Stat 134: Concepts of Probability **Fall 2018**

Quiz 1 Solution

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1

A hat contains 20 cards. 5 of these cards are black on both sides, 5 white on both sides, and 10 are black on one side and white on the other. The cards are mixed up in a hat. Then a single card is drawn and placed on a table. If the visible side of the card is black, what is the chance that the other side is white?

Solution: Remember that each card has two sides, so that once we pick a card randomly from the hat, we also randomly pick one of the sides of the card that we want face up on the table. Let B denote the event that the visible side of the card is black, and let W' denote the event that the other side is white. Then we want $\mathbb{P}(W'|B)$. Using Bayes' rule, we get

$$\mathbb{P}(W'|B) = \frac{\mathbb{P}(BW')}{\mathbb{P}(BW') + \mathbb{P}(BW'^c)}.$$

Observe that the event BW' occurs if and only if the card we drew was one of the cards with a black side and a white side, *and* the black side is face up. Clearly, this happens with probability $\frac{10}{20} \cdot \frac{1}{2} = \frac{1}{4}$. Furthermore, the event BW'^c occurs if and only if the card we drew was black on both sides; then the chance that the black side is face up is 1, hence $\mathbb{P}(BW'^c) = \frac{5}{20} = \frac{1}{4}$. It follows that

$$\mathbb{P}(W'|B) = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}.$$

2

A fair n-sided die is repeatedly rolled until one of the faces is observed twice (i.e., until the first repeat).

(a) What is the chance we stop rolling the die on the k-th roll? (You may assume $1 \leq k \leq n$.)

(b) Given we stop on the 10th roll, what is the chance the 10th roll shows a 2?

Solution:

(a) It may help to start with the first few values for k, i.e. $k = 2, 3, 4$. (Note that $k = 1$ is impossible).
$\mathbb{P}(\text{stop on 2nd roll}) = 1 \cdot \frac{1}{n}$. This follows since the first roll can be any face, but since we want to stop on the 2nd face, the second roll must be whatever face the first roll was.
$\mathbb{P}(\text{stop on 3rd roll}) = 1 \cdot \frac{n-1}{n} \cdot \frac{2}{n}$. Again, the first roll can be any face, and the second

roll can be any face except the first (since otherwise, we would stop on the second face), and finally, to stop on the third roll, we can get any of the first two faces.
Similarly, $\mathbb{P}(\text{stop on 4th roll}) = 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \frac{3}{n}$.
The generalization to any k follows easily:
$\mathbb{P}(\text{stop on } k\text{th roll}) = 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-(k-2)}{n} \cdot \frac{k-1}{n}$.

(b) Let A denote the event we stopped on the 10th roll, and let B_2 denote the event that the tenth roll shows a 2. Then we want $\mathbb{P}(B_2|A) = \frac{\mathbb{P}(B_2A)}{\mathbb{P}(A)}$. In order for the event B_2A to occur, we need top stop on the tenth roll *and* have the tenth roll be a 2. In particular, we stopped when we rolled 2 because we rolled 2 before, so the first 9 rolls contain a 2. Note that there are 9 "places" to put the 2: on the first roll, second roll, etc. Once we've decided when to roll the 2, the remaining 8 of the first 9 rolls must be distinct and not contain 2. Therefore, it folows that $\mathbb{P}(B_2A) = 9 \cdot \frac{1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-8}{n} \cdot \frac{1}{n}$. From part (a), $\mathbb{P}(A) = 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-8}{n} \cdot \frac{9}{n}$, where we have plugged in $k = 10$. It follows easily that $\mathbb{P}(B_2|A) = \frac{1}{n}$.

3

You have n coins $C_1, C_2, ..., C_n$. Each coin is weighted differently so that
$p_i = P(\text{coin } C_i \text{ comes up heads}) = \frac{1}{2i+1}$.
(a) Let r_n be the chance that there are an odd number of heads in tossing coins $C_1, C_2, ..., C_n$, once per coin. Write a formula for r_6 in terms of r_5 and p_6.
(b) Prove by induction that if the n coins are tossed, then the probability of getting an odd number of heads is $\frac{n}{2n+1}$.
Solution:

(a) By the law of total probability, $r_6 = p_6(1 - r_5) + (1 - p_6)r_5$.

(b) If just C_1 is tossed, then the probability of getting an odd number of heads is simply the chance that C_1 is heads. This is $\frac{1}{2\times 1+1} = \frac{1}{3}$. This verifies the base case.
Now, assume that for $n \geq 1, r_n = \frac{n}{2n+1}$.
We want to show that the equation holds for the case $n+1$. That is, we want to show that $r_{n+1} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+3}$. Indeed, from part (a), we can deduce that
$r_{n+1} = p_{n+1}(1 - r_n) + (1 - p_{n+1})r_n$. By assumption, it follows that

$$r_{n+1} = p_{n+1}\left(1 - \frac{n}{2n+1}\right) + (1 - p_{n+1})\frac{n}{2n+1} = \frac{1}{2n+3} \cdot \frac{n+1}{2n+1} + \frac{2n+2}{2n+3} \cdot \frac{n}{2n+1} =$$

$$= \frac{n+1}{(2n+1)(2n+3)} + \frac{2n^2+2n}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3},$$

as desired.