Stat 134: Concepts of Probability **Fall 2018**

Quiz 3 Solution

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1

10 cards are drawn from a standard deck of 52 cards. Find the probability of having exactly 2 sets of three of a kind (three cards of the same rank).

Solution: The probability of having *at least* two sets of three of a kind is given by $\frac{\binom{13}{2}\binom{4}{3}^2\binom{44}{4}}{\binom{52}{10}}$. This is because we must choose 2 out of the 13 ranks (note that order of what ranks does not matter), then for each rank, we must pick 3 out of the four cards of that rank. Since we cannot pick any of the remaining cards of these ranks, there are now $52 - 8 = 44$ cards we must pick 4 from to complete the 10-card hand. Now, observe that this counts the number of ways that these 4 cards contain a three of a kind as well. In particular, the term $\binom{13}{2}\binom{4}{3}^2\binom{44}{4}$ counts the number of ways there are 3 sets of three of a kind, once for each of the 3 ways we initially picked two ranks to be the three of a kind and the remaining 4 cards happen to contain a three of a kind. Therefore, we must subtract $3 \cdot \binom{13}{3}\binom{4}{3}^3\binom{40}{1}$ from $\binom{13}{2}\binom{4}{3}^2\binom{44}{4}$. The reasoning behind this term is the same as above. Finally, we divide by the total number of ways we can pick 10 cards from 52, and the final answer is

$$\frac{\binom{13}{2}\binom{4}{3}^2\binom{44}{4} - 3 \cdot \binom{13}{3}\binom{4}{3}^3\binom{40}{1}}{\binom{52}{10}}.$$

2

Roll a fair n-sided die twice. Call the results X and Y. Find:

(a) $P(X > Y)$

Solution #1: We use the law of total probability and condition on the value of Y, and sum over all such values. This yields

$$\mathbb{P}(X > Y) = \sum_{k=1}^{n} \mathbb{P}(X > Y, Y = k) = \sum_{k=1}^{n-1} \mathbb{P}(X > k)\mathbb{P}(Y = k) = \sum_{k=1}^{n-1} \frac{1}{n} \cdot \frac{n-k}{n} =$$

$$= \frac{1}{n^2}\sum_{k=1}^{n-1}(n-k) = \frac{n-1}{n} - \frac{1}{n^2}\sum_{k=1}^{n-1} k = \frac{n-1}{n} - \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}.$$

The first equality is the law of total probability and the second equality follows by the independence of X and Y. Note that if $Y = n$, then $X > n$ is impossible.

Solution #2: Observe that by symmetry, $\mathbb{P}(X > Y) = \mathbb{P}(X < Y)$. Therefore, $\mathbb{P}(X < Y) + \mathbb{P}(X > Y) + \mathbb{P}(X = Y) = 1$, which implies $2\mathbb{P}(X > Y) + \mathbb{P}(X = Y) = 1$, so $\mathbb{P}(X > Y) = \frac{1}{2}(1 - \mathbb{P}(X = Y)) = \frac{1}{2}(1 - \frac{1}{n}) = \frac{n-1}{2n}$.

(b) $P(\max\{X, Y\} = k)$, for $1 \le n$.
Solution: When dealing with distributions of the maximum of random variables, the strategy is to first find the probability that the maximum is less than something. (Analogously, for the minimum, first find the probability that it is larger than something.) Thus, note that $\mathbb{P}(\max\{X, Y\} \le k) = \mathbb{P}(X \le k, Y \le k) = (\frac{k}{n})^2$. It follows that

$$\mathbb{P}(\max\{X, Y\} = k) = \mathbb{P}(\max\{X, Y\} \le k) - \mathbb{P}(\max\{X, Y\} \le k - 1) = \left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2.$$

3

On a telephone wire, n birds sit arranged in a line. A noise startles them, causing each bird to look left or right at random. Calculate the expected number of birds which are not seen by an adjacent bird.
Solution: The context of the problem induces a solution using the method of indicators. Since we want the expected number of birds, we indicate on each bird not being seen by an adjacent bird. Define

$$I_j = \begin{cases} 1 \text{ if bird } j \text{ is not seen by an adjacent bird} \\ 0 \text{ otherwise} \end{cases}$$

Let X denote the number of birds not seen by an adjacent bird. Then $X = I_1 + ... + I_n$, so by linearity of expectation, $\mathbb{E}(X) = \sum_{j=1}^{n} \mathbb{E}(I_j) = \sum_{j=1}^{n} \mathbb{P}(I_j = 1)$. What does it mean when $I_j = 1$? It precisely means that bird $j - 1$ and bird $j + 1$ look away from bird j. Clearly, this happens with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. However, notice that this holds only for $2 \le j \le n - 1$. For the case $j = 1$ and $j = n$, notice that these birds are not seen only when their single adjacent birds $j = 2$ and $j = n - 1$, respectively, look away. For each case, this happens with probability $\frac{1}{2}$. Therefore,

$$\mathbb{E}(X) = \mathbb{P}(I_1 = 1) + \sum_{j=2}^{n-1} \mathbb{P}(I_j = 1) + \mathbb{P}(I_n = 1) = 1 + \frac{n-2}{4}.$$

4

Suppose that each year, Berkeley admits 12,000 students on average, with an SD of 3,000 students. Assuming that the application pool is roughly the same across years, find the smallest upper bound you can on the probability that Berkeley will admit at least 20,500 students in 2019.
Solution: Let X denote the number of Cal admits in 2019. Then $\mathbb{E}(X) = 12,000$ and $\text{Var}(X) = (3,000)^2$. By Markov's inequality, $\mathbb{P}(X \ge 20,500) \le \frac{\mathbb{E}(X)}{20{,}500} = \frac{12{,}000}{20{,}500} = \frac{120}{205}$. By Chebyshev's inequality, $\mathbb{P}(X \ge 20,500) = \mathbb{P}(X - 12,000 \ge 20,500 - 12,000) \le \mathbb{P}(|X - 12,000| \ge 8,500) \le$ $\le \frac{\text{Var}(X)}{8{,}500^2} = \frac{3{,}000^2}{8{,}500^2}$. We see that Chebyshev's inequality gives a tighter bound.