

# $n^2 + 1$ UNIT EQUILATERAL TRIANGLES CANNOT COVER AN EQUILATERAL TRIANGLE OF SIDE $> n$ IF ALL TRIANGLES HAVE PARALLEL SIDES

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**ABSTRACT.** Conway and Soifer showed that an equilateral triangle  $T$  of side  $n + \varepsilon$  with sufficiently small  $\varepsilon > 0$  can be covered by  $n^2 + 2$  unit equilateral triangles. They conjectured that it is impossible to cover  $T$  with  $n^2 + 1$  unit equilateral triangles no matter how small  $\varepsilon$  is. We make progress towards their conjecture by showing that if we require the sides of all triangles to be parallel to the sides of  $T$  (e.g.  $\triangle$  and  $\nabla$ ), then it is impossible to cover  $T$  with  $n^2 + 1$  unit equilateral triangles for any  $\varepsilon > 0$ . As the coverings of  $T$  by Conway and Soifer only involve triangles with parallel sides, our result determines the exact minimum number  $n^2 + 1$  of unit equilateral triangles with parallel sides required to cover  $T$ .

## 1. INTRODUCTION

Conway and Soifer provided two ways to cover an equilateral triangle  $T$  of side  $> n$  with  $n^2 + 2$  unit equilateral triangles (Figure 1 and 2), and conjectured that  $n^2 + 1$  unit equilateral triangles cannot cover  $T$  [3, 4].

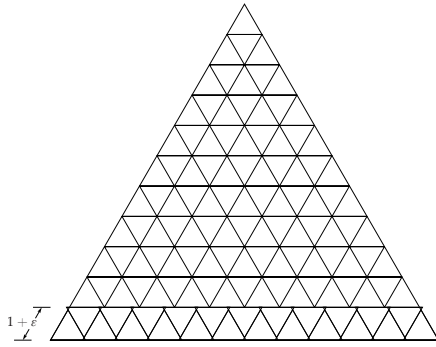


FIGURE 1

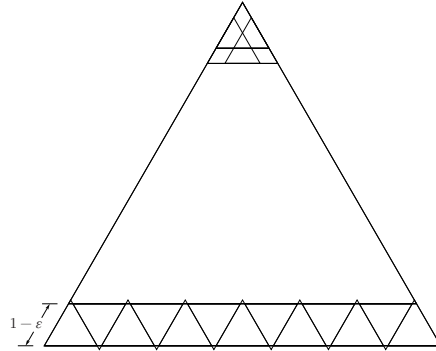


FIGURE 2

**Theorem 1** (Conway and Soifer [3, 4]).  $n^2 + 2$  unit equilateral triangles can cover an equilateral triangle  $T$  of side  $n + \varepsilon$  for a sufficiently small  $\varepsilon > 0$ .

**Conjecture 1** (Conway and Soifer [3]).  $n^2 + 1$  unit equilateral triangles cannot cover an equilateral triangle  $T$  of side  $> n$ .

Related, Karabash and Soifer showed that for every non-equilateral triangle  $T$ ,  $n^2 + 1$  triangles similar to  $T$  and with the ratio of linear sizes  $1 : (n + \varepsilon)$  can cover  $T$  [7], so the “equilaterality” is essential for Conjecture 1 to be true [3, 8]. Also, Karabash and Soifer generalized the coverings of Conway and Soifer and showed that a *trigon*<sup>1</sup> made of  $n$  unit equilateral triangles can be covered by  $n + 2$  triangles

<sup>1</sup>A connected shape formed by unit equilateral triangles with matching edges.

of side  $1 - \varepsilon$  [7]. A similar problem of covering a square of side  $n + \varepsilon$  with unit squares has been also extensively studied [5, 6, 2, 9, 1]. Still, to the best of the authors' knowledge, the original Conjecture 1 raised by Conway and Soifer hasn't been addressed directly in the literature.

Define an equilateral triangle as *vertical* if one side of the triangle is parallel to the  $x$ -axis. Note that both triangles  $\triangle$  and  $\nabla$  are vertical, and all the unit triangles used in Conway and Soifer's constructions (Figure 1 and 2) are vertical. Also, the generalized covering of trigons by Karabash and Soifer [7] only uses vertical triangles as well. Thus, it is natural to ask if one can cover the equilateral triangle of side  $> n$  with  $n^2 + 1$  vertical unit triangles. In this paper, we show that it is impossible.

**Theorem 2.**  *$n^2 + 1$  unit vertical equilateral triangles cannot cover an vertical equilateral triangle of side  $> n$ .*

Our proof generalizes to an arbitrary union  $X$  of  $n$  vertical triangles with disjoint interiors: it is impossible to cover  $X$  with  $n + 1$  vertical equilateral triangles of side  $< 1$ .

**Theorem 3.** *Let  $X$  be any union of  $n$  unit vertical equilateral triangles  $S_1, S_2, \dots, S_n$  with disjoint interiors. Then  $X$  cannot be covered by  $n + 1$  vertical equilateral triangles of sides less than one.*

To recover Theorem 2 from Theorem 3, assume by contradiction that an vertical equilateral triangle  $T$  of side  $> n$  can be covered by  $n^2 + 1$  unit vertical equilateral triangles. Shrink the covering so that  $T$  have side exactly  $n$  and the small triangles have side  $< 1$ . Then we get contradiction by Theorem 3 as  $T$  is a union of  $n^2$  unit vertical triangles with disjoint interiors.

As the coverings of  $T$  by Conway and Soifer (Figure 1 and 2) and the coverings of trigons by Karabash and Soifer only uses vertical triangles, we match the exact minimum number of unit vertical equilateral triangles required for covering.

**Corollary 1.** *The minimum number of unit vertical equilateral triangles required to cover a vertical equilateral triangle of side  $n + \varepsilon$  with a sufficiently small  $\varepsilon > 0$  is exactly  $n^2 + 2$ .*

*Also, the minimum number of unit vertical triangles required to cover a trigon made of  $n$  vertical equilateral triangles of side  $1 + \varepsilon$  with a sufficiently small  $\varepsilon > 0$  is exactly  $n + 2$ .*

## 2. PROOF OF THEOREM 3

Take the standard Cartesian  $xy$ -coordinate system of a plane. Inside the plane, take the triangular grid of unit equilateral triangles with the  $x$ -axis as one of the three axes of the triangular grid.

For every unit vertical triangle  $T$ , define its rescaled  $y$ -coordinate  $z_T$  as the  $y$ -coordinate of the horizontal side of  $T$  divided by  $\sqrt{3}/2$ . Note that  $\sqrt{3}/2$  is the height of a unit equilateral triangle, so the value of  $z_T$  is an integer for every triangle  $T$  in the triangular grid. Define the function  $\tilde{f}_T : \mathbb{R} \rightarrow \mathbb{R}$  as the following. For any  $z \neq z_T$ , the value  $\tilde{f}_T(z)$  is the length of the part of the line  $y = \sqrt{3}z/2$  covered by triangle  $T$  (the value is zero if  $T$  is disjoint from the line). The value of  $\tilde{f}_T(z_T)$  is chosen so that  $\tilde{f}_T$  is right-continuous everywhere: 1 if  $T$  is pointed upwards, and 0 if  $T$  is pointed downwards.

In this paper, let  $S^1$  be the abelian group quotient  $\mathbb{R}/\mathbb{Z}$ . For every unit vertical triangle  $T$ , define  $f_T : S^1 \rightarrow \mathbb{R}$  as the function  $f_T(t + \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \tilde{f}_T(t + n)$ . For any real number  $x$ , let  $\{x\}$  be the value in  $[0, 1)$  equal to  $x$  modulo 1. Define  $\nabla_0(x + \mathbb{Z}) = \{x\}$  and  $\Delta_0(x + \mathbb{Z}) = 1 - \{x\}$  for any  $x + \mathbb{Z} \in S^1$  with representative  $x \in \mathbb{R}$ . For every  $a \in S^1$ , define the functions  $\Delta_a, \nabla_a : S^1 \rightarrow \mathbb{R}$  as the functions

$\nabla_a(x) = \nabla_0(x-a)$  and  $\Delta_a(x) = \Delta_0(x-a)$ . If an unit vertical triangle  $T$  is pointed upwards, we have  $f_T = \Delta_{y_T}$ , and if  $T$  is pointed downwards, we have  $f_T = \nabla_{y_T}$ .

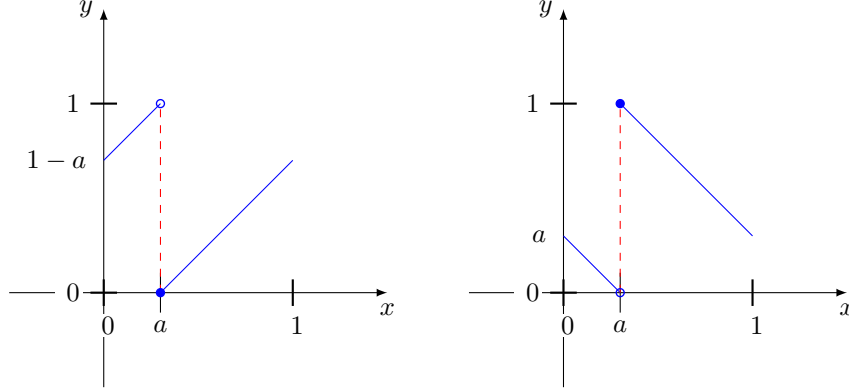


FIGURE 3. Graphs of  $\nabla_a(x)$  and  $\Delta_a(x)$  for  $a = 0.3$ .

We now prove Theorem 3 by contradiction. Assume that the union  $X$  of  $n$  unit vertical equilateral triangles  $S_1, S_2, \dots, S_n$  with disjoint interiors can be covered by  $n+1$  triangles  $T'_0, T'_1, \dots, T'_n$  of side  $< 1$ . Take arbitrary  $n+1$  triangles  $T_0, T_1, \dots, T_n$  of side 1 so that each  $T_i$  contains the smaller triangle  $T'_i$ .

Define  $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$  as the function  $\tilde{g} = \sum_{i=0}^n \tilde{f}_{T_i} - \sum_{j=1}^n \tilde{f}_{S_j}$ . Take any  $z$  different from the rescaled  $y$ -coordinates  $z_{T_i}$  and  $z_{S_j}$  of the triangles. As the triangles  $T_0, T_1, \dots, T_n$  cover the union  $X$  of disjoint triangles  $S_1, S_2, \dots, S_n$ , the total length of the parts of the line  $y = \sqrt{3}z/2$  covered by  $T_i$ 's is at least the total length of the parts of the line  $y = \sqrt{3}z/2$  covered by  $S_j$ 's. Thus we have  $\tilde{g}(z) \geq 0$ . As  $\tilde{g}$  is right-continuous, by sending the right limit we have  $\tilde{g}(z) \geq 0$  for every  $z \in \mathbb{R}$  including the case where  $z$  is equal to the rescaled  $y$ -coordinate of some triangle.

Define  $g : S^1 \rightarrow \mathbb{R}$  as  $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$  so that we have  $g(z + \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \tilde{g}(z + n)$ . Then consequently we have  $g(t) \geq 0$  for every  $t \in S^1$ . It turns out that this is sufficient to derive a contradiction. Define  $\mathcal{T}$  as the abelian group generated by all functions  $\nabla_a, \Delta_a$  with  $a \in S^1$ . Then  $g \in \mathcal{T}$  by the definition of  $g$ . We now examine the properties of  $g \in \mathcal{T}$ .

Denote the integral of any integrable function  $f : S^1 \rightarrow \mathbb{R}$  over the whole  $S^1$  as simply  $\int f$ . Say that two real numbers are equal modulo 1 if their difference is in  $\mathbb{Z}$ .

**Lemma 1.** *Any function  $f : S^1 \rightarrow \mathbb{R}$  in  $\mathcal{T}$  has the following properties.*

- $f$  is right-continuous.
- $f$  is differentiable everywhere except for a finite number of points, and the derivative is always equal to a fixed constant  $a \in \mathbb{Z}$ .
- For all  $x, y \in \mathbb{R}$ , the value  $f(y + \mathbb{Z}) - f(x + \mathbb{Z})$  is equal to  $a(y - x)$  modulo 1.
- The integral  $\int f$  is equal to  $b/2$  for some  $b \in \mathbb{Z}$  where  $b - a$  is divisible by 2.

*Proof.* Check that all the claimed properties are closed under addition and negation. Then check that the functions  $\nabla_a$  and  $\Delta_a$  with  $a \in S^1$  satisfy the claimed properties.  $\square$

We observed that  $g \in \mathcal{T}$  and  $g(t) \geq 0$  for every  $t \in S^1$ . Also, for any unit vertical triangle  $T$  we have  $\int f_T = 1/2$  so we also have  $\int g = 1/2$  by the definition  $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$ . We now use the following lemma.

**Lemma 2.** *Let  $f : S^1 \rightarrow \mathbb{R}$  be any function in  $\mathcal{T}$  such that  $\int f = 1/2$  and  $f(x) \geq 0$  for every  $x \in S^1$ . Then there is a positive odd integer  $a$  and some  $c \in [0, 1)$  such that  $f$  is either  $f(x) = \{ax + c\}$  or  $f(x) = 1 - \{ax + c\}$ .*

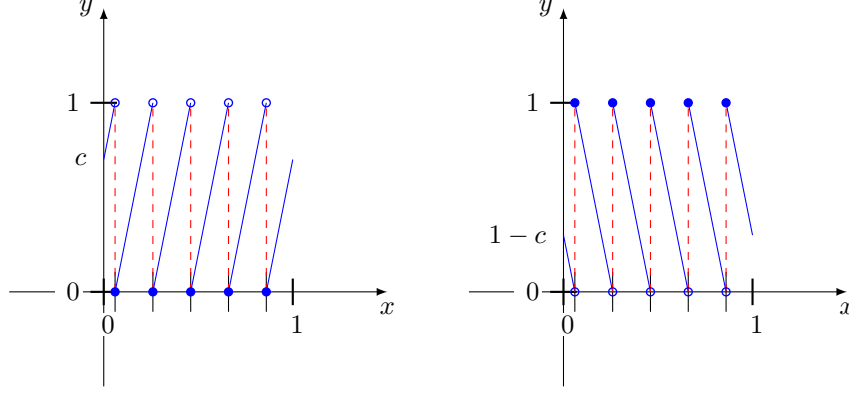


FIGURE 4. Graphs of  $x \mapsto \{ax + c\}$  and  $x \mapsto 1 - \{ax + c\}$  for  $a = 5$  and  $c = 0.7$ .

*Proof.* By Lemma 1, there is some odd number  $a \in \mathbb{Z}$  such that  $f'(x)$  is  $a$  for all  $x$  except for a finite number of values. Let  $f(0) = c$ , then by Lemma 1 again we have  $f(x)$  equal to  $ax + c$  modulo 1 for all  $x \in S^1$ . Let  $g : S^1 \rightarrow \mathbb{R}$  be the function  $g(x) = \{ax + c\}$ . Then for every  $x \in S^1$ , as the value  $f(x)$  is nonnegative and equal to  $ax + c$  modulo 1, we have  $f(x) \geq g(x) \geq 0$ . But note that the integral  $\int g$  is exactly equal to  $1/2$  (see Figure 4). So  $f$  and  $g$  should be equal almost everywhere. As  $f$  is right-continuous by Lemma 1,  $f(x)$  should be equal to the right limit  $g(x-)$  of  $g$ . If  $a > 0$ , then  $g$  is right-continuous so  $f(x) = g(x) = \{ax + c\}$ . If  $a < 0$ , then the right limit of  $g$  is  $1 - \{-ax + \{-c\}\}$  (this is the value in  $(0, 1]$  equal to  $ax + c$  modulo 1).  $\square$

We now finish the proof of Theorem 3. By Lemma 2, the discontinuities of  $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$  have to be equidistributed in  $S^1$  with a gap of  $1/a$  for some positive odd number  $a$ . But each  $T_i$  can be taken arbitrary as it contains the smaller triangle  $T'_i$  of side  $< 1$ . So take each  $T_i$  so that the rescaled  $y$ -coordinates  $z_{T_0}, z_{T_1}, \dots, z_{T_n}$  are different from  $z_{S_1}, z_{S_2}, \dots, z_{S_n}$  modulo 1 and  $z_{T_1} - z_{T_0}$  is an irrational number. Then  $g$  has discontinuities at  $z_{T_0} + \mathbb{Z}, z_{T_1} + \mathbb{Z}, \dots, z_{T_n} + \mathbb{Z} \in S^1$ , and two of them has an irrational gap. This gives contradiction and finishes the proof.

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