$n^2 + 1$ UNIT EQUILATERAL TRIANGLES CANNOT COVER AN EQUILATERAL TRIANGLE OF SIDE $> n$ IF ALL TRIANGLES HAVE PARALLEL SIDES

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Abstract. Conway and Soifer showed that an equilateral triangle T of side $n + \varepsilon$ with sufficiently small $\varepsilon > 0$ can be covered by $n^2 + 2$ unit equilateral triangles. They conjectured that it is impossible to cover T with $n^2 + 1$ unit equilateral triangles no matter how small ε is. We make progress towards their conjecture by showing that if we require the sides of all triangles to be parallel to the sides of T (e.g. $\triangle$ and $\triangledown$), then it is impossible to cover T with $n^2 + 1$ unit equilateral triangles for any $\varepsilon > 0$. As the coverings of T by Conway and Soifer only involve triangles with parallel sides, our result determines the exact minimum number $n^2 + 1$ of unit equilateral triangles with parallel sides required to cover T.

1. Introduction

Conway and Soifer provided two ways to cover an equilateral triangle T of side $> n$ with $n^2 + 2$ unit equilateral triangles (Figure 1 and 2), and conjectured that $n^2 + 1$ unit equilateral triangles cannot cover T [3, 4].

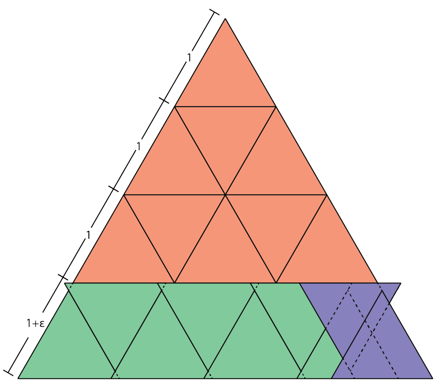

Figure 1

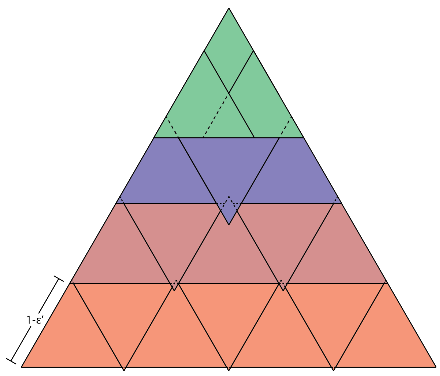

Figure 2

For Figure 1, we first cover the upper part of the original equilateral triangle of side length $n + \epsilon$ equilateral triangle of side length $n - 1$ with $(n-1)^2$ triangles (light red triangles). After that, the remaning part is a trapezoid of lengths $1 + \epsilon$, $n + \epsilon$, $1 + \epsilon$, and $n - 1$. Now put $2n - 2$ triangles from left, alternatively (green triangles), then we can check that the remaining part is a parallelogram of lengths $1 + \epsilon$ and ϵn, minus a small equilateral triangle of length ϵ on the left-upper corner. This can be covered with 2 triangles if $\epsilon \leq 1/(n+1)$ (blue triangles).
For Figure 2, we cover the large triangle from the bottom. We first cover the bottom layer with n upward triangles and $n - 1$ downward triangles, with $\epsilon' = \epsilon/(n-1)$ deviations (light red triangles). Then the resulting shape is a trapezoid of lengths $1 - \epsilon'$, $n + (n-1)\epsilon'$, $1 - \epsilon'$, and $n - 1 + n\epsilon'$, with small "bump" triangles of lengths

ϵ'. Now we stack the next bottom layer with $n-1$ upward triangles and $n-2$ downward triangles, with ϵ'' deviations. To cover the upper side of the light red trapezoid tightly, our new ϵ'' should satisfy $(n-1)+(n-2)\epsilon'' = (n-1)+n\epsilon'$, hence $\epsilon'' = n\epsilon'/(n-2)$. Continue this until you stack up the total $(n-1)$ layers, where the top layer (blue triangles) consists of 2 upward triangles and 1 downward triangle with deviation

$$\frac{n}{n-2}\frac{n-1}{n-3}\frac{n-2}{n-4}\cdots\frac{3}{1}\epsilon' = \frac{n(n-1)}{2}\epsilon' = \frac{n}{2}\epsilon.$$

The remaining part of the triangle can be covered with three triangles of unit lengths if $1+2\cdot\frac{n}{2}\epsilon \leq \frac{3}{2}$, i.e. if $\epsilon \leq \frac{1}{2n}$.

Theorem 1 (Conway and Soifer [3, 4])**.** *n^2+2 unit equilateral triangles can cover an equilateral triangle T of side $n+\varepsilon$ for a sufficiently small $\varepsilon > 0$.*

Conjecture 1 (Conway and Soifer [3])**.** *n^2+1 unit equilateral triangles cannot cover an equilateral triangle T of side $> n$.*

Related, Karabash and Soifer showed that for every non-equilateral triangle T, n^2+1 triangles similar to T and with the ratio of linear sizes $1:(n+\varepsilon)$ can cover T [7], so the "equilaterality" is essential for Conjecture 1 to be true [3, 8]. Also, Karabash and Soifer generalized the coverings of Conway and Soifer and showed that a *trigon*[1] made of n unit equilateral triangles can be covered by $n+2$ triangles of side $1-\varepsilon$ [7]. A similar problem of covering a square of side $n+\varepsilon$ with unit squares has been also extensively studied [5, 6, 2, 9, 1]. Still, to the best of the authors' knowledge, the original Conjecture 1 raised by Conway and Soifer hasn't been addressed directly in the literature.

Define an equilateral triangle as *vertical* if one side of the triangle is parallel to the x-axis. Note that both triangles $\triangle$ and $\bigtriangledown$ are vertical, and all the unit triangles used in Conway and Soifer's constructions (Figure 1 and 2) are vertical. Also, the generalized covering of trigons by Karabash and Soifer [7] only uses vertical triangles as well. Thus, it is natural to ask if one can cover the equilateral triangle of side $> n$ with n^2+1 vertical unit triangles. In this paper, we show that it is impossible.

Theorem 2. *n^2+1 unit vertical equilateral triangles cannot cover an vertical equilateral triangle of side $> n$.*

Our proof generalizes to an arbitrary union X of n vertical triangles with disjoint interiors: it is impossible to cover X with $n+1$ vertical equilateral triangles of side < 1.

Theorem 3. *Let X be any union of n unit vertical equilateral triangles $S_1, S_2, \ldots, S_n$ with disjoint interiors. Then X cannot be covered by $n+1$ vertical equilateral triangles of sides less than one.*

To recover Theorem 2 from Theorem 3, assume by contradiction that an vertical equilateral triangle T of side $> n$ can be covered by n^2+1 unit vertical equilateral triangles. Shrink the covering so that T have side exactly n and the small triangles have side < 1. Then we get contradiction by Theorem 3 as T is a union of n^2 unit vertical triangles with disjoint interiors.

As the coverings of T by Conway and Soifer (Figure 1 and 2) and the coverings of trigons by Karabash and Soifer only uses vertical triangles, we match the exact minimum number of unit vertical equilateral triangles required for covering.

[1]A connected shape formed by unit equilateral triangles with matching edges.

Corollary 1. *The minimum number of unit vertical equilateral triangles required to cover a vertical equilateral triangle of side $n+\varepsilon$ with a sufficiently small $\varepsilon > 0$ is exactly n^2+2.*

Also, the minimum number of unit vertical triangles required to cover a trigon made of n vertical equilateral triangles of side $1+\varepsilon$ with a sufficiently small $\varepsilon > 0$ is exactly $n+2$.

2. Proof of Theorem 3

Take the standard Cartesian xy-coordinate system of a plane. Inside the plane, take the triangular grid of unit equilateral triangles with the x-axis as one of the three axes of the triangular grid.

For every unit vertical triangle T, define its rescaled y-coordinate z_T as the y-coordinate of the horizontal side of T divided by $\sqrt{3}/2$. Note that $\sqrt{3}/2$ is the height of a unit equilateral triangle, so the value of z_T is an integer for every triangle T in the triangular grid. Define the function $\tilde{f}_T : \mathbb{R} \to \mathbb{R}$ as the following. For any $z \neq z_T$, the value $\tilde{f}_T(z)$ is the length of the part of the line $y = \sqrt{3}z/2$ covered by triangle T (the value is zero if T is disjoint from the line). The value of $\tilde{f}_T(z_T)$ is chosen so that $\tilde{f}_T$ is right-continuous everywhere: 1 if T is pointed upwards, and 0 if T is pointed downwards.

In this paper, let S^1 be the abelian group quotient $\mathbb{R}/\mathbb{Z}$. For every unit vertical triangle T, define $f_T : S^1 \to \mathbb{R}$ as the function $f_T(t+\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \tilde{f}_T(t+n)$. For any real number x, let $\{x\}$ be the value in $[0,1)$ equal to x modulo 1. Define $\nabla_0(x+\mathbb{Z}) = \{x\}$ and $\Delta_0(x+\mathbb{Z}) = 1-\{x\}$ for any $x+\mathbb{Z} \in S^1$ with representative $x \in \mathbb{R}$. For every $a \in S^1$, define the functions $\Delta_a, \nabla_a : S^1 \to \mathbb{R}$ as the functions $\nabla_a(x) = \nabla_0(x-a)$ and $\Delta_a(x) = \Delta_0(x-a)$. If an unit vertical triangle T is pointed upwards, we have $f_T = \Delta_{y_T}$, and if T is pointed downwards, we have $f_T = \nabla_{y_T}$.

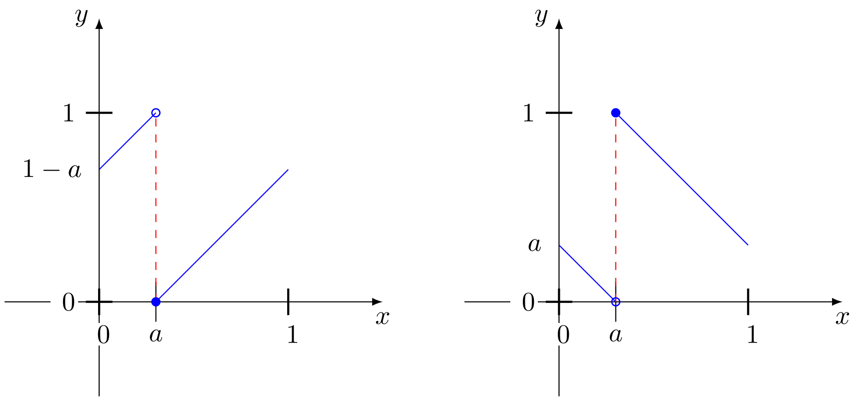

Figure 3. Graphs of $\nabla_a(x)$ and $\Delta_a(x)$ for $a = 0.3$.

We now prove Theorem 3 by contradiction. Assume that the union X of n unit vertical equilateral triangles $S_1, S_2, \ldots, S_n$ with disjoint interiors can be covered by $n+1$ triangles $T'_0, T'_1, \ldots, T'_n$ of side < 1. Take arbitrary $n+1$ triangles $T_0, T_1, \ldots, T_n$ of side 1 so that each T_i contains the smaller triangle T'_i.

Define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as the function $\tilde{g} = \sum_{i=0}^{n} \tilde{f}_{T_i} - \sum_{j=1}^{n} \tilde{f}_{S_j}$. Take any z different from the rescaled y-coordinates z_{T_i} and z_{S_j} of the triangles. As the triangles $T_0, T_1, \ldots, T_n$ cover the union X of disjoint triangles $S_1, S_2, \ldots, S_n$, the total length of the parts of the line $y = \sqrt{3}z/2$ covered by T_i's is at least the total length of the parts of the line $y = \sqrt{3}z/2$ coverved by S_j's. Thus we have $\tilde{g}(z) \geq 0$. As $\tilde{g}$

is right-continuous, by sending the right limit we have $\tilde{g}(z) \geq 0$ for every $z \in \mathbb{R}$ including the case where z is equal to the rescaled y-coordinate of some triangle.

Define $g : S^1 \to \mathbb{R}$ as $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$ so that we have $g(z + \mathbb{Z}) = \sum_{n\in\mathbb{Z}} \tilde{g}(z + n)$. Then consequently we have $g(t) \geq 0$ for every $t \in S^1$. It turns out that this is sufficient to derive a contradiciton. Define $\mathcal{T}$ as the abelian group generated by all functions ∇_a, Δ_a with $a \in S^1$. Then $g \in \mathcal{T}$ by the definition of g. We now examine the properties of $g \in \mathcal{T}$.

Denote the integral of any integrable function $f : S^1 \to \mathbb{R}$ over the whole S^1 as simply $\int f$. Say that two real numbers are equal modulo 1 if their difference is in $\mathbb{Z}$.

Lemma 1. *Any function $f : S^1 \to \mathbb{R}$ in $\mathcal{T}$ has the following properties.*

- *f is right-continuous.*
- *f is differentiable everywhere except for a finite number of points, and the derivative is always equal to a fixed constant $a \in \mathbb{Z}$.*
- *For all $x, y \in \mathbb{R}$, the value $f(y + \mathbb{Z}) - f(x + \mathbb{Z})$ is equal to $a(y - x)$ modulo 1.*
- *The integral $\int f$ is equal to $b/2$ for some $b \in \mathbb{Z}$ where $b - a$ is divisible by 2.*

Proof. Check that all the claimed properties are closed under addition and negation. Then check that the functions ∇_a and Δ_a with $a \in S^1$ satisfy the claimed properties. $\square$

We observed that $g \in \mathcal{T}$ and $g(t) \geq 0$ for every $t \in S^1$. Also, for any unit vertical triangle T we have $\int f_T = 1/2$ so we also have $\int g = 1/2$ by the definition $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$. We now use the following lemma.

Lemma 2. *Let $f : S^1 \to \mathbb{R}$ be any function in $\mathcal{T}$ such that $\int f = 1/2$ and $f(x) \geq 0$ for every $x \in S^1$. Then there is a positive odd integer a and some $c \in [0, 1)$ such that f is either $f(x) = \{ax + c\}$ or $f(x) = 1 - \{ax + c\}$.*

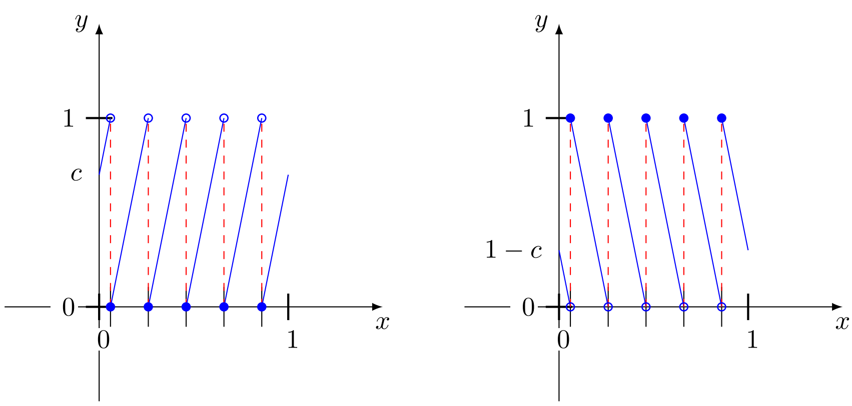

FIGURE 4. Graphs of $x \mapsto \{ax+c\}$ and $x \mapsto 1-\{ax+c\}$ for $a = 5$ and $c = 0.7$.

Proof. By Lemma 1, there is some odd number $a \in \mathbb{Z}$ such that $f'(x)$ is a for all x except for a finite number of values. Let $f(0) = c$, then by Lemma 1 again we have $f(x)$ equal to $ax + c$ modulo 1 for all $x \in S^1$. Let $g : S^1 \to \mathbb{R}$ be the function $g(x) = \{ax+c\}$. Then for every $x \in S^1$, as the value $f(x)$ is nonnegative and equal to $ax + c$ modulo 1, we have $f(x) \geq g(x) \geq 0$. But note that the integral $\int g$ is exactly equal to $1/2$ (see Figure 4). So f and g should be equal almost everywhere. As f is right-continuous by Lemma 1, $f(x)$ should be equal to the right limit $g(x-)$ of g. If $a > 0$, then g is right-continuous so $f(x) = g(x) = \{ax + c\}$. If $a < 0$, then the right limit of g is $1 - \{-ax + \{-c\}\}$ (this is the value in $(0, 1]$ equal to $ax + c$ modulo 1). □

We now finish the proof of Theorem 3. By Lemma 2, the discontinuities of $g = \sum_{i=0}^{n} f_{T_i} - \sum_{j=1}^{n} f_{S_j}$ have to be equidistributed in S^1 with a gap of $1/a$ for some positive odd number a. But each T_i can be taken arbitrary as it contains the smaller triangle T_i' of side < 1. So take each T_i so that the rescaled y-coordinates $z_{T_0}, z_{T_1}, \ldots, z_{T_n}$ are different from $z_{S_1}, z_{S_2}, \ldots, z_{S_n}$ modulo 1 and $z_{T_1} - z_{T_0}$ is an irrational number. Then g has discontinuities at $z_{T_0} + \mathbb{Z}, z_{T_1} + \mathbb{Z}, \ldots, z_{T_n} + \mathbb{Z} \in S^1$, and two of them has an irrational gap. This gives contradiction and finishes the proof.

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