n^2+1 UNIT EQUILATERAL TRIANGLES CANNOT COVER AN EQUILATERAL TRIANGLE OF SIDE $>n$ IF ALL TRIANGLES HAVE PARALLEL SIDES

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ABSTRACT. Draft. DO NOT DISTRIBUTE. In a famous short paper, Conway and Soifer asked if an equilateral triangle T of side $n+\varepsilon$ with sufficiently small $\varepsilon > 0$ can be covered by n^2+1 unit equilateral triangles, and provided two ways to cover T with n^2+2 unit triangles. We show that if we require the sides of all triangles to be parallel to the sides of T (e.g. $\triangle$ and $\triangledown$), then it is impossible to cover T with exactly n^2+1 unit equilateral triangles.

1. INTRODUCTION

In 2004, Conway and Soifer attempted to set a world record in the minimum number of words in a paper by submitting the following paper to the *American Mathematical Monthly* [1].

Can n^2+1 unit equilateral triangles cover an equilateral triangle of side $>n$, say $n+\varepsilon$?[1]

John H. Conway & Alexander Soifer

n^2+2 can:

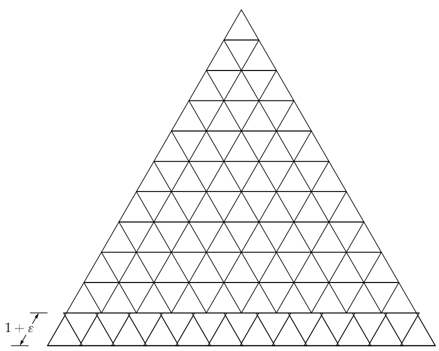

FIGURE 1.

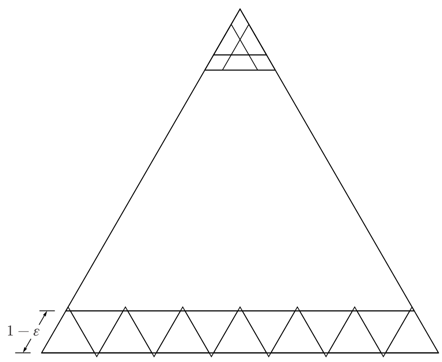

FIGURE 2.

The *American Mathematical Monthly* didn't publish the paper as-is and instead put it inside a "boxed filler" with modifications [2], but the paper is often mentioned as an example of mathematics papers with shortest length in multiple places (e.g. blogs [3, 4], online communities [5, 6, 7], and a Numberphile video).

Related, Karabash and Soifer showed that for every non-equilateral triangle T, n^2+1 triangles similar to T and with the ratio of linear sizes $1:(n+\varepsilon)$, can cover T [8]. Also, they generalized the result of Conway and Soifer and showed that a

[1]Reproduction done by authors with slight modification; for the exact we refer to [1].

trigon[2] made of n unit equilateral triangles can be covered by $n+2$ triangles of side $1-\varepsilon$ [8]. A similar problem of covering a square of side $n+\varepsilon$ with unit squares has been also extensively studied [9, 10, 11, 12, 13]. Still, to the best of the authors' knowledge, the original question in the title of the paper by Conway and Soifer haven't been addressed in the literature.

Define an equilateral triangle as *upright* if all the sides of the triangle are parallel to the three axes of a triangular grid. Note that both triangles $\triangle$ and $\bigtriangledown$ are upright, and all the unit triangles used in Figure 1 and 2 are upright. Also, the generalized covering of trigons by Karabash and Soifer [8] only uses upright triangles as well. Thus, it is natural to ask if one can cover the equilateral triangle of side $> n$ with n^2+1 upright unit triangles. In this paper, we settle this problem by showing that such covering with upright unit triangles does not exist.

The proof generalizes to an arbitrary union X of n upright triangles with disjoint interiors: it is impossible to cover X with $n+1$ equilateral triangles of side < 1.

Theorem 1. *Let X be any union of n unit upright equilateral triangles $S_1, S_2, \ldots, S_n$ with disjoint interiors. Then X cannot be covered by $n+1$ upright equilateral triangles of sides less than one.*

To recover the original problem, assume that an upright equilateral triangle T of side $> n$ can be covered by n^2+1 unit upright equilateral triangles. Rescale the covering so that T have side n and the small triangles have side < 1. Then we get contradiction by Theorem 1 as T is a union of n^2 unit triangles with disjoint interiors.

Corollary 1. *n^2+1 unit upright equilateral triangles cannot cover an upright equilateral triangle of side $> n$.*

With the covering of T by Conway and Soifer (Figure 1 and 2), and the covering of trigons by Karabash and Soifer, we match the exact minimum number of upright unit equilateral triangles for covering.

Corollary 2. *The minimum number of unit upright equilateral triangles required to cover an upright equilateral triangle of side $> n$ is exactly n^2+2. Also, the minimum number of unit upright triangles required to cover a trigon of n triangles is exactly $n+2$.*

2. Proof of Theorem 1

Take the standard Cartesian xy-coordinate system of a plane. Inside the plane, take the triangular grid of unit equilateral triangles with the x-axis as one of the three axes of the triangular grid.

For every unit upright triangle T, define its rescaled y-coordinate z_T as the y-coordinate of the horizontal side of T divided by $\sqrt{3}/2$. Note that $\sqrt{3}/2$ is the height of a unit equilateral triangle, so the value of z_T is an integer for every triangle T in the triangular grid. Define the function $\tilde{f}_T : \mathbb{R} \to \mathbb{R}$ as the following. For any $z \neq z_T$, the value $\tilde{f}_T(z)$ is the length of the part of the line $y = \sqrt{3}z/2$ covered by triangle T (the value is zero if T is disjoint from the line). The value of $\tilde{f}_T(z_T)$ is chosen so that $\tilde{f}_T$ is right-continuous everywhere: 1 if T is pointed upwards, and 0 if T is pointed downwards.

In this paper, let S^1 be the abelian group quotient $\mathbb{R}/\mathbb{Z}$. For every unit upright triangle T, define $f_T : S^1 \to \mathbb{R}$ as the function $f_T(t+\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \tilde{f}_T(t+n)$. For every $a \in S^1$, define $\tilde{a} \in [0,1)$ as the unique representator of a in the interval $[0,1)$. Define $\nabla(x) = \tilde{x}$ for all $x \in S^1$. Define $\Delta_0(0) = 1$ and $\Delta_0(x) = 1-\tilde{x}$ for all nonzero

[2]A connected shape formed by unit equilateral triangles with matching edges.

$x \in \mathbb{R}/\mathbb{Z}$. For every $a \in S^1$, define the functions $\Delta_a, \nabla_a : S^1 \to \mathbb{R}$ as the functions $\nabla_a(x) = \nabla_0(x-a)$ and $\Delta_a(x) = \Delta_0(x-a)$. If an unit upright triangle T is pointed upwards, we have $f_T = \Delta_{y_T}$, and if T is pointed downwards, we have $f_T = \nabla_{y_T}$.

We now prove Theorem 1 by contradiction. Assume that the union X of n unit upright equilateral triangles $S_1, S_2, \dots, S_n$ with disjoint interiors can be covered by $n+1$ triangles $T'_0, T'_1, \dots, T'_n$ of side < 1. Take arbitrary $n+1$ triangles $T_0, T_1, \dots, T_n$ of side 1 so that each T_i contains smaller triangle T'_i.

Define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as the function $\tilde{g} = \sum_{i=0}^n \tilde{f}_{T_i} - \sum_{j=1}^n \tilde{f}_{S_j}$. Take any z different from the rescaled y-coordinates z_{T_i} and z_{S_j} of the triangles. As the triangles $T_0, T_1, \dots, T_n$ cover the union X of disjoint triangles $S_1, S_2, \dots, S_n$, the total length of the parts of the line $y = \sqrt{3}z/2$ covered by T_i's is at least the total length of the parts of the line $y = \sqrt{3}z/2$ coverved by S_j's. Thus we have $\tilde{g}(z) \geq 0$. As $\tilde{g}$ is right-continuous, by sending the right limit we have $\tilde{g}(z) \geq 0$ for every $z \in \mathbb{R}$ including the case where z is the rescaled y-coordinate of some triangle.

Define $g : S^1 \to \mathbb{R}$ as $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$ so that we have $g(z+\mathbb{Z}) = \sum_{n \in \mathbb{Z}} \tilde{g}(z+n)$. Then consequently we have $g(t) \geq 0$ for every $t \in S^1$. It turns out that this is sufficient to derive a contradiciton. Define $\mathcal{T}$ as the abelian group generated by all functions ∇_a, Δ_a with $a \in S^1$. Then $g \in \mathcal{T}$ by the definition of g. We now examine the properties of $g \in \mathcal{T}$.

Denote the integral of any integrable function $f : S^1 \to \mathbb{R}$ over the whole S^1 as simply $\int f$. Say that two real numbers are equal modulo 1 if their difference is in $\mathbb{Z}$.

Lemma 1. *Any function $f : S^1 \to \mathbb{R}$ in $\mathcal{T}$ has the following properties.*

- *f is right-continuous.*
- *f is differentiable everywhere except for a finite number of points, and the derivative is equal to some fixed constant $a \in \mathbb{Z}$.*
- *For all $x, y \in \mathbb{R}$, the value $f(y+\mathbb{Z}) - f(x+\mathbb{Z})$ is equal to $a(y-x)$ modulo 1.*
- *The integral $\int f$ is equal to $b/2$ for some $b \in \mathbb{Z}$ where $b-a$ is divisible by 2.*

Proof. Check that all the claimed properties are closed under addition and negation. Then check that the functions ∇_a and Δ_a with $a \in S^1$ satisfies the claimed properties. □

We observed that $g \in \mathcal{T}$ and $g(t) \geq 0$ for every $t \in S^1$. Also, for any unit upright triangle T we have $\int f_T = 1/2$ so we also have $\int g = 1/2$ by the definition $g = \sum_{i=0}^n f_{T_i} - \sum_{j=1}^n f_{S_j}$. We now use the following lemma. For any real number x, let $\{x\}$ be the value in $[0,1)$ equal to x modulo 1.

Lemma 2. *Let $f : S^1 \to \mathbb{R}$ be any function in $\mathcal{T}$ such that $\int f = 1/2$ and $f(x) \geq 0$ for every $x \in S^1$. Then there is a positive odd integer a and some $c \in [0,1)$ such that f is either $f(x) = \{ax+c\}$ or $f(x) = 1 - \{ax+c\}$.*

Proof. By Lemma 1, there is some odd number $a \in \mathbb{Z}$ such that $f'(x)$ is a for all x except for a finite number of values. Let $f(0) = c$, then by Lemma 1 again we have $f(x)$ equal to $ax+c$ modulo 1 for all $x \in S^1$. Let $g : S^1 \to \mathbb{R}$ be the function $g(x) = \{ax+c\}$. Then for every $x \in S^1$, as the value $f(x)$ is nonnegative and equal to $ax+c$ modulo 1, we have $f(x) \geq g(x) \geq 0$. But note that the integral $\int g$ is exactly equal to $1/2$. So f and g should be equal almost everywhere. As f is right-continuous by Lemma 1, $f(x)$ should be equal to the right limit $g(x-)$ of g. If $a > 0$, then g is right-continuous so $f(x) = g(x) = \{ax+c\}$. If $a < 0$, then the right limit of g is $1 - \{-ax + \{-c\}\}$ (this is the value in $(0,1]$ equal to $ax+c$ modulo 1). □

We now finish the proof of Theorem 1. By Lemma 2, the discontinuities of $g = \sum_{i=0}^{n} f_{T_i} - \sum_{j=1}^{n} f_{S_j}$ have to be equidistributed in S^1 with a gap of $1/a$ for some positive odd number a. But each T_i can be taken arbitrary as it contains the smaller triangle T_i' of side < 1. So take each T_i so that the rescaled y-coordinates $z_{T_0}, z_{T_1}, \ldots, z_{T_n}$ are different from $z_{S_1}, z_{S_2}, \ldots, z_{S_n}$ modulo 1 and $z_{T_1} - z_{T_0}$ is an irrational number. Then g has discontinuities at $z_{T_0} + \mathbb{Z}, z_{T_1} + \mathbb{Z}, \ldots, z_{T_n} + \mathbb{Z} \in S^1$, and two of them has an irrational gap. This gives contradiction and finishes the proof.

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