) \geq (U, \varphi_U) \in \mathcal{T}$. Therefore by Zorn's Lemma, S has a maximal element, call it (U_m, φ_m). By the lemma above, if $U_m \subset V$, we would be able to make another extension, contradicting the maximality of U_m. Thus $U_m = V$, and the desired extension is φ_m. ■

It is clear that norms are gauges, so we have the following corollaries.

Corollary: Let V be a normed vector space, and let $W \subseteq V$ be a subspace. Let φ be a continuous linear functional on W. Then there is an extension $\tilde{\varphi}$ of φ on V such that $\left\|\tilde{\phi}\right\| = \|\varphi\|$.

Corollary: Let V be a normed vector space, and let $v_0 \in V$. Then there is $\varphi \in V'$ with $\|\varphi\| = 1$ and $\varphi(v_0) = \|v_0\|$.
Proof: Let W be the one-dimensional subspace spanned by v_0. Define $\psi(\alpha v_0) = \alpha \|v_0\|$, so that $\|\psi\| = 1$. By the previous corollary, consider φ the extension of ψ on V with $\|\varphi\| = \|\psi\| = 1$. ■

Corollary: If V is a normed vector space, and $v \in V$, then

$$\|v\| = \sup\{\|\varphi(v)\| : \varphi \in V', \ \|v\| = 1\}.$$

3.2 Quotient Spaces

Given a subspace $W \subseteq V$, we can form V/W. If V has a norm $|\cdot|$, we define the **quotient (semi-norm)** on V/W by

$$\|v + W\| := \operatorname{dist}(v, W) = \inf\{|v - w| : w \in W\}.$$

Let $\overline{v} \in V/W$. If $\|\overline{v}\| = 0$, then $v \in \overline{W}$ (the closure of W). Note that if W is closed, then $\|\cdot\|$ is a norm on V/W. Otherwise, we can see that $\|\cdot\|$ is a semi-norm.

Proposition: Let V be a normed vector space, $W \subset V$ a closed subspace. Then for any $\epsilon > 0$, there is a $v \in V$ such that $|v - w| \geq 1$ for all $w \in W$, and $|v| < 1 + \epsilon$.
Proof: Form V/W. Choose $z \in V/W$ with $\|z\| = 1$. We can find $v \in V$ such that $[v] = z$, and $\|z\| \leq |v - w|$, hence $|v - w| \geq 1$, where we use $[\cdot]$ to denote the equivalence class of an element. Since $\|[v]\| = 1$, there exists $w \in W$ with $|v - w| < 1 + \epsilon$. Let $u = v - w$. Then $\|[v]\| = \|[u]\|$, which gives the desired result. ■

3.3 Banach Spaces

Proposition: Let V be a normed vector space, and let W be a Banach space. Let $B(V, W)$ be the vector space of all bounded linear operators from V to W with the operator norm. Then $B(V, W)$ is complete and is thus a Banach space.
Proof: Let $\{T_n\}$ be a Cauchy sequence of operators in $B(V, W)$. Then for any $v \in V$, clearly $\{T_n(v)\}$ is Cauchy in W.
Indeed, $\|T_n(v) - T_m(v)\| \leq \|T_n - T_m\| \|v\|$, and $\|T_n - T_m\| \|v\| \to 0$ as $n, m \to \infty$.
Now define $T : V \to W$ by $T(v) = \lim T_n(v)$. Then for $v, v' \in V$, we have $T(v + v') = \lim T_n(v + v') = \lim(T_n v + T_n v') = T(v) + T(v')$.
Similarly, $T(\alpha v) = \alpha T(v)$, so we have $T \in B(V, W)$. Now, recall that all $\|T_n\|$ are bounded, so let $s = \sup\{\|T_n\| : n \geq 1\}$. Then for any $v \in V$, we have $\|Tv\| = \lim \|T_n v\| \leq s \|v\|$, so $\|T\| \leq s$.
Let $\epsilon > 0$. Then there is N such that $n, m > N$ implies $\|T_n - T_m\| < \epsilon$.
Then for $n > N$,

$$\|T_n(v) - T(v)\| = \lim_{m \to \infty} \|T_n(v) - T_m(v)\| < \epsilon \|v\|,$$

hence $\|T_n - T\| < \epsilon$. Therefore $T_n \to T$ in $B(V, W)$. ■

If V is a normed vector space, and if V has a countable subset $\{v_n\}$ whose linear span is dense, then when proving the Hahn-Banach Extension theorem, $W \subseteq V$, is φ is a linear functional on W, use $\{v_n\}$ to use induction to get $\hat{\varphi}$ on $W + \operatorname{span}\{v_n\}$, $\|\hat{\varphi}\| = \|\varphi\|$. By denseness, this easily extends to V.

From the proposition, $B(V, \mathbb{R}) = V'$ is a Banach space.
Recall that V is indeed isomorphic to V', but there is no natural isomorphism. However, there is a natural isomorphism from V to V''.

Indeed, by defining $J_v : V' \to \mathbb{R}$ by $J_v(\varphi) = \varphi(v)$, we claim that the map $v \mapsto J_v$ is an ismorphism from V to V''. Indeed,

$$|J_v(\varphi)| = |\varphi(v)| \leq \|\varphi\| \|v\| \implies \|J_v\| \leq \|v\| .$$

But for $v \in V$, there is a $\varphi \in V'$ with $\|\varphi\| = 1$ and $\varphi(v) = \|v\|$, thus $\|J_v\| = \|v\|$. This also means that the map $v \mapsto J_v$ is isometric.

Def: A Banach space is **reflexive** if $J : V \to V''$ is onto.

Ex: Once we know that for $1 < p < q$, the dual space of $L^p(X, \mathcal{S}, \mu)$ is $L^q(X, \mathcal{S}, \mu)$, we can then show that L^p is reflexive.
But, unless X is finite, L^1 and L^∞ are not reflexive.

3.4 Weak Topologies

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and let Z be any linear subspace of linear functionals on V.
Then the Z-topology on V is the weakest topology on V for which all $z \in Z$ are continuous. A subspace for the neighborhoods $\mathcal{O}_v$ consists of

$$\mathcal{O}_{\varphi,\epsilon} = \{v \in V : |\varphi(v)| < \epsilon\} \text{ for } \varphi \in Z.$$

Note that $\mathcal{O}_{\varphi,\epsilon}$ is convex. Then the base is $\mathcal{O}_{\varphi_1,\ldots,\varphi_n,\epsilon} = \{v \in V : |\varphi_j(v)| < \epsilon,\ 1 \leq j \leq n\}$.

Notice that $v \mapsto |\varphi(v)|$ is a semi-norm. So, more generally, if V is a vector space, and if S is a collection of semi-norms on V, we can consider the weakest topology on V for which all the seminorms are continuous.

Def: A **topological vector space** is a vector space V with a topology that makes addition and scalar multiplication continuous.

Def: A set A in a vector space is **convex** is for any two points $x, y \in A$, and for all $t \in [0, 1]$, the point $tx + (1 - t)y \in A$. That is, the line between x and y is contained in A.

Def: A topological vector space is **locally convex** if each point has a sub-base for its neighborhood system consisting of convex sets.

From above, we see that the topology determined by a family of semi-norms is locally convex.

Def: If V is a locally convex topological vector space, given any set $A \subseteq V$, its **convex hull** is the smallest convex set containing A.

The following is a useful fact which we will not prove.

Proposition: For any non-empty open set in $L^p(X, \mathcal{S}, \mu)$, its convex hull is all of $L^p(X, \mathcal{S}, \mu)$.

Def: The weakest topology on V making all linear functionals in V' continuous is called the **weak topology**.

Def: The **weak-* topology** on V' is the weakest topology determined by elements of V''.

Thus, we can see that if V is reflexive, the weak-* topology is the same as the weak topology on V'.

Alaoglu's Theorem: Let V be a normed vector space. The closed unit ball (for norm) of V' is compact for the weak-* topology.
Proof: For each $v \in V$, let $D_v = \{\alpha \in \mathbb{C} : |\alpha| \leq \|v\|\}$, so D_v is compact. Let $P = \prod_{v \in V} D_v$, so by Tychonoff's Theorem, P is compact for its product topology.
Let $B_1 = \overline{B_1(0)} \subset V'$. For $\varphi \in B_1$, we have for any $v \in V$,

$$|\varphi(v)| \leq \|\varphi\| \, \|v\| \leq \|v\| \,, \text{ so } \varphi(v) \in D_v.$$

Thus, we have a natural map $S : B_1 \to P$ by $S(\varphi) = \{\varphi(v)\}_{v \in V}$. Clearly, S is one-to-one. We show that $S(B_1)$ is closed. Suppose $f \in P$, and suppose that f is in the closure of $S(B_1)$. That is, $f(v) \in D_v \iff |f(v)| \leq \|v\|$. We claim that f is a linear functional, and thus $\|f\| \leq 1$.
Let $v_1, v_2 \in V$. Then define

$$\mathcal{O}_{f,v_1,v_2,\epsilon} = \{g \in P : |f(v_1) - g(v_1)| < \epsilon, \ |f(v_2) - g(v_2)| < \epsilon, \ |f(v_1 + v_2) - f(v_1) - f(v_2)| < \epsilon\}.$$

Since $f \in \overline{S(B_1)}$, there is $\varphi \in B_1$, such that $S(\varphi) \in \mathcal{O}_{f,v_1,v_2,\epsilon/3}$. Thus,

$$|f(v_1+v_2) - f(v_1) - f(v_2)| \leq |f(v_1) - \varphi(v_1)| + |f(v_2) - \varphi(v_2)| + |f(v_1+v_2) - \varphi(v_1+v_2)| < \epsilon.$$

Therefore $f(v_1 + v_2) = f(v_1) + f(v_2)$, and a similar argument shows that $f(\alpha v) = \alpha f(v)$, hence f is a linear functional, and also $f \in S(B_1)$, hence $S(B_1)$ is closed in P, thus $S(B_1)$ is compact. ■

Let $B_1 \subseteq V'$ be the closed unit ball, and let $C(B_1)$ be the set of continuous functions on B_1. For each $v \in V$, define $f_v \in C(B_1)$ by $f_v(\varphi) = \varphi(v)$. By definition of the weak-* topology,

f_v is continuous, i.e. $f_v \in C(B_1)$.
Note that if $\varphi \in B_1$, then $|f_v(\varphi)| = |\varphi(v)| \leq \|\varphi\| v$, so $\|f_v\|_\infty \leq \|v\|$. But by the Hahn-Banach Extension Theorem, there is a $\varphi \in B_1$ such that $\varphi(v) = \|v\|$, so $\|f_v\|_\infty = \|v\|$. This leads to the following proposition:

Proposition: The map $v \mapsto f_v$ is an isometric inclusion of V into $C(B_1)$.

Let V be a vector space, and let W be a vector space of linear functionals on V. So we can put the weak-* topology on V. Thus, we can ask what are the linear functionals on V that are continuous for W together.

Proposition: Every W-continuous linear functional on V is an element of W.

Lemma: Let $\varphi, \varphi_1, \ldots, \varphi_n$ be linear functionals on V. The following are equivalent:

(i) φ is a linear combination of $\varphi_1, \ldots, \varphi_n$.

(ii) There is a constant c such that for all $v \in V$, we have $|\varphi(v)| \leq c \max_j\{|\varphi_j(v)|\}$.

(iii) $\bigcap_{j=1}^n \ker \varphi_j \subseteq \ker\varphi$.

Proof: It is easy to see that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).
(iii) $\Longrightarrow$ (i):
Define $T : V \to \mathbb{R}^n$ by $T(v) = (\varphi_1(v), \ldots, \varphi_n(v))$.
Then $\ker T = \bigcap_{j=1}^n \ker \varphi_j \subseteq \ker \varphi$.
Then φ drops to a linear functional on $V/\bigcap_{j=1}^n \ker \varphi_j$, so there are scalars $\alpha_1, \ldots, \alpha_n$ so that $\varphi(v) = (Tv) \cdot (\alpha_1, \ldots, \alpha_n) = \alpha_1\varphi_1(v) + \cdots + \alpha_n\varphi_n(v)$.
Therefore $\varphi = \alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n$. ■

Proof of Prop: Since φ is continuous for the W-topology, $\varphi^{-1}((-1,1))$ must contain a W-topology open neighborhood of $\mathcal{O}_v$, so it contains a basis neighborhood,
$\mathcal{O}_{0,\varphi_1,\ldots,\varphi_n,\epsilon} = \{v : |\varphi_j(v)| < \epsilon,\ 1 \leq j \leq n\}$.
Then if $|\varphi_j(v)| < \epsilon$ for $1 \leq j \leq n$, then $|\varphi(v)| \leq 1$.
Define a semi-norm M on V by $M(v) = \max\{|\varphi_j(v)| : 1 \leq j \leq n\}$, so that if $M(v) < \epsilon$, then $|\varphi(v)| < 1$.
Let $v \in V$. If $M(v) \neq 0$, then $M\Big(\frac{v}{M(v)} \cdot \frac{\epsilon}{2}\Big) = \epsilon/2 < \epsilon$, so

$$\Big|\varphi\Big(\frac{v}{M(v)} \cdot \frac{\epsilon}{2}\Big)\Big| \leq 1, \text{ and } \varphi(v) \leq \frac{2M(v)}{\epsilon} = \frac{2\max\{|\varphi_j(v)| : 1 \leq j \leq n\}}{\epsilon}.$$

Otherwise, if $M(v) = 0$, then $M(tv) = 0$, so $|\varphi(tv)| = 0$ for all t, so $\varphi(v) = 0$. ■

3.5 Convexity

Let V be a topological vector space, and let $\mathcal{O}$ be an open neighborhood of 0 in V. Now for any $v \in V$, it is easily seen that there exists $s > 0$ such that $sv \in \mathcal{O}$.
Let $m_{\mathcal{O}}(v) = \inf\{s > 0 : s^{-1}v \in \mathcal{O}\}$.
If $s^{-1}v \in \mathcal{O}$, then $s^{-1}t^{-1}(tv) \in \mathcal{O}$ for $t > 0$. Thus

$$\inf\{st > 0 : (st)^{-1}(tv) \in \mathcal{O}\} = t{\cdot}\inf\{s > 0 : s^{-1}t^{-1}(tv) \in \mathcal{O}\} = t{\cdot}\inf\{s > 0 : s^{-1}v \in \mathcal{O}\} = tm_{\mathcal{O}}(v)$$

$$\implies \text{For } t > 0,\ m_{\mathcal{O}}(tv) = tm_{\mathcal{O}}(v).$$

Lemma: If $\mathcal{O}$ is convex, then $m_{\mathcal{O}}(v+w) \leq m_{\mathcal{O}}(v) + m_{\mathcal{O}}(w)$, so $m_{\mathcal{O}}$ is a Minkowski gauge.
Proof: If $s^{-1}v \in \mathcal{O}$, and $t^{-1}w \in \mathcal{O}$, then $\frac{s}{s+t} + \frac{t}{s+t} = 1$, so

$$\frac{s}{s+t}(s^{-1}v) + \frac{t}{s+t}(t^{-1}w) \in \mathcal{O} \text{ by convexity.}$$

The expression above equals $\frac{1}{s+t}(v+w)$, so

$$m_{\mathcal{O}}(v+w) \leq s+t \implies m_{\mathcal{O}}(v+w) \leq m_{\mathcal{O}}(v) + m_{\mathcal{O}}(w).$$

■

Hahn-Banach Separation Theorem: Let V be a topological vector space, and let $\mathcal{O}$ be an open convex set in V. Let C be any convex set with $\mathcal{O} \cap C = \varnothing$. Then there is a $\varphi \in V'$ and $t_0 \in \mathbb{R}$ such that $\varphi(\mathcal{O}) < t_0 \leq \varphi(C)$, i.e., for all $v \in \mathcal{O}$ and $w \in C$, we have $\varphi(v) < t_0 \leq \varphi(w)$.
Proof: Let $U = \mathcal{O} - C := \{v - w : v \in \mathcal{O}, w \in C\} = \bigcup_{w \in C} \mathcal{O} - w$.
Since $\{\mathcal{O} - w\}$ is open, it follows that U is open. Furthermore, consider $v_1 - w_1, v_2 - w_2 \in U$, and $t \in [0,1]$. Then

$$t(v_1 - w_1) + (1-t)(v_2 - w_2) = \underbrace{tv_1 + (1-t)v_2}_{\in \mathcal{O}} - (\underbrace{tw_1 + (t-1)w_2}_{\in C}) \in U,$$

so U is convex. Note that $0 \notin \mathcal{O} - C$, since they are disjoint, so choose some $v_0 \in \mathcal{O} - C$, and redefine $U := \mathcal{O} - C - v_0$. Then U is open, convex, and is an open neighborhood of 0. Consider m_u the Minkowski gauge as above. Notice that $-v_0 \notin U$, so let $Z = \text{span}\{-v_0\}$. Define φ_0 on Z by $\varphi_0(-rv_0) = r$. We want φ_0 to be dominated by m_u. For $t < 0$, we have $\varphi(t(-v_0)) < 0 \leq m_u(tv_0)$. And for $t \geq 0$, we have $\varphi(t(-v_0)) \leq m_u(tv_0) = tm_u(v_0)$.
Now, we need $\underbrace{\varphi(-v_0)}_{=1} \leq m_u(-v_0)$, but this is true since $-v_0 \notin U$.

By the Hahn-Banach Separation Theorem, there is a linear functional φ on V that extends φ_0 and is dominated by m_u. So for $v \in \mathcal{O},\ w \in C$,

$$\varphi(v - w - v_0) \leq m_u(v - w - v_0) \leq 1 \implies \varphi(v) - \varphi(w) - \underbrace{\varphi(v_0)}_{=1} \leq 1.$$

$$\implies \varphi(v) \leq \varphi(w), \quad \text{so } \sup\{\varphi(v) : v \in \mathcal{O}\} \leq \inf\{\varphi(w) : w \in C\}.$$

Let $t_0 \in \mathbb{R}$ lie between these two numbers. Then we have $\varphi(\mathcal{O}) < t_0 \leq \varphi(C)$, since $\mathcal{O}$ is open, i.e. $\varphi(v)$ for $v \in \mathcal{O}$ does not attain the supremum on $\mathcal{O}$. The proof is finished by showing φ is continuous. ■

Corollary: If V is a locally convex topological vector space, and if C is a closed convex subset of V, and if $v_0 \in V \backslash C$, then there is a $\varphi \in V'$ and $t_0 \in \mathbb{R}$ such that $\varphi(C) \leq t_0 < \varphi(v_0)$.
Proof: C^c is open and contains v_0, so there is a convex open set in C^c containing v_0, say, $\mathcal{O}$. Apply the Hahn-Banach Separation Theorem on C and $\mathcal{O}$. ■

Corollary: Let V be a normed vector space, and let S be a norm-closed convex subset of V. Then S is also closed for the weak topology on V.
Proof: Let $v_0 \in V \setminus S$. Since S is norm-closed, there is a norm-open ball B about v_0 that does not intersect S, i.e. $B \cap S = \varnothing$. So there exists $\varphi \in V'$ with $\varphi(S) \leq t_0 \leq \varphi(v_0)$. But $v_0 \in \{v : t_0 < \varphi(v)\}$, which is open for the weak topology and doesn't meet S, so v_0 is not in the weak closure of S. ■

Facts about convex sets (proofs left as exercises):

1. The intersection of any convex sets is convex.
2. Given any subset of S, there is a smallest convex set that contains S, denoted $\mathrm{conv}(S)$.
3. If S is a subset of a topological vector space, there is a smallest closed convex set that contains it, namely, the closure of $\mathrm{conv}(S)$, denoted $\overline{\mathrm{conv}}(S)$.

Def: Let C be a convex set. A **face** of C is a convex set $S \subseteq C$ such that if a point in S is in the interior of a line segment between two points in C, then these two points are in S. That is, if $v_0, v_1 \in C$ and if there is $t \in [0, 1]$ with $tv_0 + (1-t)v_1 \in S$, then $v_0, v_1 \in S$.

Def: A point of C that is a face is called an **extreme point**.

Ex: Consider the closed unit disk. Then only faces are the disk itself and the individual points on the boundary, thus they are extreme points.

Sources of Faces:
Let C be a convex set in V, and let φ be a linear functional in V.
If φ attains its maximum or minimum as a function on C, then $\{v \in C : \varphi(v) = m\}$, where m is the max or min, is a face of C.
Indeed, if $v_0, v_1 \in C$, and if $\varphi(tv_0 + (t-1)v_1) = m$, then $t\varphi(v_0) + (1-t)\varphi(v_1) = m$. Without loss of generality, suppose m is the maximum. Then $\varphi(v_0) \leq m$ and $\varphi(v_1) \leq m$, so t is either 0 or 1, and in fact $\varphi(v_0) = \varphi(v_1) = m$.

The following property's proof will be left as an exercise:
If C is convex, and if F is a face of C, and if G is a face of F, then G is a face of C.

Krein-Milman Theorem: Let V be a locally convex topological vector space, and let C be a compact convex subset of V. Let $\mathcal{E}$ be its set of extreme points. Then $\overline{\text{conv}}(\mathcal{E}) = C$.
Proof: Let $\mathcal{F}_C$ be the set of all closed compact faces of C. Put a partial order by reverse inclusion, i.e., if $F_1, F_2 \in \mathcal{F}_C$, then $F_1 \geq F_2$ if $F_1 \subseteq F_2$. Let T be a totally-ordered subset of $\mathcal{F}_C$, and let $F_0 = \bigcap_{F \in T} F$. Then F_0 is convex, closed, and nonempty by the finite intersection property. Note that $F_0 \in \mathcal{F}_C$, and $F_0 \geq$ all F in T, so $\mathcal{F}_C$ is inductively ordered. By Zorn's Lemma, there are maximal elements in $\mathcal{F}_C$, i.e. minimal for inclusion.
We claim that F_0 contains only one point. Suppose $v_1, v_2 \in F_0$, with $v_1 \neq v_2$. By Hausdorff and Hahn-Banach separation theorem, there is $\varphi \in V'$ with $\varphi(v_1) \leq \varphi(v_2)$, and φ will take its maximum on F_0.
Let $F' = \{v \in F_0 : \varphi(v) = \text{max.}\}$. Now F' is a face of C not containing v_1, $F' \in F_0$, so $F' \subset F_0$, a contradiction. Thus, F_0 contains only one point, an extreme point.
Let $D = \overline{\text{conv}}(\mathcal{E})$, and suppose $D \neq C$. Then take $v_0 \in C \setminus D$. By the Hahn-Banach Separation theorem, there is $\varphi \in V'$ with $\varphi(D) \leq t_0 < \varphi(v_0)$. Let G be the subset of C where φ takes its maximum. Then G is a closed face of C, and G contains an extreme point, and thus an extreme point of $C \setminus D$. Then by continuity of φ, this means that $D = \overline{\text{conv}}(\mathcal{E})$.
■

3.6 Hilbert Spaces

Def: A **Hilbert space** is a complete inner product space (over $\mathbb{R}$ or $\mathbb{C}$), where completeness is with respect to the norm defined by $\|v\| = \langle v, v\rangle^{1/2}$.

Throughout, we will use $\mathcal{H}$ to denote a Hilbert space.

Parallelogram Rule: For any $v, w \in \mathcal{H}$, we have $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$.
Proof: $\langle v+w, v+w\rangle + \langle v-w, v-w\rangle = 2\langle v, v\rangle + 2\langle w, w\rangle = 2(\|v\|^2 + \|w\|^2)$. ■

Proposition: Let C be a closed convex subset of $\mathcal{H}$ not containing 0. Then there is a unique point $v \in C$ that is closest to 0. That is, $\|v\| \leq \|w\|$ for all $w \in C$.
Proof: Let $m = \inf\{\|v\| : v \in C\}$.
Let $\{v_n\}$ be a sequence in C such that $\|v_n\| \to m$. Then

$$\left\|\frac{v_m - v_n}{2}\right\|^2 = \frac{1}{2}\Big(\|v_n\|^2 + \|v_m\|^2\Big) - \left\|\frac{v_n + v_m}{2}\right\|^2,$$

where we note that $\frac{v_n + v_m}{2} \in C$, by convexity of C.
Since $\|v_n\|, \|v_m\| \to m$, and $\left\|\frac{v_n+v_m}{2}\right\| \to m$ also, we have that $\left\|\frac{v_m - v_n}{2}\right\|^2 \to 0$ as $n, m \to \infty$,

so it follows that $\{v_n\}$ is Cauchy and thus converges to some $v \in C$ with $\|v\| = m$. Suppose there were another point v' inC that is also closest to 0. Then $\|v'\| = m$, and thus by the parallelogram rule, $\left\|\frac{v-v'}{2}\right\| = 0$, so that $v = v'$. ■

Corollary: If W is a subspace of $\mathcal{H}$, and $v_0 \in \mathcal{H} \setminus W$, then there is a closest point $w_0 \in W$ to v_0.

We will now show the analogous result in $\mathbb{R}^n$ where the closest point w_0 in a subspace W to another point v_0 outside the subspace is the "orthogonal projection" of v_0 onto W.

For any $w \in W$, since $w_0 \in W$, we have $\|v_0 - w_0\|^2 \leq \|v_0 - (w_0 - w)\|^2$, so

$$\|v_0 - w_0\|^2 \leq \langle (v_0 - w_0) + w, (v_0 - w_0) + w\rangle = \|v_0 - w_0\|^2 + 2\mathrm{Re}\langle v_0 - w_0, w\rangle + \|w\|^2 \implies$$

$$\implies 0 \leq 2\mathrm{Re}\langle v_0 - w_0, w\rangle + \|w\|^2 .$$

For any $t > 0$, we have $0 \leq 2\mathrm{Re}\langle v_0 - w_0, tw\rangle + \|tw\|^2$, so by letting $t \to 0$, we get $0 \leq 2\mathrm{Re}\langle v_0 - w_0, w\rangle$, and $\mathrm{Re}\langle v_0 - w_0, w\rangle = 0$. Therefore $\langle v_0 - w_0, w\rangle = 0$.

Notation: $W^\perp = \{v \in \mathcal{H} : \ \langle v, w\rangle = 0 \ \forall w \in W\}$.

$\to$ From above, $v_0 - w_0 \in W^\perp$.
Conversely, if $w_0 \in W$ satisfies $v_0 - w_0 \in W^\perp$, then w_0 is the closest point in W to v_0 :

$$\|v_0 - w\|^2 = \left\| \underbrace{(v_0 - w_0)}_{\in W^\perp} + \underbrace{(w_0 - w)}_{\in W} \right\|^2 = \|v_0 - w_0\|^2 + \|w_0 - w\|^2 ,$$

by the Pythagorean Theorem.

Dual Space to a Hilbert Space

Let $\psi \in \mathcal{H}', \psi \neq 0$, and $\mathcal{H}$ a Hilbert space. Let $W = \ker(\psi)$. Then W is a closed subspace, not equal to $\mathcal{H}$. Let $v_1 \in W^\perp$, with $\psi(v_1) = 1$. Then for any $v \in \mathcal{H}$, we have $\psi(v - \psi(v)v_1) = 0$. This means that $v - \psi(v)v_1 \in W$. Therefore

$$\langle v - \psi(v)v_1, v_1\rangle = 0 \implies \langle v, v_1\rangle = \langle \psi(v)v_1, v_1\rangle = \psi(v)\|v_1\|^2 \implies \psi(v) = \frac{\langle v, v_1\rangle}{\|v_1\|^2}.$$

Conversely, for any $v_0 \in W$, define ψ_{v_0} on $\mathcal{H}$ by $\psi_{v_0}(v) = \langle v, v_0\rangle$. One can easily check that $\|\psi_{v_0}\| = \|v_0\|$ by using Cauchy-Schwarz.
Thus, we see that the map $v_0 \mapsto \psi_{v_0}$ from $\mathcal{H}$ to $\mathcal{H}'$ shows that the dual of $\mathcal{H}$ is itself. In particular, $\mathcal{H}$ is reflexive.

* Note that if $\mathcal{H}$ is not complete, then $(\ker(\psi))^\perp$ may be empty. For example, look at $C([0,1])$ with $\psi(f) = \int_0^{1/2} f(x)\, dx$.

Chapter 4

Dual to L^p

Using the theory we have developed on Hilbert spaces, we will first show that the dual to L^1 is L^∞. Then, we will prove the Lebesgue Decomposition Theorem and the Radon-Nikodym Theorem. From these theorems, we show that the dual to L^p is L^q, where $p^{-1} + q^{-1} = 1$.

4.1 Dual to L^1

Let us first determine the dual space to $L^1(X, \mathcal{S}, \mu)$ for $\mu(X) < \infty$. In this case, we know that $L^2(X, \mathcal{S}, \mu) \subseteq L^1(X, \mathcal{S}, \mu)$.
Let $\psi \in L^1(X, \mathcal{S}, \mu)'$. For $f \in L^2$, we have $|\psi(f)| \leq \|\psi\| \, f_1 \leq \|\psi\| \, f_2 \, \|\mathbf{1}\|$. So ψ as a linear functional of L^2 is continuous, so there is $g \in L^2$ such that $\psi(f) = \langle f, g \rangle \; \forall f \in L^2$, i.e., $\psi(f) = \int f\overline{g} \, d\mu$.
Recall that we also have $\left| \int f\overline{g} \, d\mu \right| \leq \|f\|_1 \, \|\psi\|$.

<u>Claim</u>: $g \in L^\infty$.
Let $E \in S$, $\mu(E)$, and let $f = \chi_E$, so we consider $\int_E \overline{g} \, d\mu \leq \|\chi_E\|_1 \, \|\psi\| = \mu(E) \, \|\psi\|$.

$$\implies \frac{1}{\mu(E)} \int_E \overline{g} \, d\mu \leq \|\psi\| .$$

Recall that $\frac{1}{\mu(E)} \int_E \overline{g} \, d\mu$ is the average value of $\overline{g}$ on E. By the following lemma, we have $\overline{g}(x) \leq \|\psi\| < \infty$ a.e., so indeed $g \in L^\infty(X, \mathcal{S}, \mu)$.

<u>Lemma</u>: Let $f \in L^1(X, \mathcal{S}, \mu, \mathcal{B})$. Suppose $C \subseteq \mathcal{B}$ is closed, and suppose that for all $E \in \mathcal{S}$, $\mu(E) < \infty$, and $\frac{1}{\mu(E)} \int_E f \, d\mu \in C$. Then $f(x) \in C$ a.e.
<u>Proof</u>: We can take $\mathcal{B}$ to be separable, and also $\mu(X) < \infty$ in particular. Let $v \in \mathcal{B} \setminus C$. Choose $r > 0$ such that $B_r(v) \cap C = \varnothing$. Let $F = \{x : \; f(x) \in B_r(v)\}$. If $\mu(F) > 0$,

$$\left\| \frac{1}{\mu(F)} \int_F f \, d\mu - v \right\| = \left\| \frac{1}{\mu(F)} \int_F f \, d\mu - \frac{1}{\mu(F)} \int_F v \, d\mu \right\| \leq \frac{1}{\mu(F)} \int_F \|f(x) - v\| \; d\mu(x) < r,$$

which means that $\frac{1}{\mu(F)} \int_F f \, d\mu \in B_r(v) \cap C$, a contradiction. Therefore $\mu(F) = 0$.
Since $\mathcal{B}$ is separable, $\mathcal{B} \setminus C$ is open in $\mathcal{B}$ and is a countable union of balls $B_{r_j}(v_j)$ with $B_{r_j}(v_j) \cap C = \varnothing$. It follows that $\mu\Big(f^{-1}(\mathcal{B} \setminus C)\Big) = 0$. $\blacksquare$

Theorem: Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Then $(L^1)' = L^\infty$.
Proof: Let $\varphi \in L^1(X, \mathcal{S}, \mu)'$. There exists a sequence $\{F_j\}_{j=1}^\infty$ with $\mu(F_j) < \infty$ for all j. Define $E_n = \bigcup_{j=1}^n F_j$. So $E_n \uparrow X$. In other words $X = \bigcup_{j=1}^\infty F_j$.
Restrict φ to $L^1(E_n, \mathcal{S}_{E_n}, \mu_{E_n}) \subseteq L^1(X, \mathcal{S}, \mu)$. Recall that there is a $g_n \in L^\infty(E_n, \mathcal{S}_{E_n}, \mu_{E_n})$ with $\varphi(f) = \int_{E_n} g_n f \; d\mu$, with this inclusion map an isometry. But if $m > n$, for $f \in L^1(E_n) \subseteq L^1(E_m)$,

$$\varphi(f) = \int_{E_n} g_n f \; d\mu = \int_{E_m} g_m f \; d\mu, \text{ so } g_m\Big|_{E_n} = g_n \text{ a.e.}$$

Thus, up to a null set, get $g \in L^\infty(X)$ such that $g\Big|_{E_n} = g_n$ a.e.
Then for $f \in L^1(X)$, we have $\varphi(f) = \int gf \; d\mu$.
Let $f_n = \chi_{E_n} f$, so $\varphi(f_n) = \int g_n f_n \; d\mu$. By letting $n \to \infty$, we see that $\|\varphi\| = \|g\|$. ■

* Remark: We need $(X, \mathcal{S}, \mu)$ to be σ-finite. Indeed, let X be uncountable, and let $\mathcal{S}$ be the σ-ring of countable sets with counting measure μ. Then $L^1 = \ell^1$, and
$\ell^1(X)' = \ell^\infty(X) \neq L^\infty(X, \mathcal{S}, \mu)$, since, for example, $g(x) = 1$ is not $\mathcal{S}$-measurable, so $g \notin L^\infty(X, \mathcal{S}, \mu)$.

4.2 Lebesgue and Radon-Nikodym Theorems

Def: A function g is **locally measurable** for $\mathcal{S}$ if for any $E \in \mathcal{S}$, we have $\chi_E g$ is $\mathcal{S}$-measurable.

Def: Let $(X, \mathcal{S})$ be a measurable space. Let μ be a positive measure on $\mathcal{S}$, and let ν be a $\mathcal{B}$-valued measure, where $\mathcal{B}$ is a Banach space. Then ν is **absolutely continuous** with respect to μ if $E \in \mathcal{S}$ with $\mu(E) = 0$ implies $\nu(E) = 0$. We write $\nu \ll \mu$.

Def: μ and ν are **mutually singular** if there is $E \in \mathcal{S}$ such that for any $F \subseteq E$, $\mu(F) = 0$, and for any $G \subseteq E^c$, $\nu(G) = 0$. We write $\nu \perp \mu$.

Theorem: Let $(X, \mathcal{S})$ be a measurable space, and let μ, ν be measures defined on $\mathcal{S}$ that are σ-finite.

(1) **Lebesgue Decomposition Theorem:** $\nu = \nu_{\text{ac}} + \nu_{\text{s}}$, where $\nu_{\text{s}} \perp \mu$, and $\nu_{\text{ac}} \ll \mu$.

(2) **Radon-Nikodym Theorem:** If $\nu \ll \mu$, then there is a measurable $g \geq 0$ such that $\nu(E) = \int_E g \; d\mu$.

Proof: First, assume the case $\mu(X) < \infty$ and $\nu(X) < \infty$.
Observe that $\mu + \nu$ is a measure, and that for $f \in L^1(\mu + \nu)$,

$$\int |f| \, d(\mu + \nu) = \int |f| \, d\mu + \int |f| \, d\nu.$$

Define φ on $L^1(\mu + \nu)$ by $\varphi(f) = \int f \, d\nu$. Then

$$|\varphi(f)| \leq \int |f| \, d\nu = \|f\|_{1_\nu} \leq \|f\|_{1_{\mu+\nu}}.$$

So $\varphi \in (L^1(\mu + \nu))'$ with $\|\varphi\| \leq 1$, so there is a $h \in L^\infty(X, \mathcal{S}, \mu + \nu)$ such that $\varphi(f) = \int fh \, d(\mu + \nu)$ for all $f \in L^1(\mu + \nu)$, and $\|h\|_\infty \leq 1$.
Furthermore, $h \geq 0$ a.e. We can also assume $0 \leq h \leq 1$ since $h > 1$ on a null set. Let $E_s = \{x : \ h(x) = 1\}$, and define the measure $\nu_s(E) = \nu(E \cap E_s)$.
We claim that $\mu \perp \nu_s$. For any $E \subseteq E_s$,

$$\int \chi_E \, d\nu = \int \chi_E h \, d(\mu + \nu) = \int_E h \, d\mu + \int_E h \, d\nu.$$

Since $E \subseteq E_s$, we have $h = 1$ on E, so it follows that $\mu(E) = \int \chi_E \, d\mu = 0$.
For $E \subseteq E_s^c$, we have $\nu_s(E) = \nu(E \cap E_s) = 0$, since $E \cap E_s = \varnothing$.

Now, assume that ν is absolutely continuous with respect to μ. Since

$$\int f \, d\nu = \int fh \, d(\mu + \nu) = \int fh \, d\nu + \int fh \, d\mu),$$

with $0 \leq h < 1$, it follows that

$$\int f(1-h) \, d\nu = \int fh \, d\mu.$$

Let $g = \dfrac{h}{1-h}$, which may be unbounded. Let $E_n = \{x : \ 1 - h(x) \geq 1/n\}$. Then $g\chi_{E_n} \in L^\infty$, and let $E \subseteq E_n$. Then $\dfrac{\chi_E}{1-h} \in L^\infty$, and

$$\nu(E) = \int (1-h)\frac{\chi_E}{1-h} \, d\nu = \int h\frac{\chi_E}{1-h} \, d\mu = \int_E g \, d\mu.$$

For any E, we have $\nu(E) \geq \nu(E \cap E_n) = \int_{E \cap E_n} g \, d\mu \uparrow \int_E g \, d\mu$, with $g \in L^1(\mu)$. Now for the σ-finite case, choose $E_n \uparrow X$, and apply the result to each E_n. The result should then hold for all of X.
■

4.3 The Dual to L^p

Let us now finally prove that for $p > 1$, the dual to L^p is L^q, where $p^{-1} + q^{-1} = 1$.

Def: $f \in L^p$ is positive if $f \geq 0$. $\varphi \in (L^p(X))'$ is a **positive linear functional** if $\varphi(f) \geq 0$ for every positive f.

Lemma: Let $\mu(X) < \infty$, if $g \in L^1(X)$, and if there exists a c such that $\left| \int fg \, d\mu \right| \leq c \|f\|_p$ for all $f \in L^p$, then $g \in L^q$.

Proof: Note that it suffices to have $g \geq 0$. Let $g_n(x) = \begin{cases} g(x) \text{ if } g(x) \leq n. \\ n \text{ if } g(x) > n \end{cases}$

Then $g_n \in L^\infty \subseteq L^q$. For $f \in L^1 \cap L^\infty$, we have

$$\left| \int f g_n \, d\mu \right| \leq \int |f g_n| \, d\mu \leq \|f\|_p \|g_n\|_q ,$$

by Holder's Inequality and the fact that $L^1 \cap L^\infty$ is dense in L^p. But also, recall that

$$\|g_n\|_q = \sup \left\{ \left| \int f g_n \, d\mu \right| : \|f\|_p \leq 1 \right\} \leq c.$$

Since this holds for all n, it follows that $g \in L^q$. ■

Theorem: Suppose $\mu(X) < \infty$, and let $\varphi \in (L^p)'$ be positive. Then there exists $g \in L^q$ ($g \geq 0$) such that $\varphi(f) = \int fg \, d\mu$ for all $f \in L^p(X)$.

Proof: For $E \in \mathcal{S}$, define a function ν on $\mathcal{S}$ by $\nu(E) = \varphi(\chi_E)$. We claim that ν is a measure. Let $E, F \in \mathcal{S}$ be disjoint. Then

$$\nu(E \cup F) = \varphi(\chi_{E \cup F}) = \varphi(\chi_E) + \varphi(\chi_F) = \nu(E) + \nu(F).$$

Therefore ν is additive. For countable subadditivity, let $E = \bigcup_{n=1}^\infty E_n$, where $\{E_n\}$ are disjoint sets in $\mathcal{S}$. Let $F_m = \bigcup_{n=1}^m E_n$, so that $F_m \uparrow E$. Clearly, by the Monotone Convergence Theorem, we have $\chi_{F_m} \to \chi_E$, so

$$\nu(E) = \varphi\Big(\lim_{m\to\infty} \chi_{F_m} \Big) = \lim_{m\to\infty} \varphi(\chi_{F_m}) = \lim_{m\to\infty} \nu\Big(\bigcup_{n=1}^m E_n \Big) = \lim_{m\to\infty} \sum_{n=1}^m \nu(E_n) = \sum_{n=1}^\infty \nu(E_n),$$

as desired. Furthermore, observe that if $\mu(E) = 0$, then $\varphi(\chi_E) = 0$, since $\chi_E = 0$ a.e. in L^p. This means that $\nu \ll \mu$, so by the Radon-Nikodym Theorem, there exists a measurable g such that $\nu(E) = \int_E g \, d\mu$ for all $E \in \mathcal{S}$. But by the lemma above, since

$$\left| \int fg \, d\mu \right| = |\varphi(f)| \leq \|\varphi\| \, \|f\|_p ,$$

indeed $g \in L^q$. ■

Chapter 5

The Stone-Weierstrass Theorem

In this chapter, we will develop theory to prove the end result of the Stone-Weierstrass Theorem, an important result that generalizes the Weierstrass polynomial approximation theorem. First, we discuss ordered vector spaces and lattice-ordered groups. We then prove the Kakutani-Krein Theorem, which then yields the Stone-Weierstrass Theorem.

5.1 Ordered Vector Spaces

Consider V a countable group, and let $\leq$ be a partial order on V.

Def: The order $\leq$ is **compatible** with the group structure if whenever $v \leq w$, we have that $v + x \leq w + x$ implies $v \leq w$ for all x.

Now consider V a vector space with a compatible order $\leq$.

Proposition: If $v \leq w$, then $-w \leq -v$.
Proof:

$$(-w - v) + v \leq (-w - v) + w \implies -w \leq -v.$$

■

Def: Let X be partially ordered. $(X, \leq)$ is a **lattice** if for any two elements $x, y \in X$, there is a least upper bound, denoted $x \vee y$, and there is a greatest lower bound, denoted $x \wedge y$. That is,

- $x \vee y \geq x, y$ and if $w \geq x, y$, then $w \geq x \vee y$.
- $x \wedge y \leq x, y$ and if $w \leq x, y$, then $w \leq x \wedge y$.

Ex: Consider $C_{\mathbb{R}}(X)$, with $f \leq g$ iff $f(x) \leq g(x)$ for all x. It can easily be seen that $f \vee g = \max\{f, g\}$, and $f \wedge g = \min\{f, g\}$.

Ex: Slightly more tricky to check, but $L^p(X, \mathcal{S}, \mu, \mathbb{R})$ is also a lattice.

5.2 Properties of Lattice Ordered Groups & Vector Spaces

Here we will list some properties of lattice-ordered groups. We will prove some of these properties, as the rest should be easy to show.

(1) Given $x, v, w \in V$, then $x+(v \vee w) = (x+v) \vee (x+w)$, and $x+(v \wedge w) = (x+v) \wedge (x+w)$.

(2) $-(v \vee w) = (-v) \wedge (-w)$, and $-(v \wedge w) = (-v) \vee (-w)$.
Proof: $x \leq (-v) \wedge (-w)$ iff $x \leq -v$ and $x \leq -w$ iff $(-x \geq v)$ and $(-x \geq w)$ iff $-x \geq v \vee w$ iff $x \leq -(v \vee w)$. The other direction follows similarly. ■

(3) Let $v^+ := v \vee 0$, and $v^- := (-v) \vee 0$. Then $v = v^+ - v^-$.
Proof: $v^+ - v = (v \vee 0) - v = (v - v) \vee (0 - v) = 0 \vee (-v) = v^-$. ■

(4) $v^+ \wedge v^- = 0$.
Proof: $v^+ \wedge v^- = v^+ \wedge (v^+ - v) = v^+ + (0 \wedge (-v)) = v^+ - (0 \vee v) = v^+ - v^+ = 0$. ■

(5) Let $|v| := v^+ + v^-$. Then $|v| = v^+ \vee v^-$.
Proof: $v^+ + v^- = v^+ + (0 \vee (-v)) = v^+ \vee (v^+ - v) = v^+ \vee v^-$. ■

(6) If $x, v, w \geq 0$, and if $x \leq v + w$, then there exists v_1, w_1 such that $0 \leq v_1 \leq v$ and $0 \leq w_1 \leq w$ such that $x = v_1 + w_1$. This is known as the **Riesz Property**.

*A vector space lattice V is a vector space with a compatible lattice order. For a norm, we also need $\|w\| \geq \|v\|$ whenever $w \geq v \geq 0$. Also, we want $\|v\| = \||v|\|$.

Def: Let V be a lattice-ordered vector space. $C \subseteq V$ is a **cone** if $v, w \in C$ implies $v+w \in C$, and for $t > 0$, $tv \in C$.

Theorem: If V is lattice-ordered, then V' is lattice-ordered (with a norm is V is normed), where $V'_+ := \{\varphi \in V' : \varphi(v) \geq 0, \forall v \geq 0\}$ is a cone.
Proof:

(1) ($\varphi \geq 0$ means $\varphi \in V'_+$) If $\varphi \geq 0$ and $\varphi \leq 0$ (i.e. $-\varphi \geq 0$), then for any $v \in V$, $v \geq 0$, $\varphi(v) \geq 0$, and $\varphi(v) \leq 0$. Therefore $V'_+ \cap (-V'_+) = \{0\}$, since any $v \in V$ has $v = v^+ - v^-$, so $\varphi(v) = 0 \implies \varphi = 0$.

(2) For $\varphi \in V'$, consider $\varphi \vee 0$, and φ^+ on V^+ by $\varphi^+(v) = \sup\{\varphi(x) : 0 \leq x \leq v\}$. In turns out that if $v, w \geq 0$, then $\varphi^+(v + w) = \varphi^+(v) + \varphi^+(w)$.
$\rightarrow$ If $0 \leq v_1 \leq v$ and $0 \leq w_1 \leq w$, then $v_1 + w_1 \leq v + w$, so
$\varphi(v_1) + \varphi(w_1) = \varphi(v_1 + w_1) \leq \varphi^+(v_1 + w_1)$, so it follows that
$\varphi^+(v) + \varphi^+(w) \leq \varphi^+(v + w)$, by taking supremums on v_1 and w_1.
Conversely, if $0 \leq x \leq v + w$, by the Riesz Property there exists v_1, w_1 such that

$0 \leq v_1 \leq v$, $0 \leq w_1 \leq w$, and $x = v_1 + w_1$. Then
$\varphi(x) = \varphi(v_1) + \varphi(w_1) \leq \varphi^+(v) + \varphi^+(w)$. Therefore $\varphi^+(v+w) = \varphi^+(v) + \varphi^+(w)$, as desired.

(3) Now define φ^+ on all of V as follows: if $v = v_1 - v_2$, for $v_1, v_2 \geq 0$, then
$\varphi^+(v) := \varphi^+(v_1) - \varphi^+(v_2)$. We must check that this is well-defined, but we will not worry about this here. Now it is easily seen that $\varphi^+(v+w) = \varphi^+(v) + \varphi^+(w)$ for any $v, w \in V$.
We claim that φ^+ is a least-upper bound for φ and 0. Clearly, $\varphi^+ - \varphi \geq 0$, and $\varphi^+ \geq 0$. Let $\psi \in V'$, $\psi \geq \varphi$, $\psi \geq 0$. Then for $v \geq 0$, and for $0 \leq x \leq v$, we have $\varphi(x) \leq \psi(x) \leq \psi(v)$, and by taking the supremum on x, we have $\varphi^+(v) \leq \psi(v)$, so φ^+ is indeed a least upper bound for φ.
Given $\varphi, \psi \in V'$, we want $\varphi \vee \psi$, and we expect this to equal
$\psi + (\varphi - \psi) \vee 0 = \psi + (\varphi - \psi)^+$.
We know that $(\varphi - \psi) \vee 0$ is a least upper bound for $(\varphi - \psi)$ and 0, then $\psi + (\varphi - \psi)^+$ is a least upper bound for φ and ψ.
$\implies$ We *define* $\varphi \vee \psi := \psi + (\varphi - \psi)^+$.
We found that V' is an ordered vector space and has least upper bounds, so now we want to find its greatest lower bounds.
Do first for $\varphi \wedge 0$, expect it is $-((-\varphi) \vee 0)$. We show that this is the greatest lower bound for φ and 0. Clearly $-((-\varphi) \vee 0) \leq 0$.
We want $-((-\varphi) \vee 0) \leq \varphi$, i.e. $0 \leq \varphi + ((-\varphi) \vee 0) = 0 \vee \varphi$:
We also want for any $\psi \leq 0$, $\psi \leq \varphi$, then $\psi \leq -((-\varphi) \vee 0)$, i.e.
$\psi + (-\varphi \vee 0) \leq 0 \implies (\psi - \varphi) \vee \psi \leq 0$, but note that $\psi - \varphi \leq 0$.
Finally, we need $\|\varphi\| = \||\varphi|\|$, where $|\varphi| = \varphi^+ + \varphi^-$.
Consider $v \geq 0$. Then

$$|\varphi(v)| = |\varphi^+(v) - \varphi^-(v)| \leq |\varphi^+(v) + \varphi^-(v)| = |\varphi|(v) \leq \||\varphi|\| \|v\| .$$

Then for any $v \in V$,

$$|\varphi(v)| = |\varphi(v^+ - v^-)| \leq |\varphi(v^+)| + |\varphi(v^-)| \leq |\varphi|(v^+) + |\varphi|(v^-) = |\varphi|(|v|) \leq \||\varphi|\| \|v\| .$$

This shows that $\|\varphi\| \leq \||\varphi|\|$.
Conversely, consider $v \geq 0$. Let $\epsilon > 0$, so there is $0 \leq w \leq v$ with $\varphi(w) \geq \varphi^+(v) - \epsilon$, and $0 \leq z \leq v$ with $-\varphi(z) \geq \varphi^-(v) - \epsilon$. Hence $\varphi(w-z) = \varphi(w) - \varphi(z) \geq \varphi^+(v) + \varphi^-(v) - 2\epsilon$.
On the other hand, note that $w - z \leq v$ and $z - w \leq v$, so

$$(w-z)^+ \leq v, \; (z-w)^+ \leq v \implies |w - z| = (w-z)^+ \vee (z-w) \leq v.$$

Then $\varphi(w-z) \leq \varphi(|w-z|) \leq \|\varphi\| \||w-z|\| \leq \|\varphi\| v$, i.e.
$\|\varphi\| v \geq \varphi^+(v) + \varphi^-(v) - 2\epsilon = |\varphi|(v) - 2\epsilon$, so for $v \geq 0$, $\|\varphi\| \|v\| \geq |\varphi| \|v\|$.
Now for any $v \in V$,

$$\Big||\varphi|(v)\Big| = \Big||\varphi|(v^+ - v^-)\Big| = \Big||\varphi|(v^+) - |\varphi|(v^-)\Big| \leq \max\{|\varphi|(v^+), |\varphi|(v^-)\} \leq \|\varphi\| \|v\| ,$$

so we have $\|\varphi\| \geq \||\varphi|\|$, hence $\|\varphi\| = \||\varphi|\|$.

■

5.3 The Stone-Weierstrass Theorem

First, we will prove the Kakutani-Krein Theorem, which has a very theoretical proof. The Stone-Weierstrass Theorem will follow, and its proof is quite technical.

Kakutani-Krein Theorem: Let X be compact Hausdorff. Let $\mathcal{L}$ be a subspace of $C_{\mathbb{R}}(X)$ such that

(i) $\mathcal{L}$ is "stable" for the lattice operations, i.e. $f, g \in \mathcal{L} \implies f \vee g, f \wedge g \in \mathcal{L}$.

(ii) $\mathcal{L}$ **strongly separates points** of X, i.e., if $x, y \in X$ $(x \neq y)$, then for any $r, s \in \mathbb{R}$, there exists $f \in \mathcal{L}$ with $f(x) = r$ and $f(y) = s$.

Then $\mathcal{L}$ is dense in $C_{\mathbb{R}}(X)$ for $\|\cdot\|_\infty$.
Proof: Let $f \in C_{\mathbb{R}}(X)$, and let $\epsilon > 0$. We want to find $g \in \mathcal{L}$ with $|f(x) - g(x)| < \epsilon$ for all x. Now fix x, and for each y choose h_y such that $h_y(x) = f(x)$ and $h_y(y) = f(y)$. Then there is an open neighborhood U_y of y such that $h_y(z) < f(z) + \epsilon$ for all $z \in U_y$. There must be a finite subcover $U_{y_1}, \ldots, U_{y_n}$ of X.
Let $g_x = h_{y_1} \wedge \cdots \wedge h_{y_n}$. Thus $g_x(x) = f(x)$, and $g_x(y) \leq f(y) + \epsilon$. Then for each x, there is an open neighborhood V_x of x such that $g_x(z) > f(z) - \epsilon$ for $z \in V_x$. There must be a finite subcover $V_{x_1}, \ldots, V_{x_m}$ of X. Let $g = g_{x_1} \vee \cdots \vee g_{x_m}$. Then $g(y) \leq f(y) + \epsilon$, and $g(y) \geq f(y) - \epsilon$ for all $y \in X$. ■

Stone-Weierstrass Theorem: Let X be compact Hausdorff. Let $\mathcal{A}$ be a subalgebra of $C_{\mathbb{R}}(X)$ that strongly separates points of X. Then $\mathcal{A}$ is dense in $C_{\mathbb{R}}(X)$ for $\|\cdot\|_\infty$.
Proof: Let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A}$ in $C_{\mathbb{R}}(X)$ for $\|\cdot\|_\infty$. Then $\overline{\mathcal{A}}$ is a subalgebra of $C_{\mathbb{R}}(X)$ also, and we claim that $\overline{\mathcal{A}}$ is stable for the lattice operations.

Since $f \vee g = \dfrac{f + g + |f - g|}{2}$, and $f \wedge g = \dfrac{f + g - |f - g|}{2}$, it suffices to show that if $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$.
Now $|f| = \sqrt{f^2}$, and by scaling, it suffices to consider f with $\|f\|_\infty \leq \frac{1}{2}$.
Given $\delta > 0$, consider the function $t \mapsto \sqrt{t + \delta}$, for $t \in \mathbb{R}$. This function has a power series about $1/2$ converging in $(-\delta,\ 1 + \delta)$, so in particular it converges uniformly on $[0, 1]$. By truncating the power series, we get a polynomial q that approximates $\sqrt{t + \delta}$ as closely as we want. Set $p = q - q(0)$. Then we can approximate $\sqrt{t + \delta}$ on $[0, 1]$ by a polynomial p with $p(0) = 0$. We can choose δ small enough that $\sqrt{t + \delta}$ approaches $\sqrt{t}$ on $[0, 1]$. Thus we can approximate the map $t \mapsto \sqrt{t + \delta}$ by a polynomial p with $p(0) = 0$ as accurately as we want. Thus, given $\epsilon > 0$, we can find a polynomial p_ϵ with $p_\epsilon(0) = 0$ such that $|\sqrt{t} - p(t)| < \epsilon$ for all $t \in [0, 1]$.
Thus, given $f \in \overline{\mathcal{A}}$, $\|f\|_\infty \leq 1/2$, $|\sqrt{f(x)^2} - p(f(x)^2)| < \epsilon$ for all $t \in [0, 1]$, i.e. $\||f| - p(f^2)\|_\infty < \epsilon$ since $p(f^2) \in \overline{\mathcal{A}}$. Therefore $|f| \in \overline{\mathcal{A}}$. The result then follows from the Kakutani-Krein Theorem. ■

Complex Stone-Weierstrass Theorem: Let X be compact Hausdorff, and let $\mathcal{A} \subseteq C_{\mathbb{C}}(X)$ be a subalgebra that strongly separates points of X and is stable under complex conjugation, i.e. if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C_{\mathbb{C}}(X)$ for $\|\cdot\|_\infty$.

Proof: Observe that $f = \dfrac{f+\overline{f}}{2} + i \cdot \dfrac{f-\overline{f}}{2}$. The $\mathbb{R}$-valued functions are a subalgebra of $C_{\mathbb{R}}(X)$ that strongly separates points. Then Re $\mathcal{A}$ is dense in $C_{\mathbb{R}}(X)$. Similar arguments can be made for Im $\mathcal{A}$, so that $\mathcal{A}$ is dense in $C_{\mathbb{C}}(X)$. ■

Finally, we will show a version of the Stone-Weierstrass Theorem to Hausdorff spaces that are only locally compact.

Theorem: Let X be locally compact Hausdorff, but not compact. Let $\mathcal{A}$ be a subalgebra of $C_\infty(X, \mathbb{R})$ that separates points. If for each $x \in X$ there is $f \in \mathcal{A}$ with $f(x) \neq 0$, then $\mathcal{A}$ is dense in $C_\infty(X)$ for $\|\cdot\|_\infty$.
Proof: Let $\tilde{X}$ be the one-point compactification of X with x_∞ being the added point. Then $\mathcal{A} \subseteq C(X) \subseteq C(\tilde{X})$. Let $\tilde{\mathcal{A}}$ be $\mathcal{A}$ with $\mathbf{1} \in C(\tilde{X})$. Then $\tilde{\mathcal{A}}$ separates the points of $\tilde{X}$, and $\tilde{\mathcal{A}}$ contains $\mathbf{1}$, so by the Stone-Weierstrass Theorem, $\tilde{\mathcal{A}}$ is dense in $C(\tilde{X})$ for $\|\cdot\|_\infty$. Thus, given $f \in C_\infty(X)$ and $\epsilon > 0$, there is $h \in \tilde{\mathcal{A}}$ such that $\|f - h\|_\infty < \epsilon/2$. Then $|h(x_\infty)| = |h(x_\infty) - \underbrace{f(x_\infty)}_{=0}| < \epsilon/2$, so let $k = h - h(x_\infty)$, so $k(x_\infty) = 0$, i.e. $k \in \mathcal{A}$. ■

Chapter 6

Positive Radon Measures

Here we discuss an important link between functional analysis and measure theory, particularly, the link between positive linear functionals and a class of measures called the positive Radon measures. We generalize the regularity of Borel measures on $\mathbb{R}$ to a notion of regularity on locally compact Hausdorff spaces. Thus, throughout this chapter, we will let X denote a locally compacy Hausdorff space.
We will denote $C_c(X)$ to be the continuous functions on X with compact support.

6.1 Properties of Locally Compact Hausdorff Spaces

Proposition: Let $C \subseteq X$ be compact. Then there is an open set V with $C \subseteq V$ such that $\overline{V}$ is compact.
Proof: For each $x \in C$, there is an open set V_x where $\overline{V_x}$ is compact. There is a finite subcover of C: $V_{x_1}, ...V_{x_n}$. Then $C \subseteq \bigcup_{j=1}^{n} V_{x_j}$, and a finite union of compact sets is compact. It follows that $\overline{\bigcup_{j=1}^{n} V_{x_j}}$ is compact. ■

Proposition: Let $C \subseteq X$ be compact, U open, $C \subseteq U$. Then there is an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.
Proof: By the above proposition, it suffices to consider U with compact closure. Let $\tilde{X}$ be the one-point compactification. Then U^c in $\tilde{X}$ is closed and is disjoint from C. By Urysohn's Lemma, there is a continuous function $h : \tilde{X} \to [0, 1]$ with $h(U^c) = 0$, and $h(C) = 1$. Then the support of h is contained in $\overline{U}$, so $h \in C_c(X)$.
Set $V := \{x : \ h(x) > 1/2\} = h^{-1}((1/2, \infty))$, so V is open, and clearly $C \subseteq V \subseteq \overline{V} \subseteq U$. ■

Proposition: Let U be open, C compact, $C \subseteq U$. Then there is $f \in C_c(X)$ with $0 \leq f \leq 1$, where $f(C) = 1$, and $\text{supp}(f) \subseteq U$.
Proof: Find V open, $C \subseteq V \subseteq \overline{V} \subseteq U$. V^c is closed, so there is a continuous function $f : X \to [0, 1]$ such that $f(C) = 1$, $f(V^c) = 0$. Since $\text{supp}(f) \subseteq \overline{V}$, $f \in C_c(X)$. ■

Partition of Unity: Let $C \subseteq X$ be compact, and let $U_1, ..., U_n$ be open, with $C \subseteq U_1 \cup \cdots \cup U_n$. For each $1 \le j \le n$, there is $f_j \in C_c(X)$, $0 \le f_j \le 1$ where $\operatorname{supp}(f_j) \subseteq U_j$, and $(\sum_{j=1}^n f_j)(C) = 1$.
Proof: We will use the lemma below to obtain compact sets $D_j \subseteq U_j$, with $C \subseteq \bigcup_{j=1}^n D_j$. Choose g_j with $\operatorname{supp}(g_j) \subseteq U_j$, $g_j \ge 1$, on D_j. Let $h = \sum_{j=1}^n g_j$, so $f_j = g_j/k$.
Then $f_j \ge 0$, and

$$f_j \le \frac{g_j}{\sum_{j=1}^n g_j \vee \mathbf{1}} \le \frac{g_j}{\sum_{j=1}^n g_j} \le 1,$$

since $g_j \ge 0$. Since $k > 0$ always, $\operatorname{supp}(f_j) = \operatorname{supp}(g_j) \subseteq U_j$. Finally, for $x \in C$,

$$\sum_{j=1}^n f_j(x) = \frac{1}{k(x)} \sum_{j=1}^n g_j(x) = 1, \implies k(x) = \sum_{j=1}^n g_j(x)$$

since $h(x) \ge 1$ for $x \in C$. ■

Lemma: Let $C \subseteq X$ be compact, with $U_1, ..., U_n$ open, and $C \subseteq \bigcup_{j=1}^n U_j$. Then there exist compact closed sets $D_1, ..., D_n$ where $D_j \subseteq U_j$, and $C \subseteq \bigcup_{j=1}^n D_j$.
Proof: For each $x \in C$, there is $1 \le j \le n$ such that $x \in U_j$. We can choose an open V_x such that $x \in V_x \subseteq \overline{V_x} \subseteq U_j$. There must be a finite subcover $V_{x_1}, ..., V_{x_p}$. For $1 \le k \le p$, choose j_k, $1 \le j_k \le n$ such that $V_k \subseteq U_{j_k}$. Let $W_j = \bigcup_{j_k = j} V_k \subseteq U_j$. Then let $D_j = \overline{W_j} = \bigcup_{j_k = j} \overline{V_k}$. Then the D_j's satisfy the given conditions. ■

Proposition: Let $U \subseteq X$ be open, and consider $C_c(U)$. Then $\varphi\Big|_{C_c(U)}$ is continuous for $\|\cdot\|_\infty$.
Proof: Choose $h \in C_c(X)$ with $0 \le h \le 1$, and $h = 1$ on $\overline{U}$. Then for any $f \in C_c(U)$, $\|f\|_\infty h \ge f \ge -\|f\|_\infty h$. Therefore $\|f\|_\infty \varphi(h) \ge \varphi(f) \ge -\|f\|_\infty \varphi(h)$, so that $|\varphi(f)| < \varphi(h) \|f\|_\infty$, i.e. $\|\varphi\|_{C_c(U)} \le \varphi(h)$. ■

For each $U \subseteq X$ with U compact, consider the inclusion $C_\infty \supseteq C_c(U) \to C_0(X)$. (Functions of compact support must vanish at infinity.) Then we can put on $C_0(X)$ the strongest topology making all the inclusions continuous: this is called the **inductive limit topology**.

6.2 Positive Radon Measures

Given a positive linear functional φ on $C_c(X)$, define a function μ_φ on the collection of open subsets of X by

$$\mu_\varphi(U) = \sup\{\varphi(f) : \ f \in C_c(X),\ 0 \le f \le 1,\ \operatorname{supp}(f) \subseteq U\}, \text{ and } \mu_\varphi(\varnothing) = 0.$$

Properties of μ_φ

1. If $\overline{U}$ is compact, then $\mu_\varphi(U) < \infty$.
   Proof: Follows from the continuity of $\varphi\big|_{C_c(U)}$ for $\|\cdot\|_\infty$. ■

2. Monotonicity: If $U \subseteq V$, then $\mu_\varphi(U) \leq \mu_\varphi(V)$.
   Proof: Follows easily from the definition of μ_φ. ■

3. μ_φ is countably subadditive.
   Proof: Let $f \in C_c(X)$ with $0 \leq f \leq 1$, $\operatorname{supp}(f) \subseteq U = \bigcup_{j=1}^\infty U_j$.
   Notice that $\{U_j\}$ covers $\operatorname{supp}(f)$, so there is a finite subcover, and thus without loss of generality, suppose $\operatorname{supp}(f) \subseteq \bigcup_{j=1}^n U_j$. Then there is a partition of unity $\{g_j\}_{j=1}^n$, $0 \leq j \leq 1$, where $\operatorname{supp}(f_j) \subseteq U_j$, and $(\sum g_j)(\operatorname{supp}(f)) = 1$.
   Let $f_j = fg_j$, so $\operatorname{supp}(f_j) \subseteq U_j$, and $\sum_{j=1}^n f_j = f$ $(0 \leq f_j \leq 1)$. Then

   $$\varphi(f) = \varphi(\sum f_j) = \sum_{j=1}^n \varphi(f_j) \leq \sum_{j=1}^n \mu_\varphi(U_j) \leq \sum_{j=1}^\infty \mu_\varphi(U_j).$$

   By taking the supremum over all $f \in C_c(X)$ with $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subseteq U$, we get $\mu_\varphi(U) \leq \sum_{j=1}^\infty \mu_\varphi(U_j)$. ■

4. If U, V are disjoint, then $\mu_\varphi(U \cup V) = \mu_\varphi(U) + \mu_\varphi(V)$.
   Proof: From (3), we have $\mu_\varphi(U \cup V) \leq \mu_\varphi(U) + \mu_\varphi(V)$.
   Conversely, let $f \in C_c(X)$ have $0 \leq f \leq 1$, $\operatorname{supp}(f) \subseteq U$, and $g \in C_c(X)$ have $0 \leq g \leq 1$, $\operatorname{supp}(g) \subseteq V$. Then $\operatorname{supp}(f+g) \subseteq U \cup V$, and observe that since U, V are disjoint, we have $0 \leq f+g \leq 1$. Then $\mu_\varphi(U \cup V) \geq \varphi(f+g) = \varphi(f) + \varphi(g)$. By taking the supremum over all such f, g, we get $\mu_\varphi(U \cup V) \geq \mu_\varphi(U) + \mu_\varphi(V)$. ■

5. For any U, we have $\mu_\varphi(U) = \sup\{\mu_\varphi(V) : \overline{V} \subseteq U,\ \overline{V} \text{ compact}\}$.
   Proof: Given U and $0 \leq f \leq 1$, $\operatorname{supp}(f) \subseteq U$, there is $V \subseteq U$ with $\overline{V}$ compact, $\operatorname{supp}(f) \subseteq V$, since U is open. Then $\mu_\varphi(V) \geq \varphi(f)$. It follows that $\mu_\varphi(V) \geq \mu_\varphi(U)$. By monotonicity, $\mu_\varphi(V) \leq \mu_\varphi(U)$. ■

Def: A function on open sets to $[0, \infty]$ satisfying properties (1)-(4) is called a **content**.

Proposition: For a content μ, μ^* defined by $\mu^* = \inf\{\mu(U) : \ U \text{ open},\ A \subseteq U\}$ is an outer measure.
Proof: It suffices to show countable subadditivity. Let $A = \bigcup_{j=1}^\infty A_j$. Let $\epsilon > 0$. For each j, choose U_j such that $A_j \subseteq U_j$, and $\mu(U_j) \leq \mu^*(A_j) + \epsilon/2^j$. Then

$$\mu^*(A) \leq \mu\Big(\bigcup_{j=1}^\infty U_j\Big) \leq \sum_{j=1}^\infty \mu(U_j) \leq \sum_{j=1}^\infty \Big(\mu^*(A_j) + \epsilon/2^j\Big) = \sum_{j=1}^\infty \mu^*(A_j) + \epsilon.$$

■

Def: Let μ be a measure or outer measure on $(X, \mathcal{S})$. Then μ is **outer regular** if for every $E \in \mathcal{S}$, we have $\mu(E) = \inf\{\mu(U) : \ U \text{ open}, \ E \subseteq U\}$. μ is **inner regular** if $\mu(E) = \sup\{\mu(C) : \ C \subseteq E, \ C \text{ compact}\}$.

Def: μ is **inner regular on open sets** if for all open U,

$$\mu(U) = \sup\{\mu(V) : \ V \text{ open}, \ \overline{V} \text{ compact}, \ \overline{V} \subseteq U\} \ \text{(i.e., property (5) for } \mu_\varphi)$$

6.3 The Riesz-Markov Theorem

Given a positive linear functional on $C_c(X)$, we call the measure μ_φ defined in the previous section a **positive Radon measure**. We will see shortly that positive Radon measures correspond precisely to positive linear functionals.
In this section, we will prove the Riesz-Markov Theorem: If φ is a positive linear functional on $C_c(X)$, then for any $f \in C_c(X)$, we have $\varphi(f) = \int f \, d\mu_\varphi$.

Lemma 1: If $f \in C_c(X)$, and $f \geq \chi_A$ for some $A \subseteq X$, then $\varphi(f) \geq \mu_\varphi^*(A)$.
Proof: First, we will just consider the case that A is open. If $f \geq \chi_A$, then for any g with $\text{supp}(f) \subseteq A$ and $g \leq \chi_A$, and $f \geq g$, then $\varphi(f) \geq \varphi(g)$, but

$$\mu_\varphi(A) = \sup\{\varphi(g) : \ g \in C_c(X), \ 0 \leq g \leq 1, \ \text{supp}(g) \subseteq A\},$$

so $\varphi(f) \geq \mu_\varphi(A)$.
Now, let's consider any A. We have $\mu_\varphi^*(A) = \inf\{\mu_\varphi(A) : \ U \text{ open}, \ A \subseteq U\}$. Let $\epsilon > 0$. Let $U = \{x : \ f(x) > 1 - \epsilon\}$, so U is open and $A \subseteq U$. Then

$$\frac{1}{1-\epsilon} \cdot f \geq \chi_U, \quad \text{so} \quad \frac{1}{1-\epsilon} \cdot \varphi(f) \geq \mu_\varphi^*(U) \geq \mu_\varphi^*(A).$$

Letting $\epsilon \to 0$ gives the desired result. ■

Lemma 2: If $A \subseteq X$, and $\chi_A \geq f \in C_c(X)$, then $\mu_\varphi^*(A) \geq \varphi(f)$.
Proof: Consider A open. $\chi_A \geq f$. Let $f_n = (f - 1/n) \vee 0$, so $\text{supp}(f_n) = \{x : \ f(x) \geq 1/n\}$, which is closed and contained in A. Then $\varphi(f_n) \leq \mu_\varphi(A)$. But $f_n \uparrow f$ uniformly, and $\text{supp}(f_n) \subseteq \text{supp}(f) \subseteq V$, $\overline{V}$ compact, V open (for inductive limit topology). Therefore $\varphi(f_n) \uparrow \varphi(f) \implies \varphi(f) \leq \mu_\varphi(f)$.
Now for any A, proceed as in lemma 1 above to extend to this case. ■

Riesz-Markov Theorem: Let φ be a positive Radon measure on X. Define the measure μ_φ as in the previous section. Then for any $f \in C_c(X)$, we have $\varphi(f) = \int f\, d\mu_\varphi$.
Proof: It suffices to consider $f \geq 0$ and $\|f\|_\infty \leq 1$. Let $\epsilon > 0$, and choose N such that $N\epsilon \geq 1$.
For $0 \leq n \leq N$, let $f_n = f \wedge n\epsilon$, $f_0 = 0$, $f_N = f$. Note that if $n \leq m$, then $f_n \leq f_m$. For $0 \leq n \leq N-1$, let $g_n = f_{n+1} - f_n$, so that $0 \leq g_n \leq \epsilon$, and $\sum_{n=0}^{N-1} g_n = f$.
Let $K_0 = \text{supp}(f)$, $K_n = \{x : \ f(x) \geq n\epsilon\}$.

Then $g_n(x) = \begin{cases} 0, \text{ if } x \notin K_n, \\ \epsilon, x \in K_{n+1}, \\ \text{b/w } 0 \text{ and } \epsilon, \text{ if } x \in K_n \setminus K_{n+1} \end{cases}$

Then $\epsilon\chi_{K_n} \geq g_n \geq \epsilon\chi_{K_{n+1}}$, so by the lemmas, $\epsilon\mu_\varphi(K_n) \geq \varphi(g_n) \geq \epsilon\mu_\varphi(K_{n+1})$.

Then $\displaystyle\int \epsilon\chi_{K_{n+1}}\, d\mu \leq \int g_n\, d\mu \leq \int \epsilon\chi_{K_n}\, d\mu$.

$$\implies \left|\varphi(g_n) - \int g_n\, d\mu_\varphi\right| \leq \epsilon\Big(\mu_\varphi(K_n) - \mu_\varphi(K_{n+1})\Big) = \epsilon\mu_\varphi(K_n \setminus K_{n+1}),$$

so that

$$\left|\varphi(f) - \int f\, d\mu_\varphi\right| = \left|\varphi\Big(\sum_{n=0}^{N-1} g_n\Big) - \int \sum_{n=0}^{N-1} g_n\, d\mu_\varphi\right| \leq \sum_{n=0}^{N-1} \left|\varphi(g_n) - \int g_n\, d\mu_\varphi\right| \leq$$

$$\leq \sum_{n=0}^{N-1} \epsilon\mu_\varphi(K_n \setminus K_{n+1}) = \epsilon\mu_\varphi\Big(\bigcup_{n=0}^{N-1} (K_n \setminus K_{n+1})\Big) = \epsilon\mu_\varphi(K_0) = \epsilon \cdot \text{supp}(f).$$

Since $\epsilon > 0$ was arbitrary, it follows that that $\varphi(f) = \int f\, d\mu_\varphi$. ■

In fact, the measure μ_φ above is essentially unique. This will establish that $C_c(X)'$ is the space of all positive Borel measures which are outer regular and inner regular on open sets.

Theorem: If ν is a positive Borel measure on X that is outer regular and inner regular on open sets, such that $\varphi(f) = \int f\, d\nu$ for all $f \in C_c(X)$, where φ is a positive linear functional, then $\nu = \mu_\varphi$.
Proof: If K is compact, U is open, and $K \subseteq U$, then the there exists $f \in C_c(X)$, with $0 \leq f \leq 1$, $\text{supp}(f) \subseteq U$, $\chi_K \leq f \leq \chi_U$.
Then $\nu(K) \leq \int f\, d\nu \leq \nu(U)$. Since ν is inner regular on open sets, we have
$\nu(U) = \sup\Big\{ \int f\, d\nu : \ f \leq \chi_U, \ \text{supp}(f) \subseteq U\Big\} = \mu_\varphi(U)$, since $\varphi(f) = \int f\, d\nu$. ■

Consider X compact. We know that $C(X)'$ is a lattice Banach space, so any $\varphi \in C(X)'$ has $\varphi = \varphi^+ - \varphi^-$, and $\varphi^+ \wedge \varphi^- = 0$.
From the Riesz-Markov Theorem, we get regular measures μ^+, μ^- such that

$$\varphi^+(f) = \int f\, d\mu^+ \text{ and } \varphi^-(f) = \int f\, d\mu^-$$

This implies that $\varphi^+(1) = \int d\mu^+ = \mu^+(X) < \infty$, and $\varphi^-(1) = \int d\mu^- = \mu^-(X) < \infty$.
Let $\mu = \mu^+ - \mu^-$, which is a well-defined measure. Furthermore, note that $\varphi^+ - \varphi^- = 0$ implies that μ^+, μ^- are mutually singular.

To see this, consider the Lebesgue decomposition of μ^- with respect to μ^+:

$$\mu^- = \mu_{\mathrm{s}}^- + \mu_{\mathrm{ac}}^-, \text{ and } \mu_{\mathrm{ac}}^-(E) = \int_E h \, d\mu^+, \text{ if } h = 0.$$

If $h \neq 0$, let $k = h \wedge 1 \neq 0$, and let $\nu(E) = \int_E k \, d\mu^+$, $\varphi_\nu \leq \varphi$, so $\varphi_\nu \leq \varphi^+ \wedge \varphi^- = 0$, because $k \leq h$, $\varphi_\nu \leq \varphi_{\mathrm{ac}}^- \leq \varphi^-$.

Consider X locally compact but not compact, and consider $C_0(X)$, the set of continuous functions that vanish at infinity. Then $C_0(X)$ is a Banach algebra.
If $\varphi \in C_0(X)'$, and if $\varphi \geq 0$, from $\tilde{X}$ the one-point compactification of X, if $g \in C(\tilde{X})$, $g = f + r$ for $f \in C_0(X)$ and $r \in \mathbb{R}$.
We can extend φ to $\tilde{\varphi}$ on $\tilde{X}$ by $\tilde{\varphi}(f + r) = \varphi(f) + r \left\|\varphi\right\|$, and we can check that $\tilde{\varphi} \geq 0$. So there is a regular measure $\mu_{\tilde{\varphi}}$ which is positive and finite on $\tilde{X}$ with $\tilde{\varphi}(f+r) = \int (f+r) \, d\mu_{\tilde{\varphi}}$ so in particular $\tilde{\varphi}(f) = \int f \, d\mu_{\tilde{\varphi}}$ and $\mu\Big|_X$ gives φ.

Chapter 7

Product Measures

We now take some time to discuss construction of product measures. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be measure spaces. Obviously, one can theoretically define any measure on $X \times Y$. However, throughout this chapter we will develop a natural measure $\mu \otimes \nu$ on the product space that reflects the measures on X and Y. We rely on a simple and intutive case: given a rectangle with side lengths ℓ and w, we know that its measure (area) is the product ℓw.

We will also discuss important theorems (Fubini and Tonelli) on integrals over the product of measure spaces, and indeed under certain conditions, the "product" integral is equal to the iterated integrals.

7.1 Construction of the Product Measure

For $E \in \mathcal{S}$, $F \in \mathcal{T}$, we say that $E \times F$ is a **measurable rectangle**. Let $\mathcal{R}$ be the σ-algebra generated by the measurable rectangles, and let $\mathcal{P}$ be the semi-ring of measurable rectangles (technically, for this we should consider half-open rectangles).

Proposition: Define the function $\mu \otimes \nu$ on $\mathcal{P}$ by $\mu \otimes \nu := \mu(E)\ \nu(F)$. Then $\mu \otimes \nu$ is a premeasure on $\mathcal{P}$.
Proof: It suffices to show countable additivity.
Suppose $E \times F = \bigotimes_{n=1}^{\infty}(E_n \times F_n)$. Then $\chi_E(x)\ \chi_F(y) = \chi_{E\times F}(x, y) = \sum_{n=1}^{\infty} \chi_{E_n}(x)\chi_{F_n}(y)$.
Fix $x \in X$. Then

$$\sum_{n=1}^{p} \chi_{E_n}(x)\chi_{F_n}(y) \uparrow_p \sum_{n=1}^{\infty} \chi_{E_n}(x)\chi_{F_n}(y) = \chi_E(x)\chi_F(y),$$

so by the Monotone Convergence Theorem, by integrating over y, we get
$\chi_E(x)\nu(F) = \displaystyle\sum_{n=1}^{\infty} \chi_{E_n}(x)\nu(F_n)$. Now by the Monotone Convergence Theorem again and integrating over x, we get the desired result. ■

By Caratheodory's Theorem, we can simply restrict the outer measure $(\mu \otimes \nu)^*$ to the measurable sets to obtain the product measure on the σ-algebra $\mathcal{M}((\mu \otimes \nu)^*)$, which turns out to be the completion of $\mathcal{S} \otimes \mathcal{T}$.

7.2 Iterated Integrals

If $E \in \mathcal{S}$, and $F \in \mathcal{T}$, it is clear that

$$\int \chi_G \, d(\mu \otimes \nu) = (\mu \otimes \nu)(G) = \mu(E)\nu(F) = \mu(E) \int \chi_F(y) \, d\nu(y) =$$

$$= \int \mu(E)\chi_F(y) \, d\nu(y) = \int \Big(\int \chi_E(x) \, d\mu(x) \Big) \chi_F(y) \, d\nu(y) = \int \Big(\int \chi_G(x,y) \, d\mu(x) \Big) d\nu(y).$$

Our goal in this section is Fubini's Theorem. To prove this, it helps to discuss monotone classes.

<u>Def</u>: A collection $\mathcal{M}$ of subsets of X is a **<u>monotone class</u>** if whenever $A_n \in \mathcal{M}$ and $A_n \uparrow A$, then $A \in \mathcal{M}$, and also if $A_n \downarrow A$, then $A \in \mathcal{M}$.

Given a collection $\mathcal{C}$ of subsets of X, we let $\mathcal{M}(\mathcal{C})$ denote the minimal monotone class containing $\mathcal{C}$, and we let $\sigma(\mathcal{C})$ denote the minimal such σ-ring.

<u>Monotone Class Theorem</u>: If $\mathcal{R}$ is a ring, then $\mathcal{M}(\mathcal{R}) = \sigma(\mathcal{R})$.
<u>Proof</u>: Clearly $\sigma(\mathcal{R}) \subseteq \mathcal{M}(\mathcal{R})$. We must show that $\mathcal{M}(\mathcal{R}) \subseteq \sigma(\mathcal{R})$, and to do this it suffices to show that $\mathcal{M}(\mathcal{R})$ is a ring. Indeed, assuming $\mathcal{R}$ is a ring, if $E = \bigcup_{n=1}^{\infty} E_n$, then take $F_n = \bigcup_{k=1}^{n} E_k$ so that $F_n \uparrow E$.
Let $E \in \mathcal{M}(\mathcal{R})$, and let $L(E) = \{F \in \mathcal{M}(\mathcal{R}) : \ E \setminus F, \ F \setminus E, \ E \cup F \in \mathcal{M}(\mathcal{R})\}$.
We show that $L(E)$ is a monotone class:

- If $F_n \in L(E)$ and $F_n \uparrow F$, then $E \setminus F_n \downarrow E \setminus F \in \mathcal{M}(\mathcal{R})$, and $F_n \setminus E \uparrow F \setminus E \in \mathcal{M}(\mathcal{R})$, and $F_n \cup E \uparrow F \cup E \in \mathcal{M}(\mathcal{R})$, so that $F \in L(E)$. Similarly, if $F_n \downarrow F$, then $F \in L(E)$. Thus $L(E)$ is a monotone class.

If $A \in \mathcal{R}$, then $L(A)$ is a monotone class, and $\mathcal{R} \subseteq L(A)$, so $\mathcal{M}(\mathcal{R}) \subseteq L(A)$. Since $L(A) \subseteq \mathcal{M}(\mathcal{R})$, we therefore have $L(A) = \mathcal{M}(\mathcal{R})$. But for every $E, F \in \mathcal{M}(\mathcal{R})$, observe that $E \in L(F)$ iff $F \in L(E)$. Since $L(A) = \mathcal{M}(\mathcal{R})$, every $E \in \mathcal{M}(\mathcal{R})$ is in $L(A)$, and so $\mathcal{R} \subseteq L(E)$, so also $L(E) = \mathcal{M}(\mathcal{R})$, so $\mathcal{M}(\mathcal{R})$ is a ring. ■

For any $G \subset X \times Y$, let $G_x := \{y \in Y : \ (x,y) \in G\}$ and $G^y := \{x \in X : \ (x,y) \in G\}$.

<u>Theorem</u>: Let μ and ν be σ-finite measures. For $G \in \mathcal{S} \otimes \mathcal{T}$, $x \in X$, $y \in Y$, we have $G_x \in \mathcal{T}$ and $G^y \in \mathcal{S}$. The maps $x \mapsto \nu(G_x)$ and $y \mapsto \mu(G^y)$ are $\mathcal{S}$- and $\mathcal{T}$-measurable, respectively, and

$$(\mu \otimes \nu)(G) = \int \nu(G_x) \, d\mu(x) = \int \mu(G^y) \, d\nu(y).$$

<u>Proof</u>: Note that the results hold for G in the ring generated by $\{E \times F : \ E \in \mathcal{S}, \ F \in \mathcal{T}\}$. Let $\mathcal{C}$ denote the set of all subsets for which the theorem holds. Then $\mathcal{C}$ is nonempty, and

we will show that $\mathcal{C}$ is a monotone class.
If $G_n \in \mathcal{C}$, and $G_n \uparrow G$, then for $x \in X$, $y \in Y$, we have $(G_n)_x \in \mathcal{T}$, $(G_n)^y \in \mathcal{S}$, $(G_n)_x \uparrow G_x$, and $(G_n)^y \uparrow G^y$.
Then $\nu((G_n)_x) \uparrow \nu(G_x)$, so $x \mapsto \nu(G_x)$ is $\mathcal{S}$-measurable. Similarly, $y \mapsto \mu(G^y)$ is $\mathcal{T}$-measurable. By the Monotone Convergence Theorem,

$$\int \nu((G_n)_x)\, d\mu(x) \uparrow \int \nu(G_x)\, d\mu(x).$$

We also have

$$(\mu \otimes \nu)(G_n) = \int \nu((G_n)_x)\, d\mu(x) = \int \mu((G_n)^y)\, d\nu(y)$$

so that $(\mu\otimes\nu)(G_n) \uparrow (\mu\otimes\nu)(G)$. To show the decreasing case, we need σ-finiteness. Assume $\mu(X) < \infty$, $\nu(Y) < \infty$, $G_n \downarrow G$ with $G_n \in \mathcal{C}$. Then $(G_n)_x \downarrow G_x$ so $G_x \in \mathcal{T}$ and $(G_n)^y \downarrow G^y$, so $G^y \in \mathcal{S}$. As done above, we get that $x \mapsto \nu(G_x)$ is $\mathcal{S}$-measurable, and $y \mapsto \mu(G^y)$ is $\mathcal{T}$-measurable.
Since $\mu(X) < \infty$, we have

$$\int \nu((G_n)_x)\, d\mu(x) \downarrow \int \nu(G_x)\, d\mu(x) \implies (\mu \otimes \nu)(G_n) \downarrow (\mu \otimes \nu)(G).$$

In general, $X = \bigoplus_{n=1}^{\infty} X_n$ and $Y = \bigoplus_{n=1}^{\infty} Y_n$ where $\mu(X_n) < \infty$, $\nu(Y_n) < \infty$. By applying the result to each X_n and Y_n, and applying additivity, we get the desired result. ∎

Proposition: Let f be an integrable simple function on $X \times Y$. Then

$$\int_{X\times Y} f\, d(\mu \otimes \nu) = \int \Big(\int f(x,y)\, d\mu(x)\Big)\, d\nu(y) = \int \Big(\int f(x,y)\, d\nu(y)\Big) d\mu(x).$$

We need to know when $f \in \mathcal{L}^1$. It suffices to have f measurable and the map $(x,y) \mapsto \|f(x,y)\|$ is $(\mu \otimes \nu)$-integrable.

Proposition: Let μ, ν be σ-finite. Let f on $X \times Y$ be $\mathcal{S} \otimes \mathcal{T}$-measurable and $f \geq 0$. Then

$$\int f\, d(\mu \otimes \nu) = \iint f(x,y)\, d\mu(x)\, d\nu(y) = \iint f(x,y)\, d\nu(y)\, d\mu(x).$$

Proof: Let $\{f_n\}$ be an increasing sequence of integrable simple functions, with $f_n \uparrow f$ pointwise. By the Monotone Convergence Theorem, since $(f_n)^y \uparrow f^y$, we have

$$\int (f_n)^y\, d\mu \uparrow \int f^y\, d\mu \implies \int \Big(\int (f_n)^y\, d\mu\Big) d\nu \uparrow \int \Big(\int f^y\, d\mu\Big) d\nu.$$

But $\int \Big(\int (f_n)^y\, d\mu\Big) d\nu = \int f_n\, d(\mu \otimes \nu) \uparrow \int f\, d(\mu \otimes \nu)$. ∎

Corollary: If f^y is μ-integrable for a.e. y, and $f \geq 0$, and if $y \mapsto \int f^y \, d\mu$ is ν-integrable, then $f \in \mathcal{L}^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, \mathbb{R})$.

Tonelli's Theorem: Let f be a B-valued $\mathcal{S} \otimes \mathcal{T}$-measurable function. Let $g(x, y) = \|f(x, y)\|_B$. If g^y is μ-integrable for a.e. y, and if $y \mapsto \int g^y \, d\mu$ is ν-integrable, then $f \in \mathcal{L}^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, \mathbb{R})$.
Proof: By the corollary above, $g \in \mathcal{L}^1$, thus $f \in \mathcal{L}^1$. $\blacksquare$

Fubini's Theorem: If $f \in \mathcal{L}^1(X \times Y, \mathcal{S} \otimes \mathcal{T}, \mu \otimes \nu, B)$, then f^y is μ-integrable for a.e. y, and $y \mapsto \int f^y \, d\mu$ is ν-integrable, and

$$\int f \, d(\mu \otimes \nu) = \int \Big(\int f^y \, d\mu \Big) d\nu = \int \Big(\int f_x \, d\nu \Big) d\mu.$$

Proof: Let $g(x, y) = \|f(x, y)\|_B$, integrable. Let $\{f_n\}$ be integrable simple functions such that $f_n \to f$ pointwise. Then $\|f_n(x, y)\| \leq 2g(x, y)$.
By the Dominated Convergence Theorem,

$$\int f \, d(\mu \otimes \nu) = \lim_{n \to \infty} \int \underbrace{f_n}_{\text{ISF}} \, d(\mu \otimes \nu) = \lim_{n \to \infty} \int \Big(\int f_n^y \, d\mu \Big) d\nu.$$

Now $\displaystyle\int (f_n)^y \, d\mu$ for all y such that g^y is integrable is dominated by $\displaystyle\int g^y \, d\mu$. Therefore $\displaystyle\int f \, d(\mu \otimes \nu) = \int \Big(\int f^y \, d\mu \Big) d\nu$. $\blacksquare$

*Remark: The assumption that we are working with σ-finite measures is important. Here is an example where the iterated integrals and the integral with respect to the product measure are all not equal to each other:

Let $X = Y = [0, 1]$, with $\mathcal{S} = \mathcal{T} =$ the Borel σ-algebra on $[0, 1]$. Let μ be Lebesgue measure, but let ν be counting measure (hence ν is not σ-finite). Let $D = \{(x, x) : x \in [0, 1]\}$ (the diagonal of $X \times Y$). Then one can check that

- $\displaystyle\iint \chi_D \, d\mu \, d\nu = 0,$
- $\displaystyle\iint \chi_D \, d\nu \, d\mu = 1,$
- $\displaystyle\int \chi_D \, d(\mu \otimes \nu) = \infty.$

Chapter 8

Baire Spaces

In this chapter we introduce the notion of Baire spaces, which may seem to be a simple concept, but in fact has very interesting implications.

Def: Let $(X, \mathcal{T})$ be a topological space. If $\{B_n\}$ is any countable collection of open dense sets, and $\bigcap_{n=1}^{\infty} B_n$ is also dense, then $(X, \mathcal{T})$ is called a **Baire space**.

Theorem: If X is a complete metric space, then X is a Baire space.
Proof: We show that if W is a nonempty open set in X, then $W \cap \left(\bigcap_{n=1}^{\infty} U_n\right) \neq \varnothing$ whenever $\{U_n\}$ is a sequence of open dense sets in X. Now U_1 is open and dense, so $W \cap U_1$ is open and nonempty. Thus $W \cap U_1$ contains an open ball $B_{r_0}(x_0)$, where we take $0 < r_0 < 1$. For $n \geq 1$, having chosen $x_0, ..., x_{n-1}$, and $r_0, ..., r_{n-1}$, note that $B_{r_{n-1}}(x_{n-1}) \cap U_n$ is open and nonempty, so choose x_n, r_n such that $0 < r_n < 2^{-n}$, and $\overline{B_{r_n}(x_n)} \subseteq U_n \cap B_{r_{n-1}}(x_{n-1})$. Then if $n, m \geq N$, we have $x_n, x_m \in B_{r_N}(x_N)$, and since $r_n \to 0$ as $n \to \infty$, $\{x_n\}$ is Cauchy, so it converges to some $x \in X$. Since $x_n \in B_{r_N}(x_N)$ if $n \geq N$, we have
$x \in \overline{B_{r_N}}(x_N) \subseteq U_N \cap B_{r_1}(x_1) \subseteq U_N \cap W$ for all N. Therefore $x \in W \cap \left(\bigcap_{n=1}^{\infty} U_n\right)$. ■

Def: A **nowhere dense set** is a set whose closure has empty interior.

Proposition: If X is a Baire space, then it is not a countable union of nowhere dense sets.
Proof: Let $\{E_n\}$ be a sequence of nowhere dense sets in X. Then $\{\overline{E_n}^c\}$ is a sequence of open dense sets. Then $\bigcap_{n=1}^{\infty} \overline{E_n}^c$ is dense in X. In particular, $\bigcap_{n=1}^{\infty} \overline{E_n}^c \neq \varnothing$, so we have $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n} \neq X$. ■

*Remark: Since the conclusion of the theorem above is purely topological, it also holds for spaces homeomorphic to a Baire space.

Def: If $E \subseteq X$ is a counatable union of nowhere dense sets, then we say E is **meager**.

Ex: Continuous, nowhere differentiable functions exist.
Let $E_n = \{f \in C([0,1]) : \exists\, x_0 \in [0,1] \text{ such that } |f(x) - f(x_0)| \leq n|x - x_0| \ \forall x \in [0,1]\}$.