Stat 134: Concepts of Probability **Fall 2018**

Quiz 2 Solution

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1

Southwest overbooks its flights, knowing some people will not show up. Suppose for a plane that seats 350 people, 380 tickets are sold. Based on previous data, the airline claims that each passenger has a 90% chance of showing up. Approximately, what is the chance that at least one empty seat remains? (There are no assigned seats.)

Solution: Notice that there is one empty seat if and only if 349 or less people show up, since there are exactly 350 seats on the plane. Let X equal the number of people that show up. View each person as a trial, and success means they show up. For each person, this happens with probability 90%. It follows that $X \sim$Binomial $(380, 0.9)$, since there are 380 people total. Then we simply want to approximate $\mathbb{P}(X \leq 349)$. Observe that $\mu = 380 \cdot 0.9 = 342$, and $\sigma = \sqrt{380 \cdot 0.9 \cdot 0.1} = \sqrt{34.2} < 3$. It easily follows that the normal approximation is appropriate (not Poisson), and we get

$$\mathbb{P}(X \leq 349) \approx \Phi\Big(\frac{349 - 342 + 1/2}{\sqrt{34.2}}\Big).$$

2

A lot of 250 items is inspected by the following two-stage plan. i) A first sample of 10 items is drawn without replacement. If all are good the lot is passed; if two or more are bad the lot is rejected. ii)If the first sample contains just one bad item, a second sample of 20 more items is drawn without replacement (from the remaining 240 items) and the lot is rejected if three or more of these are bad. Otherwise it is accepted. Suppose there are 20 bad items in the lot. Write down an expression for the the probability that the lot is accepted.

Solution: Notice that there are two cases when the lot is accepted: (i) when the first sample of 10 items is all good, **or** (ii) when the first sample has exactly one bad item and the next sample of 20 items has 2 or less bad items. It follows that

$$\mathbb{P}(\text{accepted}) = \mathbb{P}(\text{1st sample all good}) + \mathbb{P}(\text{1 bad item in 1st sample}) \cdot \mathbb{P}(\leq 2 \text{ bad items in the 2nd sample}).$$

Note that we sample *without* replacement from a population with good and bad items, and we wish to count the number of good/bad items in our sample. To find $\mathbb{P}(\text{1st sample all good})$, observe that our sample size is 10, and there are 230 good items in the lot. Therefore, we simply want to pick 10 items that are good, so

$$\mathbb{P}(\text{1st sample all good}) = \frac{\binom{230}{10}}{\binom{250}{10}}.$$

Similarly, to find $\mathbb{P}$(1 bad item in 1st sample), we want to pick exactly 9 good items from the 230 good items, then 1 bad item from the 20 bad items. Thus,

$$\mathbb{P}(\text{1 bad item in 1st sample}) = \frac{\binom{230}{9}\binom{20}{1}}{\binom{250}{10}}.$$

Finally, to find $\mathbb{P}(\leq$ 2 bad items in the 2nd sample), we sum over $k = 0, 1, 2$, the chance there are k bad items in the 2nd sample. Note that this probability is found *given* we already drew a sample of 10 items with exactly one bad item in that sample, hence there are 19 bad items and 221 good items total. The 2nd sample is of size 20. Therefore, it is easily seen that

$$\mathbb{P}(\leq 2 \text{ bad items in the 2nd sample}) = \sum_{k=0}^{2} \frac{\binom{19}{k}\binom{221}{20-k}}{\binom{240}{20}}.$$

This follows from the hypergeometric distribution. The final answer is thus

$$\mathbb{P}(\text{accepted}) = \frac{\binom{230}{10}}{\binom{250}{10}} + \frac{\binom{230}{9}\binom{20}{1}}{\binom{250}{10}} \cdot \Big[\sum_{k=0}^{2} \frac{\binom{19}{k}\binom{221}{20-k}}{\binom{240}{20}}\Big].$$

3

Suppose you and I each have a box of 800 marbles. In my box, 3 of the marbles are black, while 5 of your marbles are black. We each draw 600 marbles with replacement from our own boxes. Approximately, what is the chance you and I draw the same number of black marbles?
Solution: We draw the marbles with replacement, so we can view each of our 600 draws as a trial, with success if we draw a black marble. Since the draws are done with replacement, even the black marbles can return to the box! (i.e., it is possible we draw more than 3 marbles each) Let X denote the number of black marbles I draw in 600 draws, and let Y denote the number of black marbles you draw in 600 draws. It is easily seen that $X \sim \text{Binomial}(600, \frac{3}{800})$, and $Y \sim \text{Binomial}(600, \frac{5}{800})$. We want $\mathbb{P}(X = Y)$. There is too much randomness in the event $\{X = Y\}$. Thus, we condition on $X = k$, and sum over all possible values for which X can equal k. Then

$$\mathbb{P}(X = Y) = \sum_{k=0}^{600} \mathbb{P}(X = k, Y = k) = \sum_{k=0}^{600} \mathbb{P}(X = k)\mathbb{P}(Y = k),$$

since clearly our marble draws are independent.
We must approximate this probability. It is easily seen that since the probabilities of success in each trial are really small, the Poisson approximation is appropriate (One can easily check that σ_X and σ_Y are less than 3). The means are given by $\mu_X = 600 \cdot \frac{3}{800} = \frac{9}{4}$, and $\mu_Y = 600 \cdot \frac{5}{800} = \frac{15}{4}$. Therefore the approximation is simply

$$\mathbb{P}(X = Y) \approx \sum_{k=0}^{600} e^{-9/4} \cdot \frac{(9/4)^k}{k!} \cdot e^{-15/4} \cdot \frac{(15/4)^k}{k!}.$$

4

In a class of 45 students, each student is given a bag of Smarties chocolates. Each bag of Smarties contains 13 chocolates, of which 4 are red. Students eat their chocolates one at a time, chosen

at random. What is the chance that 5 or more students eat all the red chocolates last?
Solution: View each student as a trial, with success if they eat all the red chocolates last. Note that this is equivalent to them eating all the non-red chocolates first. For a particular student, the probability they eat the 9 non-red chocolates first is just
$p := \frac{9}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} \cdots \frac{1}{5} = \frac{9!4!}{13!} = \binom{13}{9}^{-1}$. Let X denote the number of students who eat all their red chocolates last. Knowing that $X \sim$Binomial$(45, p)$, the answer is simply

$$\mathbb{P}(X \geq 5) = \sum_{k=5}^{45} \binom{45}{k} p^k (1-p)^{n-k}.$$