Math 202A: Topology and Analysis I Lecture Notes

Joseph Pagadora
Instructor: Marc Rieffel

Fall 2018

Contents

Chapter 1

Metric Spaces

1.1 Fundamentals

Def: Let X be a set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ that satisfies:

(a) $d(x, y) = d(y, x)$ for any $x, y \in X$

(b) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$

(c) $d(x, y) = 0$ iff $x = y$

If a function d satisfies (a), (b) above, and $d(x, x) = 0$ for all $x \in X$, then d is a **semi-metric**.

Ex: On $\mathbb{C}^n$, the following are common metrics:

- $d_p(v, w) = \left(\sum\limits_{j=1}^{n} |v_j - w_j|^p \right)^{1/p}$ for $p \geq 1$
- $d_\infty(v, w) = \sup\{|v_j - w_j| : 1 \leq j \leq n\}$

(Verify that these are metrics.)

Fact: If $S \subseteq X$, and d is a metric on X, then d is a metric on S.

Def: Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ such that:

(a) $\|cv\| = |c| \cdot \|v\|$ for $c \in \mathbb{R}$ or $\mathbb{C}$ and $v \in V$

(b) $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$

(c) $\|v\| = 0$ implies $v = 0$

A function that satisfies only (a) and (b) above is called a **seminorm**.

Remark: Any norm $\|\cdot\|$ on X induces the metric $d(x, y) := \|x - y\|$.

Ex: Let V be the space of continuous functions on $[0, 1]$. Then
$\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$ is a norm on V.
It can also be shown that $\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ is a norm on V.

Def: Let (X, d_x) and (Y, d_y) be metric spaces. A function $f : X \to Y$ is **isometric** if $d_y(f(v), f(w)) = d_x(v, w)$ for all $v, w \in X$.

- Note that all isometries are injective.

Ex: If $S \subseteq X$, and $f : S \to X$ is defined by $f(x) = x$ (inclusion), then f is an isometry.

If f is onto, then f is viewed as an isomorphism between (X, d_x) and (Y, d_y). f^{-1} is also an isomorphism.

Def: A function $f : X \to Y$ is **Lipschitz** if there is a constant $k \geq 0$ such that $d_y(f(x_1), f(x_2)) \leq k \cdot d_x(x_1, x_2)$. The smallest such constant is the **Lipschitz constant** for f.

Def: $f : X \to Y$ is **uniformly continuous** if for any $\epsilon > 0$, there exists $\delta > 0$ such that $d_y(f(x_1), f(x_2)) < \epsilon$ whenever $d_x(x_1, x_2) < \delta$.

- It is easy to see that if f is Lipschitz, then it is uniformly continuous.

Def: $f : X \to Y$ is **continuous at x_0** if for any $\epsilon > 0$, there exists $\delta > 0$ such that $d_y(f(x), f(x_0)) < \epsilon$ whenever $d_x(x, x_0) < \delta$. We say f is **continuous** if it is continuous at every $x \in X$.

Def: A sequence $\{x_n\}$ in X **converges** to $x^* \in X$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $d(x_n, x^*) < \epsilon$.

Proposition: A function $f : X \to Y$ is continuous iff $x_n \to x$ implies $f(x_n) \to f(x)$.

Proof: *Exercise.* ■

Def: $S \subseteq X$ is **dense** in X if for any $x \in X$ and $\epsilon > 0$, there exists $s \in S$ such that $d(x, s) < \epsilon$.

Proposition: Let S be dense in X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that $f(s) = g(s)$ for all $s \in S$. Then $f = g$ on X.

Proof: Let $x \in X \setminus S$, and let $\epsilon > 0$. Then there exists $\delta > 0$ and $s \in S$ such that $d(f(x), f(s)) < \epsilon/2$, and $d(g(x), g(s)) < \epsilon/2$ for $d(x, s) < \delta$, by continuity and density. Then

$$d(f(x), g(x)) \leq d(f(x), f(s)) + d(g(s), g(x)) < \epsilon/2 + \epsilon/2 = \epsilon,$$

since $f(s) = g(s)$. Thus $d(f(x), g(x)) = 0$, so $f(x) = g(x)$. ■

Def: A sequence $\{x_n\}$ is **Cauchy** if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$.
A metric space in which every Cauchy sequence converges is **complete**.

Ex: Consider $(\mathbb{Q}, |\cdot|)$. We know there exists a Cauchy sequence converging to $\sqrt{2} \in \mathbb{R}$, but in this metric space, $\sqrt{2}$ is not an element, so this sequence does not converge, hence this metric space is not complete.

1.2 Completion of a Metric Space

Proposition: If $f : X \to Y$ is uniformly continuous, and $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Proof: *Exercise.* ■

Def: Let (X, d) be a metric space. A complete metric space $(\overline{X}, \overline{d})$, together with an isometric function $f : X \to \overline{X}$ with dense range is a **completion** of (X, d).

- Remark: Completions are unique up to isomorphism.

Proposition: If $((Y_1, d_1), f_1)$ and $((Y_2, d_2), f_2)$ are completions of (X, d), then there exists an onto isometry (metric space isomorphism) $g : Y_1 \to Y_2$ with $f_2 = g \circ f_1$. This can be visualized by the following commutative diagram:

$$\begin{tikzcd} & Y_1 \arrow[dd, "g"] \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & Y_2 \end{tikzcd}$$

Every metric space has a completion, and the proof will be constructive. The completion will be defined using equivalence classes of Cauchy sequences. We will need the following lemmas to support the construction.

Lemma 1: If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in X, then the sequence $\{d(s_n, t_n)\}$ in $\mathbb{R}$ converges.
Proof: *Exercise. Hint:* $\{d(s_n, t_n)\}$ is a Cauchy sequence in a complete metric space. ■

Lemma 2: Let $\mathrm{Cau}(X)$ denote the set of all Cauchy sequences in X. Then the relation $\{s_n\} \sim \{t_n\}$ iff $d(s_n, t_n) \to 0$ is an equivalence relation.

Proof: Reflexivity and symmetry are trivial. Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$. Then $d(s_n, t_n) \leq d(s_n, r_n) + d(r_n, t_n)$ for all $n \in \mathbb{N}$. The result follows immediately. ■

Lemma 3: Let $\overline{X}$ be the set of all equivalence classes of $\mathrm{Cau}(X)$ under the equivalence relation above. Then $\overline{d} : \overline{X} \to [0, \infty)$ defined by $\overline{d}(\{s_n\}, \{t_n\}) := \lim_{n\to\infty} d(s_n, t_n)$ is a metric on $\overline{X}$.

Proof: First, note that by Lemma 1, $\overline{d}$ is always defined. Since we are dealing with equivalence classes, we must show that $\overline{d}$ is also well-defined. Let $\xi, \eta \in \overline{X}$, and let $\{x_n\}, \{s_n\} \in \xi$, and $\{y_n\}, \{t_n\} \in \eta$. We have $\lim d(x_n, s_n) = \lim d(y_n, t_n) = 0$. Thus, $d(s_n, t_n) \leq d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n)$. For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that both $d(s_n, x_n) < \epsilon/2$ and $d(y_n, t_n) < \epsilon/2$ for $n \geq N$. Then $|d(s_n, t_n) - d(x_n, y_n)| < \epsilon$. It follows that $\overline{d}(\xi, \eta) = \lim d(x_n, y_n) = \lim d(s_n, t_n)$, so that $\overline{d}$ is indeed well-defined.

Symmetry is trivial. The triangle inequality follows from the proof to Lemma 2. If $\overline{d}(\xi, \eta) = 0$, then for any $\{x_n\} \in \xi, \{y_n\} \in \eta$, we have $\lim d(x_n, y_n) = 0$, so in particular, $\{y_n\} \in \xi$, hence $\xi = \eta$. ■

Theorem: Let (X, d_x) and (Y, d_y) be metric spaces with Y complete. If $S \subseteq X$ is dense, and $f : S \to Y$ is uniformly continuous, then there exists a unique continuous extension $\overline{f} : X \to Y$ of f. In fact, $\overline{f}$ is uniformly continuous.

Proof: (Existence only) For $x \in X$, choose a Cauchy sequence $\{s_n\}$ in S converging to x. Then $\{f(s_n)\}$ is Cauchy in Y, so it converges to a point $p \in Y$. Set $\overline{f}(x) := p$. We show that $\overline{f}$ is well-defined. Indeed, if $\{t_n\} \in \mathrm{Cau}(S)$ and converges to x, then we have $\lim d_x(s_n, t_n) = 0$, implying that $\lim d_y(f(s_n), f(t_n)) = 0$. Therefore $\lim d_y(f(t_n), p) = 0$, so $\{f(t_n)\}$ converges to p also. It remains to show continuity, which is left as an exercise. ■

Theorem: Every metric space (X, d) has a completion.

Proof: As in Lemma 3, $(\overline{X}, \overline{d})$ is a completion of (X, d). We embed X in $\overline{X}$ by the isometry $\iota : X \to \overline{X}$ defined by $\iota(x) := [\{x, x, x, ...\}]$, where $[\cdot]$ denotes the corresponding equivalence class. Note that $\overline{d}\Big|_X = d$, i.e., $\overline{d}(\iota(x), \iota(y)) = d(x, y)$.
It remains to show that $\overline{d}$ has dense range, and that $(\overline{X}, \overline{d})$ is complete.

- Let $\xi \in \overline{X}, \epsilon > 0, \{x_n\} \in \xi$. There exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Then $\overline{d}(\iota(x_N), \xi) = \lim\limits_{n\to\infty} d(x_N, x_n) < \epsilon$. Therefore $\overline{d}$ has dense range by considering $\iota(x_N)$.
- Let $\{\xi_n\}$ be a Cauchy sequence in $\overline{X}$. For each $m \in \mathbb{N}$, pick $x_m \in X$ such that $\overline{d}(\iota(x_m), \xi_m) < 1/m$. Then $\{x_m\}$ is a Cauchy sequence, and it follows that $\{\xi_m\}$ converges to the equivalence class of $\{x_m\}$.

■

Remark on functions:
Denote $C([0,1])$ the space of continuous functions on $[0,1]$. Consider the metric space $C([0,1])$ induced by the norms $\|\cdot\|_\infty$ or $\|\cdot\|_p$. This space is not complete. It is easy to come up with a sequence of continuous functions converging under these norms to a function that is not continuous.

Vector Space Remark:
Let V be a vector space with norm $\|\cdot\|$. Consider V^∞, the space of all sequences of elements in V. This is also a vector space. It can be shown that $\mathrm{Cau}(V)$ is a sebspace of V^∞.
Now let $\mathcal{N}(V)$ denote the set of all Cauchy sequences in V converging to 0. Then $\mathcal{N}(V)$ is a subspace of $\mathrm{Cau}(V)$. If $\{v_n\}$ and $\{w_n\}$ are equivalent Cauchy sequences, then $\|v_n - w_m\| \to 0$, so $\{v_n - w_n\} \in \mathcal{N}(V)$. Thus $\overline{V}$ is in fact the quotient space $\mathrm{Cau}(V)/\mathcal{N}(V)$.

Fact: Any two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite dimensional vector space are **equivalent**, meaning that there are constants $c, C > 0$ such that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ for all x. If a function is continuous with respect to a particular norm, then it is easily seen that it is continuous with respect to any equivalent norm.

1.3 Openness

Def: If (X, d) is a metric space, the **open ball** of radius r around x is $B_r(x) := \{y \in X : d(x, y) < r\}$.

Def: We can rephrase continuity as $f : X \to Y$ is continuous at x_0 if for any $\epsilon > 0$, there exists $\delta > 0$ such that $f\Big(B_\delta(x_0)\Big) \subseteq B_\epsilon(f(x_0))$.

If $y \in B_\epsilon(f(x_0))$ and $y = f(x)$ for some $x \in X$, let $\epsilon' = \epsilon - d(y, f(x_0)) > 0$. Then $B_{\epsilon'}(y) \subseteq B_\epsilon(f(x_0))$, so there exists $\delta' > 0$ such that $f\Big(B_{\delta'}(x)\Big) \subseteq B_\epsilon(f(x_0))$.
If $x_1 \in f^{-1}\Big(B_\epsilon(f(x))\Big)$, there is an open ball $B_{\delta'}(x)$ such that $B_{\delta'}(x_1) \subseteq f^{-1}\Big(B_\epsilon(f(x))\Big)$. Thus $f^{-1}\Big(B_\epsilon(f(x))\Big)$ is a union of open balls in X. Similarly, $f^{-1}(B_\epsilon(y))$ is a union of open balls in X.

Def: A subset $\mathcal{O} \subseteq X$ (where X is a metric space) is **open** if it is a union of open balls in X.

By the arguments above, it can be shown that $f : X \to Y$ is continuous iff for any open ball $B_\epsilon(y) \subseteq Y$, we have that $f^{-1}(B_\epsilon(y))$ is open in X.

Set theoretic facts:
Let $f : X \to Y$ be a function, and let $\{A_\alpha\}$ be a family of subsets of X. Then:

- $f^{-1}\Big(\bigcup_\alpha A_\alpha\Big) = \bigcup_\alpha f^{-1}(A_\alpha)$
- $f^{-1}\Big(\bigcap_\alpha A_\alpha\Big) = \bigcap_\alpha f^{-1}(A_\alpha)$
- $f^{-1}\Big(A_1 \setminus A_2) = f^{-1}(A_1) \setminus f^{-1}(A_2)$
- $f\Big(\bigcup_\alpha A_\alpha\Big) = \bigcup_\alpha f(A_\alpha)$
- $f\Big(\bigcap_\alpha A_\alpha\Big) \subseteq \bigcap_\alpha f(A_\alpha)$
- $f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$

Let $\mathcal{O} \subseteq Y$ be open. Then $f^{-1}(\mathcal{O}) = f^{-1}\Big(\bigcup_{U \subseteq \mathcal{O}, U \text{ open ball}} U\Big) = \bigcup_{U \subseteq \mathcal{O}, U \text{ open ball}} f^{-1}(U)$.
Thus, if $f : X \to Y$ is continuous, and $\mathcal{O}$ is open in Y, then $f^{-1}(\mathcal{O})$ is open in X.

Theorem: If (X, d) is a metric space, and τ_d is the collection of all open sets, then:

(1) If $\{\mathcal{O}_\alpha\}$ is an arbitrary collection of subsets in τ_d, then $\bigcup_\alpha \mathcal{O}_\alpha$ is open.

(2) If $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is a finite collection of subsets in τ_d, then $\bigcap_{j=1}^{n} \mathcal{O}_j$ is open.

(3) $X \in \tau_d$ (X is open)

Proof:

(1) Trivial

(2) Let $x \in \bigcap_{j=1}^{n} \mathcal{O}_j$. For each j, there exists $\delta_j > 0$ such that $B_{\delta_j}(x) \subseteq \mathcal{O}_j$. Let $\delta = \min\{\delta_j : 1 \leq j \leq n\}$. Then $B_\delta(x) \subseteq \mathcal{O}_j$ for each j. This yields the desired result.

(3) Let $\epsilon > 0$. Then $X = \bigcup_{x \in X} B_\epsilon(x)$. The result follows from (1).

■

By convention, $\varnothing \in \tau_d$ ($\varnothing$ is open).

Chapter 2

Introduction to Topology

2.1 Fundamentals

Def: Let X be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ that satisfies:

(1) For any arbitrary family $\{\mathcal{O}_\alpha\} \subseteq \mathcal{T}$, we have $\bigcup\limits_{\alpha} \mathcal{O}_\alpha \in \mathcal{T}$.

(2) For $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{T}$, we have $\bigcap\limits_{j=1}^{n} \mathcal{O}_j \in \mathcal{T}$.

(3) $X \in \mathcal{T}, \varnothing \in \mathcal{T}$.

Def:

- If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is a **topological space**. The sets in $\mathcal{T}$ are called **open**, and the complements of the sets in $\mathcal{T}$ are **closed**.
- For any $A \subseteq X$, there is a smallest closed set containing A, namely, the intersection of all closed sets containing A (by DeMorgan's Laws, the arbitrary intersection of closed sets is closed). We denote such a set by the **closure** of A, denoted $\overline{A}$.
- If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is **weaker/coarser** than $\mathcal{T}_2$. $\mathcal{T}_2$ is **stronger/finer** than $\mathcal{T}_1$.

Proposition: For any sets A, B in a topological space, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof: *Exercise.* ■

Let $\mathcal{C}$ be a collection of topologies on X. It can be shown that $\bigcap\limits_{\mathcal{T} \in \mathcal{C}} \mathcal{T}$ is a topology on X. From this fact, it follows that for any non-empty collection $\mathcal{S}$ of subsets of X, there is a smallest topology on X that contains $\mathcal{S}$.

Def: Let $\mathcal{B} \subseteq \mathcal{T}$. We say that $\mathcal{B}$ is a **base** for $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of elements in $\mathcal{B}$.

Proposition: Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X. If $\mathcal{B}$ has the property that for any $U, V \in \mathcal{B}$, $U \cap V$ is a union of elements of $\mathcal{B}$, then the collection of unions of elements of $\mathcal{B}$, then the collection of unions of elements in $\mathcal{B}$ is a topology for which $\mathcal{B}$ is a base.

Proof: *Exercise.* ■

Def: Let $\mathcal{S}$ be any collection of subsets of X such that $\bigcup_{V \in \mathcal{S}} V = X$, and the set of finite intersections of elements of $\mathcal{S}$ is a base for a topology, then $\mathcal{S}$ is a **sub-base** for that topology.

Ex: The standard metric topology on $\mathbb{R}^n$ has the base $\{B_r(x) : r > 0,\ x \in \mathbb{R}^n\}$.
A sub-base for $\mathbb{R}$ with the metric topology is the collection of open rays: $(-\infty, a)$ and (b, ∞).
Ex: Given a metric space (X, d), we will always assume that it has a standard topology whose base consists of the open balls: $\{B_r(x) : r > 0,\ x \in X\}$.

2.2 Continuity

Def: Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces. A function $f : X \to Y$ is **continuous** for $\mathcal{T}_x$ and $\mathcal{T}_y$ if for any $U \in \mathcal{T}_y$, we have $f^{-1}(U) \in \mathcal{T}_x$.

- It is clear that any composition of continuous functions on topological spaces is continuous.
- For any topological space $(X, \mathcal{T})$, the identity map $\iota : X \to X$ is continuous.
- The collection of topological spaces with continuous functions between them is a category.

Def: $f : X \to Y$ is **homeomorphic** if f is continuous, bijective, and has a continuous inverse.

Proposition: $f : (X, \mathcal{T}_x) \to (Y, \mathcal{T}_y)$ is continuous iff for a base or sub-base $\mathcal{C}$ for $\mathcal{T}_y$, we have $f^{-1}(U) \in \mathcal{T}_x$ for $U \in \mathcal{C}$.

Proof: The forward direction is obvious. For the converse, we prove the case for $\mathcal{C}$ a base. The case for $\mathcal{C}$ a sub-base follows similarly (add on finite intersections).
Suppose $f^{-1}(U) \in \mathcal{T}_x$ for any $U \in \mathcal{C}$. Let $V \in \mathcal{T}_y$. Since $\mathcal{C}$ is a base, there is a collection of open sets $\mathcal{B} \subseteq \mathcal{C}$ such that $V = \bigcup_{A \in \mathcal{B}} A$. Then $f^{-1}(V) = f^{-1}\Big(\bigcup_{A \in \mathcal{B}} A\Big) = \bigcup_{A \in \mathcal{B}} f^{-1}(A) \in \mathcal{T}_x$, since each $f^{-1}(A)$ is open. ■

- Let X be a set and let $(Y_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces, and for each α, let $f_\alpha : X \to Y_\alpha$ be a function. Then there is a smallest topology on X for which each f_α is continuous, namely, the smallest topology having as sub-base all sets $f_\alpha^{-1}(U)$, where $U \in \mathcal{T}_\alpha$ for all α.

- Let $(X, \mathcal{T}_x)$ be a topological space, and let Y be a set with $f : X \to Y$ a function. What is the strongest topology on Y making f continuous?

  – We need that if $A \subseteq Y$ is open, then $f^{-1}(A) \in \mathcal{T}_x$.
  Let $\mathcal{T} := \{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_x\}$. Then $\mathcal{T}$ is easily seen to be a topology on Y. This is called the **quotient topology** on Y for f.

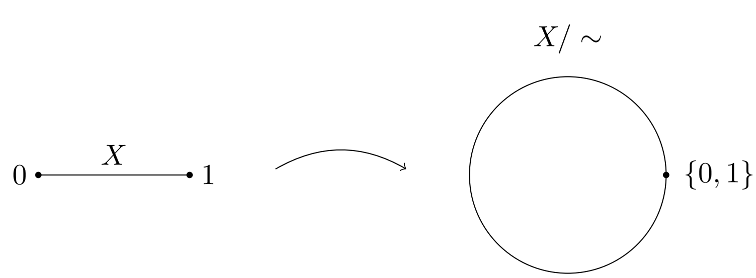
- Note that if $y \notin f(X)$, then $f^{-1}(\{y\}) = \varnothing$, so $\{y\}$ is open. Also, $f^{-1}(\{y\}^c) = X$, so $\{y\}$ is also closed. Therefore, on $f(X)^c$, the quotient topology is discrete. Thus, we usually require $f : X \to Y$ to be onto.

2.3 Quotient and Product Topologies

Let $f : X \to Y$ be onto, and define the equivalence relation on X by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Conversely, given any set X, and an equivalence relation on X, let Y be the set of equivalence classes for $\sim$. We often denote Y by $X/\sim$. Thus, given a topology on X, we get the **quotient topology** on $X/\sim$.

Given a collection $\{(X_\alpha, \mathcal{T}_\alpha)\}$ of topological spaces and a set Y, and for each α a function $f_\alpha : X_\alpha \to Y$, the strongest topology on Y making all f_α continuous is the intersection of all quotient topologies for each f_α. This is called the **final topology**.

Ex: Let $X = [0,1]$. Define the equivalence relation $s \sim t$ iff $s = t$, and have $0 \sim 1$. That is, $\{0,1\}$ is an equivalence class.



Define $f : X \to \{z \in \mathbb{C} : |z| = 1\}$ by $f(t) = e^{2\pi i t}$, for $t \in [0, 1]$.
Note that f is continuous but f^{-1} is not: there is a discontinuity at $1 \in \mathbb{C}$. However, the corresponding function $f : X/\sim \to \{z \in \mathbb{C} : |z| = 1\}$ is a homeomorphism with the usual topology from $\mathbb{C}$.

Ex: Let $X_1 = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$, the closed unit disk in $\mathbb{R}^2$. Let X_2 be a copy of X_1. Consider the disjoint union $X_1 \cup X_2$, and consider the following equivalence relation. Let x, y be interior points of the same disk. Then $x \sim y$ iff $x = y$. If x is a boundary point of X_1, and y is a boundary point of X_2, then $x \sim y$ iff they correspond to the same point in the plane. Similarly as in the previous example, one can visualize that $X_1 \cup X_2/\sim$ is homeomorphic to the 2-sphere.

Def: Let X be a set, and let $\{Y_\alpha\}$ be a family of topological spaces. For each α, let $f_\alpha : X \to Y_\alpha$ be a function. The corresponding **weak topology** on X is the smallest topology making each f_α continuous.
Fact: The weak topology has as sub-base all sets of the form $f_\alpha^{-1}(U)$, where U is open in Y_α.

Def: Given $(X, \mathcal{T})$, and a subset $Y \subseteq X$, the topology on Y induced by $\mathcal{T}$ is the **relative topology**, which has $\{Y \cap \mathcal{O} : \mathcal{O} \in \mathcal{T}\}$ as open sets.

Proposition: If $A \subseteq X$ is closed, and $C \subseteq A$ is closed in the relative topology of A, then C is closed in X.

Proof: $A \setminus C$ is open in A. There is an open set $\mathcal{O} \in \mathcal{T}$ such that $A \setminus C = A \cap \mathcal{O} \in \mathcal{T}$, so $C = A \cap \mathcal{O}^c$ which is closed in X. ■

Def: Let $\{(X_\alpha, \mathcal{T}_\alpha\}$ be a collection of topological spaces indexed by A. The **product** space is defined by $\prod\limits_{\alpha \in A} X_\alpha := \{h : A \to \bigcup\limits_{\alpha} X_\alpha \mid h(\alpha) \in X_\alpha, \ \forall \alpha\}$.
The **α-th projection map** is $\pi_\alpha : \prod\limits_{\beta \in A} X_\beta \to X_\alpha$, defined by $\pi_\alpha(h) = h(\alpha)$.

Def: The **product topology** on $\prod\limits_{\alpha \in A} X_\alpha$ is the weakest topology making all projections continuous. That is, it is the weak topology with respect to all the projection maps.

Fact: In general, the product topology will have as a base all sets of the form $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha \in \mathcal{T}_\alpha$, and also $U_\alpha = X_\alpha$ for all but finitely-many α.

Proposition: Consider $f_\alpha : X \to Y_\alpha$ for $\alpha \in A$. Let $\mathcal{T}_x$ be the corresponding weak topology on X. Let $(Z, \mathcal{T}_z)$ be a topological space, and let $g : Z \to X$. Then g is continuous iff $f_\alpha \circ g$ is continuous for all α.

Proof: Suppose $f_\alpha \circ g$ is continuous for all α. It suffices to check on the sub-base. Let $\mathcal{O} \in \mathcal{T}_\alpha$. Then $g^{-1}(f_\alpha^{-1}(\mathcal{O})) = (f_\alpha \circ g)^{-1}(\mathcal{O})$ is open, hence g is continuous.
Conversely, if g is continuous, then $(f_\alpha \circ g)^{-1}(\mathcal{O}) = g^{-1}(f_\alpha^{-1}(\mathcal{O}))$ is open since $f_\alpha^{-1}(\mathcal{O})$ is open in $\mathcal{T}_x$, thus $(f_\alpha \circ g)$ is continuous. ∎

2.4 Special Topological Spaces

Def:

- A topological space X is **Hausdorff** if for any two distinct points $x_1, x_2 \in X$, there are disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $x_1 \in \mathcal{O}_1$, and $x_2 \in \mathcal{O}_2$.
- X is **normal** if for any two disjoint closed sets C_1, C_2, there are disjoint open sets $\mathcal{O}_1, \mathcal{O}_2$ such that $C_1 \subseteq \mathcal{O}_1$, and $C_2 \subseteq \mathcal{O}_2$.
- A topological space is **metrizable** if its topology comes from a metric, i.e., its base consists of open balls from some metric.

Clearly, every metrizable space with more than one element is Hausdorff. Suppose the topology is induced by a metric d, and take two distinct points x, y. Let $r = d(x, y)$. Then the open balls $B_{r/3}(x)$ and $B_{r/3}(y)$ are disjoint. More is true:

Proposition: Every metrizable topological space is normal.

Proof: It suffices to consider a metric space (X, d). Let C_1, C_2 be disjoint closed subsets of X. For each $x \in C_1$ choose $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq C_2^c$, and for each $y \in C_2$, choose $\epsilon_y > 0$ such that $B_{\epsilon_y}(y) \subseteq C_1^c$.
Let $\mathcal{O}_1 = \bigcup_{x \in C_1} B_{\epsilon_x/3}(x)$, and $\mathcal{O}_2 = \bigcup_{y \in C_2} B_{\epsilon_y/3}(y)$. Clearly, $\mathcal{O}_1$ and $\mathcal{O}_2$ are open.
Since $C_1 \subseteq C_2^c$, and $C_2 \subseteq C_1^c$, we have $C_1 \subseteq \mathcal{O}_1$ and $C_2 \subseteq \mathcal{O}_2$. Toward contradiction, suppose $z \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then there are $x' \in C_1$ and $y' \in C_2$ such that $z \in B_{\epsilon_{x'}/3}(x')$ and $z \in B_{\epsilon_{y'}/3}(y')$. But then

$$d(x', y') \leq d(x', z) + d(z, y') < \frac{\epsilon_{x'}}{3} + \frac{\epsilon_{y'}}{3} \leq \frac{2}{3} \max(\epsilon_{x'}, \epsilon_{y'}).$$

This implies that $C_1 \cap C_2 \neq \varnothing$, a contradiction, hence $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$, as desired. ∎

We now develop two important results: Urysohn's Lemma and Tietze's Theorem. Given topological spaces, there may not be many continuous functions between them, but in the case of normal spaces, these results demonstrate their abundance.

Urysohn's Lemma: Let $(X, \mathcal{T})$ be a normal topological space. Then for any two disjoint closed sets $C_0, C_1 \subseteq X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ for $x \in C_0$, and $f(x) = 1$ for $x \in C_1$.

Urysohn's Lemma is easy for metric spaces. Let d denote the metric, and let A, B be disjoint closed subsets. For *any* non-empty subset E, we can define $\rho_E(x) = \inf\{d(x, y) : y \in E\}$, which can be shown to be continuous. Furthermore, $\rho_E(x) = 0$ iff $x \in \overline{E}$.
Define $f(x) = \dfrac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$. One can easily check that this function yields the desired result of Urysohn's Lemma for metric spaces.

Lemma: Let $(X, \mathcal{T})$ be a normal space, and let C be a closed subset. Let $\mathcal{O}$ be an open subset such that $C \subseteq \mathcal{O}$. Then there exists an open set U such that $C \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$.

Proof: C and $\mathcal{O}^c$ are disjoint closed sets, so there are disjoint open sets U, V such that $C \subseteq U$ and $\mathcal{O}^c \subseteq V$. Then $C \subseteq U \subseteq V^c \subseteq \mathcal{O}$. V^c is a closed set containing U; it therefore contains the closure $\overline{U}$, so that $C \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$. ■

Proof of Urysohn's Lemma: By the lemma, there is an open set $\mathcal{O}_{1/2}$ such that $C_0 \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq C_1^c$. Applying the lemma again, there are open sets $\mathcal{O}_{1/4}$ and $\mathcal{O}_{3/4}$ such that $C_0 \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4} \subseteq C_1^c$. Then there are open sets $\mathcal{O}_{1/8}, \mathcal{O}_{3/8}, \mathcal{O}_{5/8}, \mathcal{O}_{7/8}$ such that $C_0 \subseteq \mathcal{O}_{1/8} \subseteq \overline{\mathcal{O}}_{1/8} \subseteq \mathcal{O}_{1/4} \subseteq \cdots \subseteq \overline{\mathcal{O}}_{7/8} \subseteq C_1^c$. Continuing the pattern, for each dyadic rational number (numbers of the form $k2^{-n}$, for $n \in \mathbb{N}$, $0 < k \leq 2^n - 1$), we get an open set $\mathcal{O}_{k2^{-n}}$, and if s, t are dyadic rationals in $(0, 1)$ such that $s < t$, then $\overline{\mathcal{O}}_s \subseteq \mathcal{O}_t$.
Define $f : X \to [0, 1]$ by $f(x) = \inf\{r : r \text{ is a dyadic rational, } x \in \mathcal{O}_r\}$.
Clearly, if $x \in C_0$, then $x \in \mathcal{O}_{2^{-n}}$ for any $n \in \mathbb{N}$, so it follows that $f(x) = 0$. On the other hand, if $x \in C_1$, then $x \notin \mathcal{O}_{k2^{-n}}$ for any n, k, hence $f(x) = 1$ on C_1. Thus, it remains to show that f is continuous. Recall that it suffices to consider the sub-base of open rays.

For $a \leq 0$ and $b \geq 1$, we get $f^{-1}((-\infty, a)) = f^{-1}((b, \infty)) = \varnothing$. For $a > 1$ and $b < 0$, $f^{-1}((-\infty, a)) = f^{-1}((b, \infty)) = X$.
Suppose $0 \leq a < 1$. If $x \in X$ and $f(x) < a$, then there is a dyadic rational r such that $f(x) < r < a$, so $x \in \mathcal{O}_r$, so $f^{-1}((-\infty, a)) = \bigcup_{r<a} \mathcal{O}_r$, which is open.
Similarly, suppose $0 \leq b < 1$. If $f(x) > b$, then there is a dyadic rational r such that $b < r < f(x)$, so $x \notin \mathcal{O}_r$, so there is a dyadic rational s such that $b < s < r$, so $\overline{\mathcal{O}}_s \subseteq \mathcal{O}_r$, so $x \notin \overline{\mathcal{O}}_s$, so $x \in \overline{\mathcal{O}}_s^c$, which is open. Then $f^{-1}((b, \infty)) = \bigcup_{s<b} \overline{\mathcal{O}}_s^c$, which is open. ■

Before we discuss Tietze's Theorem, we digress to talk about Banach spaces.

Def: A **Banach space** is a complete, normed vector space.

Let X be a set, and let V be a normed vector space. Let $B(X,V)$ denote the set of all bounded functions from X to V, that is, functions whose range is contained in an open ball. Then it can easily be checked that $B(X,V)$ is a vector space for pointwise operations, and that $\|f\|_\infty := \sup\{\|f(x)\|_V : x \in X\}$ is a norm on $B(X,V)$.

Proposition: If V is a Banach space, then $B(X,V)$ with $\|\cdot\|_\infty$ is a Banach space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $B(X,V)$. For each $x \in X$, the sequence $\{f_n(x)\}$ is Cauchy in V, so by the completeness of V, call the limit $f(x) = \lim f_n(x)$. It is easy to see that since all the f_n's are bounded, the limit f is bounded as well. We need to show that $f_n \to f$ in norm. Let $\epsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that for $n,m \geq N_1$, we have $\|f_n - f_m\|_\infty < \epsilon/2$. For fixed $x \in X$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have $\|f_n(x) - f(x)\| < \epsilon/2$. Then for $n \geq \max(N_1, N_2)$, we have $\|f_n(x) - f(x)\|_\infty \leq \|f_n - f_{n+1}\|_\infty + \|f_{n+1}(x) - f(x)\| < \epsilon$. Therefore $\|f_n - f\| < \epsilon$. ■

Proposition: Let $(X, \mathcal{T})$ be a topological space, and let $C_b(X,V)$ be the set of bounded continuous functions from X to V. Then $C_b(X,V)$ is a closed subspace.

Proof: *Exercise.* ■

Tietze Extension Theorem: Let $(X, \mathcal{T})$ be a normal topological space, and let A be a closed subset of X. Let $f : A \to \mathbb{R}$ be continuous. Then f has a continuous extension $\tilde{f} : X \to \mathbb{R}$, i.e., $\tilde{f}|_A = f$. If $f : A \to [a,b]$, then we can arrange the extension $\tilde{f} : X \to [a,b]$.

Proof: First, we will prove the case $f : A \to [0,1]$. For E_0, F_0 disjoint closed sets in X, by Urysohn's Lemma, let $h_{E_0,F_0} : X \to [0,1]$ be a continuous function such that $h_{E_0,F_0}|_{E_0} = 0$ and $h_{E_0,F_0}|_{F_0} = 1$.
Let $f_0 = f$, and let $A_0 = \{x \in A : f_0(x) \leq 1/3\}$, $B_0 = \{x \in A : f_0(x) \geq 2/3\}$.
Clearly A_0 and B_0 are disjoint. Let $g_1 = \frac{1}{3} h_{A_0,B_0}$.
Now let $f_1 = f_0 - g_1|_A$. That is, $f_1 : A \to [0, 2/3]$, and $g_1 : X \to [0, 1/3]$.
Inductively, let $f_n : A \to [0, (2/3)^n]$. Let $A_n = \{x \in A : f(x) \leq \frac{1}{3}(2/3)^n\}$,
$B_n = \{x \in A : f(x) \geq \frac{2}{3}(2/3)^n\}$, with $g_{n+1} = \frac{1}{3}\left(\frac{2}{3}\right)^n h_{A_n,B_n}$, so
$g_{n+1} : X \to \left[0, \frac{1}{3}\left(\frac{2}{3}\right)^n\right]$. Let $f_{n+1} = f_n - g_{n+1}|_A$, so $f_{n+1} : A \to \left[0, \frac{1}{3}\left(\frac{2}{3}\right)^{n+1}\right]$.
Note that $\|g_n\|_\infty = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Let $g = \sum_{n=1}^{\infty} g_n$. We will show that the sequence of partial sums is Cauchy in $C_b(X, \mathbb{R})$, thus $\sum_{n=1}^{\infty} g_n$ converges.
Let $k_n = \sum_{j=1}^{n} g_j$. For $m < n$, consider $k_n - k_m = \sum_{j=m+1}^{n} g_j$.

Then $\|k_n - k_m\|_\infty \leq \sum_{j=m+1}^{n} \|g_j\|_\infty = \sum_{j=m+1}^{n} \frac{1}{3}\left(\frac{2}{3}\right)^{j-1}$.
Clearly, for large enough n, m, we can get this arbitrarily small. Therefore g is well-defined and continuous, by the previous proposition. Then

$$f_n = f_{n-1} - g_n = f_{n-2} - g_{n-1} - g_n = \cdots = f_0 - \sum_{j=1}^{n} g_j,$$

so $\|f_n\|_\infty = \left(\frac{2}{3}\right)^n$, so $\|f_n\|_\infty \to 0$, thus $f - g|_A = 0$, i.e., $g|_A = f$.
Finally, we want to check that the range of g is contained in $[0,1]$. Note that

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \leq \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n} = \frac{1}{3}\cdot\frac{1}{1-2/3} = 1.$$

Therefore $0 \leq g(x) \leq 1$ for all $x \in X$.

Now suppose that $f : A \to \mathbb{R}$ is unbounded. Let h be a homeomorphism of $\mathbb{R}$ with $(0,1)$. Let $g = h \circ f$, so $g : A \to (0,1) \subset [0,1]$. By the arguments above, we can find an extension $\tilde{g} : X \to [0,1]$. Let $D = \tilde{g}^{-1}(\{0,1\})$. Since $\tilde{g}$ is continuous, D is closed in X and is disjoint from A. By Urysohn's Lemma, there exists a continuous function $k : X \to [0,1]$ such that $k|_D = 0$ and $k|_A = 1$. Define $\tilde{f} = \tilde{g}k$ (pointwise product). Then the function $h^{-1} \circ \tilde{f}$ is a continuous extension of f to X. ■

Chapter 3

Compactness

3.1 Fundamentals

Def:

- Let X be a set, and let $\mathcal{C}$ be a collection of subsets of X. We say $\mathcal{C}$ **covers** X if $\bigcup_{A \in \mathcal{C}} A = X$.
- If $\mathcal{C}$ is a cover of X, and $\mathcal{D} \subseteq \mathcal{C}$ is also a cover of X, then $\mathcal{D}$ is a **subcover** of X.
- For a topological space $(X, \mathcal{T})$, an **open cover** is a cover of X that is contained in $\mathcal{T}$.

Def: $(X, \mathcal{T})$ is **compact** if every open cover of X has a finite subcover.

Def: Let $\mathcal{F}$ be a collection of subsets of X. Then $\mathcal{F}$ has the **finite intersection property** if the intersection of any finite collection of sets in $\mathcal{F}$ is nonempty.

Proposition: $(X, \mathcal{T})$ is compact iff it has the property that whenever $\mathcal{F}$ is a collection of closed subsets of X with the finite intersection property, then $\bigcap_{A \in \mathcal{F}} A \neq \varnothing$.

Proof: *Exercise.* ■

Proposition: Let $(X, \mathcal{T})$ be a topological space. Then $A \subseteq X$ is compact for the relative topology iff for any open cover of A, there is a finite subcover of A.

Proof: The open sets in the relative topology are exactly the sets of the form $A \cap \mathcal{O}$, where $\mathcal{O} \in \mathcal{T}$. ■

Proposition: If $(X, \mathcal{T}_x)$ is compact, and $(Y, \mathcal{T}_y)$ is a topological space, and $f : X \to Y$ is continuous, then $f(X)$ is compact in Y.

Proof: Let $\mathcal{C} \subseteq \mathcal{T}_y$ be an open cover of $f(X)$. Then $\{f^{-1}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{C}}$ is an open cover of X, so there is a finite subcover $\{f^{-1}(\mathcal{O}_1), ..., f^{-1}(\mathcal{O}_n)\}$ of X. It follows that $f^{-1}(\mathcal{O}_1) \cup \cdots \cup f^{-1}(\mathcal{O}_n) = f^{-1}(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n) = X$, so $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ is an open cover of $f(X)$. $\blacksquare$

Proposition: If $(X, \mathcal{T})$ is compact, and $A \subseteq X$ is closed, then A is compact.

Proof: Let $\mathcal{C}$ be an open cover of A. Since A^c is open, $\mathcal{C} \cup \{A^c\}$ covers X, and since $(X, \mathcal{T})$ is compact, there is a finite subcover of X, so clearly there is a finite subcover for A. $\blacksquare$

Proposition: If $(X, \mathcal{T})$ is Hausdorff, then any compact subset is closed.

Proof: Let $A \subseteq X$ be compact, and let $x^* \notin A$. For any $y \in A$, there are disjoint open sets $\mathcal{O}_y, U_y$ such that $y \in \mathcal{O}_y$ and $x^* \in U_y$. Then $\{\mathcal{O}_y : y \in A\}$ is an open cover of A, so by compactness there is a finite subcover $\{\mathcal{O}_{y_1}, \ldots, \mathcal{O}_{y_n}\}$. Let $U = \bigcap_{j=1}^{n} U_{y_j}$. Then U is open and covers A^c and is also disjoint from A. It follows that $U = A^c$, and so A is closed. $\blacksquare$

Recall from standard real analysis the fact that in $\mathbb{R}^n$, a subset is compact iff it is closed and bounded (Heine-Borel).

Def: $(X, \mathcal{T})$ is **regular** if for any closed set $A \subseteq X$ and any $x \notin A$, there are disjoint open sets $\mathcal{O}, U$ such that $A \subseteq \mathcal{O}$ and $x \in U$.

3.2 Tychonoff's Theorem

We now develop an important compactness theorem of Tychonoff, which says that a product of compact spaces is compact in the product topology. In order to start, we need some set theory.

Axiom of Choice: Given any family of non-empty sets, there is a set containing an element from each of these sets.

We will see later that the axiom of choice is in fact equivalent to Tychonoff's Theorem.

Def: A **partially-ordered set** P is a set with a partial order $\leq$, which is a relation that satisfies

(i) $x \leq x$ for all $x \in P$

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$

(iii) If $x \leq y$, and $y \leq x$, then $x = y$.

A **totally/linearly-ordered** set satisfies the extra condition

(iv) For any $x, y \in P$, either $x \leq y$ or $y \leq x$.

Ex: If X is a set, consider its power set $\mathcal{P}(X)$. Then the relation $A \leq B$ iff $A \subseteq B$ is a partial order on $\mathcal{P}(X)$ but not a total order.
Ex: In the plane $\mathbb{R}^2$, the relation $x \leq y$ iff $\|x\|_2 \leq \|y\|_2$ is *not* a partial order, since two points could have the same norm, but be unequal.
Ex: The usual relation $\leq$ on $\mathbb{R}$ is a total order.

Def:

- A **chain** in P is a totally-ordered subset of P.
- A **maximal element** in P is an element $x \in P$ such that if $y \geq x$, then $y = x$.
- An **upper bound** for a subset $A \subseteq P$ is an element $x \in P$ such that $y \leq x$ for all $y \in A$.
- P is **inductively ordered** if every chain in P has an upper bound.

Zorn's Lemma: If P is inductively ordered, then every chain C has a maximal element b for C with $a \leq b$ for all $a \in C$.

Zorn's Lemma seems quite obscure, but it is incredibly practical in many important results in mathematics and particularly in analysis.

Tychonoff's Theorem: Let $\{(X_\alpha, \mathcal{T}_\alpha)\}$ be a family of compact spaces indexed by A. Then $X = \prod_\alpha X_\alpha$ with the product topology is compact.

Proof: We will show compactness of X through the finite intersection property. Let $\mathcal{C}$ be a collection of closed subsets of X with the finite intersection property. We wish to show that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.

Let $\Theta = \{\mathcal{D} \subseteq \mathcal{P}(X) : \mathcal{C} \subseteq \mathcal{D},\ \mathcal{D}$ has finite intersection property$\}$.
Then Θ is partially ordered by set inclusion. To show that Θ is inductively ordered, let $\Phi \subseteq \Theta$ be a chain, and let $\mathcal{E} = \bigcup_{\mathcal{D}\in\Phi} \mathcal{D}$. We show that $\mathcal{E}$ has the finite intersection property so that $\mathcal{E} \in \Theta$ and is thus an upper bound for Φ.
Let $Z_1, \ldots, Z_n \in \mathcal{E}$. Then there are $\mathcal{D}_1, \ldots, \mathcal{D}_n \in \Phi$ with $Z_j \in \mathcal{D}_j$. Since Φ is totally ordered, one of the $\mathcal{D}_j$'s will be the largest, say, $\mathcal{D}_n$, and each $\mathcal{D}_j \subseteq \mathcal{D}_n$. Then $Z_1, \ldots, Z_n \in \mathcal{D}_n$. But $\mathcal{D}_n$ has the finite intersection property, so $\bigcap\limits_{j=1}^{n} Z_j \neq \varnothing$. Clearly $\mathcal{C} \subseteq \mathcal{E}$, so $\mathcal{E} \in \Theta$, and $\mathcal{E}$ is an upper bound for Φ. Thus Θ is inductively ordered, as desired.

By Zorn's Lemma, Θ contains a maximal element, say, $\mathcal{D}^*$, which will have the following properties:

- If $Z_1, Z_2 \in \mathcal{D}^*$, then $Z_1 \cap Z_2 \in \mathcal{D}^*$.
  Indeed, for $Y_1, \ldots, Y_n \in \mathcal{D}^*$, we have $(Z_1 \cap Z_2) \cap (Y_1 \cap \cdots \cap Y_n) \neq \varnothing$. Therefore $\mathcal{D}^* \cup \{Z_1 \cap Z_2\}$ has the finite intersection property. But by maximality of $\mathcal{D}^*$, in fact $\mathcal{D}^* = \mathcal{D}^* \cup \{Z_1 \cap Z_2\}$, therefore $Z_1 \cap Z_2 \in \mathcal{D}^*$.

- For $Y \subseteq X$, if $Y \cap Z \neq \varnothing$ for all $Z \in \mathcal{D}^*$, then $Y \in \mathcal{D}^*$.
  Indeed, for $Z_1, \ldots, Z_n \in Z$, we have $Y \cap (Z_1 \cap \cdots \cap Z_n) = \bigcap\limits_{j=1}^{n} (Y \cap Z_j) \neq \varnothing$, and by maximality, $Y \in \mathcal{D}^*$.

For any $\mathcal{D} \in \Theta$ and for any $\alpha \in A$, we claim that $\mathcal{F}_\alpha := \{\pi_\alpha(Y) :\ Y \in \mathcal{D}\}$ has the finite intersection property. By the finite intersection property of $\mathcal{D}$, for any $Z_1, \ldots, Z_n \in \mathcal{D}$, there exists $x \cap Z_1, \ldots \cap Z_n$. Then $\pi_\alpha(x) \in \pi_\alpha(Z_1) \cap \cdots \cap \pi_\alpha(Z_n)$, thus F_α has the finite intersection property. It follows that $\left\{\overline{\pi_\alpha(Z)} :\ Z \in \mathcal{D}\right\}$ has the finite intersection property. This is a collection of closed subsets with the finite intersection property in X_α, which is compact, so $\bigcap\limits_{Z\in\mathcal{D}} \overline{\pi_\alpha(Z)} \neq \varnothing$.
Apply this to each $\mathcal{D} \subseteq \mathcal{D}^*$. For each $\alpha \in A$, by the axiom of choice, pick $x_\alpha \in X_\alpha$.
Let $x = (x_\alpha) \in \prod_{\alpha\in A} X_\alpha$. We claim that $x \in \bigcap\limits_{C\in\mathcal{C}} C$.
It suffices to show that if $\mathcal{O}$ is an open set containing x, then $\mathcal{O} \cap C \neq \varnothing$ for all $C \in \mathcal{C}$. It further suffices to have $\mathcal{O}$ be a basis element for the product topology. That is, suppose $x \in \mathcal{O} = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod\limits_{\alpha\neq\alpha_i\ \forall i} X_\alpha$.
Note that $x_{\alpha_j} \in U_{\alpha_j}$ for $j = 1, \ldots, n$. Then $x_{\alpha_j} \in \overline{\pi_{\alpha_j}(Z)}$ for all $Z \in \mathcal{D}^*$, so $U_{\alpha_j} \cap \pi_\alpha(Z) \neq \varnothing$ for all $Z \in \mathcal{D}^*$ and for all $1 \leq j \leq n$. So $\pi_{\alpha_j}^{-1}(U_{\alpha_j}) \cap Z \neq \varnothing$ for all $Z \in \mathcal{D}^*$, so by maximality, $\pi_{\alpha_j}^{-1}(U_{\alpha_j}) \in \mathcal{D}^*$, thus $\bigcap\limits_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j}) \in \mathcal{D}^*$, so $\mathcal{O} \in \mathcal{D}^*$.
This proves that $\bigcap_{C\in\mathcal{C}} C \neq \varnothing$, hence X is compact. ■

Ex: Here is an interesting application of Tychonoff's Theorem.
Let $\mathcal{H}$ be a Hilbert space (complete inner product space) with its norm induced by its inner product. Given $\eta \in \mathcal{H}$, define $\phi_\eta : \mathcal{H} \to \mathbb{R}$ by $\phi_\eta(\xi) = \langle \eta, \xi \rangle$, and thus we can put on $\mathcal{H}$ the weakest topology making all $\{\phi_\eta\}_{\eta\in\mathcal{H}}$ continuous. The closed unit ball $B := \{\xi \in \mathcal{H} :\ \|\xi\| \leq 1\}$ is compact with the relative weak topology. The proof of this uses

Tychonoff's Theorem. To start, for each $\eta \in \mathcal{H}$, let $D_\eta = \{r \in \mathbb{R} : |r| \leq \|\xi\|\}$, with the usual topology. Then form $\prod_{\eta \in \mathcal{H}} D_\eta$, and one can show that this product is equal to B. Clearly each D_η is compact, so B is thus compact.

Tychonoff's Theorem uses the axiom choice for its proof, but in fact, it is equivalent to the axiom of choice as well! Note that the axiom of choice essentially says that the product of non-empty sets is non-empty.

Theorem: Let $\{X_\alpha\}_{\alpha \in A}$ be any collection of non-empty sets. Then without the axiom of choice, and assuming Tychonoff's Theorem, $\prod_{\alpha \in A} X_\alpha \neq \varnothing$.

Proof: Let $X = \bigcup_{\alpha \in A} X_\alpha$. Now let ω be some set not in $\bigcup_{\alpha \in A} X_\alpha$. For each α, let $Y_\alpha = X_\alpha \cup \{\omega\}$, and define the topology $\mathcal{T}_\alpha$ for Y_α by $\mathcal{T}_\alpha = \{X_\alpha, \{\omega\}, Y_\alpha, \varnothing\}$. Clearly, $(Y_\alpha, \mathcal{T}_\alpha)$ is compact. Then $Y := \prod_{\alpha \in A} Y_\alpha$ with the product topology is compact. For each α, let $C_\alpha = \pi_\alpha^{-1}(X_\alpha)$, where $\pi_\alpha : Y \to Y_\alpha$ is the standard projection map. Note that C_α is closed.
We will show that $\{C_\alpha\}$ has the finite intersection property. Given $C_{\alpha_1}, \ldots, C_{\alpha_n}$, where $x_{\alpha_j} \in X_{\alpha_j}$, define $y \in Y$ by

$$y_\alpha = \begin{cases} x_{\alpha_j}, & \text{if } \alpha = \alpha_j, \\ \omega, & \text{if } \alpha \neq \alpha_j \ \forall j \end{cases}$$

Then $y \in \bigcap_{j=1}^n C_{\alpha_j}$, so $\{C_\alpha\}$ has the finite intersection property, as desired. By compactness, $\bigcap_{\alpha \in A} C_\alpha \neq \varnothing$. Let $z \in \bigcap_{\alpha \in A} C_\alpha$. Then $z \in X_\alpha$ for all α, so $z \in \prod_{\alpha \in A} X_\alpha$. Thus we can deduce the axiom of choice from Tychonoff's Theorem. ■

3.3 Compact and Hausdorff

Proposition: If $(X, \mathcal{T})$ is compact and Hausdorff, then it is normal.

Proof: Let C, D be disjoint closed sets of X. Since X is compact, C and D are also compact, and since X is Hausdorff, it follows that it is regular. Therefore, for any $x \in D$, there are disjoint open sets $\mathcal{O}_x, U_x$ with $x \in U_x$ and $C \subseteq \mathcal{O}_x$. Then $\{U_x\}_{x \in D}$ is an open cover for D, so there is a finite subcover. That is, there are points $x_1, \ldots, x_n \in D$ such that $\{U_{x_1}, \ldots, U_{x_n}\}$ covers D. Then $U := \bigcup_{j=1}^n U_{x_j}$ is an open set containing D, and $\mathcal{O} := \bigcap_{j=1}^n \mathcal{O}_{x_j}$ is an open set containing C, and $U \cap C = \varnothing$. ■

Proposition: Let X be a set, and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X with $\mathcal{T}_1 \supseteq \mathcal{T}_2$. Then

(i) If $(X, \mathcal{T}_1)$ is compact, so is $(X, \mathcal{T}_2)$.

(ii) If $(X, \mathcal{T}_2)$ is Hausdorff, so is $(X, \mathcal{T}_1)$.

Proof: *Exercise.* ■

Corollary: If $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are both compact and Hausdorff, then $\mathcal{T}_1 = \mathcal{T} = 2$.

Proof: If C is closed for $\mathcal{T}$, then it is compact for $\mathcal{T}_1$, so is compact for $\mathcal{T}_2$, so is closed for $\mathcal{T}_2$. ■

Proposition: Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be compact Hausdorff spaces. If $f : X \to Y$ is continuous and bijective, then it is a homeomorphism.

Proof: It suffices to show that $f(C)$ is closed for any closed set $C \subseteq X$. Since X is compact, C is compact, so $f(C)$ is compact by continuity of f. Now since Y is Hausdorff, it follows that $f(C)$ is closed. ■

3.4 Compactness for Metric Spaces

We now study specifically the importance of compactness for metric spaces, starting with a simple necessary and sufficient condition for a metric space to be compact, and leading to the Arzela-Ascoli theorem.

First, observe that if (X, d) is a compact metric space, and if $A \subseteq X$ is dense in X, then the balls $B_\epsilon(y)$ for $\epsilon > 0$ and $y \in A$ form an open cover for X, so there is a finite subcover.

Def: A metric space (X, d) is **totally bounded** if for any $\epsilon > 0$, there is a finite collection of open balls of radius ϵ that covers X.

Proposition: Let (X, d) be a metric space, and let $A \subseteq X$. If A is totally bounded, then so is $\overline{A}$.

Proof: Let $\epsilon > 0$. Then there are points $y_1, \ldots, y_n \in A$ such that $\{B_{\epsilon/2}(y_j)\}_{j=1}^n$ covers A. For each $z \in \overline{A}$, there exists $y \in A$ such that $z \in B_{\epsilon/2}(y)$, and there is some j such that $y \in B_{\epsilon/2}(y_j)$. Therefore $z \in B_\epsilon(y_j)$, so that $\{B_\epsilon(y_j)\}_{j=1}^n$ covers $\overline{A}$. ■

Proposition: Let (X, d) be a metric space. If X is compact, then it is complete.

Proof: We prove the contrapositive. Suppose X is not complete. Let $\{x_n\}$ be a Cauchy sequence that does not converge. For any $x \in X$, there exists $\epsilon_x > 0$ such that for any $N \in \mathbb{N}$, there exists $n \geq N$ such that $d(x_n, x) \geq \epsilon_x$. But since $\{x_n\}$ is Cauchy, there is $M \in \mathbb{N}$ such that for any $n, m \geq M$, we have $d(x_n, x_m) < \epsilon_x$. Pick $M_x > M$ such that there is $n_x \geq M_x$ with $d(x, x_{n_x}) \geq \epsilon_x$. So for $n > M_x$, we have $d(x, x_n) \geq \epsilon/2$. Therefore for each $x \in X$, $B_{\epsilon_x}(x)$ contains at most a finite number of elements in the sequence $\{x_n\}$. Clearly, the balls $\{B_{\epsilon_x}(x)\}$ cover X, and no finite subcollection can cover X. ■

Compactness clearly implies totally bounded, but the converse is not true. The following theorem says that a metric space must be both totally bounded *and* complete. For example, the non-compact space $(0, 1)$ is totally bounded, but not complete. Furthermore, $\mathbb{R}$ is complete, but not totally bounded.

Theorem: Let (X, d) be a complete metric space. If X is totally bounded, then it is compact.

Proof: Let $\mathcal{C}$ be an open cover of X, and let $B_1^1, \ldots, B_n^1$ be a finite cover of X by closed balls of radius 1. Toward contradiction, suppose X is not compact, so at least one of these closed balls, denote it by A^1, has no finite subcover. Let $B_1^2, \ldots, B_{n_2}^2$ be closed balls of radius 1/2 that cover A^1. One of these, say, B_*^2 has no finite subcover. Let $A^2 = A^1 \cap B_*^2$. Let $B_1^3, \ldots, B_{n_3}^3$ be balls of radius 1/4 that cover A^2. One of these has no finite subcover, etc. By continuing this process, we get a sequence $\{A^n\}$ of non-empty closed sets with $A^{n+1} \subseteq A^n$ for all n. Furthermore, by construction, $\operatorname{diam}(A^n) \to 0$.
For each n, choose $x_n \in A^n$. Then $\{x_n\}$ is a Cauchy sequence, and by completeness, there is $x \in X$ such that $x_n \to x$. Since $\mathcal{C}$ is a cover, there is $\mathcal{O} \in \mathcal{C}$ with $x \in \mathcal{O}$, and then there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{O}$. There is also $N \in \mathbb{N}$ such that for $n \geq N$, $x_n \in B_{\epsilon/2}(x)$. For large n, we can get $\operatorname{diam}(A_n) < \epsilon/2$, $A^n \subseteq B_\epsilon(x_n) \subseteq \mathcal{O}$, so A^n is covered by $\mathcal{C}$, a contradiction. ■

Recall that for a set X and normed vector space V, $B(X, V)$ denotes the set of bounded functions from X to V.

Def: Let X be a set, and let (Y, d) be a metric space. A function $f : X \to Y$ is **bounded** if its range is bounded in Y.

Let $B^*(X, Y)$ denote the set of bounded functions from X to Y (Y is not necessarily a normed vector space). One can verify that $d_\infty(f, g) = \sup\limits_{x \in X}\{d(f(x), g(x))\}$ is a metric on $B^*(X, Y)$. Furthermore, one can check that the bounded continuous functions, denoted by $C_b(X, Y)$, is a closed subspace of $B^*(X, Y)$. Given a collection $\mathcal{F} \subseteq C_b(X, Y)$, when is $\mathcal{F}$ totally bounded?

- Suppose it is. Then for $\epsilon > 0$, there are $g_1, \ldots, g_n \in C_b(X, Y)$ such that the balls $B_\epsilon(g_j)$ cover $\mathcal{F}$. Let $x \in X$. Then for each j, there is $\mathcal{O}_j \subseteq \mathcal{T}_x$ such that $x \in \mathcal{O}_j$ and $y \in \mathcal{O}_j$ imply $d(g_j(x), g_j(y)) < \epsilon$.
  Let $\mathcal{O}_x = \bigcap\limits_{j=1}^{n} \mathcal{O}_j \in \mathcal{T}_x$, where clearly $x \in \mathcal{O}_x$, and if $y \in \mathcal{O}_x$, then $d(g_j(x), g_j(y)) < \epsilon$ for all $j = 1, \ldots, n$. For any $f \in \mathcal{F}$ there is j such that $d_\infty(f, g) < \epsilon$. Then for any $y \in \mathcal{O}_x$, we have
  $$d(f(x), f(y)) \leq d(f(x), g(x)) + d(g(x), g(y)) + d(g(y), f(y)) < 3\epsilon.$$

Then for each x and each $\epsilon' > 0$, there is an open set $\mathcal{O}_x$ such that if $y \in \mathcal{O}_x$, then $d(f(x), f(y)) < \epsilon'$ for all $f \in \mathcal{F}$.

Def: Let $(X, \mathcal{T})$ be a topological space, and let (Y, d) be a metric space. Let $\mathcal{F} \subseteq C(X, Y)$ (a collection of continuous functions from X to Y). $\mathcal{F}$ is **equicontinuous at x** if for any $\epsilon > 0$, there exists an open set $\mathcal{O}_x$ in X such that for all $f \in \mathcal{F}$ and any $x' \in \mathcal{O}_x$, we have $d(f(x), f(x')) < \epsilon$.
$\mathcal{F}$ is **equicontinuous** if it is equicontinuous at every $x \in X$.

Continuing the previous argument, note that $d(f(x), g_j(x)) < \epsilon$, i.e., the open balls $B_\epsilon(g_j(x))$ cover $\{f(x) : f \in \mathcal{F}\}$, so $\mathcal{F}$ is **"pointwise totally bounded."**

Arzela-Ascoli Theorem: If $(X, \mathcal{T})$ is compact, (M, d) is a metric space, and $\mathcal{F} \subseteq C_b(X, Y)$, and if $\mathcal{F}$ is equicontinuous and pointwise totally bounded, then $\mathcal{F}$ is totally bounded for d_∞.

Proof: Let $\epsilon > 0$. Since $\mathcal{F}$ is equicontinuous, for each $x \in X$ there is an open set $\mathcal{O}_x$ with $x \in \mathcal{O}_x$ such that for $y \in \mathcal{O}_x$, we have $d(f(x), f(y)) < \epsilon$ for all $f \in \mathcal{F}$. Since X is compact, there are points $x_1, \ldots, x_n$ such that $X \subseteq \bigcup_{j=1}^n \mathcal{O}_{x_j}$. For each j, $\{f(x) : f \in \mathcal{F}\}$ is totally bounded since $\mathcal{F}$ is pointwise totally bounded. Let $S_j \subseteq \{f(x_j) : f \in \mathcal{F}\} \subseteq M$ be a finite subset of M such that the balls of radius ϵ about the points of S_j cover $\{f(x_j) : f \in \mathcal{F}\}$. Let $S = \bigcup_{j=1}^n S_j$, and let $\Psi = \{\psi : \{1, \ldots, n\} \to S\}$, a finite set. Let $B_\psi = \{f \in \mathcal{F} : \ d(f(x_j), \psi(j)) < \epsilon, \ \forall j\}$. Then $\mathcal{F} = \bigcup_{\psi \in \Psi} B_\psi$.
Let ψ be given. Let $f, g \in B_\psi$, and let $x \in X$, so that $x \in \mathcal{O}_{x_j}$ for some j. Then

$$d(f(x), g(x)) \leq d(f(x), f(x_j)) + d(f(x_j), g(x_j)) + d(g(x_j), g(x)) \leq$$

$$\leq d(f(x), f(x_j)) + d(f(x_j), \psi(j)) + d(\psi(j), g(x_j)) + d(g(x_j), g(x)) < 4\epsilon.$$

Therefore B_ψ is contained in the ball of radius 4ϵ about any of its points. Since Ψ is finite, it follows at once that $\mathcal{F}$ is totally bounded for d_∞. ■

Corollary: Let M be a complete metric space, so that $C(X, M)$ is complete. Then $\mathcal{F} \subseteq C(X, M)$ is compact iff $\mathcal{F}$ is equicontinuous, pointwise totally bounded, and closed in $C(X, M)$.

3.5 Locally Compact Spaces

Although compact spaces form a nice class of topological spaces with many properties, there are many important spaces that share a similar structure: these are the locally compact spaces. Even more important are the locally compact Hausdorff spaces, which we discuss.

Def: A topological space $(X, \mathcal{T})$ is **locally compact** if for each $x \in X$, there is an open set $\mathcal{O}$ containing x such that $\overline{\mathcal{O}}$ is compact.

Proposition: Let $(X, \mathcal{T})$ be locally compact, and let C be a compact subset of X. Then there exists an open set $\mathcal{O}$ such that $C \subseteq \mathcal{O}$, and $\mathcal{O}$ is compact.

Proof: For each $x \in C$, there is $\mathcal{O}_x \in \mathcal{T}$ such that $x \in \mathcal{O}_x$ and $\overline{\mathcal{O}}_x$ is compact. $\{\mathcal{O}_x\}_{x \in X}$ covers C, so there is a finite subcover, say, $\mathcal{O}_1, \ldots, \mathcal{O}_n$. Then $C \subseteq \overline{\mathcal{O}}_1 \cup \cdots \cup \overline{\mathcal{O}}_n = \overline{\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n}$, which is compact. ■

From here, let LCH mean locally compact Hausdorff.

Proposition: Let $(X, \mathcal{T})$ be a LCH space, let $C \subseteq X$ be compact, and let $\mathcal{O} \in \mathcal{T}, C \subseteq \mathcal{O}$. Then there is an open set U such that $C \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$ and $\overline{U}$ is compact.

Proof: We know that we can find an open set V such that $C \subseteq V$ and $\overline{V}$ is compact. Let $W = V \cap \mathcal{O}$. Then $C \subseteq W$, and W is open. Furthermore, $\overline{W}$ is compact, and so the relative topology of $\overline{W}$ makes $\overline{W}$ compact and Hausdorff, hence normal. Let $B = \overline{W} \setminus W$, so B is closed and disjoint from C, so by normality, there are disjoint open sets U, Z such that $C \subseteq U$ and $B \subseteq Z$. Then $U \subseteq Z^c \cap \overline{W}$, so $\overline{U} \subseteq Z^c \cap \overline{W}$, so $\overline{U} \subseteq B^c = (\overline{W} \setminus W)^c \cap \overline{W} = W$. Thus $\overline{U} \subseteq W \subseteq \mathcal{O}$ and $\overline{U}$ is compact. ■

Def: For a continuous function f on X to a normed vector space, its **support** is the set $\operatorname{supp}(f) := \overline{\{x : f(x) \neq 0\}}$.
f has **compact support** if its support is compact.

Let V be a normed vector space. We will let $C_c(X, V)$ denote the set of all continuous functions from X to V with compact support.

Proposition: Let $(X, \mathcal{T})$ be a LCH space. Let $C \subseteq X$ be compact, and let $\mathcal{O}$ be open with $C \subseteq \mathcal{O}$. Then there is a continuous function $f : X \to [0, 1]$ such that $f = 1$ on C, and $f = 0$ outside $\mathcal{O}$, and f has compact support.

Proof: There exists $U \in \mathcal{T}$ such that $C \subseteq U$ and $\overline{U}$ is compact, and $\overline{U} \subseteq \mathcal{O}$. There also exists $V \in \mathcal{T}$ with $C \subseteq V \subseteq \overline{V} \subseteq U$. Let $B = \overline{U} \setminus V$, so that B is closed and disjoint from C. Since $\overline{U}$ is compact and Hausdorff, we apply Urysohn's Lemma to obtain a continuous function $f : \overline{U} \to [0, 1]$ such that $f = 1$ on C, and $f = 0$ on B. For $x \notin U$, set $f(x) = 0$. Since $U \subseteq \mathcal{O}$, it follows that $f = 0$ outside $\mathcal{O}$, as desired. It remains to show that $f : X \to [0, 1]$ is continuous, but this is left as an exercise. ■

Remark: $C_c(X) \subseteq C_b(X)$, which denotes the set of bounded continuous functions. Equip $C_b(X)$ with the norm $\|\cdot\|_\infty$, so that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$, making $C_b(X)$ a Banach algebra. One can verify that the closure of $C_c(X)$ in $C_b(X)$ is the space of all continuous functions that vanish at ∞, denote $C_\infty(X)$.
f **vanishes at ∞** if for any $\epsilon > 0$ there exists a compact K such that $\|f(x)\| < \epsilon$ for $x \notin K$.

Chapter 4

Measure Theory

4.1 Fundamentals

Def: Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X. We say $\mathcal{M}$ is a **ring** if

(1) $E \cup F \in \mathcal{M}$ for any $E, F \in \mathcal{M}$.

(2) $E \setminus F \in \mathcal{M}$ for any $E, F \in \mathcal{M}$.

$\mathcal{M}$ is an **algebra/field** if $\mathcal{M}$ is a ring, and $X \in \mathcal{M}$.

Note that (1) and (2) imply that if $E, F \in \mathcal{M}$, then $E \cap F \in \mathcal{M}$.

Ex: Clearly, $\mathcal{P}(X)$ is a ring and an algebra on X.
Ex: The set $\mathcal{M}$ of all finite subsets of X is a ring, but it is not an algebra if X is infinite.
Ex: If $X = \mathbb{R}$, then $\mathcal{M} := \{[a, b) : \ a < b\}$ is a ring.

Def: $\mathcal{F}$ is a **σ-ring** on X if

(1) $\mathcal{F}$ is a ring

(2) If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of elements in $\mathcal{F}$, then $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{F}$.

$\mathcal{F}$ is a **σ-algebra/field** if it is a σ-ring, and $X \in \mathcal{F}$.

Note that σ-rings are also closed under countable intersections.

Proposition: Let X be a set, and let $\{\mathcal{F}_\alpha\}$ be a collection of rings/algebras/σ-rings/σ-algebras on X. Then $\bigcap_\alpha \mathcal{F}_\alpha$ is also a ring/algebra/σ-ring/σ-algebra, respectively.

Proof: *Exercise.* ■

Proposition: For any family of sets $\mathcal{A} \subseteq \mathcal{P}(X)$, there is a smallest ring/algebra/σ-ring/σ-algebra that contains $\mathcal{A}$.

Proof: The smallest such ring/algebra/σ-ring/σ-algebra is in fact the intersection of all rings/algebras/σ-rings/σ-algebras that contain $\mathcal{A}$. ■

The smallest such ring/algebra/σ-ring/σ-algebra containing $\mathcal{A}$ is the ring/algebra/σ-ring/σ-algebra **generated** by $\mathcal{A}$.
The smallest σ-ring generated by $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A})$.

Def: If $(X, \mathcal{T})$ is a topological space, then the σ-algebra generated by $\mathcal{T}$ is called the **Borel σ-algebra**.
If $(X, \mathcal{T})$ is a locally compact space, let $\mathcal{C}$ denote the collection of all compact sets. The σ-ring generated by $\mathcal{C}$ is called the **Borel σ-ring**.

Ex: Let X be an uncountable set with the discrete topology. Then compact sets are finite sets, so the Borel σ-ring consists of all countable subsets of X.

Def: Let $\mathcal{A}$ be a collection of subsets of a set X. A function $\mu : \mathcal{A} \to [0, \infty]$ is **additive** if for all disjoint $E, F \in \mathcal{A}$ such that $E \cup F \in \mathcal{A}$, we have $\mu(E \cup F) = \mu(E) + \mu(F)$.
μ is **countable additive** if whenever $\{E_n\}_{n=1}^{\infty}$ is a countable collection of mutually disjoint sets such that $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{A}$, we have $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum\limits_{n=1}^{\infty} \mu(E_n)$.

Def: A **measure** on X consists of a σ-ring $\mathcal{S}$ on X, together with a function $\mu : \mathcal{S} \to [0, \infty]$ that is countably additive.

Ex: Let X be a set with the σ-field $\mathcal{P}(X)$. Then $\mu(E) = |E|$, where $|E| = \infty$ if E is infinite, is a measure (we could also have the σ-field be all countable subsets). This measure is called *counting measure*.

4.2 Borel Measures on $\mathbb{R}$

We will develop Borel measures in more generality. That is, let $\alpha : \mathbb{R} \to \mathbb{R}$ be any function that is non-decreasing and left continuous, i.e., for any t, we have $\lim\limits_{\epsilon \downarrow 0} \alpha(t - \epsilon) = \alpha(t)$. If we weren't general, we would have used $\alpha(t) = t$. These will lead to the "usual" measure on $\mathbb{R}$.

Let $P = \{[a, b) : a < b \in \mathbb{R}\}$, and define μ on P by $\mu([a, b)) = \alpha(b) - \alpha(a)$. We will show that μ is countably additive.

Def: Let X be a set, P a collection of subsets. Then P is a **pre-ring/semi-ring** if

(1) $E, F \in P$ implies $E \cap F \in P$.

(2) $E, F \in P$ implies $E \setminus F$ is a finite disjoint union of elements in P.

Then it is easy to see that $P = \{[a, b) : a < b\}$ is a pre-ring on $\mathbb{R}$.

Def: For a pre-ring P, a function $\mu : P \to [0, \infty]$ is a **premeasure** on P if it is countably additive.

Theorem: The function μ on P defined above is countably additive. That is, μ is a pre-measure.

Proof: Let $\{[a_j, b_j)\}_{j=1}^{\infty}$ be disjoint, and let $[a, b) = \bigcup_{j=1}^{\infty}[a_j, b_j)$ be their union.
We will first show that $\sum_{j=1}^{\infty} \mu([a_j, b_j)) \leq \mu([a, b))$. To show this, let $n \in \mathbb{N}$. Then $\sum_{j=1}^{n} \mu([a_j, b_j)) = \sum_{j=1}^{n}[\alpha(b_j) - \alpha(a_j)]$. Without loss of generality, suppose $a_1 < a_2 < \cdots < a_n$. Since the intervals are disjoint, $b_j \leq a_{j+1}$ for all j, so $\alpha(b_j) \leq \alpha(a_{j+1})$. Then

$$\sum_{j=1}^{n} \mu([a_j, b_j)) \leq [\alpha(b_n) - \alpha(a_n)] + [\alpha(a_n) - \alpha(a_{n-1})] + \cdots + [\alpha(a_2) - \alpha(a_1)] \leq \alpha(b) - \alpha(a).$$

Since this holds for any n, it holds for the limit.

Conversely, let $\epsilon > 0$. Let $\{\epsilon_j\}$ be a sequence such that $\sum_{j=1}^{\infty} \epsilon_j < \epsilon/2$, and let b' be a number such that $\alpha(b') + \epsilon/2 \geq \alpha(b)$. For each j, choose $a'_j < a_j$ such that $\alpha(a'_j) + \epsilon_j \geq \alpha(a_j)$.
Then $[a, b'] \subseteq [a, b) = \bigcup_{j=1}^{\infty}[a_j, b_j) \subseteq \bigcup_{j=1}^{\infty}(a'_j, b_j)$. Since $[a, b']$ is compact, and $\bigcup_{j=1}^{\infty}(a'_j, b_j)$ is an open cover, there must be a finite subcover. Relabel these intervals: $(a'_1, b_1), \ldots, (a'_n, b_n)$, such that $a_0 \in (a'_1, b_1)$, and $b_j \in (a'_{j+1}, b_{j+1})$, so $a'_{j+1} < b_j$, and also $b' \leq b_n$. Then

$$\sum_{j=1}^{\infty}[\alpha(b_j) - \alpha(a'_j)] = \alpha(b_n) + \alpha(b_{n-1}) - \alpha(a'_n),$$

where $\alpha(b_{n-1}) - \alpha(a'_n) \geq 0$. Then

$$\begin{aligned} \alpha(b) - \alpha(a) \ &\leq\ \alpha(b') + \frac{\epsilon}{2} - \alpha(a) \ \leq\ \alpha(b_n) + \frac{\epsilon}{2} - \alpha(a) \ \leq\ \alpha(b_n) - \alpha(a_1) + \frac{\epsilon}{2} \ \leq \\ &\leq \alpha(b_n) + [\alpha(b_{n-1}) - \alpha(a'_n)] + [\alpha(b_{n-2}) - \alpha(a'_{n-1})] + \cdots + \alpha(a'_1) + \frac{\epsilon}{2} \ = \\ &= \sum_{j=1}^{n}[\alpha(b_j) - \alpha(a'_j)] \ \leq\ \sum_{j=1}^{n}[\alpha(b_j) - \alpha(a_j) + \epsilon_j] + \frac{\epsilon}{2} \ \leq\ \sum_{j=1}^{\infty} \mu([a_j, b_j)) + \epsilon. \end{aligned}$$

Therefore, $\mu([a, b)) \leq \sum_{j=1}^{\infty} \mu([a_j, b_j))$. ■

Note that if P is a pre-ring, then for $E, F_1, F_2 \in P$, there exist $\{G_{jk}\}_{j,k=1}^{n_1,n_2} \subseteq P$, all disjoint, such that

$$(E \setminus F_1) \setminus F_2 = \Big(\bigcup_{j=1}^{n_1} G_j\Big) \setminus F_2 = \bigcup_{j=1}^{n_1} \bigcup_{k=1}^{n_2} G_{jk}.$$

Then for $F_1, \ldots, F_\ell \in P$, we have $E \setminus \bigcup_{i=1}^{\ell} F_i = \bigcup_{j=1}^{p} G_j$, where $G_j \in P$.

Proposition: If $\mu : P \to [0, \infty]$, and μ is finitely additive, and if $E \subseteq \bigcup_{j=1}^{n} F_j$, where the F_j's are disjoint, and $E, F_j \in P$, then $\mu(E) \leq \sum_{j=1}^{n} \mu(F_j)$.

Proof:

$$\bigcup_{j=1}^{n} F_j = E \cup \Big(\bigcup_{j=1}^{n} F_j \setminus E\Big) = E \cup \Big(\bigcup_{j=1}^{n_1} \bigcup_{k=1}^{n_2} G_{jk}\Big),$$

for some $\{G_{jk}\} \subseteq P$ disjoint. Furthermore, E and $\bigcup_{j=1}^{n_1} \bigcup_{k=1}^{n_2} G_{jk}$ are disjoint, so

$$\sum_{j=1}^{n} \mu(F_j) = \mu(E) + \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \mu(G_{jk}) \geq \mu(E).$$

■

Corollary: If $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Def: A function μ satisfying $\mu(E) \leq \mu(F)$ whenever $E \subseteq F$ is called **monotone**.

For a family of sets $\mathcal{F}$, we say $\mu : \mathcal{F} \to \mathbb{R}$ is **countably subadditive** if whenever $E \subseteq \bigcup_{j=1}^{\infty} F_j$, where $E, F_j \in \mathcal{F}$, then $\mu(E) \leq \sum_{j=1}^{\infty} \mu(F_j)$.

Proposition: If (P, μ) is a pre-measure, then μ is countably subadditive.

Proof: Suppose $E \subseteq \bigcup_{j=1}^{\infty} F_j$. Then $E = E \cap \Big(\bigcup_{j=1}^{\infty} F_j\Big) = \bigcup_{j=1}^{\infty} (E \cap F_j)$.
Note that $\mu(E \cap F_j) \leq \mu(F_j)$, so it suffices to show the result for $E = \bigcup_{j=1}^{\infty} F_j$.
Define $H_1 = F_1$, and for $n > 1$, define $H_n = F_n \setminus \bigcup_{j=1}^{n-1} F_j$. Then clearly $H_1, H_2, \ldots$ are all disjoint, and clearly $\bigcup_{j=1}^{\infty} H_j = \bigcup_{j=1}^{\infty} F_j$. Now for each j, there are sets $\{G_{jk}\}_{k=1}^{n_j}$ such that $H_j = \bigcup_{k=1}^{n_j} G_{jk}$, so

$$\mu(E) = \mu\Big(\bigcup_{j=1}^{\infty} H_j\Big) = \sum_{j=1}^{\infty} \mu(H_j) = \sum_{j=1}^{\infty} \mu\Big(F_j \setminus \bigcup_{k=1}^{j-1} F_k\Big) =$$

$$= \sum_{j=1}^{\infty} \mu\Big(\sum_{k=1}^{n_j} G_{jk}\Big) = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \mu(G_{jk}) \leq \sum_{j=1}^{\infty} \mu(F_j).$$

■

4.3 Outer Measures

Def: Let $\mathcal{F}$ be a family of subsets of X. Let $A \subseteq X$. Then A is **countably covered** by $\mathcal{F}$ if there is a sequence $\{F_j\}$ of subsets of $\mathcal{F}$ with $A \subseteq \bigcup_{j=1}^{\infty} F_j$.

Let $\mathcal{H}(\mathcal{F})$ denote the collection of all subsets of X that are countably covered by $\mathcal{F}$. Properties of $\mathcal{H}(\mathcal{F})$:

(i) $\mathcal{H}(\mathcal{F})$ is a σ-ring

(ii) $\mathcal{H}(\mathcal{F})$ is **hereditary**, i.e., if $A \in \mathcal{H}(\mathcal{F})$, and $B \subseteq A$, then $B \in \mathcal{H}(\mathcal{F})$.

Given $\mathcal{H}(\mathcal{F})$, let $\mu : \mathcal{F} \to [0, \infty]$ be any function. Define $\mu^* : \mathcal{H}(\mathcal{F}) \to [0, \infty]$ by

$$\mu^*(A) = \inf \Big\{ \sum_{j=1}^{\infty} \mu(F_j) : \ A \subseteq \bigcup_{j=1}^{\infty} F_j, \ F_j \in \mathcal{F} \Big\}.$$

Def: Let $\mathcal{H}$ be a hereditary σ-ring of subsets on X. A function $\nu : \mathcal{H} \to [0, \infty]$ is an **outer measure** if

(i) $\nu(\varnothing) = 0$

(ii) ν is monotone

(iii) ν is countably subadditive

Proposition: $\mu^* : \mathcal{H}(\mathcal{F}) \to [0, \infty]$ defined as above is an outer measure.

Proof: Clearly, $\mu^*(\varnothing) = 0$, and monotonicity is trivial.
Suppose $A \subseteq \bigcup_{j=1}^{\infty} B_j$. Let $\epsilon > 0$, and suppose $\{\epsilon_j\}_{j=1}^{\infty}$ satisfies $\sum_{j=1}^{\infty} \epsilon_j < \epsilon$.
For each j, there is a collection $\{B_{jk}\}_{k=1}^{\infty} \subseteq \mathcal{F}$ such that $B_j \subseteq \bigcup_{k=1}^{\infty} B_{jk}$ and $\mu^*(B_j) + \epsilon_j > \sum_{k=1}^{\infty} \mu(B_{jk})$. Then

$$\mu^*(A) \le \mu^*\Big(\bigcup_{j=1}^{\infty} B_j\Big) \le \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(B_{jk}) < \sum_{j=1}^{\infty} \Big(\mu^*(B_j) + \epsilon_j\Big) < \sum_{j=1}^{\infty} \mu^*(B_j) + \epsilon.$$

It follows that $\mu^*(A) \le \sum_{j=1}^{\infty} \mu^*(B_j)$, so μ^* is countably subadditive. ■

Proposition: Let (P, μ) be a pre-measure. Then μ^* agrees with μ on P.

Proof: Let $E \in P$. Clearly, $\mu^*(E) \le \mu(E)$. However, if $E \subseteq \bigcup_{j=1}^{\infty} E_j$, then $\mu(E) \le \sum_{j=1}^{\infty} \mu(E_j)$, so $\mu^*(E) \ge \mu(E)$, hence $\mu^*(E) = \mu(E)$. ■

4.4 From Outer Measure to Measure

Def: Let ν be an outer measure on a hereditary σ-ring $\mathcal{H}$. Then $E \in \mathcal{H}$ is **ν-measurable** if for any $A \in \mathcal{H}$,

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c).$$

Note that by subadditivity, we always have $\nu(A) \leq \nu(A \cap E) + \nu(A \cap E^c)$.

From now on, let $\oplus$ denote a disjoint union.
Let $\mathcal{M}(\nu)$ denote the set of all ν-measurable sets in $\mathcal{H}$.

Caratheodory's Theorem $\mathcal{M}(\nu)$ is a σ-ring, and ν restricted to $\mathcal{M}(\nu)$ is a measure.

Proof: First, we show that $\mathcal{M}(\nu)$ is a ring.
Let $E, F \in \mathcal{M}(\nu)$. Let $A \in \mathcal{H}$. Then

$$\nu(A \cap (E \cup F)) + \nu(A \cap (E \cup F)^c) =$$

$$= \nu\Big((A \cap E) \oplus (A \cap E^c \cap F)\Big) + \nu\Big((A \cap E^c) \cap F^c\Big) \leq$$

$$\leq \nu(A \cap E) + \nu((A \cap E^c) \cap F) + \nu((A \cap E^c) \cap F^c) = \nu(A \cap E) + \nu(A \cap E^c) = \nu(A),$$

so $E \cup F \in \mathcal{M}(\nu)$. On the other hand,

$$\nu\Big(A \cap (E \cap F^c)\Big) + \nu\Big(A \cap (E^c \cup F)\Big) =$$

$$= \nu\Big((A \cap E) \cap F^c\Big) + \nu\Big((A \cap E^c) \oplus (A \cap E \cap F)\Big) \leq$$

$$\leq \nu(A \cap E \cap F^c) + \nu(A \cap E^c) + \nu(A \cap E \cap F) = \nu(A \cap E) + \nu(A \cap E^c) = \nu(A),$$

so $E \setminus F \in \mathcal{M}(\nu)$. Therefore $\mathcal{M}(\nu)$ is a ring.

We now show that ν is finitely additive on $\mathcal{M}(\nu)$.
Let $E, F \in \mathcal{M}(\nu)$ be disjoint, and let $A = E \cup F$. Then E "splits" A:

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c) = \nu(E) + \nu(F).$$

Furthermore, for E, F disjoint and for any $A \in \mathcal{H}$,

$$\nu\Big((A \cap E) \cup (A \cap F)\Big) = \nu(A \cap E) + \nu(A \cap F).$$

Finally, we show that $\mathcal{M}(\nu)$ is a σ-ring.
Let $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\nu)$. Since $\mathcal{M}(\nu)$ is a ring, we can construct, as done previously, disjoint sets $\{F_j\}_{j=1}^{\infty}$ such that $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j$, where $F_j \in \mathcal{M}(\nu)$. Let $A \in \mathcal{H}$. For any $m \geq 1$, we know that

$$\nu(A) = \nu\Big(A \cap \Big(\bigcup_{j=1}^{m} F_j\Big)\Big) + \nu\Big(A \cap \Big(\bigcup_{j=1}^{m} F_j\Big)^c\Big) \geq \sum_{j=1}^{m} \nu(A \cap F_j) + \nu\Big(A \cap \Big(\bigcup_{j=1}^{m} F_j\Big)^c\Big) \geq$$

$$\geq \nu\Big(\bigcup_{j=1}^{m}(A\cap F_j)\Big)+\nu\Big(A\cap\Big(\bigcup_{j=1}^{m}F_j\Big)^c\Big)=\nu\Big(A\cap\Big(\bigcup_{j=1}^{m}F_j\Big)\Big)+\nu\Big(A\cap\Big(\bigcup_{j=1}^{m}F_j\Big)^c\Big).$$

In particular, this holds for $A=\bigcup_{j=1}^{\infty}F_j$, so

$$\nu(A)=\sum_{j=1}^{\infty}\nu(A\cap F_j)=\sum_{j=1}^{\infty}\nu(F_j),$$

so ν restricted to its ν-measurable sets $\mathcal{M}(\nu)$ is a measure. ■

Proposition: Let (P,μ) be a pre-measure. From $\mathcal{H}(P)$ define the outer measure μ^* as done previously. Let $\mathcal{M}(\mu^*)$ denote the σ-ring of μ^*-measurable sets contained in $\mathcal{H}(P)$. then $P\subseteq\mathcal{M}(\mu^*)$.

Proof: Let $E,F\in P$, so that there are disjoint sets $G_1,\dots,G_n\in P$ such that $E\setminus F=\bigcup_{j=1}^{n}G_j$. Then $\mu^*(E\setminus F)\leq\sum_{j=1}^{n}\mu(G_j)$. By measurability,

$$\mu^*(E)=\mu^*(E\cap F)+\sum_{j=1}^{n}\mu^*(G_j)\geq\mu^*(E\cap F)+\mu^*(E\setminus F).$$

Therefore $\mu(E)\geq\mu(E\cap F)+\mu^*(E\setminus F)$. Now we must show that, given $A\in\mathcal{H}(P)$ and $E\in P$, then $\mu^*(A)\geq\mu^*(A\cap E)+\mu^*(A\setminus E)$.
Let $\epsilon>0$. There exists $\{F_j\}_{j=1}^{\infty}\subseteq P$ such that $A\subseteq\bigcup_{j=1}^{\infty}F_j$, and $\mu^*(A)+\epsilon>\sum_{j=1}^{\infty}\mu(F_j)$. Notice that $A\cap E\subseteq\bigcup_{j=1}^{\infty}(F_j\cap E)$, and $A\setminus E\subseteq\bigcup_{j=1}^{\infty}(F_j\setminus E)$. Therefore

$$\mu^*(A)+\epsilon\geq\sum_{j=1}^{\infty}\mu(F_j)=\sum_{j=1}^{\infty}\Big(\mu^*(F_j\cap E)+\mu^*(F_j\setminus E)\Big)\geq\mu^*(A\cap E)+\mu^*(A\setminus E).$$

It follows that $\mu^*(A)\geq\mu^*(A\cap E)+\mu^*(A\setminus E)$, so E is μ^*-measurable. ■

Proposition: Let (H,ν) be any outer measure. If $A\in\mathcal{H}$, and $\nu(A)=0$, then $A\in\mathcal{M}(\nu)$.

Proof: Let $B\in\mathcal{H}$. Then $\nu(B\cap A)=0$, since $B\cap A\subseteq A$, and also $\nu(B\cap A^c)\leq\nu(B)$, since $B\cap A^c\subseteq B$. Measurability of A follows immediately. ■

In other words, sets of zero outer measure are measurable.

Def: A measure $(\mathcal{S},\mu)$ is **complete** if whenever $E\subseteq\mathcal{S}$, and $\mu(E)=0$, then for all $A\subseteq E$, we have $A\in\mathcal{S}$ and $\mu(A)=0$.
Given a complete measure, sets whose measure is zero are called **null sets**.

- For a complete measure, the null sets form a hereditary σ-ring.

- For any outer measure (H, ν) the corresponding measure on $\mathcal{M}(\nu)$ is complete (from the last proposition).
- For a pre-measure (P, μ), μ^* on $\mathcal{S}(P)$ may not be complete. However, $\mathcal{S}(P) \subseteq \mathcal{M}(\mu^*)$, where $\mathcal{M}(\mu^*)$ is complete.
- It's easy to see that for $P = \{[a, b) : a < b\}$, that $\mathcal{S}(P)$ is exactly the Borel σ-algebra on $\mathbb{R}$ (countable operations on open intervals can yield these half-open intervals).
- Extensions of measures are in general not unique.

Def: Let $\mathcal{F}$ be a family of subsets of X, and let $\nu : \mathcal{F} \to [0, \infty]$. Then $A \in \mathcal{F}$ is **σ-finite** if there is $\{B_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$ such that $A \subseteq \bigcup_{j=1}^{\infty} B_j$, and $\nu(B_j) < \infty$ for all j.

We say that $(\mathcal{F}, \nu)$ is σ-finite if every $A \in \mathcal{F}$ is σ-finite. If $X \in \mathcal{F}$ and X is σ-finite, then we say that $(\mathcal{F}, \nu)$ is totally σ-finite.

Lemma: Let (P, μ) be a pre-measure. Let $\mathcal{S}$ be any σ-ring with $P \subseteq \mathcal{S} \subseteq \mathcal{M}(\mu^*)$. Let $\nu : \mathcal{S} \to [0, \infty]$ and let ν be countably subadditive, monotone, and $\nu|_P = \mu$. Then $\nu(A) \leq \mu^*(A)$ for all $A \in \mathcal{S}$ such that $A \in \mathcal{H}(P)$.

Proof: For any $\{E_j\}_{j=1}^{\infty} \subseteq P$ such that $A \subseteq \bigcup_{j=1}^{\infty} E_j$, we have

$$\nu(A) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

Since $\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(F_j) : \ A \subseteq \bigcup_{j=1}^{\infty} F_j \right\}$, it follows that $\nu(A) \leq \mu^*(A)$. ■

Theorem: Let (P, μ) be a σ-finite pre-measure. Then for any σ-ring $\mathcal{S}$ such that $P \subseteq \mathcal{S} \subseteq \mathcal{M}(\mu^*)$, and any measure ν on $\mathcal{S}$ such that $\nu|_P = \mu$, we have $\nu = \mu^*|_{\mathcal{S}}$.

Proof: Let $A \in \mathcal{S}$.

- Step 1: Suppose $A \subseteq E$, $E \in P$, $\mu(E) < \infty$. Then $E = A \oplus (E \setminus A)$, so by measurability of A, we have $\nu(E) = \nu(A) + \nu(E \setminus A) \leq \mu^*(A) + \mu^*(E \setminus A) = \mu^*(E) = \mu(E) = \nu(E)$. Thus $\nu(A) + \nu(E \setminus A) = \mu^*(A) + \mu^*(E \setminus A)$. But $\nu(A) \leq \mu^*(A)$, and $\nu(E \setminus A) \leq \mu^*(E \setminus A)$, and since $\mu^*(A) \leq \mu^*(E) < \infty$, and $\mu^*(E \setminus A) \leq \mu^*(E) < \infty$, it follows that in fact, $\nu(A) = \mu^*(A)$, and $\nu(E \setminus A) = \mu^*(E \setminus A)$.
- Step 2: For general $A \in \mathcal{S}$, there is $\{E_j\} \subseteq P$, $A \subseteq \bigcup_{j=1}^{\infty} E_j$, and $\mu(E_j) < \infty$ for all j. Without loss of generality, we can let $\{E_j\}$ be disjoint. Then $A = \bigcup_{j=1}^{\infty} (A \cap E_j)$, so $\nu(A) = \sum_{j=1}^{\infty} \nu(A \cap E_j)$, but $A \cap E_j \subseteq E_j$, so by step 1, $\nu(A \cap E_j) = \mu^*(A \cap E_j)$, implying that $\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A \cap E_j)$.

■

Def: On $\mathbb{R}$, define $\alpha(x) = x$, and consider the pre-measure μ_α defined by $\mu_\alpha([a,b)) = b-a$, which can be extended to $\mathcal{M}(\mu_\alpha^*)$. Then μ_α restricted to $\mathcal{M}(\mu_\alpha^*)$ is **Lebesgue measure** on $\mathbb{R}$.

- Lebesgue measure is translation invariant, i.e., if $A \in \mathcal{M}(\mu_\alpha^*)$, and $r_0 \in \mathbb{R}$, then if $r_0 + A = \{r_0 + a : a \in A\}$, we have $\mu_\alpha(r_0 + A) = \mu_\alpha(A)$.
- Above, we can let α be any non-decreasing, left-continuous function. Then the resulting measure is the general **Lebesgue-Stieltjes** measure.

Proposition: (Continuity from below): Let $(\mathcal{S}, \mu)$ be a measure. If $\{E_j\} \subseteq \mathcal{S}$, and $E_j \subseteq E_{j+1}$ for all j, and if $E = \bigcup_{j=1}^\infty E_j$, then $\mu(E_j) \to \mu(E)$.

Proof: Observe that $E = E_1 \oplus \left(\bigoplus_{j=1}^\infty E_{j+1} \setminus E_j\right)$, so
$\mu(E) = \mu(E_1) + \sum_{j=1}^\infty \mu(E_{j+1} \setminus E_j) = \lim\limits_{m\to\infty} \mu(E_{m+1})$. ∎

Observe that if E_j is a decreasing sequence of subsets, i.e., $E_{j+1} \subseteq E_j$ for all j, and if $E = \bigcap_{j=1}^\infty E_j$, then the analogue of the above proposition for intersections is not necessarily true. Consider $E_j = [j, \infty)$. However, we can say the following:

Proposition: (Continuity from above): If $\{E_j\} \subseteq \mathcal{S}$, and $E_{j+1} \subseteq E_j$ for all j, with at least one $\mu(E_j) < \infty$, and if $E = \bigcap_{j=1}^\infty E_j$, then $\mu(E_j) \to \mu(E)$.

Proof: Note that $\mu(E) < \infty$, and furthermore, suppose $\mu(E_N) < \infty$. Then for all $n \geq N$, $\mu(E_n) < \infty$. Then the proof follows in a similar manner as above. ∎

4.5 Non-Measurable Sets

The construction of non-measurable sets often makes use of the axiom of choice. The following is a well-known example.
Ex: Let $T = \{z \in \mathbb{C} : \ |z| = 1\}$. Let $H = \{z \in T : \exists n \in \mathbb{Z} \text{ such that } z^n = 1\}$. Then T is a compact group under multiplication, and H is a subgroup, so we can form the cosets of H. Let A be a set that contains exactly one element from each coset of H. By rotation invariance, $\mu(hA) = \mu(A)$ for any $h \in H$. Also, since H is countable, $T = \bigoplus_{h\in H} hA$, so $\mu(T) = \sum_{h\in H} \mu(hA) = \sum_{h\in H} \mu(A)$.
Clearly, $0 < \mu(T) < \infty$. If $\mu(A) = 0$, then this would yield $\mu(T) = 0$, a contradiction. But if $\mu(A) > 0$, then $\mu(T) = \infty$, a contradiction. Therefore A is not measurable.

Chapter 5

Integration

5.1 Fundamentals

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let B be a Banach space.
To start simply, suppose $\mathcal{S}$ is a ring, and μ is finitely-additive.
Let $f = b\chi_E$, for $b \in \mathbb{R}$, and $E \in \mathcal{S}$. That is, $f(x) = \begin{cases} b, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$.

Def: Given $(X, \mathcal{S})$, a function $f : X \to B$ is a **simple $\mathcal{S}$-measurable function** if its range is finite, and for each $b \in \mathrm{Im}(f)$, we have $\{x : f(x) = b\} \in \mathcal{S}$. Then f can be written as $f = \sum_{j=1}^{n} b_j \chi_{E_j}$, where the b_j's are distinct, and the E_j's are disjoint.

It is clear that if $f = \sum_{j=1}^{\infty} b_j \chi_{E_j}$, where the E_j's are disjoint, but the b_j's are not all distinct, then f is simple $\mathcal{S}$-measurable.

Proposition: If f, g are simple $\mathcal{S}$-measurable, then so is $f + g$.

Proof: Suppose $f = \sum_{j=1}^{\infty} b_j \chi_{E_j}$, and for simplicity, $g = c\chi_F$. The general case follows easily.
Let $E_{n+1} = F \setminus \bigoplus_{j=1}^{n} E_j$, and let $b_{n+1} = 0$, so $f = \sum_{j=1}^{n+1} b_j \chi_{E_j}$ with $F \subseteq \bigoplus_{j=1}^{n+1} E_j$.
Therefore $F = \bigoplus_{j=1}^{n+1} (F \cap E_j)$.
Then $f = \sum_{j=1}^{n+1} b_j (\chi_{F \cap E_j} + \chi_{E_j \setminus F})$, so that

$$f + g = \sum_{j=1}^{n+1} (b_j + c)\chi_{E_j \cap F} + \sum_{j=1}^{n} b_j \chi_{E_j \setminus F}.$$

Then $f + g$ is simple $\mathcal{S}$-measurable. ■

Def: The **integral** (with respect to the measure μ) of a simple $\mathcal{S}$-measurable function $f = \sum_{j=1}^n b_j \chi_{E_j}$ is

$$\int f \, d\mu = \sum_{j=1}^n b_j \mu(E_j).$$

Simple measurable function will be abbreviated as SMF.

Def: If a simple measurable function has a finite integral with respect to the measure μ, it is called a **simple μ-integrable function**, abbreviated as SIF.

The following facts have easy proofs.

Proposition:

(1) If f, g are SIFs and $c \in \mathbb{R}$, then $\displaystyle\int (cf + g) \, d\mu = c \int f \, d\mu + \int g \, d\mu$.

(2) If f is a SMF, then the map $x \mapsto \|f(x)\|_B$ is a SMF.

(3) If f is a real-valued SIF, and if $f \geq 0$, then $\displaystyle\int f \, d\mu \geq 0$.

(4) If f, g are real-valued SIFs with $f \geq g$, then $\displaystyle\int f \, d\mu \geq \int g \, d\mu$.

Proof: *Exercise.* ∎

Observe that the simple measurable functions form a vector space (closed under addition and scalar multiplication). Also, $f \mapsto \displaystyle\int f \, d\mu$ is a linear functional.

Def: The **L^1 norm** of a measurable function is $\|f\|_1 := \displaystyle\int \|f(x)\|_B \, d\mu(x)$.

Proposition: If f, g are B-valued SIFs, then $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

Proof:

$$\int \|f(x) + g(x)\|_B \, d\mu(x) \leq \int \|f(x)\|_B \, d\mu(x) + \int \|g(x)\|_B \, d\mu(x) = \|f\|_1 + \|g\|_1.$$

∎

5.2 Towards $\mathcal{L}^1$

Let $\mathrm{SIF}(S, \mu)$ denote the vector space of simple μ-integrable functions.

- If f is a SIF, and $r \in \mathbb{R}$, then $\|rf\|_1 = |r| \cdot \|f\|_1$. Therefore $\|\cdot\|_1$ is a seminorm on $\mathrm{SIF}(S, \mu)$.
- For any $f : X \to B$, let $\mathcal{C}(f) = \{x : f(x) \neq 0\}$. If f is a SIF, then $\mathcal{C}(f) \in \mathcal{S}$.
- Let $\mathcal{N} = \{f \in \mathrm{SIF}(S, \mu) : \ \mu(\mathcal{C}(f)) = 0\}$. Then $\mathcal{N}$ is a vector subspace of $\mathrm{SIF}(S, \mu)$.
- If $f \in \mathcal{N}$, then clearly $\displaystyle\int f \, d\mu = 0$. Conversely, if $f \in \mathrm{SIF}(S, \mu)$, and $\|f\|_1 = 0$, then $\sum \|b_j\| \, \chi_{E_j} = 0$ iff $\mu(E_j) = 0$ whenever $b_j \neq 0$, implying that $f \in \mathcal{N}$.
- Thus, on $\mathrm{SIF}(S, \mu)/\mathcal{N}$, $\|\cdot\|_1$ becomes a norm, and $f \mapsto \displaystyle\int f \, d\mu$ is well-defined. Therefore, $\mathrm{SIF}(S, \mu)/\mathcal{N}$ is a normed vector space, so we can then consider its abstract completion (recall this is done by equivalence classes of Cauchy sequences).
- We will try to find a useful "concrete realization" of the elements of the completion, which we will see is the space $L^1(X, \mathcal{S}, \mu)$.
- Let $\{b_n\}$ be a Cauchy sequence in B, and let $E \in \mathcal{S}$ with $0 < \mu(E) < \infty$. Then $\{b_n \chi_E\}$ is a Cauchy sequence in $\mathrm{SIF}(S, \mu)/\mathcal{N}$.
- Suppose $\{E_j\}$ is a sequence of disjoint sets with $0 < \mu(E_j) < 2^{-j}$. Let $E = \bigoplus_{j=1}^{\infty} E_j$ and let $F_n = \bigoplus_{j=1}^{n} E_j$.

  Let $b \in B, \ b \neq 0$. Suppose $\{b\chi_{F_n}\}$ is a Cauchy sequence. Then $\displaystyle\int b\chi_{F_n} \, d\mu = b \, \mu(F_n)$, so that $\displaystyle\int b\chi_{F_n} \, d\mu \to \int b\chi_E \, d\mu = b \, \mu(E)$.

Def: Given a measurable space $(X, \mathcal{S})$, let $f : X \to B$. We say that f is **$\mathcal{S}$-measurable** if there is a sequence of SMFs $\{f_n\}$ converging pointwise to f.

Def: Let $(X, \mathcal{S}, \mu)$ be a measure space. $A \subseteq X$ is a **null set** if there is a set $E \in \mathcal{S}$ such that $A \subseteq E$ and $\mu(E) = 0$. The μ-null sets form a σ-ring.

We say $f_n \to f$ μ-almost everywhere (a.e.) if $f_n \to f$ pointwise, except on a null set.

Def: Let B be a Banach space. Let $f : X \to B$ (or at least f is defined on $X \setminus A$, for a null set A). We say that f is **μ-measurable** if there is a sequence $\{f_n\}$ of B-valued SMFs such that $f_n \to f$ μ-almost everywhere.

- The $\mathcal{S}$-measurable or μ-measurable functions form a vector space.

- The function $x \mapsto \|f(x)\|$ is $\mathcal{S}$-or-μ-measurable.
- If f, g are measurable real-valued functions, then $\max(f, g)$ and $\min(f, g)$ are measurable.
- If f, g are measurable, f is real-valued and g is B-valued, then fg is measurable.

Let $\mathcal{M}(X, \mathcal{S}, B)$ denote the vector space of $\mathcal{S}$-measurable functions from X to B.
Let $\mathcal{M}(X, \mathcal{S}, \mu, B)$ denote the vector space of μ-measurable functions from X to B.

Note: If $\{f_n\}$ is a sequence of SMFs, and if $f_n \to f$ pointwise, then the closure of the range of f contains a countably-dense subset.

Def: Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is **separable** if it contains a countably-dense subset, or its closure contains a countably-dense subset.

Then if f is measurable, $\text{range}(f)$ is separable.

Proposition: If $\{f_n\}$ is a sequence of functions such that $\text{range}(f_n)$ is separable for all n, and if $f_n \to f$ pointwise, then $\text{range}(f)$ is separable.

Proof: Let $E = \overline{\left(\bigcup_{n=1}^{\infty} \text{range}(f_n)\right)}$. Then clearly E is separable, and furthermore $\text{range}(f) \subseteq E$, so $\text{range}(f)$ is separable. ■

Observe that if f is a B-valued SMF, and if $\mathcal{O} \subseteq B$ is open with $0 \notin \mathcal{O}$, then $f^{-1}(\mathcal{O}) \in \mathcal{S}$.

Lemma: Let $\{f_n\}$ be a sequence of B-valued functions, and assume that each f_n has the property that $f_n^{-1}(\mathcal{O}) \in \mathcal{S}$ for every open $\mathcal{O} \subseteq B$, $0 \notin \mathcal{O}$. If $f_n \to f$ pointwise, then f has this same property.

Proof: Let $x \in X$. Then $x \in f^{-1}(\mathcal{O})$ iff $f(x) \in \mathcal{O}$.
For each n, let $\mathcal{O}_n = \{b \in \mathcal{O} : \text{ distance}(b, \mathcal{O}^c) > 1/n\}$.
Observe that $\overline{O_n} \subseteq \mathcal{O}_{n+1}$. Then $f(x) \in \mathcal{O}$ iff there exists n such that $f(x) \in \mathcal{O}_n$, and there is $K > 0$ such that for all $k \geq K$, $f_k(x) \in \mathcal{O}_n$ iff there exist n, K such that $x \in \bigcap\limits_{k \geq K} f_k^{-1}(\mathcal{O}_n)$ iff $\bigcup\limits_{n \in \mathbb{N}} \bigcup\limits_{K \in \mathbb{N}} \bigcap\limits_{k \geq K} f_k(\mathcal{O}_n) \in \mathcal{S}$. This implies that $x \in f^{-1}(\mathcal{O})$ iff $x \in \bigcup\limits_{n \in \mathbb{N}} \bigcup\limits_{K \in \mathbb{N}} \bigcap\limits_{k \geq K} f_k^{-1}(\mathcal{O})$, thus $f^{-1}(\mathcal{O}) \in \mathcal{S}$. ■

Corollary: If $\{f_n\}$ is a sequence of measurable functions and $f_n \to f$ pointwise, then for any open $\mathcal{O} \in B$ with $0 \notin \mathcal{O}$, we have $f^{-1}(\mathcal{O}) \in \mathcal{S}$. That is, if f is $\mathcal{S}$-measurable, then f satisfies the above property.

Theorem: Let $f : X \to B$ satisfy (i) for any open $\mathcal{O} \in B,\ \ 0 \notin \mathcal{O}$, we have $f^{-1}(\mathcal{O}) \in \mathcal{S}$, (ii) range$(f)$ is separable. Then f is $\mathcal{S}$-measurable.

Proof: Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of points in B that is dense in the range of f. For i, j, let $C_{j,i} = \{x : \|f(x) - b_j\| < 1/j\} = f^{-1}\Big(B_{1/j}(b_i)\Big) \setminus \{0\}$, where $f(x) \neq 0$. Order the pairs (i, j) lexicographically, i.e.,

$$(j, i) \leq (k, n) \text{ if } \begin{cases} j < k, \text{ or} \\ j = k \text{ and } i < n \end{cases} .$$

This is a total order. Given n, consider the pairs $(i, j) \leq (n, n)$, and disjointize the C_{ji}'s.
Set $E_{ji} = C_{ji} \setminus \bigcup\{C_{k\ell} :\ (j, i) < (k, \ell) \leq (n, n)\} \subseteq \mathcal{S}$.
Let $f_n = \sum_{j,i=1}^{n} b_i \chi_{E_{ji}}$ be a SMF. We claim that $f_n \to f$ pointwise. Suppose x satisfies $f(x) \neq 0$. Let $\epsilon > 0$. Choose j_0 such that $j_0^{-1} < \epsilon$.
Then choose i_0 such that $\|f(x) - b_{i_0}\| < j_0^{-1} >$ Let $N = \max(i_0, j_0)$, and let $n \geq N$. To show $\|f_n(x) - f(x)\| < \epsilon$, given $n \geq N$ we claim that $x \in C_{j_0,i_0}$.
Let $(\ell, k) = \max\{(j, i) :\ x \in C_{ji},\ j \leq n,\ i \leq n\}$.
Then $x \in E_{\ell k}$, so $\|f(x) - b_k\| < \ell^{-1} \leq j_0^{-1} < \epsilon$.
Therefore $f_n(x) = b_k$, so $\|f(x) - f_n(x)\| < \epsilon$. ■

Before continuing, it is important to introduce and discuss the different modes of convergence of functions.

Egoroff's Theorem: Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of μ-measurable functions. Let $E \in \mathcal{S},\ \ \mu(E) < \infty$. Assume that $f_n \to f$ pointwise on E. Then for any $\epsilon > 0$, there is a subset $F \subseteq E$, with $F \in \mathcal{S}$, such that $\mu(E \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F. That is, given $\delta > 0$, there is N such that for $n > N$, we have $\|f_n(x) - f(x)\| < \delta$ for all $x \in F$.

Proof: For m, k, let $G_m^k = \{x \in E :\ \|f(x) - f_k(x)\| > m^{-1}\} \in \mathcal{S}$.
Let $F_m^n = \bigcup\limits_{k \geq n} G_m^k \in\ Sm$.

Fix m. As $n \to \infty,\ \ F_m^n \downarrow \varnothing$. Since $F_m^n \subseteq E,\ \ \mu(E) < \infty$, we have $\mu(F_m^n) \to 0$.
Let $\epsilon > 0$. For each m, choose n_m such that $\mu(F_m^{n_m}) < \epsilon 2^{-m}$.
Let $H = \bigcup\limits_{m=1}^{\infty}$. Then $\mu(E) < \epsilon$, and let $F = E \setminus H$, so that $\mu(E \setminus F) < \epsilon$. We claim that $\{f_n\}$ converges uniformly to f on F.
Let $\delta > 0$. Choose m_0 such that $m_0^{-1} < \delta$. Then for $x \in F,\ \ x \notin H$, so $x \notin F_{m_0}^{n_{m_0}}$, so for all $k \geq n_{m_0}$, we have $\|f(x) - f_k(x)\| \leq m_0^{-1} < \delta$. ■

Note: The result does not hold for $\mu(E) = \infty$. Let $X = \mathbb{R}$ with Lebesgue measure. Then $f_n = \chi_{[n,n+1]}$ converges pointwise to $f = 0$, but not uniformly.

Def: Given $(X, \mathcal{S}, \mu)$ and measurable function $\{f_n\}$, f, and $E \in \mathcal{S}$, we say that $\{f_n\}$ converges to f **almost uniformly** on E if for any $\epsilon > 0$, there is $F \in \mathcal{S}$, $F \subseteq E$, such that $\mu(E \setminus F) < \epsilon$, and $f_n \to f$ uniformly on F.

Note: Almost uniform convergence implies converges a.e.

Proposition: If $\{f_n\}$ converges to f uniformly on E, then $f_n \to f$ pointwise except possibly on a null set.

Proof: For each n, let $F_n \subseteq E$ be such that $\mu(E \setminus F_n) < 1/n$, and $f_m \to f$ uniformly on F_n. Let $G = \bigcup_{n=1}^{\infty} F_n$, so $E \setminus G \subseteq E \setminus F_n$ for each n, $\mu(E \setminus G) = 0$. But $f_m \to f$ uniformly, hence pointwise, on each F_n so $f_m \to f$ pointwise on G. ■

Def: Let $(X, \mathcal{S}, \mu)$ be a measure space, and let B be a Banach space. Let $E \in \mathcal{S}$. Let $\{f_n : X \to B\}$ be a sequence of μ-measurable functions. We say that $\{f_n\}$ is **almost uniformly Cauchy** on E if for any $\epsilon > 0$, there is $F \subseteq E$ such that $\mu(E \setminus F) < \epsilon$, and $\{f_n\}$ is Cauchy on F, i.e., for any $\delta > 0$, there is N such that for $m, n \geq N$, we have $\|f_n(x) - f_m(x)\| < \delta$ for all $x \in F$.

Proposition: If $\{f_n\}$ is almost uniformly Cauchy on E, then there is f on E such that $f_n \to f$ almost uniformly.

Proof: Given $\epsilon > 0$, find $F \subseteq E$ with $\mu(E \setminus F) < \epsilon$, and $\{f_n\}$ is uniformly Cauchy on F. By completeness, there is a limist f, but this depends on f. Take F_n, $\mu(E \setminus F_n) < 1/n$, so we get f defined on all of $G := \bigcup_{n=1}^{\infty} F_n$. Therefore $\mu(E \setminus G) = 0$. ■

Def: Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions, and let f be a measurable function. We say that $\{f_n\}$ **converges in measure** to f if for any $\epsilon > 0$,

$$\mu\Big(\{x : \|f(x) - f_n(x)\| \geq \epsilon\}\Big) \to 0 \text{ as } n \to \infty.$$

Proposition: If $\{f_n\}$ converges to f almost uniformly on E, then $\{f_n\}$ converges to f in measure.

Proof: Let $\epsilon, \delta > 0$. Choose $N \in \mathbb{N}$ such that for $n \geq N$, we have $\mu\Big(\{x : \|f(x) - f_n(x)\| \geq \epsilon\}\Big) < \delta$. Let $F \subseteq E$ satisfy

(i) $\mu(E \setminus F) < \delta$

(ii) $f_n \to f$ uniformly on F.

Now "update" N so that for $n \geq N$, we also $\|f_n(x) - f(x)\| < \epsilon$ for all $x \in F$. Then for $n \geq N$, we have

$$\{x \in E : \|f(x) - f_n(x)\| \geq \epsilon\} \subseteq E \setminus F \implies \mu\Big(\{x \in E : \|f(x) - f_n(x)\| \geq \epsilon\}\Big).$$

■

Proposition: Suppose there are two measurable functions f, g such that $f_n \to f$ and $f_n \to g$ in measure on E. Then $f = g$ a.e.

Proof: Let $\epsilon > 0$. Then for $n \in \mathbb{N}$, the set $\{x \in E : \|f(x) - g(x)\| \geq \epsilon\}$ is contained in the set $\{x \in E : \|f_n(x) - f(x)\| \geq \frac{\epsilon}{2}\} \cup \{x \in E : \|f_n(x) - g(x)\| \geq \frac{\epsilon}{2}\}$. By monotonicity and subadditivity of measure, it follows that
$\mu\Big(\{x \in E : \|f(x) - g(x)\| > 0\}\Big) = 0$, by letting $\epsilon \to 0$, since $f_n \to f$ and $f_n \to g$ converge in measure. ■

Def: $\{f_n\}$ is **Cauchy in measure** on E if for any $\epsilon > 0$,

$$\mu\Big(\{x : \|f_n(x) - f_m(x)\| \geq \epsilon\}\Big) \to 0 \text{ as } n, m \to \infty.$$

Proposition: Let $\{f_n\}$ be a sequence of SIFs that is Cauchy for $\|\cdot\|_1$. Then $\{f_n\}$ is Cauchy in measure.

Proof: Given $\epsilon > 0$, let $A_\epsilon = \{x \in E : \|f_n(x) - f_m(x)\| \geq \epsilon\}$. Then

$$\mu(A_\epsilon) = \int \chi_{A_\epsilon}(x)\, d\mu(x) \leq \frac{1}{\epsilon} \int \|f_n(x) - f_m(x)\|\, d\mu(x) \to 0$$

as $n, m \to \infty$. ■

Riesz-Weyl Theorem: If $\{f_n\}$ is a sequence Cauchy in measure on E, then there is a subsequence that is almost uniformly Cauchy.

Proof: Define $\{n_k\}$ as follows:
Let $n_1 = 1$. Given n_{k-1}, consider $E_{m,\ell} = \{x \in E : \|f_m(x) - f_\ell(x)\| \geq 2^{-k}\}$.
Then choose n_k such that for $m, \ell > n_k$, we have $\mu(E_{m,\ell}) < 2^{-k}$ and $n_k > n_{k-1}$. We now show that $\{f_{n_k}\}$ is uniformly Cauchy.
Let $\epsilon > 0$. Choose K such that $\sum_{k=K}^{\infty} 2^{-k} < \epsilon$.

Let $F = E \setminus \bigcup_{k=K}^{\infty} \{x : \|f_{n_k}(x) - f_{n_{k+1}}(x)\| \geq 2^{-k}\}$.
For $k \geq K$, we have $\mu\Big(\{x : \|f_{n_k}(x) - f_{n_{k+1}}(x)\| \geq 2^{-k}\}\Big) < 2^{-k}$, so $\mu(E \setminus F) < \sum_{k=K}^{\infty} 2^{-k} < \epsilon$.
It remains to show that $\{f_{n_k}\}$ is Cauchy on F.
Let $\delta > 0$. Choose $N > K$ such that $\sum_{n=N}^{\infty} 2^{-n} < \delta$.
Then for $j > \ell > N$, and for $x \in F$,

$$\left\|f_{n_j}(x) - f_{n_\ell}(x)\right\| \leq \sum_{i=\ell}^{j-1} \|f_{n_{i+1}}(x) - f_{n_i}(x)\| \leq \sum_{i=\ell+1}^{j} 2^{-i} < \delta.$$

■

Proposition: Let $\{f_n\}$ be Cauchy in measure, and suppose it has a subsequence $\{f_{n_k}\}$ that converges almost uniformly to some f. Then $\{f_n\}$ converges to f in measure.

Proof: Given $\epsilon > 0$, consider $\{x \in E : \|f(x) - f_n(x)\| > \epsilon\}$. Observe that this set is contained in
$\{x \in E : \|f(x) - f_{n_k}(x)\| > \epsilon/2\} \cup \{x \in E : \|f_{n_k}(x) - f_n(x)\| > \epsilon/2\}$. Since $f_{n_k} \to f$ almost uniformly, we can choose N such that for $k \geq N$, and for $\delta > 0$, we have
$\mu\Big(\{x \in E : \|f(x) - f_{n_k}(x)\| > \epsilon/2\}\Big) < \delta/2$, and then we can choose $N_2 \geq N$ such that
$\mu\Big(\{x \in E : \|f_{n_k}(x) - f_n(x)\| > \epsilon/2\}\Big) < \delta/2$. Thus for $n > N_2$, we have
$\mu\Big(\{x \in E : \|f(x) - f_n(x)\| > \epsilon\}\Big) < \delta$, ■

Let $\{f_n\}$ be a sequence of SIFs, Cauchy for $\|\cdot\|_1$. Then $\{f_n\}$ is Cauchy in measure, and it has a subsequence almost uniformly Cauchy, hence converging to some f almost uniformly. Finally, $f_n \to f$ in measure, where f is unique a.e.

Proposition: If $\{f_n\}$ and $\{g_n\}$ are Cauchy for $\|\cdot\|_1$, and are "equivalent" in the sense that $\|f_n - g_n\|_1 \to 0$, and if $f_n \to f$ in measure, then so does $\{g_n\}$.

Proof: Let $\{h_n\}$ be the sequence $\{f_1, g_1, f_2, g_2, \ldots\}$. Then clearly h_n is Cauchy for $\|\cdot\|_1$ and thus has a subsequence, namely, $\{f_n\}$, converging to f in measure, and so $h_n \to f$ in measure. It follows that $g_n \to f$ in measure. ■

On the space $\mathcal{M}(X, \mathcal{S}, \mu, B)$ of μ-measurable, B-valued functions, define $f \sim g$ iff $f = g$ a.e.

Def: $\{f_n\}$ is **mean Cauchy** if it is Cauchy for $\|\cdot\|_1$.

Lemma: If $\{h_n\}$ is a mean Cauchy sequence converging to 0 almost uniformly, then $\|h_n\|_1 \to 0$.

Proof: Let $\epsilon > 0$. Choose N such that for $m, n \geq N$, $\|h_n - h_m\| < \epsilon$.
Let $E = \{x : \ h_N(x) \neq 0\}$, so $\mu(E) < \infty$. Now find $F \subseteq E$ with $\mu(F) \neq 0$ so that $\mu(E \setminus F) < \frac{\epsilon}{1+\|h_N\|_\infty}$, and $\{h_n\}$ converges uniformly to 0 on F. For given $n \geq N$,

$$\int_{E^c} \|h_n(x)\| \ d\mu(x) = \int_{E^c} \|h_n(x) - h_N(x)\| \ d\mu(x) \leq \|h_n - h_N\|_1 < \epsilon.$$

Choose $N_1 > N$ so that for $n > N_1$ and $x \in F$, by uniform convergence, $\|h_n(x)\| < \frac{\epsilon}{\mu(F)}$, so for $n > N$, $\displaystyle\int_F \|h_n(x)\| \ d\mu(x) \leq \int_F \frac{\epsilon}{\mu(F)} \ d\mu(x) = \epsilon$. Therefore,

$$\int_{E \setminus F} \|h_n(x)\| \ d\mu(x) \leq \int_{E \setminus F} \|h_n(x) - h_N(x)\| \ d\mu(x) + \int_{E \setminus F} \|h_N(x)\| \ d\mu(x) \leq$$

$$\leq \|h_n - h_N\|_1 + \mu(E \setminus F) \|h_N\|_\infty < 2\epsilon + \frac{\epsilon}{1 + \|h_N\|_\infty} \cdot \|h_N\|_\infty < 4\epsilon.$$

■

Proposition: If $\{f_n\}$ and $\{g_n\}$ are mean Cauchy sequences, and if they both converge to the a.e.-same function in measure, then $\{f_n\}$ and $\{g_n\}$ are equivalent Cauchy sequences.

Proof: $\{f_n\}$ and $\{g_n\}$ have subsequences that converge almost uniformly, and the limit is unique a.e. It suffices to show that the subsequences are equivalent Cauchy sequences. Let $h_k = f_{n_k} - g_{n_k}$. Then $\{h_k\}$ is a mean Cauchy sequence that converges uniformly to 0. But by the lemma, it follows that $\{f_{n_k}\}$ and $\{g_{n_k}\}$ are equivalent Cauchy sequences. ■

Theorem: Let $f \in \mathcal{M}(X, \mathcal{S}, \mu, B)$. The following are equivalent. There is a mean Cauchy sequence $\{f_n\}$ of SIFs that converges to f

(i) in measure

(ii) almost uniformly

(iii) almost everywhere

Proof: (i) $\implies$ (ii) by Riesz-Weyl. The other two implications are exercises. ■

5.3 The Space $\mathcal{L}^1$

Def: Let $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ be the subset of $\mathcal{M}(X, \mathcal{S}, \mu, B)$ of **μ-integrable** functions, where a function is μ-integrable if it satisfies any of properties (i), (ii), (iii) above.

That is, for $f \in \mathcal{L}^1(X, \mathcal{S}, \mu)$, there is a mean Cauchy sequence of SIFs that converges to f in measure, almost uniformly, and almost everywhere.

Def: Let $\{f_n\}$ be a mean Cauchy sequence converging to f. Then the **integral** of f is

$$\int f \, d\mu := \lim_{n\to\infty} \int f_n \, d\mu.$$

The following propositions have easy proofs.

Proposition: If $f, g \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, and $a \in \mathbb{R}$, then:

(i) $\displaystyle\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu.$

(ii) $\displaystyle\int af \, d\mu = a \int f \, d\mu.$

(iii) If $f \in \mathcal{L}^1$, then the map $x \mapsto \|f(x)\|$ is in $\mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$.

(iv) If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$, and if $f \geq 0$ a.e., then $\displaystyle\int f \, d\mu \geq 0.$

(v) If $f, g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$, and if $f \geq g$ a.e., then $\displaystyle\int f \, d\mu \geq \int g \, d\mu.$

(vi) If $f, g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$, then $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

(vii) If $f \in \mathcal{L}^1$, and $\{f_n\}$ is a mean Cauchy sequence of SIFs converging to f, then $\|f_n\|_1 \to \|f\|_1$.
Thus, $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is complete for $\|\cdot\|_1$.

(viii) If $E, F \in \mathcal{S}$ and $E \cap F = \varnothing$, then $\displaystyle\int_{E\cap F} f \, d\mu = \int_E f \, d\mu + \int_F f \, d\mu.$

(ix) If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, and $f \geq 0$, and if $E, F \in \mathcal{S}$ and $F \subseteq E$, then $\displaystyle\int_F f \, d\mu \leq \int_E f \, d\mu.$

Proof: *Exercise.* ■

Def: The **indefinite integral** of f, denoted μ_f is the function on $\mathcal{S}$ defined by

$$\mu_f(E) = \int_E f \, d\mu = \int f\chi_E \, d\mu.$$

In fact, μ_f is a B-valued measure, i.e., countably additive (proven later).

Proposition: If $f \in \mathcal{L}^1$, then $\mathcal{C}(f) := \{x \in X : f(x) \neq 0\}$ is σ-finite.

Proof: Let $\{f_n\}$ be a mean Cauchy sequence of SIFs converging to f a.e. Then $\mu(\{x \in X : f_n(x) \neq 0\}) < \infty$ for all n. Let $D = \bigcup_{n=1}^{\infty}\{x \in X : f_n(x) \neq 0\}$, so D is σ-finite. Clearly, $\mathcal{C}(f) \subseteq D$, except possibly on a null set. ■

Thus, we can see that for $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, the indefinite integral μ_f is indeed a B-valued measure.

Proposition: If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then for any $\epsilon > 0$, there exists $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\left\| \int_{X \setminus E} f \, d\mu \right\| < \epsilon$.

Proof: Find a SIF g such that $\|f - g\|_1 = \int_{X \setminus E} \|f(x) - g(x)\| \, d\mu(x) < \epsilon$, where $E = \{x : g(x) \neq 0\}$. ■

Proposition: (Absolute continuity) If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then $\|\mu_f(E)\| < \epsilon$.

Proof: Choose a SIF g such that $\|f - g\|_1 < \epsilon/2$.
Then $\|f\|_1 \leq \|f - g\|_1 + \|g\|_1$. Therefore if $\mu(E) < \dfrac{\epsilon}{2\|g\|_\infty}$, then

$$\|\mu_f(E)\| = \left\| \int_E f \, d\mu \right\| \leq \frac{\epsilon}{2} + \left\| \int_E g \, d\mu \right\| < \frac{\epsilon}{2} \cdot \mu(E) \cdot \|g\|_\infty < \epsilon.$$

■

Lebesgue-Dominated Convergence Theorem: Let $f_n \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ for each n, and suppose $f_n \to f$ a.e. If there is a $\mathbb{R}$-valued $g \in \mathcal{L}^1$ such that $\|f_n(x)\|_B \leq g(x)$ a.e. for all n, then $\{f_n\}$ is a mean-Cauchy sequence.

Proof: Let $\epsilon > 0$. Find E such that $\mu(E) < \infty$, and $\int_{X\setminus E} g\,d\mu < \epsilon/6$. Then

$$\int_{X\setminus E} \|f_n(x) - f_m(x)\|_B \; d\mu(x) \le \int_{X\setminus E} \|f_n(x)\|_B + \|f_m(x)\|_B \; d\mu(x) \le \int_{X\setminus E} 2g(x)\,d\mu(x) < \frac{\epsilon}{3}.$$

By absolute continuity of μ_g, choose $\delta > 0$ such that if $\mu(F) < \delta$, then $\mu_g(F) < \epsilon/6$.
By Egoroff's Theorem, there exists $G \subseteq E$ such that $\mu(E \setminus G) < \delta$, and $\{f_n\}$ converges uniformly to f on G. Then

$$\int_{E\setminus G} \|f_n(x) - f_m(x)\| \; d\mu(x) \le 2\int_{E\setminus G} g(x)\,d\mu(x) = 2\mu_g(E \setminus G) < \frac{\epsilon}{3}.$$

Then choose N such that for $n, m \ge N$, we have $\|f_n(x) - f_m(x)\| \le \dfrac{\epsilon}{3\mu(G)}$, where $x \in G$.
Thus, $\int_G \|f_n - f_m\| \; d\mu < \epsilon/3$.
Therefore if $m, n \ge N$, then $\|f_n - f_m\|_1 < \epsilon$. ■

Proposition: Let f be a B-valued μ-measurable function. If there is $g \in \mathcal{L}^1$ that is $\mathbb{R}$-valued, with $\|f(x)\| \le g(x)$ a.e., then $f \in \mathcal{L}^1$.

Proof: Since f is measurable, there is a sequence $\{f_n\}$ of SMFs converging to f a.e. Choose increasing E_n's with $\mu(E_n) < \infty$, and $\bigcup_{n=1}^{\infty} E_n = \{x : \; g(x) \ne 0\}$.
Set $h_n(x) = \begin{cases} f_n(x), & \text{if } \|f_n(x)\| < 2g(x), \; x \in E_n \\ 0, & \text{o.w.} \end{cases}$
Since $\{x : \; g(x) \ne 0\}$ is σ-finite, we see that the support of h_n is measurable, thus h_n is a SIF. Clearly $h_n \to f$ a.e., and by definition, $\|h_n(x)\| < 2g(x)$. By dominated convergence, f is mean-Cauchy, hence $f \in \mathcal{L}^1$. ■

Monotone Convergence Theorem: Let $\{f_n\}$ be a sequence of real-valued integrable functions on E with $f_n \ge 0$. If $f_{n+1}(x) \ge f_n(x)$ a.e. for all n, and $\sup_n \int f_n\,d\mu < \infty$, then $f_n \in \mathcal{L}^1$, and $\{f_n\}$ is a mean-Cauchy sequence that converges to a finite $f \in \mathcal{L}^1$ a.e.

Proof: $\int f_{n+1} \; d\mu \; \ge \; \int f_n \; d\mu$, so the sequence $\left\{\int f_n \; d\mu\right\}_{n=1}^{\infty}$ is monotonically increasing, but since $\sup_n \int f_n \; d\mu < \infty$, it follows that it must converge. For a.e. x, we have that $\{f_n(x)\}$ is monotonically increasing, so let $f(x) = \lim f_n(x)$. Clearly, $\int f_n\,d\mu \le \int f\,d\mu$ for all n.
For the reverse inequality, let ϕ be any SIF such that $\phi \le f$. Let $0 < \alpha < 1$, and define $E_n = \{x : \; f_n(x) \ge \alpha\phi(x)\}$. Then

$$\int f_n\,d\mu \ge \int_{E_n} f_n\,d\mu \ge \int_{E_n} \alpha\phi\,d\mu = \alpha\int_{E_n} \phi\,d\mu.$$

Note that since $f_n \uparrow f$, it follows that also $E_n \uparrow E$ a.e. Thus $\lim \int f_n \, d\mu \geq \alpha \int \phi \, d\mu$. Now since $\alpha \in (0,1)$ was arbitrary, we have $\lim \int f_n \, d\mu \geq \int \phi \, d\mu$. Finally, since ϕ was an arbitrary SIF with $\phi \leq f$, it is easily seen that $\lim \int f_n \, d\mu \geq \int f \, d\mu$. ■

Finally, we will show that the indefinite integral of an integrable function is a measure by establishing countable additivity.

Proposition: Let $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$. Then μ_f is a B-valued measure on X.

Proof: We will show countable additivity.
Let $E = \bigoplus_{n=1}^{\infty}$. Let $\epsilon > 0$. There exists a sequence of SIFs $\{f_n\}$ converging in mean to f. Then choose N such that for $n \geq N$, we have $\|f_n - f\|_1 < \epsilon$.
Note that for SIFs f_n, μ_{f_n} is clearly a measure:

$$\mu_{f_n}(E) = \sum_{j=1}^{\infty} \mu_{f_n}(E_j) = \lim_{m \to \infty} \sum_{j=1}^{m} \mu_{f_n}(E_j).$$

Then for $m > N$,

$$\left\| \mu_f(E) - \sum_{j=1}^{\infty} \mu_f(E_j) \right\| \leq \|\mu_f(E) - \mu_{f_m}(E)\| + \left\| \mu_{f_m}(E) - \sum_{j=1}^{\infty} \mu_f(E_j) \right\|.$$

We have the bounds

- $\|\mu_f(E) - \mu_{f_m}(E)\| = \left\| \int_E f - f_m \, d\mu \right\| \leq \|f - f_m\|_1 < \epsilon$,
- $\left\| \mu_{f_m}(E) - \sum_{j=1}^{\infty} \mu_f(E_j) \right\| = \left\| \sum_{j=1}^{\infty} [\mu_{f_m}(E_j) - \mu_f(E_j)] \right\| = \left\| \lim_{m \to \infty} \sum_{j=1}^{\infty} \int_{E_j} f_m - f \, d\mu \right\| =$
  $= \left\| \lim_{m \to \infty} \int_{\bigoplus_{j=1}^{m} E_j} f_m - f \, d\mu \right\| \leq \|f_n - f\|_1 < \epsilon.$

It follows that $\mu_f(E) = \sum_{j=1}^{\infty} \mu_f(E_j)$, as desired. ■