

$n^2 + 1$ UNIT EQUILATERAL TRIANGLES CANNOT COVER AN EQUILATERAL TRIANGLE OF SIDE $> n$ IF ALL TRIANGLES HAVE PARALLEL SIDES

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ABSTRACT. **Draft. DO NOT DISTRIBUTE.** In a famous short paper, Conway and Soifer asked if an equilateral triangle T of side $n + \varepsilon$ with sufficiently small $\varepsilon > 0$ can be covered by $n^2 + 1$ unit equilateral triangles, and provided two ways to cover T with $n^2 + 2$ unit triangles. We show that if we require the sides of all triangles to be parallel to the sides of T (e.g. \triangle and ∇), then it is impossible to cover T with exactly $n^2 + 1$ unit equilateral triangles.

1. INTRODUCTION

In 2004, Conway and Soifer attempted to set a world record in the minimum number of words in a paper by submitting the following paper to the *American Mathematical Monthly* [1].

**Can $n^2 + 1$ unit equilateral triangles cover an
equilateral triangle of side $> n$, say $n + \varepsilon$?¹**

John H. Conway & Alexander Soifer

$n^2 + 2$ can:

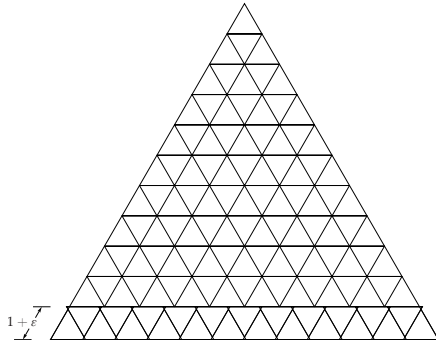


FIGURE 1

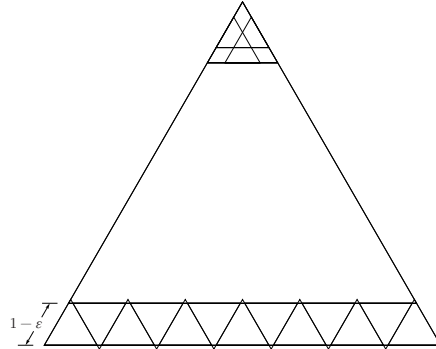


FIGURE 2

The *American Mathematical Monthly* didn't publish the paper as-is and instead put it inside a "boxed filler" with modifications [2], but the paper is often mentioned as an example of mathematics papers with shortest length in multiple places (e.g. blogs [3, 4], online communities [5, 6, 7], and a Numberphile video).

Related, Karabash and Soifer showed that for every non-equilateral triangle T , $n^2 + 1$ triangles similar to T and with the ratio of linear sizes $1 : (n + \varepsilon)$, can cover

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¹Reproduction done by authors with slight modification; for the exact we refer to [1].

T [8]. Also, they generalized the result of Conway and Soifer and showed that a *trigon*² made of n unit equilateral triangles can be covered by $n + 2$ triangles of side $1 - \varepsilon$ [8]. A similar problem of covering a square of side $n + \varepsilon$ with unit squares has been also extensively studied [9, 10, 11, 12, 13]. Still, to the best of the authors' knowledge, the original question in the title of the paper by Conway and Soifer haven't been addressed in the literature.

Define an equilateral triangle as *upright* if all the sides of the triangle are parallel to the three axes of a triangular grid. Note that both triangles \triangle and ∇ are upright, and all the unit triangles used in Figure 1 and 2 are upright. Also, the generalized covering of trigons by Karabash and Soifer [8] only uses upright triangles as well. Thus, it is natural to ask if one can cover the equilateral triangle of side $> n$ with $n^2 + 1$ upright unit triangles. In this paper, we settle this problem by showing that such covering with upright unit triangles does not exist.

The proof generalizes to an arbitrary union X of n upright triangles with disjoint interiors: it is impossible to cover X with $n + 1$ equilateral triangles of side < 1 .

Theorem 1. *Let X be any union of n unit upright equilateral triangles S_1, S_2, \dots, S_n with disjoint interiors. Then X cannot be covered by $n + 1$ upright equilateral triangles of sides less than one.*

To recover the original problem, assume that an upright equilateral triangle T of side $> n$ can be covered by $n^2 + 1$ unit upright equilateral triangles. Rescale the covering so that T have side n and the small triangles have side < 1 . Then we get contradiction by Theorem 1 as T is a union of n^2 unit triangles with disjoint interiors.

Corollary 1. *$n^2 + 1$ unit upright equilateral triangles cannot cover an upright equilateral triangle of side $> n$.*

With the covering of T by Conway and Soifer (Figure 1 and 2), and the covering of trigons by Karabash and Soifer, we match the exact minimum number of upright unit equilateral triangles for covering.

Corollary 2. *The minimum number of unit upright equilateral triangles required to cover an upright equilateral triangle of side $> n$ is exactly $n^2 + 2$. Also, the minimum number of unit upright triangles required to cover a trigon of n triangles is exactly $n + 2$.*

2. PROOF OF THEOREM 1

Take the standard Cartesian xy -coordinate system of a plane. Inside the plane, take the triangular grid of unit equilateral triangles with the x -axis as one of the three axes of the triangular grid.

For every unit upright triangle T , define its rescaled y -coordinate z_T as the y -coordinate of the horizontal side of T divided by $\sqrt{3}/2$. Note that $\sqrt{3}/2$ is the height of a unit equilateral triangle, so the value of z_T is an integer for every triangle T in the triangular grid. Define the function $\tilde{f}_T : \mathbb{R} \rightarrow \mathbb{R}$ as the following. For any $z \neq z_T$, the value $\tilde{f}_T(z)$ is the length of the part of the line $y = \sqrt{3}z/2$ covered by triangle T (the value is zero if T is disjoint from the line). The value of $\tilde{f}_T(z_T)$ is chosen so that \tilde{f}_T is right-continuous everywhere: 1 if T is pointed upwards, and 0 if T is pointed downwards.

In this paper, let S^1 be the abelian group quotient \mathbb{R}/\mathbb{Z} . For every unit upright triangle T , define $f_T : S^1 \rightarrow \mathbb{R}$ as the function $f_T(t + \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \tilde{f}_T(t + n)$. For every $a \in S^1$, define $\tilde{a} \in [0, 1)$ as the unique representator of a in the interval $[0, 1)$.

²A connected shape formed by unit equilateral triangles with matching edges.

Define $\nabla(x) = \tilde{x}$ for all $x \in S^1$. Define $\Delta_0(0) = 1$ and $\Delta_0(x) = 1 - \tilde{x}$ for all nonzero $x \in \mathbb{R}/\mathbb{Z}$. For every $a \in S^1$, define the functions $\Delta_a, \nabla_a : S^1 \rightarrow \mathbb{R}$ as the functions $\nabla_a(x) = \nabla_0(x-a)$ and $\Delta_a(x) = \Delta_0(x-a)$. If an unit upright triangle T is pointed upwards, we have $f_T = \Delta_{y_T}$, and if T is pointed downwards, we have $f_T = \nabla_{y_T}$.

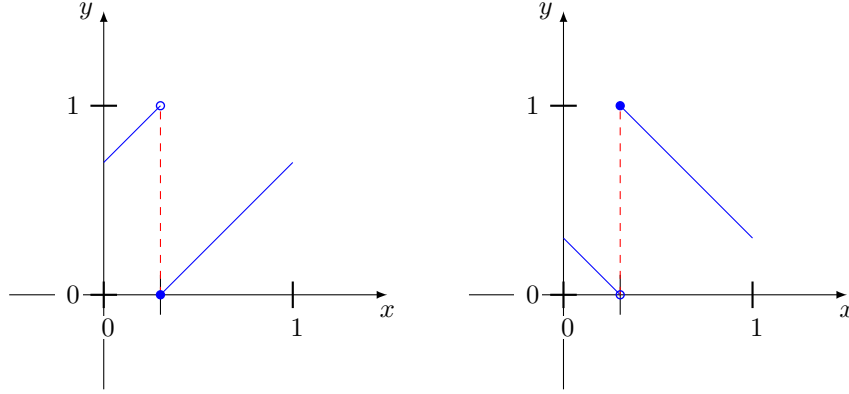


FIGURE 3. Graphs of $\nabla_a(x)$ and $\Delta_a(x)$ for $a = 0.3$.

We now prove Theorem 1 by contradiction. Assume that the union X of n unit upright equilateral triangles S_1, S_2, \dots, S_n with disjoint interiors can be covered by $n+1$ triangles T'_0, T'_1, \dots, T'_n of side < 1 . Take arbitrary $n+1$ triangles T_0, T_1, \dots, T_n of side 1 so that each T_i contains smaller triangle T'_i .

Define $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ as the function $\tilde{g} = \sum_{i=0}^n \tilde{f}_{T_i} - \sum_{j=1}^n \tilde{f}_{S_j}$. Take any z different from the rescaled y -coordinates z_{T_i} and z_{S_j} of the triangles. As the triangles T_0, T_1, \dots, T_n cover the union X of disjoint triangles S_1, S_2, \dots, S_n , the total length of the parts of the line $y = \sqrt{3}z/2$ covered by T_i 's is at least the total length of the parts of the line $y = \sqrt{3}z/2$ covered by S_j 's. Thus we have $\tilde{g}(z) \geq 0$. As \tilde{g} is right-continuous, by sending the right limit we have $\tilde{g}(z) \geq 0$ for every $z \in \mathbb{R}$ including the case where z is the rescaled y -coordinate of some triangle.

Define $g : S^1 \rightarrow \mathbb{R}$ as $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$ so that we have $g(z + \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \tilde{g}(z + n)$. Then consequently we have $g(t) \geq 0$ for every $t \in S^1$. It turns out that this is sufficient to derive a contradiction. Define \mathcal{T} as the abelian group generated by all functions ∇_a, Δ_a with $a \in S^1$. Then $g \in \mathcal{T}$ by the definition of g . We now examine the properties of $g \in \mathcal{T}$.

Denote the integral of any integrable function $f : S^1 \rightarrow \mathbb{R}$ over the whole S^1 as simply $\int f$. Say that two real numbers are equal modulo 1 if their difference is in \mathbb{Z} .

Lemma 1. *Any function $f : S^1 \rightarrow \mathbb{R}$ in \mathcal{T} has the following properties.*

- f is right-continuous.
- f is differentiable everywhere except for a finite number of points, and the derivative is equal to some fixed constant $a \in \mathbb{Z}$.
- For all $x, y \in \mathbb{R}$, the value $f(y + \mathbb{Z}) - f(x + \mathbb{Z})$ is equal to $a(y - x)$ modulo 1.
- The integral $\int f$ is equal to $b/2$ for some $b \in \mathbb{Z}$ where $b - a$ is divisible by 2.

Proof. Check that all the claimed properties are closed under addition and negation. Then check that the functions ∇_a and Δ_a with $a \in S^1$ satisfies the claimed properties. \square

We observed that $g \in \mathcal{T}$ and $g(t) \geq 0$ for every $t \in S^1$. Also, for any unit upright triangle T we have $\int f_T = 1/2$ so we also have $\int g = 1/2$ by the definition $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$. We now use the following lemma. For any real number x , let $\{x\}$ be the value in $[0, 1)$ equal to x modulo 1.

Lemma 2. *Let $f : S^1 \rightarrow \mathbb{R}$ be any function in \mathcal{T} such that $\int f = 1/2$ and $f(x) \geq 0$ for every $x \in S^1$. Then there is a positive odd integer a and some $c \in [0, 1)$ such that f is either $f(x) = \{ax + c\}$ or $f(x) = 1 - \{ax + c\}$.*

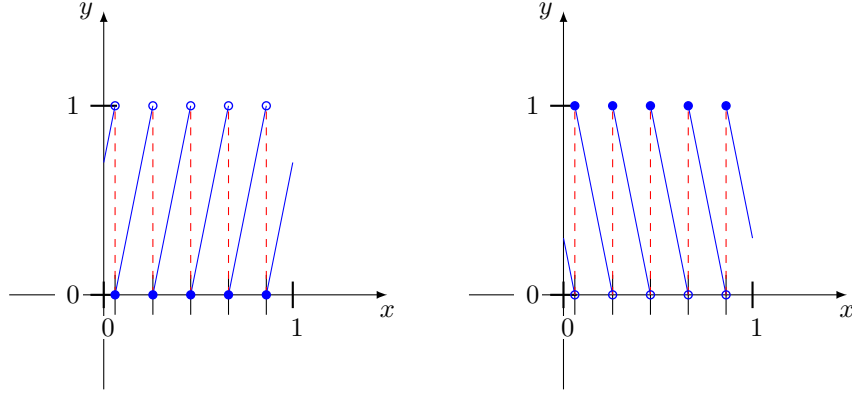


FIGURE 4. Graphs of $x \mapsto \{ax + c\}$ and $x \mapsto 1 - \{ax + c\}$ for $a = 5$ and $c = 0.7$.

Proof. By Lemma 1, there is some odd number $a \in \mathbb{Z}$ such that $f'(x)$ is a for all x except for a finite number of values. Let $f(0) = c$, then by Lemma 1 again we have $f(x)$ equal to $ax + c$ modulo 1 for all $x \in S^1$. Let $g : S^1 \rightarrow \mathbb{R}$ be the function $g(x) = \{ax + c\}$. Then for every $x \in S^1$, as the value $f(x)$ is nonnegative and equal to $ax + c$ modulo 1, we have $f(x) \geq g(x) \geq 0$. But note that the integral $\int g$ is exactly equal to $1/2$. So f and g should be equal almost everywhere. As f is right-continuous by Lemma 1, $f(x)$ should be equal to the right limit $g(x-)$ of g . If $a > 0$, then g is right-continuous so $f(x) = g(x) = \{ax + c\}$. If $a < 0$, then the right limit of g is $1 - \{-ax + \{-c\}\}$ (this is the value in $(0, 1]$ equal to $ax + c$ modulo 1). \square

We now finish the proof of Theorem 1. By Lemma 2, the discontinuities of $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$ have to be equidistributed in S^1 with a gap of $1/a$ for some positive odd number a . But each T_i can be taken arbitrary as it contains the smaller triangle T'_i of side < 1 . So take each T_i so that the rescaled y -coordinates $z_{T_0}, z_{T_1}, \dots, z_{T_n}$ are different from $z_{S_1}, z_{S_2}, \dots, z_{S_n}$ modulo 1 and $z_{T_1} - z_{T_0}$ is an irrational number. Then g has discontinuities at $z_{T_0} + \mathbb{Z}, z_{T_1} + \mathbb{Z}, \dots, z_{T_n} + \mathbb{Z} \in S^1$, and two of them has an irrational gap. This gives contradiction and finishes the proof.

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