n^2+1 UNIT EQUILATERAL TRIANGLES CANNOT COVER AN EQUILATERAL TRIANGLE OF SIDE $>n$ IF ALL TRIANGLES HAVE PARALLEL SIDES

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Abstract. Conway and Soifer asked if an equilateral triangle T of side $n+\varepsilon$ with sufficiently small $\varepsilon>0$ can be covered by n^2+1 unit equilateral triangles, and provided two ways to cover T with n^2+2 unit triangles. We show that if we require the sides of all triangles to be parallel to the sides of T (e.g. $\triangle$ and $\bigtriangledown$), then it is impossible to cover T with exactly n^2+1 unit equilateral triangles.

1. Introduction

Conway and Soifer showed that n^2+2 unit equilateral triangles can cover an equilateral triangle T of side $>n$, and asked if n^2+1 unit equilateral triangles can cover T [3, 4]. Figure 1 and 2 depict the coverings of T with n^2+2 triangles found by Conway and Soifer respectively [8].

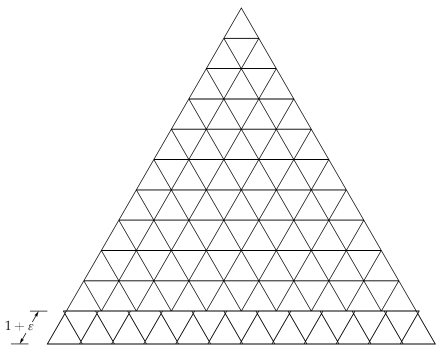

Figure 1

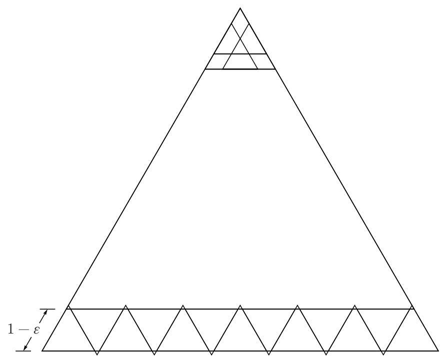

Figure 2

Related, Karabash and Soifer showed that for every non-equilateral triangle T, n^2+1 triangles similar to T and with the ratio of linear sizes $1:(n+\varepsilon)$, can cover T [7]. Also, they generalized the result of Conway and Soifer and showed that a *trigon*[1] made of n unit equilateral triangles can be covered by $n+2$ triangles of side $1-\varepsilon$ [7]. A similar problem of covering a square of side $n+\varepsilon$ with unit squares has been also extensively studied [5, 6, 2, 9, 1]. Still, to the best of the authors' knowledge, the original question raised by Conway and Soifer haven't been addressed directly in the literature.

Define an equilateral triangle as *vertical* if one side of the triangle is parallel to the x-axis. Note that both triangles $\triangle$ and $\bigtriangledown$ are vertical, and all the unit triangles used in Figure 1 and 2 are vertical. Also, the generalized covering of trigons by Karabash and Soifer [7] only uses vertical triangles as well. Thus, it is natural to ask if one can cover the equilateral triangle of side $>n$ with n^2+1 vertical unit

[1] A connected shape formed by unit equilateral triangles with matching edges.

triangles. In this paper, we show that such covering with vertical unit triangles does not exist.

The proof generalizes to an arbitrary union X of n vertical triangles with disjoint interiors: it is impossible to cover X with $n+1$ equilateral triangles of side < 1.

Theorem 1. *Let X be any union of n unit vertical equilateral triangles $S_1, S_2, \ldots, S_n$ with disjoint interiors. Then X cannot be covered by $n+1$ vertical equilateral triangles of sides less than one.*

To recover the original problem, assume by contradiction that an vertical equilateral triangle T of side $> n$ can be covered by n^2+1 unit vertical equilateral triangles. Shrink the covering so that T have side exactly n and the small triangles have side < 1. Then we get contradiction by Theorem 1 as T is a union of n^2 unit triangles with disjoint interiors.

Corollary 1. *n^2+1 unit vertical equilateral triangles cannot cover an vertical equilateral triangle of side $> n$.*

With the covering of T by Conway and Soifer (Figure 1 and 2), and the covering of trigons by Karabash and Soifer, we match the exact minimum number of vertical unit equilateral triangles required for covering.

Corollary 2. *The minimum number of unit vertical equilateral triangles required to cover an vertical equilateral triangle of side $> n$ is exactly n^2+2. Also, the minimum number of unit vertical triangles required to cover a trigon of n triangles is exactly $n+2$.*

2. Proof of Theorem 1

Take the standard Cartesian xy-coordinate system of a plane. Inside the plane, take the triangular grid of unit equilateral triangles with the x-axis as one of the three axes of the triangular grid.

For every unit vertical triangle T, define its rescaled y-coordinate z_T as the y-coordinate of the horizontal side of T divided by $\sqrt{3}/2$. Note that $\sqrt{3}/2$ is the height of a unit equilateral triangle, so the value of z_T is an integer for every triangle T in the triangular grid. Define the function $\tilde{f}_T : \mathbb{R} \to \mathbb{R}$ as the following. For any $z \neq z_T$, the value $\tilde{f}_T(z)$ is the length of the part of the line $y = \sqrt{3}z/2$ covered by triangle T (the value is zero if T is disjoint from the line). The value of $\tilde{f}_T(z_T)$ is chosen so that $\tilde{f}_T$ is right-continuous everywhere: 1 if T is pointed upwards, and 0 if T is pointed downwards.

In this paper, let S^1 be the abelian group quotient $\mathbb{R}/\mathbb{Z}$. For every unit vertical triangle T, define $f_T : S^1 \to \mathbb{R}$ as the function $f_T(t+\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \tilde{f}_T(t+n)$. For any real number x, let $\{x\}$ be the value in $[0,1)$ equal to x modulo 1. Define $\nabla_0(x+\mathbb{Z}) = \{x\}$ and $\Delta_0(x+\mathbb{Z}) = 1-\{x\}$ for any $x+\mathbb{Z} \in S^1$ with representative $x \in \mathbb{R}$. For every $a \in S^1$, define the functions $\Delta_a, \nabla_a : S^1 \to \mathbb{R}$ as the functions $\nabla_a(x) = \nabla_0(x-a)$ and $\Delta_a(x) = \Delta_0(x-a)$. If an unit vertical triangle T is pointed upwards, we have $f_T = \Delta_{y_T}$, and if T is pointed downwards, we have $f_T = \nabla_{y_T}$.

We now prove Theorem 1 by contradiction. Assume that the union X of n unit vertical equilateral triangles $S_1, S_2, \ldots, S_n$ with disjoint interiors can be covered by $n+1$ triangles $T'_0, T'_1, \ldots, T'_n$ of side < 1. Take arbitrary $n+1$ triangles $T_0, T_1, \ldots, T_n$ of side 1 so that each T_i contains the smaller triangle T'_i.

Define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as the function $\tilde{g} = \sum_{i=0}^{n} \tilde{f}_{T_i} - \sum_{j=1}^{n} \tilde{f}_{S_j}$. Take any z different from the rescaled y-coordinates z_{T_i} and z_{S_j} of the triangles. As the triangles $T_0, T_1, \ldots, T_n$ cover the union X of disjoint triangles $S_1, S_2, \ldots, S_n$, the total length of the parts of the line $y = \sqrt{3}z/2$ covered by T_i's is at least the total length of the parts of the line $y = \sqrt{3}z/2$ coverved by S_j's. Thus we have $\tilde{g}(z) \geq 0$. As $\tilde{g}$

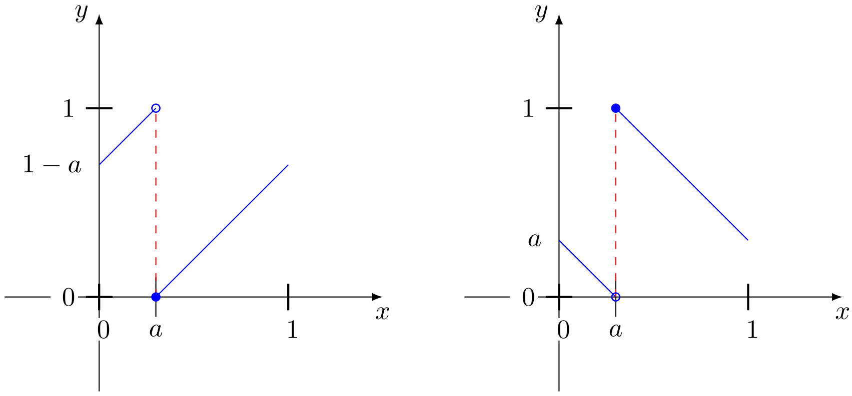

FIGURE 3. Graphs of $\nabla_a(x)$ and $\Delta_a(x)$ for $a = 0.3$.

is right-continuous, by sending the right limit we have $\tilde{g}(z) \geq 0$ for every $z \in \mathbb{R}$ including the case where z is equal to the rescaled y-coordinate of some triangle.

Define $g : S^1 \to \mathbb{R}$ as $g = \sum_{i=0}^{n} f_{T_i} - \sum_{j=1}^{n} f_{S_j}$ so that we have $g(z + \mathbb{Z}) = \sum_{n\in\mathbb{Z}} \tilde{g}(z + n)$. Then consequently we have $g(t) \geq 0$ for every $t \in S^1$. It turns out that this is sufficient to derive a contradiciton. Define $\mathcal{T}$ as the abelian group generated by all functions ∇_a, Δ_a with $a \in S^1$. Then $g \in \mathcal{T}$ by the definition of g. We now examine the properties of $g \in \mathcal{T}$.

Denote the integral of any integrable function $f : S^1 \to \mathbb{R}$ over the whole S^1 as simply $\int f$. Say that two real numbers are equal modulo 1 if their difference is in $\mathbb{Z}$.

Lemma 1. *Any function $f : S^1 \to \mathbb{R}$ in $\mathcal{T}$ has the following properties.*

- *f is right-continuous.*
- *f is differentiable everywhere except for a finite number of points, and the derivative is always equal to a fixed constant $a \in \mathbb{Z}$.*
- *For all $x, y \in \mathbb{R}$, the value $f(y + \mathbb{Z}) - f(x + \mathbb{Z})$ is equal to $a(y - x)$ modulo 1.*
- *The integral $\int f$ is equal to $b/2$ for some $b \in \mathbb{Z}$ where $b - a$ is divisible by 2.*

Proof. Check that all the claimed properties are closed under addition and negation. Then check that the functions ∇_a and Δ_a with $a \in S^1$ satisfy the claimed properties. □

We observed that $g \in \mathcal{T}$ and $g(t) \geq 0$ for every $t \in S^1$. Also, for any unit vertical triangle T we have $\int f_T = 1/2$ so we also have $\int g = 1/2$ by the definition $g = \sum_{i=0}^{n} f_{T_i} - \sum_{j=1}^{n} f_{S_j}$. We now use the following lemma.

Lemma 2. *Let $f : S^1 \to \mathbb{R}$ be any function in $\mathcal{T}$ such that $\int f = 1/2$ and $f(x) \geq 0$ for every $x \in S^1$. Then there is a positive odd integer a and some $c \in [0, 1)$ such that f is either $f(x) = \{ax + c\}$ or $f(x) = 1 - \{ax + c\}$.*

Proof. By Lemma 1, there is some odd number $a \in \mathbb{Z}$ such that $f'(x)$ is a for all x except for a finite number of values. Let $f(0) = c$, then by Lemma 1 again we have $f(x)$ equal to $ax + c$ modulo 1 for all $x \in S^1$. Let $g : S^1 \to \mathbb{R}$ be the function $g(x) = \{ax + c\}$. Then for every $x \in S^1$, as the value $f(x)$ is nonnegative and equal to $ax + c$ modulo 1, we have $f(x) \geq g(x) \geq 0$. But note that the integral $\int g$ is exactly equal to $1/2$ (see Figure 4). So f and g should be equal almost everywhere. As f is right-continuous by Lemma 1, $f(x)$ should be equal to the right limit $g(x-)$

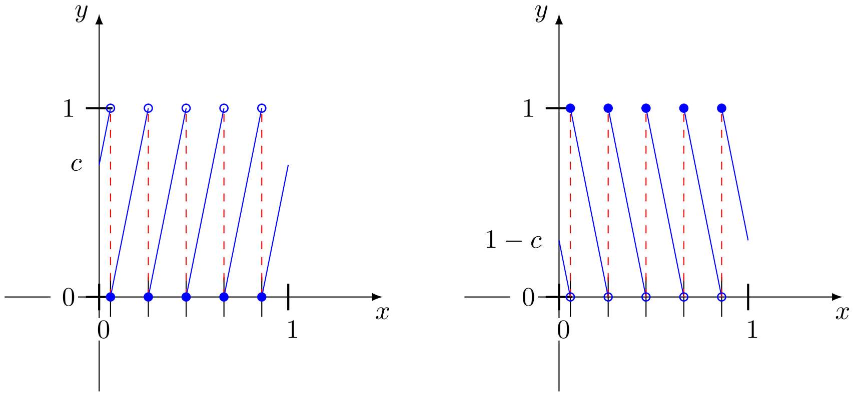

FIGURE 4. Graphs of $x \mapsto \{ax+c\}$ and $x \mapsto 1-\{ax+c\}$ for $a=5$ and $c=0.7$.

of g. If $a>0$, then g is right-continuous so $f(x)=g(x)=\{ax+c\}$. If $a<0$, then the right limit of g is $1-\{-ax+\{-c\}\}$ (this is the value in $(0,1]$ equal to $ax+c$ modulo 1). □

We now finish the proof of Theorem 1. By Lemma 2, the discontinuities of $g=\sum_{i=0}^{n} f_{T_i} - \sum_{j=1}^{n} f_{S_j}$ have to be equidistributed in S^1 with a gap of $1/a$ for some positive odd number a. But each T_i can be taken arbitrary as it contains the smaller triangle T_i' of side <1. So take each T_i so that the rescaled y-coordinates $z_{T_0}, z_{T_1}, \dots, z_{T_n}$ are different from $z_{S_1}, z_{S_2}, \dots, z_{S_n}$ modulo 1 and $z_{T_1}-z_{T_0}$ is an irrational number. Then g has discontinuities at $z_{T_0}+\mathbb{Z}, z_{T_1}+\mathbb{Z}, \dots, z_{T_n}+\mathbb{Z} \in S^1$, and two of them has an irrational gap. This gives contradiction and finishes the proof.

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