Stat 134: Review of Discrete Distributions

1 Symmetry/Exchangeability

An urn initially contains b black balls and w white balls. In each trial, a ball is drawn at random from the urn and then it is returned to the urn together with d additional balls of the same color as the drawn ball. Let X_i denote the color of the ball in the i-th trial.

(a) Let $X_1, ..., X_n$ be a particular sequence of draws with exactly j black balls and nj white balls. What is the probability of this sequence of draws?

(b) In 100 trials, what is the probability that the 25th draw is a black ball and the 75th draw is a white ball?

(c) In n trials, what is the expected number of times that a black ball is drawn?

(d) Now suppose that when you draw a ball, you simply don't replace it. Use the method of indicators to find the expected number of draws before your first black marble. Argue by symmetry to find the expected number of draws before your r-th black marble.

2 Standard Questions about a Standard Deck of Cards

For parts (a) and (b), draw cards with replacement.
(a) What is the distribution (and its parameters) of the number of cards drawn until your k-th ace?

(b) Suppose you draw 10 cards. What is the distribution (and its parameters) of the number of aces in these 10 cards?

(c) Suppose you repeatedly draw 5 cards at a time (this is done without replacement), then replace them. What is the distribution (and its parameters) of the number of 5-card hands you draw until the k-th time your hand contains a three of a kind (Including full house and four of a kind)?

3 Intuitive Distributions

Let $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(n, p)$ be independent.
(a) Let $Z = X + Y$. With little calculation, find the distribution of Z.

(b) Suppose you know that $Z = m$. With little calculation, find the conditional distribution of X.

(c) Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Let $Z = \min\{X, Y\}$. With little calculation, find the distribution of Z.

4 Poisson Practice

Suppose customers arrive at a rate of λ arrivals per unit time. Let $t > 0$, and consider the time intervals $[0, t), [t, 2t), \ldots, [(n-1)t, nt)$.
(a) For each $1 \le j \le n$, let X_j denote the number of arrivals in the time interval $[(j-1)t, jt)$. What is the distribution of X_j?

(b) Suppose the unit time is one hour, and suppose a customer arrives at 9:10am. What is the probability that the next customer arrives before 10:00am?

(c) Let B_k denote the number of time intervals with k arrivals. Find the distribution of B_k.

(d) Let $k_1, ..., k_m \in \mathbb{N}$ (where $\mathbb{N} = \{0, 1, 2, ...\}$) such that $k_i \neq k_j$ for any i, j. Find the distribution of $B_{k_1} + \cdots + B_{k_m}$ (cf. Problem 3(a)).

(e) (Hard) Let $N_1, ..., N_m$ be a partition of $\mathbb{N}$, i.e. they are disjoint and $N_1 \cup \cdots \cup N_m = \mathbb{N}$. Let $C_j = \sum_{k \in I_j} B_j$, for $j = 1, \ldots, m$. Show that the joint distribution of $(C_1, \ldots, C_m)$ is $\text{Multinomial}(n, (p_1, \ldots, p_m))$, where $p_j = \sum_{k \in I_j} \mathbb{P}(X_1 = k)$.