Moving a Sofa I: A New Upper Bound of Sofa Area

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Contents

Abstract

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1 Introduction

1.1 Moving Sofa Problem

In 1966, Leo Moser posed the following question, now known as the *moving sofa problem*.

> What is the planar shape of maximum area μ that can be moved around a right-angled corner in a hallway of unit width?

More precisely, define a planar set L as the union $L = L_H \cup L_V$ of sets $L_H = (-\infty, 0] \times [-1, 0]$ and $L_V = [-1, 0] \times (-\infty, 0]$ representing the horizontal and vertical side of L respectively. A *moving sofa* S can be defined as a planar subset of L_H that can be moved inside L by a continuous rigid motion to a subset of L_V. The maximum area μ of a moving sofa remains unknown, despite decades of attempts and partial progresses.

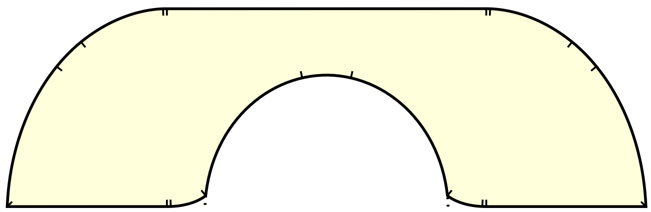

Figure 1: Gerver's sofa S_G.

The largest known area of a moving sofa was improved over decades by many people, until Gerver constructed in 1994 a moving sofa S_G with the area $\mu_G := 2.2195\cdots$ which still holds the currently best record.

Gerver conjectured that his sofa attains the maximum possible area.

Conjecture 1.1. *(Gerver's conjecture) Gerver's sofa S_G attains the maximum area of a moving sofa.*

Indeed, numerical experiments by Gibbs [5] give polygonal shapes resembling the shape of Gerver's sofa, suggesting that Gerver's conjecture is indeed true. Gerver also derived his sofa from local optimality considerations.

It is known that there exists a moving sofa which attains the maximum area μ_M [4]. The best known bounds on μ_M so far can be summarized as the following.

$$2.2195\cdots \leq \mu_M \leq 2.37$$

The area of Gerver's sofa immediately gives the lower bound $2.2195\dots$ of μ_M. On the other hand, the work of Romik and Kallus [7] shows the best upper bound 2.37 of μ_M known so far. They devised an algorithm that computes an upper bound μ_U of μ_M that converges μ_M when ran indefinitely. However, this does not solve the moving sofa problem, as it is impossible to determine the value that μ_U converges to by running the algorithm for a finite time. They proved a concrete upper bound 2.37 by running their implementation `SofaBounds` of their algorithm.

The *rotation angle* of a moving sofa is the clockwise angle that it rotates as it moves from L_H to L_V inside L. Gerver's sofa has the rotation angle $\pi/2$. Let ω be the rotation angle of an arbitrary sofa of maximum area. [4] showed that $\omega \in [\pi/3, \pi/2]$, and [7] improved this by showing that $\omega = [\arcsin(84/85), \pi/2]$ where $\arcsin(84/85) = 81.203\dots^\circ$.

Conjecture 1.2. *There is a moving sofa of maximum area μ that rotates counterclockwise by an angle of 90° in its angle.*

He viewed the movement from the frame of reference of the sofa, in which the sofa is fixed and the hallway is rotated and translated. Following the description suggested by Romik [11], Gerver essentially considered a moving sofa S as the intersection (see Figure 2) of a unit-width strip $H = \mathbb{R} \times [-1, 0]$ and a continuum of hallways L_t for $t \in [0, \pi/2]$ with inner corner $\mathbf{x}(t)$ rotated counterclockwise by an angle of t. Assuming five particular stages of the sofa movement, Gerver solved for the local optimality condition on $\mathbf{x}(t)$ where perturbing $\mathbf{x}$ does not increase the area of sofa to derive his sofa S_G.

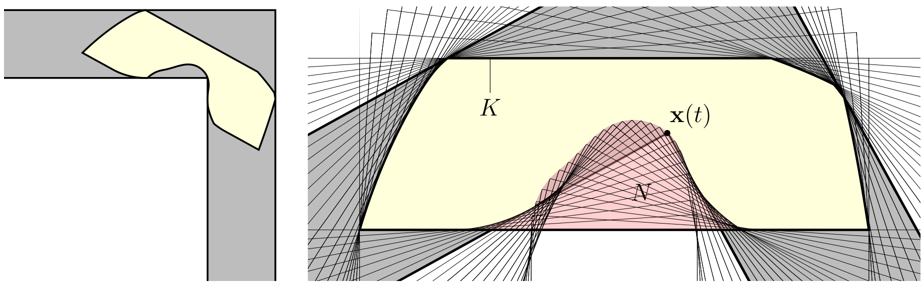

Figure 2: The movement of a moving sofa S in perspective of the hallway (left) and the sofa (right). The sofa S is monotone and consists of the cap K subtracted by its niche N.

In this paper, we propose the *injectivity hypothesis* which is a weaker conjecture implied by Gerver's conjecture. The hypothesis essentially states that there exists a moving sofa S of maximum area such that the followings hold.

- S rotates 90 degrees clockwise as it moves from L_H to L_V.
- The trajectory $\mathbf{x}(t)$ of the inner corner does not self-intersect nor goes below the bottom side of the sofa.

Assuming the injectivity hypothesis, we prove a new upper bound $\mu_M \leq 1+\pi^2/8 = 2.2338\ldots$ which is much closer to the area of Gerver's sofa. Moreover, we show that if Gerver's conjecture is true *a priori*, which is suggested by the experiments in [5], then a modification of an algorithm by Romik and Kallus [7] is able to terminate in finite time and prove the injectivity hypothesis without relying on the a priori assumption.

In Section 1, we define *monotone sofas* as a special subset of monotone sofas, and show that there is a sofa with maximum area which is monotone. A monotone sofa always consists of a convex set called the *cap* subtracted by its *niche* (see Figure 2). With this, we turn the moving sofa problem to maximizing the *sofa area functional* $\mathcal{A} : \mathcal{K} \to \mathbb{R}$ defined on the space of caps. In Section 2, we state the full statement of the injectivity hypothesis. In Section 3, assuming the injectivity hypothesis, we construct a concave upper bound $\mathcal{A}_1 : \mathcal{K} \to \mathbb{R}$ of $\mathcal{A}$ and show that the maximum value of $\mathcal{A}_1$ is $1 + \pi^2/8 = 2.2337\ldots$.

2 Monotone sofas

2.1 Notations and conventions

2.1.1 Summary

We set up basic notions and conventions that will be throughly used in the rest of the document.

2.1.2 Details

Definition 2.1. In this paper, a *shape* is a nonempty compact subset of $\mathbb{R}^2$.

Denote the area (Borel measure) of a shape X as $|X|$. For any subset X of $\mathbb{R}^2$, denote the topological closure and interior as $\overline{X}$ and X° respectively. For a subset X of $\mathbb{R}^2$ and a vector v in $\mathbb{R}^2$, define the set $X + v = \{x + v : x \in X\}$. For any two subsets X, Y of $\mathbb{R}^2$, the set $X + Y = \{x + y : x \in X, y \in Y\}$ is the Minkowski sum of X and Y.

In the introduction, we gave a definition of a moving sofa S as a subset of L_V. However, the condition $S \subseteq L_V$ is too restrictive for our future use, so we will also call any translation of such S as a *moving sofa* as well without loss of generality.

Definition 2.2. The planar *hallway* $L = L_H \cup L_V$ is the union of sets $L_H = (-\infty, 0] \times [-1, 0]$ and $L_V = [-1, 0] \times (-\infty, 0]$, each representing the horizontal and vertical side of L respectively.

Definition 2.3. A *moving sofa* S is a connected, nonempty and compact subset of $\mathbb{R}^2$, such that a translation of S is a subset of L_H that can be moved inside L by a continuous rigid motion to a subset of L_V.

We use the convention $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. For any function f on S^1 and any $t \in \mathbb{R}$, we define $f(t)$ as the value $f(t + 2\pi\mathbb{Z})$. That is, a real value coerces to a value in S^1 when used as an argument of a function that takes a value in S^1. We denote an interval of S^1 by its lift in $\mathbb{R}$. More precisely, for any $t_1 \in \mathbb{R}$ and $t_2 \in (t_1, t_1 + 2\pi]$, the intervals $(t_1, t_2]$ and $[t_1, t_2)$ of $\mathbb{R}$ are also used to denote the corresponding intervals of S^1 mapped under $\mathbb{R} \to S^1$. Likewise, for any $t_1 \in \mathbb{R}$ and $t_2 \in [t_1, t_1 + 2\pi)$, the interval $[t_1, t_2]$ of $\mathbb{R}$ is used to denote the corresponding interval of S^1 mapped under $\mathbb{R} \to S^1$.

For any function $f : \mathbb{R} \to \mathbb{R}$ or $f : S^1 \to \mathbb{R}$, $f(t-)$ denotes the left limit of f at t and $f(t+)$ denotes the right limit of f at t. For any function $f : X \to \mathbb{R}$ defined on some open subset X of either $\mathbb{R}$ and S^1, and $t \in X$, define $\partial^+ f(t)$ and $\partial^- f(t)$ as the right and left differentiation of f at t if they exists.

We denote the integral of a measurable function f with respect to a measure μ on a set X as either $\int_{x \in X} f(x)\, \mu(dx)$ or $\langle f, \mu \rangle_X$. The latter notation is used especially when we want to emphasize that the integral is bi-linear with respect to both f and μ.

We define tangent lines and corresponding half-planes of an arbitrary shape $S \subset \mathbb{R}^2$ as the following.

Definition 2.4. For any angle t in $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ or $\mathbb{R}$, define the unit vectors $u_t = (\cos t, \sin t)$ and $v_t = (-\sin t, \cos t)$.

Definition 2.5. For any shape S, define its *support function* $p_S : S^1 \to \mathbb{R}$ as the value $p_S(t) = \sup\{p \cdot u_t : p \in S\}$.

Definition 2.6. For any angle t in S^1, and a value $h \in \mathbb{R}$, define the line $l(t, h)$ with the *normal angle* t and the distance h from the origin as the following.

$$l(t, h) = \left\{p \in \mathbb{R}^2 : p \cdot u_t = h\right\}$$

Definition 2.7. For any shape S and angle $t \in S^1$, define the *tangent line* $l_S(t)$ of S with *normal angle* t as the line $l_S(t) = l(t, p_S(t))$.

Definition 2.8. For any angle t in S^1, and a value $h \in \mathbb{R}$, define the half-plane $H(t, h)$ with the boundary $l(t, h)$ as the following. We say that the half-plane $H(t, h)$ has the *normal angle* t.

$$H(t, h) = \left\{p \in \mathbb{R}^2 : p \cdot u_t \leq h\right\}$$

Definition 2.9. For any shape S and angle $t \in S^1$, define the *tangent half-plane* $H_S(t)$ of S with *normal angle* t as the line $H_S(t) = H(t, p_S(t))$.

Definition 2.10. A *convex body* K is a nonempty, compact, and convex subset of $\mathbb{R}^2$ [^We follow page 8 of Schneider for this definition. Note that the other authors often also include the condition that K° is nonempty.].

Definition 2.11. For any convex body K and $t \in S^1$, define the *edge* $e_K(t)$ of K as the intersection of K with the tangent line $l_K(t)$. Let $v_K^+(t)$ and $v_K^-(t)$ be the endpoints of the line segment $e_K(t)$ such that $v_K^+(t)$ is positioned farthest in the direction of v_t and $v_K^-(t)$ is positioned farthest in the opposite direction of v_t.

It is possible that the edge $e_K(t)$ can be a single point. In such case, the tangent line $l_K(t)$ touches K at the single point $v_K^+(t) = v_K^-(t)$.

For every convex body K with nonempty interior, there is a minimum closed subset Π_K of S^1 such that K is the intersection of half-planes $H_K(t)$ for $t \in \Pi_K$ (see). Call this set Π_K the *angular support* of K (see for the full definition).

2.2 Tangent hallway

2.2.1 Summary

Following the ideas of Gerver, we show that any moving sofa S is essentially a connected subset of the intersection of two strips and rotations of hallway. We define the *tangent hallways* of S that rotates around S, containing S and touching S in the outer walls.

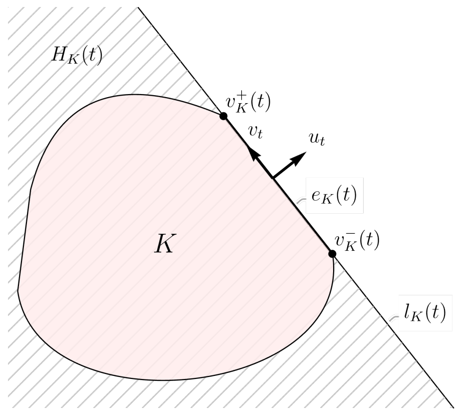

Figure 3: A convex body K with its edge, vertices, tangent line, and half-plane.

2.2.2 Details

Gerver's main insight for deriving his sofa S_G was that a moving sofa is essentially a connected common subset of rotating hallways. Let S be a moving sofa with rotation angle $\omega \in [0, \pi/2]$. By looking at the movement of S inside the hallway L in the perspective of S, Gerver observed that the fixed S should be contained inside the hallways rotating around S (line xx-yy of [4]). Furthermore, he actually showed that for a connected shape S, being a common subset of rotating hallways is also *sufficient* for S to be a moving sofa (line xx-yy of [4]). We state this observation of Gerver as a separate Theorem 2.1.

Definition 2.12. Define the unit-width horizontal and vertical strips $H = \mathbb{R} \times [-1, 0]$ and $V = [-1, 0] \times \mathbb{R}$ respectively.

Definition 2.13. Denote the rotation map of $\mathbb{R}^2$ along the origin by a counterclockwise angle of θ as R_θ.

Theorem 2.1. *Let $\omega \in [0, \pi/2]$ be an arbitrary angle. For a connected shape S, the followings are equivalent.*

1. *S is a moving sofa with rotation angle ω.*
2. *S is contained in a translation of H and $R_\omega(V)$. Also, for every $t \in [0, \omega]$, S is contained in a translation of $R_t(L)$, the hallway rotated counterclockwise by an angle of t.*

Gerver's idea for proving Theorem 2.1 was to use the hallways that touch the shape S in the outer walls. We call such hallway the *tangent hallway* of S.

Definition 2.14. For any shape S and angle $t \in S^1$, define the *tangent hallway* $L_S(t)$ of S with angle t as the following.

$$L_S(t) = R_t(L) + (p_S(t) - 1)\, u_t + (p_S(t + \pi/2) - 1)\, v_t$$

Proposition 2.2. *For any shape S and angle $t \in S^1$, the tangent hallway $L_S(t)$ is the unique rigid transformation of L such that the outer walls of $L_S(t)$ are $l_S(t)$ and $l_S(t + \pi/2)$, each corresponding to the outer walls $x = 1$ and $y = 1$ of L respectively.*

Assume that a shape S is contained in a translation L' of $R_t(L)$, so that L' is a hallway rotated counterclockwise by an angle of t. Then we can push L' towards S in the directions $-u_t$ and $-v_t$ until the outer walls of new $L' = L_S(t)$ touch S. The pushed hallway $L_S(t)$ still contains S because the directions $-u_t$ and $-v_t$ push the inner walls of L' forward or keep them stationary. Summarizing, we get the following.

Proposition 2.3. *For any shape S contained in a translation of the hallway $R_t(L)$ rotated counterclockwise by an angle of $t \in S^1$, the tangent hallway $L_S(t)$ with angle t contains S.*

Using tangent hallways, we give a complete proof of Theorem 2.1 for clarity as both Gerver's statement and proof of Theorem 2.1 are in between the lines of the proof of Theorem 1 in [4].

Proof. (of Theorem 2.1)

($\Rightarrow$) Consider the movement of S inside the hallway L. For any angle $t \in [0, \omega]$, there is a moment where the sofa S is rotated clockwise by an angle of t and contained in L. Looking at this in perspective of the sofa, S is contained in some translation of L rotated *counterclockwise* by arbitrary $t \in [0, \omega]$. Likewise, by looking at the initial (resp. final) position of S inside L_H (resp. L_V) in perspective of S, the set S is contained in both a translation of H and $R_\omega(V)$.

($\Leftarrow$) The position of the tangent hallway $L_S(t)$ depends only on the values $p_S(t)$ and $p_S(t + \pi/2)$ of the support function p_S of S. As p_S is continuous, the tangent hallway $L_S(t)$ moves continuously with respect to $t \in [0, \omega]$. Let f_t be the rigid transformation that sends $L_S(t)$ to L. Then the transformation $f_t(S)$ of S moves continuously with respect to t, and is inside L by Proposition 2.3. Because S is contained in a translation of H, the width of S measured in the direction of v_0 is at most 1. So S is contained in the horizontal side of $L_S(0)$, and by mapping it under f_0 we have $f_0(S) \subseteq L_H$. Likewise, S is contained in a translation of $R_\omega(V)$, so the width of S measured in the direction of v_ω is at most 1. Thus S is contained in the vertical side of $L_S(\omega)$, and by mapping it under f_ω we have $f_\omega(S) \subseteq L_V$. $\square$

Like Gerver did in [4], we will also use the tangent hallways $L_S(t)$ as the hallways containing our sofa S. So we give names to the parts of $L_S(t)$ that will be used thoroughly. We first name the parts of the hallway L.

Definition 2.15. Let $\mathbf{x} = (-1, -1)$ and $\mathbf{y} = (0, 0)$ be the inner and outer corner of L respectively.

Let a and b be the lines $x = 0$ and $y = 0$ representing the outer walls of L passing through $\mathbf{y}$. Let c and d be the half-lines $(-\infty, -1] \times \{-1\}$ and

$\{-1\} \times (-\infty, -1]$ representing the inner walls of L emanating from the inner corner $\mathbf{x}$.

Let $Q^+ = (-\infty, 0]^2$ be the closed quarter-plane bounded by outer walls a and c. Let $Q^- = (-\infty, -1)^2$ be the open quarter-plane bounded by inner walls b and d, so that $L = Q^+ \setminus Q^-$.

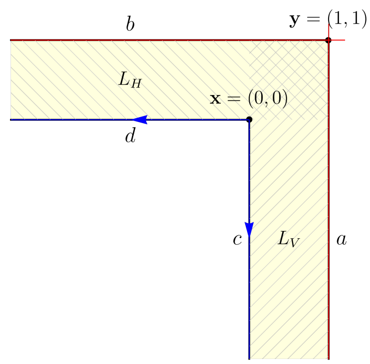

Figure 4: The standard hallway L and its parts.

Then we name the parts of tangent hallway $L_S(t)$.

Definition 2.16. For any shape S and angle $t \in S^1$, let $\mathbf{x}_S(t), \mathbf{y}_S(t), a_S(t), b_S(t), c_S(t), d_S(t), Q_S^+(t), Q_S^-(t)$ be the parts of $L_S(t)$ corresponding to the parts $\mathbf{x}, \mathbf{y}, a, b, c, d, Q^+, Q^-$ of L respectively.

So we have $L_S(t) = Q_S^+(t) \setminus Q_S^-(t)$ and $Q_S^+(t) = H_S(t) \cap H_S(t + \pi/2)$ in particular. Also we have the following representations of the parts purely in terms of the supporting function p_K.

$$\mathbf{x}_S(t) = (p_K(t) - 1)u_t + (p_K(t + \pi/2) - 1)v_t \tag{1}$$
$$\mathbf{y}_S(t) = p_K(t)u_t + p_K(t + \pi/2)v_t \tag{2}$$
$$a_S(t) = l_S(t) = l(t, p_K(t)) \tag{3}$$
$$b_S(t) \subseteq l(t, p_K(t) - 1) \tag{4}$$
$$c_S(t) = l_S(t + \pi/2) = l(t + \pi/2, p_K(t + \pi/2)) \tag{5}$$
$$d_S(t) \subseteq l(t + \pi/2, p_K(t + \pi/2) - 1) \tag{6}$$

2.3 Monotonization

2.3.1 Summary

We show that any moving sofa S can be enlarged to a *monotone sofa* $\mathcal{M}(S)$, which is the intersection of two strips and the tangent hallways. So we can assume that a moving sofa of maximum area is a monotone sofa. We also describe the structure of $\mathcal{M}(S)$ as the cap $\mathcal{C}(S)$ of S subtracted by the niche $\mathcal{N}(S)$ of S.

2.3.2 Overview

The next idea of Gerver was to enlarge a moving sofa S of rotation angle ω by intersecting the stripes H, $R_\omega(V)$ and the tangent hallways $L_S(t)$ for each $t \in [0, \omega]$ containing S. The intersection then immediately satisfies the second condition of Theorem 2.1. We call this intersection as the *monotonization* of S. Without loss of generality, here we translate S before taking the intersection so that it satisfies $p_S(\omega) = p_S(\pi/2) = 1$. By doing so we can assume that S is contained in H and $R_\omega(V)$ touching the upper edges of two stripes.

Definition 2.17. Let S be any moving sofa with rotation angle ω such that $p_S(\omega) = p_S(\pi/2) = 1$. Define the *monotonization* of S as the intersection

$$\mathcal{M}(S) = H \cap R_\omega(V) \cap \bigcap_{0 \leq t \leq \omega} L_S(t).$$

Corollary 2.4. *For any moving sofa S with rotation angle ω satisfying $p_S(\omega) = p_S(\pi/2) = 1$, its monotonization $\mathcal{M}(S)$ contains S.*

Gerver claimed in [4] that the monotonization $\mathcal{M}(S)$ of a moving sofa S is also a moving sofa because $\mathcal{M}(S)$ satisfies the second condition of Theorem 2.1. However, he did not show that the set $\mathcal{M}(S)$ is connected, which is essential for the set $\mathcal{M}(S)$ to be a moving sofa. We will fill the gap in his argument by proving that $\mathcal{M}(S)$ is indeed connected.

Theorem 2.5. *Let S be a moving sofa with rotation angle ω satisfying $p_S(\omega) = p_S(\pi/2) = 1$. Then the monotonization $\mathcal{M}(S)$ is a moving sofa with rotation angle ω containing S.*

Definition 2.18. A moving sofa is *monotone* if it is a monotonization of some sofa.

Because monotonizing a moving sofa never decreases its area (Corollary 2.4), we can safely assume that a sofa of maximum area is monotone once Theorem 2.5 is proved.

2.3.3 Geometric Structure of Monotonization

To prove Theorem 2.5, we need to first analyze the geometric structure of the monotonization $\mathcal{M}(S)$ in depth. This structure, consisting of a *cap* and its *niche* as depicted below, is very important and will be used thoroughly for the rest of the argument.

We now give full details of the structure of $\mathcal{M}(S)$. First define the following polygons.

Definition 2.19. For any angle $\omega \in [0, \pi/2]$, define the *parallelogram* $P_\omega = H \cap R_\omega(V)$ with *rotation angle* ω. If $\omega = \pi/2$, then P_ω is the horizontal strip H. Otherwise, Let $O = (0, 0)$ be the origin, and note that if $\omega < \pi/2$ then O is the upper-right corner of P_ω. Define the point $o_\omega = (-\tan(\omega/2), -1)$, so that for any $\omega < \pi/2$ the point $o_\omega = l(\omega, -1) \cap l(\pi/2, -1)$ is the lower-left corner of P_ω.

Finally, define the *fan* $F_\omega = H(\pi + \omega, 1) \cap H(3\pi/2, 1)$ as the convex cone pointed at o_ω bounded by the bottom edges of P_ω.

Let S be any moving sofa of rotation angle $\omega \in [0, \pi/2]$. By breaking down each $L_S(t)$ into $Q_S^+(t) \setminus Q_S^-(t)$, we get the following representation of $\mathcal{M}(S)$ as a subtraction of two sets.

$$\begin{aligned} \mathcal{M}(S) &= P_\omega \cap \bigcap_{0 \le t \le \omega} L_S(t) \\ &= \left(P_\omega \cap \bigcap_{0 \le t \le \omega} Q_S^+(t) \right) \setminus \left(F_\omega \cap \bigcup_{0 \le t \le \omega} Q_S^-(t) \right) \end{aligned} \tag{7}$$

The set

$$K = P_\omega \cap \bigcap_{0 \le t \le \omega} Q_S^+(t) \tag{8}$$

is a convex set circumscribed inside the parallelogram P_ω with the outer walls $a_S(t)$ and $c_S(t)$ of $L_S(t)$ as tangent lines. We will soon call K the *cap of S* (Theorem 2.6).

Imagine the set K as a block of clay and rotate it inside the hallway L in the clockwise angle of ω while touching the outer walls a and c of L. As K rotates around, the inner corner of L will carve out the set

$$N = F_\omega \cap \bigcup_{0 \le t \le \omega} Q_S^-(t), \tag{9}$$

from K and the clay will form the monotonization $\mathcal{M}(S) = K \setminus N$. We will soon call N the *niche* of K ().

We now rigorously define what a *cap* is.

Definition 2.20. A set Θ is an *angle set* with *rotation angle* $\omega \in [0, \pi/2]$ if $\{0, \omega\} \subseteq \Theta \subseteq [0, \omega]$ and Θ is closed.

Definition 2.21. Let Θ be an angle set with rotation angle ω. A *cap* K with *angle set* Θ (and *rotation angle* ω) is a convex body such that the followings hold.

1. $p_K(\omega) = p_K(\pi/2) = 0$ and $p_K(\pi + \omega) = p_K(3\pi/2) = 1$.
2. K is an intersection of half-planes with normal angles in $\Pi = \Theta \cup (\Theta + \pi/2) \cup \{\pi + \omega, 3\pi/2\}$.

Definition 2.21 is generalized to an arbitrary set Θ of angles for future use. Geometrically, the first condition of Definition 2.21 states that K is contained in the parallelogram P_ω and touches all sides of P_ω, and the second condition of Definition 2.21 states that the sides of K have normal angles in Π.

Name the set K in Equation (8) as $\mathcal{C}(S)$.

Definition 2.22. Let S be any moving sofa with rotation angle $\omega \in [0, \pi/2]$ with $p_S(\omega) = p_S(\pi/2) = 0$. Define

$$\mathcal{C}(S) = P_\omega \cap \bigcap_{0 \leq t \leq \omega} Q_S^+(t).$$

We show that $K = \mathcal{C}(S)$ satisfies Definition 2.21, justifying that K is the cap of S.

Theorem 2.6. *The set $\mathcal{C}(S)$ in Definition 2.22 is a cap with angle set $[0, \omega]$ containing S as a subset. With this, call $\mathcal{C}(S)$ as the* cap *of S.*

Proving Theorem 2.6 is not hard but needs some definition that will be used in other proofs as well.

Definition 2.23. For every $\omega \in [0, \pi/2]$, define M_ω as the convex hull of the points $O, -u_\omega, -v_0$, and o_ω.

Geometrically, M_ω is a subset of P_ω enclosed by the perpendicular legs from O to the bottom sides of P_ω. We also introduce the following terminologies.

Definition 2.24. Say that a point p_1 is *further than* (resp. *strictly further than*) the point p_2 *in the direction* of vector $v \in \mathbb{R}^2$ if $p_1 \cdot v \geq p_2 \cdot v$ (resp. $p_1 \cdot v > p_2 \cdot v$).

Definition 2.25. Say that a set $X \subseteq \mathbb{R}^2$ is *closed in the direction of* vector $v \in \mathbb{R}^2$ if, for any $x \in X$ and $\lambda \geq 0$, we have $x + \lambda v \in X$.

We prepare the following lemma.

Lemma 2.7. *The set $\mathcal{C}(S)$ in Definition 2.22 contains M_ω.*

Proof. From $p_S(\omega) = p_S(\pi/2) = 0$, we can take points q and r of S so that q is on the line $l(\pi/2, 0)$ further than O in the direction of $-u_0$, and r is on the line $l(\omega, 0)$ further than O in the direction of $-v_\omega$. Take arbitrary $t \in [0, \omega]$. Because $Q_S^+(t)$ is a right-angled convex cone with normal vectors u_t and v_t containing q and r, $Q_S^+(t)$ also contains O. Because $Q_S^+(t)$ contains O and is closed in the direction of $-u_t$ and $-v_t$, $Q_S^+(t)$ contains M_ω as a subset. $\square$

Now we go back to proving Theorem 2.6.

Proof. (of Theorem 2.6) Let $K = \mathcal{C}(S)$. That $S \subseteq K$ is an immediate consequence of Corollary 2.4 and $L_K(t) \subseteq Q_K^+(t)$. By Definition 2.22 and Lemma 2.7 we have $M_\omega \subseteq K \subseteq P_\omega$. So the first condition of Definition 2.21 is satisfied. The second condition of Definition 2.21 comes from that P_ω is the intersection of four half-planes with normal angles $\omega, \pi/2, \pi + \omega, 3\pi/2$, and $Q_S^+(t)$ is an intersection of two half-planes with normal angles t and $\pi/2 + t$. $\square$

The tangent hallways of a moving sofa S are exactly same as the tangent hallways of its cap $K = \mathcal{C}(S)$.

Proposition 2.8. *Let S be any moving sofa with rotation angle $\omega \in [0, \pi/2]$ such that $p_S(\omega) = p_S(\pi/2) = 0$. Let $K = \mathcal{C}(S)$ be its cap. Then the support functions p_S and p_K of S and K agree on the set $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$. Consequently, the tangent hallways $L_S(t)$ and $L_K(t)$ of S and K are equal for any $t \in [0, \omega]$.*

We omit the proof of Proposition 2.8. With Proposition 2.8, we have $Q_S^-(t) = Q_K^-(t)$ in particular. So we can define the set N (equation Equation (9)) purely in terms of the cap $K = \mathcal{C}(S)$ as the following.

Definition 2.26. Let K be any cap with rotation angle $\omega \in [0, \pi/2]$ and angle set $\Theta \subseteq [0, \omega]$. Define the *niche* of K as the following.

$$\mathcal{N}(K) = F_\omega \cap \bigcup_{t \in \Theta} Q_K^-(t)$$

By Equation (7) we have the following structure of a monotonization of a moving sofa.

Theorem 2.9. *Let S be any moving with rotation angle ω satisfying $p_S(\omega) = p_S(\pi/2) = 1$. Then the monotonization of S is $\mathcal{M}(S) = K \setminus \mathcal{N}(K)$ where $K = \mathcal{C}(S)$ is the cap of S with angle set $[0, \omega]$ and $\mathcal{N}(K)$ is the niche of K.*

2.3.4 Connectedness of Monotonization

We now show the connectedness of the monotonization $\mathcal{M}(S)$ of a moving sofa S as promised.

Proof. (of Theorem 2.5)

We show that $\mathcal{M}(S)$ is connected. Fix an arbitrary point p in $\mathcal{M}(S)$. Take the line l_θ passing p that makes an arbitrary angle $\theta \in [\omega, \pi/2]$ with the x-axis. Observe that $l_\theta \cap \mathcal{M}(S)$ is a nonempty line segment because $\bigcup_{t \in [0,\omega]} Q_S^-(t)$ is closed in the direction of $-u_\theta$. So the line l_θ meets S for any θ, then p is connected to S inside $\mathcal{M}(S)$ and the proof is done. Our goal now is to prove that there is some $\theta \in [\omega, \pi/2]$ such that l_θ meets S. Assume by contradiction that for any $\theta \in [\omega, \pi/2]$ the line l_θ is disjoint from S.

As $S \subseteq \mathcal{M}(S) \subseteq \mathcal{C}(S)$ and the support function of S and $\mathcal{C}(S)$ agrees on the set $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$, the support function of $\mathcal{M}(S)$ also agrees with that of S on $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$. As $p \in \mathcal{M}(S)$, the line $l_{\pi/2}$ is either equal to $l_{\mathcal{M}(S)}(0) = l_S(0)$ or strictly on the left side of $l_S(0)$. The line $l_{\pi/2}$ can't be equal to $l_S(0)$ because otherwise, $l_{\pi/2}$ contains some point of S contradicting our assumption. So the line $l_{\pi/2}$ is strictly on the left side of $l_S(0)$, and there is a point of S strictly on the right side of $l_{\pi/2}$.

Likewise, as $p \in \mathcal{M}(S)$, the line l_ω that passes through p is either equal to $l_{\mathcal{M}(S)}(\omega + \pi/2) = l_S(\omega + \pi/2)$ or strictly on the right side of $l_S(\omega + \pi/2)$. The line l_ω cannot be equal to $l_S(\omega + \pi/2)$ as we assumed that l_ω is disjoint from S. So the line l_ω is strictly on the right side of $l_S(\omega + \pi/2)$, and there is a point of S strictly on the left side of l_ω.

As the line l_θ is disjoint from S for any $\theta \in [\omega, \pi/2]$, the set S is inside the set $X = \mathbb{R}^2 \setminus \bigcup_{\theta \in [\omega, \pi/2]} l_\theta$. Note that X has exactly two connected components X_L and X_R on left and right respectively. As there is a point of S strictly on the right side of $l_{\pi/2}$, $S \cap X_R$ is nonempty. As there is also a point of S strictly on the left side of l_ω, $S \cap X_L$ is also nonempty. We get contradiction as S is a connected subset of X.

As the set $\mathcal{M}(S)$ is connected, and satisfies the second condition of Theorem 2.1 by definition, $\mathcal{M}(S)$ is a moving sofa with rotation angle ω. By Corollary 2.4, the moving sofa $\mathcal{M}(S)$ contains S. We observed that the support function of $\mathcal{M}(S)$ agrees with that of S on $\Theta \cup (\Theta + \pi/2)$, so the cap of $\mathcal{M}(S)$ is equal to the cap of S. □

In the proof of Theorem 2.5 we showed that the support function of $\mathcal{M}(S)$ is equal to that of S on $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$. So a sofa and its monotonization shares the same cap.

Corollary 2.10. *The monotonization $\mathcal{M}(S)$ of any sofa S with rotation angle $\omega \in [0, \pi/2]$ shares the same cap with S. That is, $\mathcal{C}(\mathcal{M}(S)) = \mathcal{C}(S)$.*

Consequently, any monotone sofa S is equal to its cap $K = \mathcal{C}(S)$ subtracted by its niche $\mathcal{N}(K)$ by Theorem 2.9 and Theorem 2.5. So monotization $S \mapsto \mathcal{M}(S)$ could be said as a 'projection' to monotone sofas in the sense that $\mathcal{M}$ is a surjective idempotent map that fixes monotone sofas.

2.4 Cap contains niche

2.4.1 Summary

We show that the niche $\mathcal{N}(S)$ of a monotone sofa S is contained in the cap $\mathcal{C}(S)$ of S. The proof is a bit involved, but the benefit is that the area of S is now $|S| = |K| - |\mathcal{N}(K)|$ where the main difficulty of understanding the area is concentrated at $\mathcal{N}(K)$.

2.4.2 Details

The shape of Gerver's sofa resembles a retro-style telephone handset, with the bottom side of the cap K_G of Gerver's sofa carved out by its niche $\mathcal{N}(K_G)$. We will show that any monotone sofa S shares the same property.

Theorem 2.11. *For any monotone sofa S, the cap $K = \mathcal{C}(S)$ contains the niche $\mathcal{N}(K)$.*

Theorem 2.11 is important as it allows us to understand the area of a monotone sofa $|S| = |K| - |\mathcal{N}(K)|$ by the area of cap and niche separately.

To prove Theorem 2.11, we need to prepare a handful of geometric definitions. Define the *vertices* of a cap K.

Definition 2.27. Let K be a cap with angle set Θ and rotation angle ω. For any $t \in \Theta$, define the vertices $A^+_K(t) = v^+_K(t)$, $A^-_K(t) = v^-_K(t)$, $C^+_K(t) = v^+_K(t + \pi/2)$, and $C^-_K(t) = v^-_K(t + \pi/2)$ of K.

Note that the outer wall $a_K(t)$ (resp. $c_K(t)$) of $L_K(t)$ touches the cap K at the vertices $A_K^+(t)$ and $A_K^-(t)$ (resp. $C_K^+(t)$ and $C_K^-(t)$) respectively. We also define the *upper boundary* of a cap K.

Definition 2.28. For any cap K with rotation angle ω, define the *upper boundary* δK of K as the set $\delta K = \bigcup_{t \in [0,\omega+\pi/2]} e_K(t)$.

Note that for a cap K with angle set Θ, the upper boundary δK is exactly the points touched by the outer walls $a_K(t)$ and $c_K(t)$ of tangent hallways $L_K(t)$ touch K for every $t \in \Theta$. Also note that unless K is the vertical line segment $\{0\} \times [0, 1]$, the cap K has nonempty interior and δK is a Jordan arc connecting the endpoints $A_K^-(0)$ and $C_K^+(\omega)$. We also give name to the convex polygons $F_\omega \cap Q_K^-(t)$ whose union over all $t \in \Theta$ constitutes the niche $\mathcal{N}(K)$.

Definition 2.29. For any cap K with rotation angle ω, define $T_K(t) = F_\omega \cap Q_K^-(t)$ as the *wedge* of K with angle $t \in [0, \omega]$.

Proposition 2.12. *For any cap K with angle set Θ and rotation angle ω, we have $\mathcal{N}(K) = \cup_{t\in\Theta} T_K(t)$.*

We give names to parts of the wedge $T_K(t)$.

Definition 2.30. For any cap K with rotation angle ω and $t \in (0, \omega)$, define $W_K(t)$ as the intersection of lines $b_K(t)$ and $l(\pi, 0)$. Define $w_K(t) = (A_K^-(0) - W_K(t)) \cdot u_0$ as the signed distance from point $W_K(t)$ and the vertex $A_K^-(0)$ along the line $l(\pi, 0)$.

Likewise, define $Z_K(t)$ as the intersection of lines $d_K(t)$ and $l(\omega, 0)$. Define $z_K(t) = (C_K^+(\omega) - Z_K(t)) \cdot v_\omega$ as the signed length between $Z_K(t)$ and the vertex $C_K^+(\omega)$ along the line $l(\omega, 0)$.

Note that if $T_K(t)$ contains the origin $(0, 0)$, then the points $W_K(t)$ and $Z_K(t)$ are the leftmost and rightmost point of $\overline{T_K(t)}$ respectively. We show that $w_K(t), z_K(t) \geq 0$; this implies that the endpoints $W_K(t)$ and $Z_K(t)$ of $T_K(t)$ are inside K.

Lemma 2.13. *Let K be any cap with rotation angle ω. For any angle $t \in (0, \omega)$, we have $w_K(t), z_K(t) \geq 0$.*

Proof. To show that $w_K(t) \geq 0$, we need to show that the point $v_K^-(0)$ is further than the point $W_K(t)$ in the direction of u_0. The point $q = a_K(t) \cap l(\pi/2, 1)$ is further than $W_K(t) = b_K(t) \cap l(\pi/2, 0)$ in the direction of u_0, because the lines $a_K(t)$ and $b_K(t)$ form the boundary of a unit-width vertical strip rotated counterclockwise by t. The point $A_K^-(t)$ is further than $q = a_K(t) \cap a_K(\pi/2)$ in the direction of u_0 because K is a convex body. Finally, the point $A_K^-(0)$ is further than $A_K^-(t)$ in the direction of u_0 again because K is a convex body. Summing up, the points $W_K(t), q, A_K^-(t), A_K^-(0)$ are aligned in the direction of u_0, completing the proof. A symmetric argument in the direction of v_ω proves $z_K(t) \geq 0$. $\square$

We also prepare a lemma for proving Theorem 2.11.

Definition 2.31. Any line l of $\mathbb{R}^2$ divides the plane into two half-planes. If l is not parallel to the y-axis, call the *left side* (resp. *right side*) of l as the closed half-plane with boundary l containing the point $-Nu_0$ (resp. Nu_0) for sufficiently large N. We mention the corresponding open half-plane as *strictly left side* of l.

Likewise, if l is not parallel to the x-axis, call the *upper side* (resp. *lower side*) of l as the closed half-plane with boundary l containing the point Nv_0 (resp. $-Nv_0$) for sufficiently large N.

Lemma 2.14. *Fix any cap K with rotation angle $\omega \in [0, \pi/2]$ and an angle $t \in (0, \omega)$. If the inner corner $\mathbf{x}_K(t)$ is in K, then the wedge $T_K(t)$ is a subset of K.*

Proof. Assume that $\mathbf{x}_K(t) \in K$.

If $\omega = \pi/2$, then $T_K(t)$ is the triangle with vertices $W_K(t)$, $\mathbf{x}_K(t)$, and $Z_K(t)$ in counterclockwise order. Note also that $W_K(t)$ is further in the x-axis direction than $Z_K(t)$. As $w_K(t), z_K(t) \geq 0$, this implies that all three vertices of $T_K(t)$ are in K.

If $\omega < \pi/2$, we divide the proof into four cases on whether the origin $(0, 0)$ lies strictly below the lines $b_K(t)$ and $d_K(t)$ or not respectively.

- If $(0, 0)$ lies on or above both $b_K(t)$ and $d_K(t)$, then the corner $\mathbf{x}_K(t)$ is outside the interior F_ω° of fan F_ω, and so $T_K(t)$ is an empty set.
- If $(0, 0)$ lies on or above $b_K(t)$ but lies strictly below $d_K(t)$, then $T_K(t)$ is a triangle with vertices $\mathbf{x}_K(t)$, $Z_K(t)$ and the intersection $p = l(\omega, 0) \cap b_K(t)$. In this case, the point p is in the line segment connecting $Z_K(t)$ and $(0, 0)$. Also, as $z_K(t) \geq 0$ (Lemma 2.13) the point $Z_K(t)$ lies in the segment connecting $C_K^+(\omega)$ and the origin $(0, 0)$. So the points $\mathbf{x}_K(t), Z_K(t), p$ are in K and by convexity of K we have $T \subseteq K$.
- The case where $(0, 0)$ lies strictly below $b_K(t)$ but lies on or above $d_K(t)$ can be handed by an argument symmetric to the previous case.
- If $(0, 0)$ lies strictly below both $b_K(t)$ and $d_K(t)$, then $T_K(t)$ is a quadrilateral with vertices $\mathbf{x}_K(t)$, $Z_K(t)$, $W_K(t)$ and $(0, 0)$. As $w_K(t) \geq 0$ (resp. $z_K(t) \geq 0$) by Lemma 2.13, the point $W_K(t)$ (resp. $Z_K(t)$) is in the line segment connecting $(0, 0)$ and $A_K^-(0)$ (resp. $C_K^+(\omega)$). So all the vertices of $T_K(t)$ are in K, and $T_K(t)$ is in K by convexity.

□

Now we prove that $\mathcal{N}(K) \subseteq K$ if K is the cap of a monotone sofa. In fact, we identify the exact condition where $\mathcal{N}(K) \subseteq K$ for a general cap K that may or may not be a cap of a monotone sofa.

Theorem 2.15. *Let K be any cap with angle set Θ and rotation angle ω. Then the followings are all equivalent.*

1. $\mathcal{N}(K) \subseteq K$
2. $\mathcal{N}(K) \subseteq K \setminus \delta K$

3. For any $t \in \Theta$, either $\mathbf{x}(t) \notin F_\omega^\circ$ or $\mathbf{x}(t) \in K$.
4. The set $S = K \setminus \mathcal{N}(K)$ is connected.

Proof. The conditions 1 and 2 are equivalent because the niche $\mathcal{N}(K)$ is open in the subset topology of F_ω, and the set $K \setminus \delta K$ is the interior of K in the subset topology of F_ω.

($1 \Rightarrow 3$) We will prove the contraposition and assume $\mathbf{x}_K(t) \in F_\omega^\circ \setminus K$. Then a neighborhood of $\mathbf{x}_K(t)$ is inside F_ω and disjoint from K, so a portion of $T_K(t)$ is outside K, showing $\mathcal{N}(K) \not\subseteq K \setminus \delta K$.

($3 \Rightarrow 1$) If $\mathbf{x}(t) \notin F_\omega^\circ$, then $T_K(t)$ is an empty set. If $\mathbf{x}(t) \in K$, then by Lemma 2.14 we have $T_K(t) \subseteq K$.

($2 \Rightarrow 4$) As δK is disjoint from $\mathcal{N}(K)$, we have $\delta K \subseteq S$. We show that S is connected. First, note that δK is connected. Next, take any point $p \in S$. Take the half-line r starting from p in the upward direction v_0. Then r touches a point in δK as $p \in K$. Moreover, r is disjoint from $\mathcal{N}(K)$ as the set $\mathcal{N}(K) \cup (\mathbb{R}^2 \setminus F_\omega)$ is closed in the direction $-v_0$. Now $r \cap K$ is a line segment inside S that connects the arbitrarily chosen point $p \in S$ to a point in δK. So S is connected.

($4 \Rightarrow 3$) Assume by contradiction that $\mathbf{x}(t) \in F_\omega^\circ \setminus K$ for some $t \in \Theta$. We first show that the vertical line l passing through $\mathbf{x}(t)$ is disjoint from S. The ray with initial point $\mathbf{x}(t)$ and direction v_0 is disjoint from K as the set $F_\omega^\circ \setminus K$ is closed in the direction of v_0. The ray with initial point $\mathbf{x}(t)$ and direction $-v_0$ is not in S because $\mathbf{x}(t)$ is the corner of $Q_K^-(t)$ and $Q_K^-(t)$ is closed in the direction of $-v_0$. So the vertical line l passing through $\mathbf{x}(t)$ does not overlap with S.

Now separate the horizontal strip H into two chunks by the vertical line l passing through $\mathbf{x}(t)$. As S is connected, S should lie either strictly on left or strictly on right of l. As $\mathbf{x}(t)$ lies strictly inside F_ω, the point $W_K(t)$ is strictly further than $\mathbf{x}(t)$ in the direction of u_0, and by Lemma 2.13 the point $A_K^-(0)$ is further than $W_K(t)$ in the direction of u_0. So the endpoint $A_K^-(0)$ of K lies strictly on the right side of l. Similarly, the points $Z_K(t)$ is strictly further than $\mathbf{x}_K(t)$ in the direction of $-u_0$, and by Lemma 2.13 the point $C_K^+(\omega)$ is further than $W_K(t)$ in the direction of $-u_0$. So the endpoint $C_K^+(\omega)$ of K lies strictly on the left side of l. As the endpoints $A_K^-(0)$ and $C_K^+(\omega)$ are in $\mathcal{M}(K)$ by Lemma 2.13, and the line l separates the two points, l divides $\mathcal{M}(K)$ and so $\mathcal{M}(K)$ is disconnected. □

Now Theorem 2.11 is an immediate consequence of Theorem 2.5 and Theorem 2.15, completing the proof.

2.5 Sofa area functional

2.5.1 Summary

We change the moving sofa problem to the maximization of the *sofa area functional* $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ on the convex space $\mathcal{K}_\omega$ of caps. A drawback is that the new problem is subtly harder from the original moving sofa problem, as the space

$\mathcal{K}_\omega$ is strictly larger than the moving sofas. Also, the function $\mathcal{A}$ by itself is likely not a convex function. However, the benefit is that the domain $\mathcal{K}_\omega$ of $\mathcal{A}$ is now a convex space, allowing convex upper bounds of $\mathcal{A}$ that are amenable to convex analysis in future.

2.5.2 Details

Also note that the area of S depends solely on the cap K. So with Theorem 2.11, we can turn the moving sofa problem into maximizing the sofa area functional $\mathcal{A}(K) = |K| - |\mathcal{N}(K)|$ of an arbitrary cap K.

Definition 2.32. For any angle $\omega \in [0, \pi/2]$ and a closed set of angles Θ such that $\{0, \omega\} \subseteq \Theta \subseteq [0, \omega]$, define $\mathcal{K}_\Theta$ as the *space of all caps* with the *angle set* Θ. For any angle $\omega \in [0, \pi/2]$, denote the space $\mathcal{K}_{[0,\omega]}$ as simply $\mathcal{K}_\omega$.

Definition 2.33. For any angle $\omega \in [0, \pi/2]$ and a closed angle set $\{0, \omega\} \subseteq \Theta \subseteq [0, \omega]$, define the *sofa area functional* $\mathcal{A} : \mathcal{K}_\Theta \to \mathbb{R}$ on the space of caps $\mathcal{K}_\Theta$ as $\mathcal{A}(K) = |K| - |\mathcal{N}(K)|$.

Fix an arbitrary angle $\omega \in [0, \pi/2]$. Define $\mathcal{M}_\omega$ as the subset of $\mathcal{K}_\omega$ consisting of the caps of monotone sofas. By Theorem 2.11, maximizing the area of a moving sofa with rotation angle ω is exactly same as maximizing the sofa area functional $\mathcal{A}$ on the subset $\mathcal{M}_\omega$ of $\mathcal{K}_\omega$. Note that $\mathcal{M}_\omega$ is a proper subset of $\mathcal{K}_\omega$. For example, if the cap $K \in \mathcal{K}_\omega$ is too wide in width, then the niche $\mathcal{N}(K)$ may extrude out of K so that K cannot be the cap of a monotone sofa. But in practice we will basically optimize $\mathcal{A}(K)$ for all $K \in \mathcal{K}_\omega$ and show that the maximum is attained by the cap K_G of Gerver's sofa.

Conjecture 2.16. *The cap $K = K_G$ of Gerver's sofa S_G attains the maximum value of the area functional $\mathcal{A}(K)$ over all rotation angle $\omega \in [0, \pi/2]$ and caps $K \in \mathcal{K}_\omega$.*

So Conjecture 2.16 is a generaliztion of Gerver's conjecture. Extending the domain of the problem as in Conjecture 2.16 actually will make the problem easier as we will later see that the space $\mathcal{K}_\omega$ is a *convex space*, while we don't know if the set $\mathcal{M}_\omega$ is convex or not. This however risks that

In Conjecture 2.16, we turned the moving sofa problem into optimizing the sofa area functional $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ on the space of caps $\mathcal{K}_\omega$. Our main observation here is that the space of caps $\mathcal{K}_\omega$ is a *convex space*. A set $\mathcal{K}$ is a convex space if it is equipped with a convex combination operation $c_\lambda(-, -) : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ for every $\lambda \in [0, 1]$ such that it satisfies a set of axioms matchiing the properties of convex-linear combination in a real vector space (e.g. Axioms 1-4 of [10]). Any real vector space V is a convex space with $c_\lambda(v_1, v_2) = (1-\lambda)v_1 + \lambda v_2$. For the space of caps $\mathcal{K}_\Theta$, define their convex combination as $c_\lambda(K_1, K_2) = (1 - \lambda)K_1 + \lambda K_2$ using the Minkowski sum. Then it is rountine to check that the combination is also a cap with rotation angle ω, and that the combination satisfies the axioms of convex space.

2.5.3 Calculus of Variation

Many values on the cap $K \in \mathcal{K}_\omega$ are convex-linear.

Definition 2.34. A function $f : \mathcal{K} \to V$ from a convex space $\mathcal{K}$ to a convex space V is *convex-linear* if $f(c_\lambda(K_1, K_2)) = c_\lambda(f(K_1), f(K_2))$ for all $K_1, K_2 \in \mathcal{K}$ and $\lambda \in [0, 1]$. Call a functional $f : \mathcal{K} \to \mathbb{R}$ on $\mathcal{K}$ a *linear functional* on $\mathcal{K}$ if it is convex-linear.

The support function p_K is convex-linear with respect to $K \in \mathcal{K}_\omega$. For an arbitrary angle $t \in S^1$, the vertices $A^+_K(t), A^-_K(t), C^+_K(t), C^-_K(t)$ (Definition 2.27) are all convex-linear with respect to $K \in \mathcal{K}_\omega$. Likewise, the corners $\mathbf{x}_K(t)$ and $\mathbf{y}_K(t)$ are convex-linear with respect to $K \in \mathcal{K}_\omega$.

In this paper, for a convex space $\mathcal{K}$ call a function $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ *convex-bilinear* if the maps $K \mapsto g(K_1, K)$ and $K \mapsto g(K, K_2)$ are convex-linear for any $K_1, K_2 \in \mathcal{K}$. Call a functional $h : \mathcal{K} \to \mathbb{R}$ a *quadratic functional* on $\mathcal{K}$ if $h(K) = g(K, K)$ for some convex-bilinear function g. Note that in our definitions, a linear functional f on $\mathcal{K}$ is also quadratic because $f(K) = g(K, K)$ where $g(K_1, K_2) = f(K_1)$ is convex-bilinear. We will soon see that the area $|K|$ of a cap $K \in \mathcal{K}_\omega$ is a quadratic functional on $\mathcal{K}_\omega$.

2.6 Angular extension

2.6.1 Summary

We describe a process that takes a cap K of rotation angle ω and gives a cap K' of larger rotation angle $\omega' > \omega$. We provide a sufficient condition in which $\mathcal{A}(K') \geq \mathcal{A}(K)$. In particular, if the enlarged rotation angle is $\omega' = \pi/2$, then it is always that $\mathcal{A}(K') \geq \mathcal{A}(K)$.

2.6.2 Outline

Given a cap K of rotation angle ω and angle set Θ, here we will construct a cap $K^{\omega'}$ of rotation angle $\omega' > \omega$ and angle set $\Theta \cup \{\omega'\}$. We will call $K^{\omega'}$ the *angular extension* of K with rotation angle ω'.

Definition 2.35. Let K be any cap of angle set Θ and rotation angle ω. Let $\omega' \in (\omega, \pi/2]$ be an angle. Define $K^{\omega'}$ as the following set.

$$K^{\omega'} = F_{\omega'} \cap \bigcap_{t \in \Theta} Q^+_K(t)$$

Theorem 2.17. *In Definition 2.35, the set $K' := K^{\omega'}$ is a cap with angle set $\Theta' := \Theta \cup \{\omega'\}$ and rotation angle ω'. With this, call $K^{\omega'}$ the* angular extension *of K with rotation angle ω'.*

Proof. We first show that K' is a cap with angle set Θ' by checking that K' satisfies the two conditions of Definition 2.21. The point $o_{\omega'}$ is in the segment connecting o_ω and $-v_0$, so we have $o_{\omega'} \in M_\omega$ (Definition 2.23) and thus $o_{\omega'} \in Q^+_K(t)$

for all $t \in \Theta$ (Lemma 2.7). Now we have $o_{\omega'} \in K^{\omega'} \subseteq F_{\omega'}$ (Definition 2.35) which implies the first condition of Definition 2.21. By Definition 2.35, K' is the intersection of half-planes with normal angles in $\Theta' \cup (\Theta' + \pi/2) \cup \{\pi + \omega', 3\pi/2\}$, so the second condition of Definition 2.21 is true. ☐

We will show that the cap K and its angular extension $K^{\omega'}$ share most of the values (Definition 2.16,) we defined over a cap.

Theorem 2.18. *For any cap K with angle set Θ and rotation angle ω, its angular extension K' with rotation angle $\omega' > \omega$ satisfies the following properties.*

1. *We have $p_K(t) = p_{K'}(t)$ for all $t \in [0, \omega] \cup [\pi/2, \pi/2 + \omega]$.*
2. *Consequently, $?_K(t) = ?_{K'}(t)$ for all $t \in \Theta$ and $? = L, a, b, c, d, \mathbf{x}, \mathbf{y}, Q^+, Q^-, w, W$.*

Note however that we will likely have $z_K(t) \neq z_{K'}(t)$ and $Z_K(t) \neq Z_{K'}(t)$. Next, we will prove a sufficient condition where the sofa area functional $\mathcal{A}(K^{\omega'})$ of angular extension $K^{\omega'}$ is greater than or equal to the sofa area functional $\mathcal{A}(K)$ of the original K.

Theorem 2.19. *Let K be any cap of angle set Θ and rotation angle ω. Let $K' := K^{\omega'}$ be the angular extension of K with rotation angle ω'. If $F_\omega \setminus F_{\omega'} \subseteq \mathcal{N}(K)$ then we have $\mathcal{A}(K) \leq \mathcal{A}(K^{\omega'})$. In particular, if $\omega' = \pi/2$ then we have $\mathcal{A}(K) \leq \mathcal{A}(K^{\omega'})$ unconditionally.*

Theorem 2.19 will be very useful later in extending an argument on sofas of rotation angle $\pi/2$ to sofas of rotation angle $[\omega, \pi/2]$ for an explicit $\omega < \pi/2$.

2.6.3 Proofs

(Work in progress)

The following proof justify that $K^{\omega'}$ is indeed a cap with rotation angle ω'.

Proof. (of Theorem 2.18) We show that K' is a cap with angle set $\Theta' = \Theta \cup \{\omega'\}$ by checking that K' satisfies the two conditions of Definition 2.21. The point $o_{\omega'}$ is in the segment connecting o_ω and $-v_0$, so we have $o_{\omega'} \in M_\omega$ (Definition 2.23) and thus $o_{\omega'} \in Q_K^+(t)$ for all $t \in \Theta$ (Lemma 2.7). Now we have $o_{\omega'} \in K^{\omega'} \subseteq F_{\omega'}$ (Definition 2.35) which implies the first condition of Definition 2.21. By Definition 2.35, K' is the intersection of half-planes with normal angles in $\Theta' \cup (\Theta' + \pi/2) \cup \{\pi + \omega', 3\pi/2\}$, so the second condition of Definition 2.21 is true. ☐

We analyze the triangular shape $F_\omega \setminus F_{\omega'}$ using elementary geometry.

Lemma 2.20. *For any pair of angles $\omega < \omega' \leq \pi/2$, the set $F_\omega \setminus F_{\omega'}$ is empty if $\omega' = \pi/2$, or the triangular region with vertices o_ω, $o_{\omega'}$ and $p_{\omega,\omega'} := l(\omega, -1) \cap l(\omega', -1)$ strictly under the edge $l(\omega', -1)$ connecting $o_{\omega'}$ and $p_{\omega,\omega'}$ if $\omega' < \pi/2$. Moreover, if $\omega' < \pi/2$ then the vertex o'_ω is strictly in the edge of M_ω connecting o_ω and $-v_0$, and the vertex $p_{\omega,\omega'}$ is strictly in the edge of M_ω connecting o_ω and $-u_\omega$. In particular, $F_\omega \setminus F_{\omega'}$ is a subset of M_ω strictly under the line connecting the vertices $-u_\omega$ and $-v_0$ of M_ω.*

Proof. Since $\omega' = \pi/2$ implies that $F_\omega \setminus F_{\omega'}$ is empty, we only need to prove the case where $\omega' < \pi/2$. Draw an arc c of the unit circle centered at O connecting the points $-v_0$ and $-u_\omega$ inside M_ω. The edges of F_ω are the tangent lines of c at $-v_0$ and $-u_\omega$. The line $l(\omega', -1)$ is the tangent line of c at $-u_{\omega'}$ intersecting the edges of F_ω at $o_{\omega'}$ and $p_{\omega,\omega'}$ respectively. So the vertex o'_ω is strictly in the segment connecting o_ω and $-v_0$, and the vertex $p_{\omega,\omega'}$ is strictly in the segment connecting o_ω and $-u_\omega$. As F_ω is the region on or above both the lines $l(\pi/2, -1)$ and $l(\omega, -1)$, we have

Also, and the set $F_\omega \setminus F_{\omega'}$ is is the region of F_ω under the tangent line of c at $-u_{\omega'}$. □

Lemma 2.21. *Let K be any cap of angle set Θ and rotation angle ω. Let $K' := K^{\omega'}$ be the angular extension of K with rotation angle ω'. Then we have the followings.*

1. *$\mathcal{N}(K) \setminus \mathcal{N}(K') \subseteq F_\omega \setminus F_{\omega'} = K \setminus K' \subseteq T$ where T is the triangle with vertices o_ω, $o_{\omega'}$, and $-u_\omega$.*
2. *$\mathcal{N}(K') \setminus \mathcal{N}(K) \subseteq K' \setminus K \subseteq F_{\omega'} \setminus F_\omega$*

Proof. We show $K \setminus K' = F_\omega \setminus F_{\omega'}$. By definition of K and K' we have □

Lemma 2.22. *Let K be any cap of angle set Θ and rotation angle ω, and $K' := K^{\omega'}$ be the angular extension of K with rotation angle ω'. Then $\delta K \subseteq \delta K'$.*

Proof. We have $K \setminus (K \setminus K') \subseteq K'$. Any point in $F_\omega \setminus F_{\omega'}$ is superseded by the point O in M_ω in the direction of u_t and v_t. So

On the other hand, by Definition 2.35 we have $p_{K'}$ dominated by xx.

□

We will soon show that

Next, we check $p_K(t) = p_{K'}(t)$ for all $t \in \Theta \cup (\Theta + \pi/2)$.

K' is defined as the intersection (Definition 2.35) of half-planes with normal angles in the set . Define M_θ as the

Let K be any cap of angle set Θ and rotation angle ω. Let $K' := K^{\omega'}$ be the angular extension of K with rotation angle ω'. The caps K and K' differ slightly in its left side, so we need to match K with K'.

1. K and K' share the same positions of $\mathbf{x}_K(t) = \mathbf{x}_{K'}(t)$ for all $t \in \Theta$.
2. $\mathcal{N}(K') \setminus \mathcal{N}(K) \subseteq K' \setminus K \subseteq F_{\omega'} \setminus F_\omega$

3. If $K \setminus K' = F_\omega \setminus F_\omega^{\omega'} \subseteq \mathcal{N}(K)$, then we have $\mathcal{A}(K) \leq \mathcal{A}(K')$.
   1. $\mathcal{A}(K') - \mathcal{A}(K) = |K'| - |K| - (\mathcal{N}(K') - \mathcal{N}(K))$
   2. $= |K' \setminus K| - |K \setminus K'| + |\mathcal{N}(K) \setminus \mathcal{N}(K')| - |\mathcal{N}(K') \setminus \mathcal{N}(K)|$
   3. $= |K' \setminus K| - |\mathcal{N}(K') \setminus \mathcal{N}(K)| - |K \setminus K'| + |\mathcal{N}(K) \setminus \mathcal{N}(K')|$

Let K be any sofa with angle set Θ_K and rotation angle $\omega \in [\theta_{n-1}, \pi/2]$.

1. Take last angle of K. Remove any edge cut by $L_K(\omega)$ (no contribution to niche, and takes only cap off).
2. Then replace the rotation angle to ω'. Locations
3. K.

SofaDesigner: applies to Θ.

3 Injectivity Hypothesis

3.1 Statement

We now state the *injectivity hypothesis* in full details which is a much weaker version of Gerver's conjecture. Observe that the cap K_G of Gerver's sofa S_G satisfies the following properties.

Definition 3.1. Say that a cap K is *well-behaved* if it satisfies the following conditions.

1. K has rotation angle $\pi/2$.
2. The inner corner $\mathbf{x}_K(t)$ of the rotating hallway $L_S(t)$ is injective with respect to $t \in [0, \pi/2]$.
3. For any $t \in (0, \pi/2)$, we have $\mathbf{x}_K(t) \in H(\pi/2, 0)$. That is, $\mathbf{x}_K(t)$ is always on or above the x-axis.

So the cap of Gerver's sofa is well-behaved. But also note that the condition is much more general and there will be much more monotone sofas with well-behaving caps. The injectivity hypothesis states that there exists a sofa of maximum area with a well-behaved cap.

Conjecture 3.1. *(Injectivity Hypothesis) There is a monotone sofa $S = S_{max}$ with a well-behaved cap attaining the maximum possible area.*

3.2 Proof near Gerver's Sofa

Here, we show that for any cap K with rotation angle $\pi/2$ that is sufficiently close to the cap K_G of Gerver's sofa, if K maximizes the sofa area functional $\mathcal{A}$ then K has to be well-behaved. Using this, we will also show that if Gerver's conjecture is true *a priori*, then by running the branch-and-bound algorithm described in [7] for sufficiently long, the algorithm can produce a proof of injectivity theorem without relying on the a priori assumption.

Give the space of caps $\mathcal{K}_{\pi/2}$ the metric $d(K_1, K_2) = \sup_{t \in [0,\pi]} |p_{K_2}(t) - p_{K_1}(t)|$. We omit the proof but mention that the metric is strongly equivalent to the Hausdorff distance of two caps. It can be also shown that the sofa area functional $\mathcal{A}$ is continuous with respect to this metric (TODO: add citation). We will prove the following.

Theorem 3.2. *There exists an open neighborhood U of the cap K_G of Gerver's sofa such that the following holds. For every cap $K \in U$ that maximizes the sofa area functional $\mathcal{A}(K)$, the cap is well-behaved.*

Note that if the open set U in the statement is the whole space $\mathcal{K}_{\pi/2}$, then the statement of Theorem 3.2 is a slightly stronger version of Gerver's conjecture. So in a sense, Theorem 3.2 proves the injectivity hypothesis near the cap K_G of Gerver's sofa.

Proof. (Sketch; need to expand)

- Take $\delta > 0$ sufficiently small so that, for every $t \in [0, \delta]$, the ball of radius 0.001 centered at $\mathbf{x}_{K_G}(t)$ is contained in the niche $\mathcal{N}(K_G)$ of Gerver's sofa. This is possible as the ball of radius 0.01 centered at $\mathbf{x}_{K_G}(0)$ is contained inside $\mathcal{N}(K_G)$.
- As $K \to K_G$ in the metric d as $\epsilon \to 0$, the value $g_K^+(t)$ for all $t \in [\delta, \pi/2]$ and $h_K^+(t)$ for all $t \in [0, \pi/2-\delta]$ converge uniformly to that of K_G, because the curvature of δK_G at $A_{K_G}(t)$ (resp. $C_{K_G}(t)$) for all $t \in [\delta, \pi/2]$ (resp. $t \in [0, \pi/2 - \delta]$) is bounded.
- We now bound $g_K^+(t)$ for $t \in [0, \delta]$. There exists a K sufficiently close to K_G such that $g_K^+(0)$ is far away from $g_{K_G}^+(0)$ (picture needed). But we will show that such a
- Take an arbitrary K (how?). Make a new cap $\hat{K}$ as the cap of the union of K and a segment on the line $y = 1$. Then $\mathcal{A}(\hat{K}) \geq \mathcal{A}(K)$ always (how?). Moreover, we can now bound $g_{\hat{K}}^+(t)$ for $t \in [0, \delta]$ from below.
- Use Theorem 4.16 to bound the derivative of $\mathbf{x}_K$.
  - Show that $\mathbf{x}_K'(t) \cdot v_{\pi/4} > 0$ for every $t \in [0, \pi/4]$.
  - Now we only need to show that $t \in [0, \pi/4]$ and $t \in [\pi/4, \pi/2]$ do not overlap.
  - $\mathbf{x}_K'(t) \cdot u_0 < 0$ for every $t \in [\pi/4, 3\pi/4]$.
- So the injectivity hypothesis holds.

□

So **??** and Theorem 3.2 shows that the only possible candidate of $K \in \mathcal{K}_{\pi/2}$ near K_G that maximizes $\mathcal{A}(G)$ is the cap K_G of Gerver's sofa. This is a rigorous justification that Gerver's sofa satisfies the local optimality condition.

We now mention the theoretical possibility of proving injectivity hypothesis using computer assistance. Assume *a priori* that Gerver's sofa happens to be the unique sofa of maximum area, which is suggested by numerical experimentations in [5]. Our goal now is to find a rigorous proof without relying on the assumption. Here, it is important that we won't use the a priori assumption as a part of our proof.

Kallus and Romik [7] devised an algorithm that can approximate $\mathcal{A}$ uniformly from above when ran for sufficiently long time[^note that we change the description by omitting their work on arbitrary rotation angle; we fix the rotation angle as $pi/2$ because we plan to deal with rotation angle later in a different way]. More precisely, their algorithm produces a finite partition $\mathcal{P}$ of $\mathcal{K}_{\pi/2}$ into boxes and an upper bound $\mathcal{A}_{fin}$ which is constant at each box $B \in \mathcal{P}$. Each loop of the algorithm breaks a single box into two smaller boxes and provide an improved upper bound $\mathcal{A}_{fin}$ of $\mathcal{A}$ for each new box. So by running the algorithm indefinitely, the upper bound $\mathcal{A}_{fin}$ converges to $\mathcal{A}$ uniformly.

So if K_G is the unique maximizer of $\mathcal{A}$, then their software will be able to rigorously prove the following without relying on our assumption: the maximizer of $\mathcal{A}$ should be contained in U, where U is a small neighborhood of K_G that shrinks over time as we run the software. We can run the software until U shrinks enough to be contained inside the neighborhood of K_G in Theorem 3.2. While [7] by alone cannot identify a single maximizer of U in finite time, it can shrink U sufficiently small. Then, with Theorem 3.2 we would be able to rigorously prove that any maximizer of $\mathcal{A}$ should be in U, so it should be well-behaved, proving the injectivity hypothesis.

Realistically, the size of the neighborhood U required to make this line of proof work is too small for their implementation `SofaBounds` to compute. To this, we developed a new algorithm involving exact analysis of finite intersection of hallways, and implemented as a program called `SofaDesigner`. Using the software, we are in the process of proving Conjecture 3.1 completely, which will be reported subsequently if successful.

4 Upper Bound A1

4.1 Space of Caps

The sofa area functional $\mathcal{A}(K)$ with respect to cap $K \in \mathcal{K}_\omega$ is very likely not a quadratic functional on K. But we will soon construct upper bounds $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ that are concave quadratic functionals on cap space $\mathcal{K}_\omega$. Then finding the global maximum of $\mathcal{A}_1$ and $\mathcal{A}_2$ on a convex subset of $\mathcal{K}$ can be done using the calculus of variation as the following. Suppose that we want to find the maximizer K of a concave quadratic functional $f(K)$ on a convex space $\mathcal{K}$. Then in particular, for any value $K' \in \mathcal{K}$ the value $f(c_\lambda(K, K'))$ should always attain its maximum at $\lambda = 0$. So the following value should be nonpositive.

Definition 4.1. For any convex space $\mathcal{K}$ with convex combination $c_\lambda(-,-)$ and a quadratic functional $f : \mathcal{K} \to \mathbb{R}$ and $K \in \mathcal{K}$, define the following.

$$Df(K; K') = \frac{d}{d\lambda}\bigg|_{\lambda=0} f(c_\lambda(K, K'))$$

Definition 4.1 can be seen as a generalization of the Gateaux derivative to functionals on a general convex space. For any quadratic functional f and a fixed $K \in \mathcal{K}$ the value $Df(K; K')$ is always a linear functional of K'.

Lemma 4.1. *Let f be a quadratic functional on a convex space $\mathcal{K}$, so that $f(K) = h(K, K)$ for a convex-bilinear map $h : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$. Then we have the* *following for any $K, K' \in \mathcal{K}$.*

$$Df(K; K') = h(K, K') + h(K', K) - 2h(K, K)$$

So in particular, the map $Df(K; -) : \mathcal{K} \to \mathbb{R}$ is always well-defined and a linear functional.

Proof. We have the following computation by bilinearlity of h.

$$\begin{aligned} f(c_\lambda(K, K')) &= h(c_\lambda(K, K'), c_\lambda(K, K')) \\ &= (1-\lambda)^2 h(K, K') + \lambda(1-\lambda)\left(h(K, K') + c_\lambda(K', K)\right) + \lambda^2 h(K', K') \end{aligned}$$

Now take derivative at $\lambda = 0$. ☐

To prove that K maximizes a concave quadratic functional $f(K)$ on $\mathcal{K}$, we only need to prove that $Df(K; -)$ is a nonpositive linear functional on $\mathcal{K}$.

Theorem 4.2. *For any concave quadratic functional f on a convex space $\mathcal{K}$ with convex combination $c_\lambda(-, -)$, the value $K \in \mathcal{K}$ maximizes $f(K)$ if and only if the linear functional $Df(K; -)$ is nonpositive.*

Proof. The 'only if' part is obvious. Assume an arbitrary $K \in \mathcal{K}$ such that $Df(K; -)$ is always nonpositive. Take any $K' \in \mathcal{K}$. Observe that $f(c_\lambda(K, K'))$ is a polynomial $p(\lambda)$ of $\lambda \in [0, 1]$ by **??**. Because f is concave, the polynomial $p(\lambda)$ is also concave with respect to λ and the quadratic coefficient of $p(\lambda)$ is nonpositive. The linear coefficient of $p(\lambda)$ is $Df(K; K')$ which is nonpositive as well. So $p(\lambda)$ is monotonically decreasing with respect to λ and we have $f(K) \geq f(K')$ as desired. ☐

For the rest of this paper, we will maximize $f = \mathcal{A}_1$ or $\mathcal{A}_2$ by essentially solving the equation $Df(K_{\max}; -) = 0$ for $K_{\max}$; while we technically need to solve for the inequality $Df(K_{\mathrm{m}}; -) \geq 0$, modulo minor details it turns out that we are essentially solving for the equality condition. Then the maximum value $\mathcal{A}_1(\mathcal{K}_{\max,1})$ and $\mathcal{A}_2(\mathcal{K}_{\max,2})$ will immediately give an upper bound of the sofa area functional $\mathcal{A}$. In fact, the maximum value $\mathcal{A}_1(\mathcal{K}_{\max,1}) = 1 + \pi^2/8 = 2.2337\ldots$ of $\mathcal{A}_1$ turns out to be very close to the area $2.2195\ldots$ of Gerver's sofa. The first upper bound $\mathcal{A}_1$ is used to construct a finer bound $\mathcal{A}_2$, and then we will show that the maximizer of $\mathcal{A}_2$ is Gerver's sofa.

4.2 Surface Area Measure

The *surface area measure* β_K of a convex body K [13] [12] is a measure on S^1 that will be a key ingredient in our calculus of variation. Essentially, β_K measures the side lengths of K. If K is a convex polygon, then β_K is a discrete measure such that the measure $\beta_K(\{t\})$ at point t is the length of the edge $e_K(t)$. If K has a boundary parametrized smoothly by $v_K(t)$ for $t \in S^1$, where $v_K(t)$ is the point that the tangent line $l_K(t)$ touches K, then we have $\beta_K(dt) = R(t)dt$ where $R(t) = \|v'(t)\|$ is the radius of curvature (the inverse of curvature) of the boundary of K at $v(t)$. We gather various properties of β_K that will be used thoroughly in the rest of the paper.

For two-dimensional convex bodies K, β_K is convex-linear with respect to K.

Theorem 4.3. *The surface area measure β_K is convex-linear with respect to K.*

This comes from the fact that in dimension $n = 2$, the surface area measure is equal to *the mixed area measure* with $n-1 = 1$ argument, and that the mixed area measure is convex-linear with respect to each argument (e.g. from (5.27), p284 of [12]).

We also can represent the area of a convex body, or mixed volume of two convex bodies using surface area measure. Note that for any measurable function f on a space X and a measure β on X, we denote the integral of f with respect to β as $\langle f, \beta \rangle_X = \int_{x \in X} f(x)\, \beta(dx)$.

Theorem 4.4. *The mixed volume $V(K_1, K_2)$ of any two convex bodies K_1 and K_2 can be represented as the following.*

$$V(K_1, K_2) = \langle p_{K_1}, \beta_{K_2} \rangle_{S^1} / 2$$

Consequently, the area $|K|$ of any convex body K can be represented as the following.

$$|K| = \langle p_K, \beta_K \rangle_{S^1} / 2$$

So the area $|K|$ of a cap $K \in \mathcal{K}_\omega$ is a quadratic functional on $\mathcal{K}_\omega$ in particular. The proof of Theorem 4.4 can be found in Theorem 5.1.7 in p280 of [12].

Because β_K measures the side lengths (both discrete and differential) of a convex body K, we have the following theorem.

Theorem 4.5. *For any convex body K, the following* vertex formulas *hold. For any $a < b$ such that $b \leq a + 2\pi$, the followings are true.*

$$\int_{(a,b]} v_t\, \beta_K(dt) = v_K^+(b) - v_K^+(a)$$

$$\int_{[a,b]} v_t\, \beta_K(dt) = v_K^+(b) - v_K^-(a)$$

In short, $dv_K^+(t) = v_t \beta_K(dt)$.

We elaborate the meaning of $dv_K^+(t) = v_t \beta_K(dt)$. Here, the function $v_K^+ : S^1 \to \mathbb{R}^2$ is right-continuous so the pair of Lebesgue-Stieltjes measures $dv_K^+(t)$ of $v_K^+(t)$ exists for each x and y coordinate, and each would be equal to $\cos(t)\beta_K(dt)$ and $-\sin(t)\beta_K(dt)$. A rigorous proof of Theorem 4.5 is done in Appendix A.

Gauss-Minkowski theorem states that any convex set K, up to translation, corresponds one-to-one to a measure $\beta = \beta_K$ on S^1 such that $\int_{S^1} u_t\, \beta(dt) = 0$. Using this correspondence, we establish a correspondence between any cap $K \in \mathcal{K}_\omega$ and its surface area measure β_K.

Theorem 4.6. *For any cap K with rotation angle ω and angle set $\{0, \omega\} \subseteq \Theta \subseteq [0, \omega]$, the surface area measure β_K of K on S^1 has support in $\Theta \cup (\Theta + \pi/2) \cup \{\pi + \omega, 3\pi/2\}$ and satisfies the followings.*

$$\int_{t\in\Theta} \cos(t)\, \beta_K(dt) = 1 \qquad \int_{t\in\Theta+\pi/2} \cos(\omega + \pi/2 - t)\; \beta_K(dt) = 1$$

Proof. β_K has support in $\Theta \cup (\Theta + \pi/2) \cup \{\pi + \omega, 3\pi/2\}$ by (TODO: some theorem in Appendix A). We have $\left(A^+_K(\omega) - A^-_K(0)\right) \cdot v_0 = 1$ because for a cap K, the point $A^+_K(\omega)$ is on the line $y = 1$ and $A^-_K(0)$ is on the line $y = 0$. So by (TODO: some theorem in Appendix A), we get the first equation. The second equation can be obtained by a symmetric argument. □

Theorem 4.7. *Take arbitrary angle $\omega \in [0, \pi/2]$ and angle set $\{0, \omega\} \subseteq \Theta \subseteq [0, \omega]$. Conversely to Theorem 4.6, let β be a measure on $\Theta \cup (\Theta + \pi/2)$ that satisfies the following equations.*

$$\int_{t\in\Theta} \cos(t)\, \beta(dt) = 1 \qquad \int_{t\in\Theta+\pi/2} \cos(\omega + \pi/2 - t)\; \beta(dt) = 1$$

Then the followings hold.

- *If $\omega = \pi/2$, then there is a cap K with rotation angle ω and angle set Θ such that $\beta_K|_{\Theta\cup(\Theta+\pi/2)} = \beta$. This is unique up to horizontal translation: all the other caps K' satisfying the same condition is a horizontal translation of K.*
- *If $\omega < \pi/2$, then there is a unique cap K with rotation angle ω and angle set Θ such that $\beta_K|_{\Theta\cup(\Theta+\pi/2)} = \beta$.*

Proof. We first observe that there is a unique extension of β on the set $\Pi = \Theta \cup (\Theta+\pi/2) \cup \{\pi+\omega, 3\pi/2\}$ such that $\int_{t\in\Pi} v_t\, \beta(dt) = 0$. Let $A = \int_{t\in\Theta} \sin(t)\, \beta(dt) \geq 0$, then we have $\int_{t\in\Theta} v_t\, \beta(dt) = -Au_0 + v_0$ by the first condition. Likewise, if we let $B = \int_{t\in\Theta+\pi/2} \sin(\omega + \pi/2 - t)\, \beta(dt) \geq 0$, then we have $\int_{t\in\Theta+\pi/2} v_t\, \beta(dt) = Bv_\omega - u_\omega$. By solving for $\int_{t\in\Pi} v_t\, \beta(dt) = 0$, we get the unique extension of β to Π with $\int_{t\in\Pi} v_t\, \beta(dt) = 0$ as the followings.

- If $\omega = \pi/2$, then $\beta(\{\pi + \omega\}) = \beta(\{3\pi/2\}) = \int_{t\in\Theta\cup\Theta+\pi/2} \sin(t)\, \beta(dt) \geq 0$
- If $\omega < \pi/2$, then $\beta(\{\pi + \omega\}) = B + v_\omega \cdot p_\omega > 0$ and $\beta(\{3\pi/2\}) = A + u_0 \cdot p_\omega \geq 0$.

Now we use the Gauss-Minkowski theorem (TODO: some theorem in Appendix A). If $\omega = \pi/2$, then there is a unique convex body K up to translation so that β_K has support in Π and $\beta_K = \beta$ on Π. The equations on β imply that the width of K measured in the direction of v_0 is 1. So among all possible translations of K, the body that is a cap ($p_K(\pi/2) = 1$ and $p_K(3\pi/2) = 0$) is unique up to horizontal translation. If $\omega < \pi/2$, then there is a unique convex body K such that β_K has support in Π, $\beta_K = \beta$ on Π, and $p_K(\omega) = p_K(\pi/2) = 1$. The equations on β is equivalent to that K has width 1 in the direction of u_ω and v_0. So this K is a cap. □

4.3 Definition

The main difficulty of analyzing the sofa area functional $\mathcal{A}(K)$ of a cap K is in the area $|\mathcal{N}(K)|$ of the niche. The niche of Gerver's sofa consists of three curves: the large arc carved out by the inner corner $\mathbf{x}_{K_G}(t)$, and the two smaller convex curves with tangent lines $b_{K_G}(t)$ and $d_{K_G}(t)$ trimming out a small region at the both ends of the large arc.

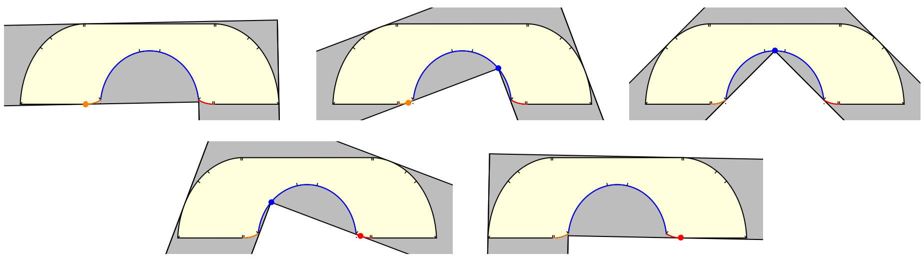

Figure 5: The five stages of movement of Gerver's sofa in perspective of the fixed sofa and moving tangent hallways. The points where the sofa and the inner walls touch are marked as dots (so there is no contact when there is no dot). Orange, blue, and yellow colors represent the points of the sofa touching the left inner wall $d_{K_G}(t)$, inner corner $\mathbf{x}_{K_G}(t)$, and right inner wall $b_{K_G}(t)$ respectively.

Gerver derived his sofa by assuming *a priori* that the niche of a sofa of maximum area consists of such three curves. We circumvent the difficulty by considering only the region bounded by the path $\mathbf{x}_K(t)$ of the inner corner for all $t \in [0, \pi/2]$, ignoring the area trimmed by $b_K(t)$ and $d_K(t)$. If the cap K is well-behaved, then the area enclosed by $\mathbf{x}_K(t)$ can be expressed as an integral using Green's theorem. We use this integral to define $\mathcal{A}_1(K)$ as the area of K subtracted by the area enclosed by $\mathbf{x}_K(t)$. So for any well-behaved cap K, the value $\mathcal{A}_1(K)$ becomes an upper bound of $\mathcal{A}(K)$.

We now rigorously define the upper bound $\mathcal{A}_1$ and prove its claimed properties. First define the *curve area functional* of an arbitrary rectifiable curve $\mathbf{x}$.

Definition 4.2. For two points $(a, b), (c, d) \in \mathbb{R}^2$, denote their cross product as $(a, b) \times (c, d) = ad - bc \in \mathbb{R}$.

Definition 4.3. Let Γ be any rectifiable curve equipped with a parametrization $\mathbf{x} : [a, b] \to \mathbb{R}^2$ of bounded variation. Write $\mathbf{x}(t) = (x(t), y(t))$. Define the *curve area functional* $\mathcal{I}(\mathbf{x})$ of Γ as the following.

$$\mathcal{I}(\mathbf{x}) = \frac{1}{2} \int_a^b \mathbf{x}(t) \times d\mathbf{x}(t) = \frac{1}{2} \int_a^b x(t)dy(t) - y(t)dx(t)$$

Also, write $\mathcal{I}(p, q)$ for the area functional of the line segment $\mathbf{x}$ connecting point p to q. Then we have $\mathcal{I}(p, q) = 1/2 \cdot (p \times q)$.

Note that the value of $\mathcal{I}(\mathbf{x})$ does not change even if we take a different parametrization of the curve Γ. Note also that $\mathbf{x}(t)$ is of bounded variation, so the Lebesgue-Stieltjes measure $d\mathbf{x}(t) = (dx(t), dy(t))$ exists and the integral is well-defined. If $\mathbf{x}$ is a Jordan curve oriented counterclockwise (resp. clockwise), by Green's theorem $\mathcal{I}(\mathbf{x})$ is the exact area of the region enclosed by $\mathbf{x}$ (resp. the area with a negative sign). If $\mathbf{x}$ is not closed, $\mathcal{I}(\mathbf{x})$ measures the signed area of the region bounded by the line segment connecting origin and $\mathbf{x}(a)$, the curve $\mathbf{x}$, and then the line segment connecting $\mathbf{x}(b)$ and the origin in order. If γ is the concatenation of two curves α and β then $\mathcal{I}(\gamma) = \mathcal{I}(\alpha) + \mathcal{I}(\beta)$; this will useful in decomposing the area enclosed by the concatenation of multiple curves.

Using the curve area functional, define the functional $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$ as the following.

Definition 4.4. For any cap $K \in \mathcal{K}_\omega$, define $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K)$.

Note that we have

$$\mathbf{x}_K(t) = (p_K(t) - 1)u_t + (p_K(t + \pi/2) - 1)v_t$$

and p_K is Lipschitz, so $\mathbf{x}_K$ is also Lipschitz and rectifiable. Thus the value $\mathcal{I}(\mathbf{x}_K)$ is well-defined. As both $|K|$ and $\mathcal{I}(\mathbf{x}_K)$ are quadratic functionals on K by Theorem 4.4, Definition 4.3 and that $\mathbf{x}_K$ is convex-linear with respect to K, the functional $\mathcal{A}_1(K)$ is also quadratic with respect to K.

W now show that $\mathcal{A}_1(K)$ is an upper bound of the area functional $\mathcal{A}(K)$ assuming the injectivity hypothesis. Our key observation is the following.

Lemma 4.8. *Let $K \in \mathcal{K}_{\pi/2}$ be arbitrary. Let $\mathbf{z} : [t_0, t_1] \to \mathbb{R}^2$ be any simple open curve (that is, curve with $t_0 < t_1$ and injective $\mathbf{z}$) inside the set $F_{\pi/2} \cap \bigcup_{0 \leq t \leq \pi/2} \overline{Q_K^-(t)}$ such that $\mathbf{z}(t_0)$ and $\mathbf{z}(t_1)$ are both on the line $y = 0$. Then we have $\mathcal{I}(\mathbf{z}) \leq |\mathcal{N}(K)|$.*

Proof. First, we define the set $\mathcal{N}_0$ which is bounded from above by $\mathbf{z}$ and from below by the line $y = 0$ as the following. Take an arbitrary $\epsilon > 0$. Construct the closed curve Γ_ϵ which is the concatenation of the following curves in order: the curve $\mathbf{z}(t)$, the vertical segment from $\mathbf{z}(t_1)$ to $\mathbf{z}(t_1) - (0, \epsilon)$, the horizontal segment from $\mathbf{z}(t_1) - (0, \epsilon)$ to $\mathbf{z}(t_0) - (0, \epsilon)$, and then the vertical segment from $\mathbf{z}(t_0) - (0, \epsilon)$ to $\mathbf{z}(t_0)$. The curve Γ_ϵ is a Jordan curve because $\mathbf{z}$ is a simple curve above the x-axis and we have $\mathbf{z}(t_0) \neq \mathbf{z}(t_1)$. By Jordan Curve Theorem, the curve Γ_ϵ encloses an open set $\mathcal{N}_\epsilon$.

Define $\mathcal{N}_0$ as the intersection $F_{\pi/2} \cap \mathcal{N}_\epsilon$. The set is independent of the choice of $\epsilon > 0$: this can be shown rigorously by exhibiting a continuous deformation of $\mathbb{R}^2$ that fixes the half-plane $y \geq 0$ and shrinks the half-plane $y \leq 0$ linearly so that it sends the line $y = -\epsilon_1$ to $y = -\epsilon_2$ for arbitrary $\epsilon_1, \epsilon_2 > 0$. Moreover, $\mathcal{N}_\epsilon$ is the disjoint union of $\mathcal{N}_0$ and the rectangle below the x-axis of width $|\mathbf{z}(t_1) - \mathbf{z}(t_0)| \cdot u_0$ and height ϵ.

We have $|\mathcal{N}_\epsilon| = |\mathcal{I}(\Gamma_\epsilon)|$ by Green's theorem on Γ_ϵ (note that this holds regardless of the orientation of Γ_ϵ). By taking $\epsilon \to 0$, we have $|\mathcal{N}_0| = |\mathcal{I}(\mathbf{z})|$.

We now show that $\mathcal{N}_0 \subseteq \mathcal{N}(K)$ which finishes the proof. Take any $p \in \mathcal{N}_0$. Take the ray r emanating from p in the direction v_0, then it should cross some point $q \neq p$ in the curve $\mathbf{z}$. As $\mathbf{z}$ is inside the set $F_{\pi/2} \cap \bigcup_{0 \leq t \leq \pi/2} \overline{Q_K^-(t)}$, the point q is contained in $F_{\pi/2} \cap \overline{Q_K^-(t)}$ for some $0 < t < \pi/2$ ($t \neq 0, \pi/2$ because q is strictly above the line $y = 0$ and $Q_K^-(t)$ is on or below the line $y = 0$ if $t = 0, \pi/2$). Because p is in $F_{\pi/2}$ and strictly below the point q, it should be that p is contained in $F_{\pi/2} \cap Q_K^-(t)$. So the point p is in the niche $\mathcal{N}(K)$, and we have $\mathcal{N}_0 \subseteq \mathcal{N}(K)$. $\square$

We can freely choose $\mathbf{z}$ to bound $|\mathcal{N}(K)|$ from below as long as $\mathbf{z}$ is inside the set $F_{\pi/2} \cap \bigcup_{0 \leq t \leq \pi/2} \overline{Q_K^-(t)}$. Right now, we simply choose $\mathbf{z} = \mathbf{x}_K$ and get the following.

Theorem 4.9. *For any well-behaved cap K, we have $\mathcal{A}(K) \leq \mathcal{A}_1(K)$.*

For the next upper bound $\mathcal{A}_2$, we will choose a more refined version of $\mathbf{z}$ that matches the boundary of $\mathcal{N}(K)$ more closer by including the two small areas trimmed at the end of the curve $\mathbf{x}_K$.

4.4 Concavity

We now show that $\mathcal{A}_1$ is concave. We use the following criterion to show the concavity of $\mathcal{A}_1$.

Theorem 4.10. *Let f be a quadratic functional on a convex space $\mathcal{K}$ with convex combination $c_\lambda(-,-)$ for all $\lambda \in [0,1]$. If there is a linear functional $g : \mathcal{K} \to \mathbb{R}$ and a convex-linear map $h : \mathcal{K} \to V$ to a real vector space V with inner product $\langle -,- \rangle_V$ such that $f(K) = g(K) - \langle h(K), h(K) \rangle_V$ for every $K \in \mathcal{K}$, then f is a concave function on $\mathcal{K}$.*

Proof. It suffices to show the case where $g = 0$. Then in the general case, $f - g$ is a concave function on $\mathcal{K}$ so consequently f is. For the case $g = 0$, take arbitrary $K_1, K_2 \in \mathcal{K}$. Fixing K_1 and K_2, observe that $f(c_\lambda(K_1, K_2))$ is a quadratic polynomial with respect to $\lambda \in [0,1]$ with leading coefficient $-\|h(K_2) - h(K_1)\|_K^2$ by expanding the term $h(c_\lambda(K_1, K_2))$ with respect to the inner product $\langle -,- \rangle_V$. This shows the concavity of f along the line segment connecting K_1 and K_2. Since K_1 and K_2 are chosen arbitrarily, this proves the concavity of f. $\square$

That is, we express $\mathcal{A}_1$ as a linear functional subtracted by a 'sum of squares'. We do this by using *Mamikon's theorem* [9]. The theorem calculates the area of the region swept by tangent segments of a convex body. Here, we use a generalization of his theorem that works for arbitrary convex bodies and rectifiable curves.

Theorem 4.11. *(Mamikon) Let K be an arbitrary convex body. Let $t_0, t_1 \in \mathbb{R}$ be the angles such that $t_0 < t_1 \leq t_0 + 2\pi$. Let p (resp. q) be any point on* *the edge $e_K(t_0)$ (resp. $e_K(t_1)$). Let $\mathbf{x}$ be the counterclockwise segment of the boundary of K from p to q (more precisely, $\mathbf{x}$ is the concatenation of the line segment from p to $v_K^+(t_0)$, the union $\bigcup_{t\in(t_0,t_1)} e_K(t)$, then the line from $v_K^-(t_1)$ to q). Let $\mathbf{y} : [t_0, t_1] \to \mathbb{R}^2$ be a rectifiable curve such that $\mathbf{y}(t)$ is always on the tangent line $l_K(t)$, so that there is a function $f : [t_0, t_1] \to \mathbb{R}$ such that $\mathbf{y}(t) = v_K^+(t) + f(t)v_t$ for all $t \in [t_0, t_1]$. Then the following holds.*

$$I(\mathbf{y}) + I(\mathbf{y}(t_1), q) - I(\mathbf{x}) - I(\mathbf{y}(t_0), p) = \frac{1}{2}\int_{t_0}^{t_1} f(t)^2\, dt$$

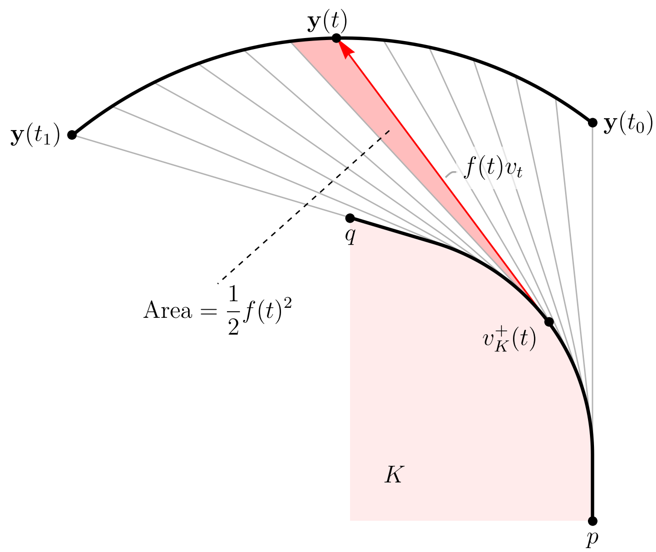

Figure 6: An illustration of Theorem 4.11.

We also need the following notion.

Definition 4.5. Let K be any convex body. Let l be any line with normal vector u_θ for some $\theta \in S^1$. For any $t \in S^1 \setminus \{\theta, \theta+\pi\}$, let $\tau_{K,l}^+(t)$ (resp. $\tau_{K,l}^-(t)$) be the unique real number such that the point $v_K^+(t)+\tau_{K,l}^+(t)v_t$ (resp. $v_K^-(t)+\tau_{K,l}^-(t)v_t$) is the intersection $l \cap l_K(t)$.

This is a variant of Mamikon's theorem where $\mathbf{y}(t)$ parametrizes a segment of the tangent line of K.

Theorem 4.12. *Let K be an arbitrary convex body. Let $t_0, t_1 \in \mathbb{R}$ be the angles such that $t_0 < t_1 < t_0+\pi$. Let p (resp. q) be any point on the edge $e_K(t_0)$ (resp. $e_K(t_1)$). Let $\mathbf{x}$ be the counterclockwise segment of the boundary of K from p to q (more precisely, $\mathbf{x}$ is the concatenation of the line segment from p to $v_K^+(t_0)$, the union $\bigcup_{t\in(t_0,t_1)} e_K(t)$, then the line from $v_K^-(t_1)$ to q). Let $r = l_K(t_0) \cap l_K(t_1)$.*

The following holds.

$$I(r,q) - I(\mathbf{x}) - I(r,p) = \frac{1}{2}\int_{t_0}^{t_1} \tau^{+}_{K,l_K(t_1)}(t)^2\, dt$$

We define the *arm lengths* of tangent hallways of a cap K.

Definition 4.6. Let $K \in \mathcal{K}_\omega$ be arbitrary. For any $t \in [0,\omega]$, let $g_K^+(t)$ (resp. $g_K^-(t)$) be the real value such that $\mathbf{y}_K(t) = A_K^+(t) + g_K^+(t)v_t$ (resp. $\mathbf{y}(t) = A_K^-(t) + g_K^-(t)v_t$). Let $h_K^+(t)$ (resp. $h_K^-(t)$) be the value such that $\mathbf{y}(t) = C_K^+(t) + h_K^+(t)u_t$ (resp. $\mathbf{y}(t) = C_K^-(t) + h_K^-(t)u_t$).

Observe that both $g_K^\pm(t)$ and $h_K^\pm(t)$ are convex-linear with respect to K. We are now ready to show the concavity of $\mathcal{A}_1$.

Theorem 4.13. *$\mathcal{A}_1$ is concave on $\mathcal{K}_{\pi/2}$.*

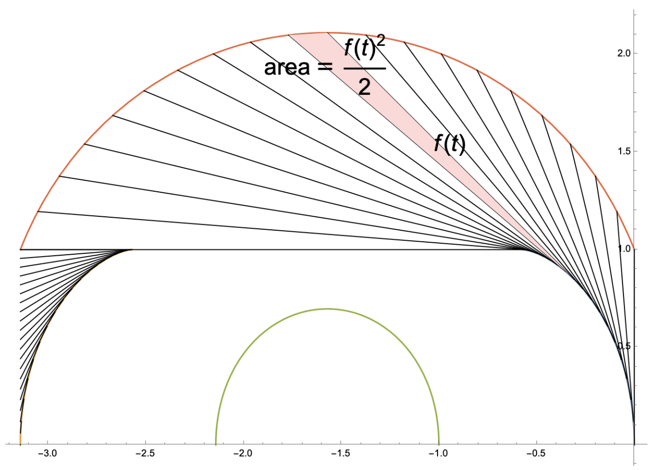

Figure 7: Expressing $\mathcal{A}_1$ as sum-of-squares via Mamikon's theorem.

Proof. Take an arbitrary $K \in \mathcal{K}_{\pi/2}$. Define the segment $\mathbf{b}_1$ of the upper boundary δK from $A_K^-(0)$ to $C_K^+(0)$. Define the segment $\mathbf{b}_2$ of δK from $C_K^+(0)$ to $C_K^+(\pi/2)$. Then we have $|K| = I(\mathbf{b}_1) + I(\mathbf{b}_2)$.

We observe that $\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{x}_K)$ is a linear functional on K. It comes from the following calculation using $\mathbf{y}_K(t) = \mathbf{x}_K(t) + u_t + v_t$.

$$\begin{aligned}
\mathcal{I}(\mathbf{y}_K) &= \frac{1}{2}\int_0^{\pi/2} \mathbf{y}_K(t) \times d\mathbf{y}_K(t) \\
&= \frac{1}{2}\int_0^{\pi/2} (\mathbf{x}_K(t) + u_t + v_t) \times d(\mathbf{x}_K(t) + u_t + v_t) \\
&= \mathcal{I}(\mathbf{x}_K) + \frac{1}{2}\left(\int_0^{\pi/2} (u_t+v_t) \times d\mathbf{x}_K(t) + \int_0^{\pi/2} \mathbf{x}_K(t) \times d(u_t+v_t) + \int_0^{\pi/2} (u_t+v_t) \times d(u_t+v_t)\right)
\end{aligned}$$

So it remains to show that $|K| - \mathcal{I}(\mathbf{y}_K)$ is a concave function with respect to K.

We stitch the following instances of Mamikon's theorem. First, by Theorem 4.11 on the curve $\mathbf{b}_1$ and $\mathbf{y}_K(t)$ for $t \in [0, \pi/2]$, we get the following after some rearrangement.

$$\begin{aligned}
\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{b}_1) &= \frac{1}{2}\int_0^{\pi/2} h_K^+(t)^2 \, dt - \mathcal{I}(\mathbf{y}_K(\pi/2), C_K^+(0)) + \mathcal{I}(\mathbf{y}_K(0), A_K^-(0)) \\
&= \frac{1}{2}\int_0^{\pi/2} h_K^+(t)^2 \, dt + \frac{1}{2} g_K^+(\pi/2) - \frac{1}{2} p_K(0)
\end{aligned}$$

Second, by Theorem 4.12 on the curve $\mathbf{b}_2$ and angles $\pi/2$, π we get the following where $\tau(t) = \tau^+_{K, l_K(\pi)}(t)$ on $t \in [\pi/2, \pi)$. Note that the function τ is convex-linear with respect to K.

$$\begin{aligned}
-I(\mathbf{b}_2) &= \frac{1}{2}\int_{\pi/2}^{\pi} \tau(t)^2 \, dt - I(\mathbf{y}_K(\pi/2), C_K^+(\pi/2)) - I(C_K^+(0), \mathbf{y}_K(\pi/2)) \\
&= \frac{1}{2}\int_{\pi/2}^{\pi} \tau(t)^2 \, dt - \frac{1}{2} p_K(\pi) - \frac{1}{2} g_K^+(\pi/2)
\end{aligned}$$

Add the two equations together to obtain the following.

$$I(\mathbf{y}_K) - |K| = \frac{1}{2}\int_0^{\pi/2} h_K^+(t)^2 \, dt + \frac{1}{2}\int_{\omega}^{\pi} \tau(t)^2 \, dt - \frac{1}{2}\left(p_K(0) + p_K(\pi)\right)$$

We observed that h_K^+ and τ are convex-linear with respect to K. So by **??** the value $\mathcal{I}(\mathbf{y}_K) - |K|$ is concave. $\square$

4.5 Directional Derivative

In this section, we calculate the directional derivative $D\mathcal{A}_1(K; -)$ of $\mathcal{A}_1$ at any $K \in \mathcal{K}_{\pi/2}$. As $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K)$, we calculate the directional derivative of $|K|$ and $\mathcal{I}(\mathbf{x}_K)$ separately. We have the following general result for the area $|K|$ of any convex body K.

Theorem 4.14. *Let K and K' be arbitrary convex bodies. Then we have the following.*

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} |(1-\lambda)K + \lambda K'| = \langle p_{K'} - p_K, \beta_K \rangle_{S^1}$$

Proof. For any convex body K we have $|K| = V(K, K)$ where V is the mixed volume of two planar convex bodies. So by applying Lemma 4.1 to $|K| = V(K, K)$ and using that $V(K, K') = V(K', K)$, we have the following.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} |(1-\lambda)K + \lambda K'| = 2V(K', K) - 2V(K, K)$$

By applying Theorem 4.4 we get the result. $\square$

We also have the following general result for the curve area functional $\mathcal{I}(\mathbf{x})$ of any curve $\mathbf{x}$.

Theorem 4.15. *Let $\mathbf{x}_1, \mathbf{x}_2 : [a, b] \to \mathbb{R}^2$ be two rectifiable curves. Then the following holds.*

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{I}((1-\lambda)\mathbf{x}_1+\lambda\mathbf{x}_2) = \left[\int_a^b (\mathbf{x}_2(t) - \mathbf{x}_1(t)) \times d\mathbf{x}_1(t)\right] + \mathcal{I}(\mathbf{x}_1(b), \mathbf{x}_2(b)) - \mathcal{I}(\mathbf{x}_1(a), \mathbf{x}_1(a))$$

Proof. Consider the bilinear form $\mathcal{J}(\mathbf{x}_1, \mathbf{x}_2) = \int_a^b \mathbf{x}_1(t) \times d\mathbf{x}_2(t)$ on general rectifiable curves $\mathbf{x}_1$ and $\mathbf{x}_2$. Apply Lemma 4.1 to $2\mathcal{I}(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbf{x})$ to get the following.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} 2\mathcal{I}((1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2) = \mathcal{J}(\mathbf{x}_1, \mathbf{x}_2) + \mathcal{J}(\mathbf{x}_2, \mathbf{x}_1) - 2\mathcal{J}(\mathbf{x}_1, \mathbf{x}_1)$$

Using the integration by parts, we have the following.

$$\int_a^b \mathbf{x}_1(t) \times d\mathbf{x}_2(t) = \mathbf{x}_1(b) \times \mathbf{x}_2(b) - \mathbf{x}_1(a) \times \mathbf{x}_2(a) + \int_a^b \mathbf{x}_2(t) \times d\mathbf{x}_1(t)$$

Plug this back in to get the result. □

To compute the directional derivative of $\mathcal{I}(\mathbf{x})$ from $\mathbf{x} = \mathbf{x}_K$ to $\mathbf{x} = \mathbf{x}_{K'}$ for caps $K, K' \in \mathcal{K}_\omega$, we need to calculate the left and right derivative of $\mathbf{x}_K(t)$ with respect to t.

Theorem 4.16. *For any cap $K \in \mathcal{K}_\omega$, the right derivative of the outer and inner corner $\mathbf{y}_K(t)$ exists for any $0 \le t < \omega$ and is equal to the following.*

$$\partial^+\mathbf{y}_K(t) = -g_K^+(t)u_t + h_K^+(t)v_t \qquad \partial^+\mathbf{x}_K(t) = -(g_K^+(t) - 1)u_t + (h_K^+(t) - 1)v_t$$

Likewise, $\mathbf{v}_K^-(t)$ exists for all $0 < t \le \omega$ and is equal to the following.

$$\partial^-\mathbf{y}_K(t) = -g_K^-(t)u_t + h_K^-(t)v_t \qquad \partial^+\mathbf{x}_K(t) = -(g_K^-(t) - 1)u_t + (h_K^-(t) - 1)v_t$$

Proof. Fix an arbitrary cap K and omit the subscript K. Take any $0 \le t < \omega$ and set $s = t + \delta$ for sufficiently small and arbitrary $\delta > 0$. We simplify $\partial^+\mathbf{y}(t) = \lim_{\delta\to 0^+}(\mathbf{y}(s) - \mathbf{y}(t))/\delta$. Define $A_{t,s} = a(t) \cap a(s)$. Since $A_{t,s}$ is on the lines $a(t)$ and $a(s)$, it satisfies both $A_{t,s} \cdot u_t = \mathbf{y}(t) \cdot u_t$ and $A_{t,s} \cdot u_s = \mathbf{y}(s) \cdot u_s$. Rewrite $u_s = \cos\delta \cdot u_t + \sin\delta \cdot v_t$ on the second equation and we have

$$A_{t,s} \cos\delta \cdot u_t + A_{t,s} \sin\delta \cdot v_t = \cos\delta(\mathbf{y}(s) \cdot u_t) + \sin\delta(\mathbf{y}(s) \cdot v_t).$$

Group by $\cos\delta$ and $\sin\delta$ and substitute $A_{t,s}\cdot u_t$ with first equation then

$$\cos\delta(\mathbf{y}(s)\cdot u_t - \mathbf{y}(t)\cdot u_t) = \sin\delta(A_{t,s}(s)\cdot v_t - \mathbf{y}(s)\cdot v_t).$$

Divide by δ and send $\delta\to 0^+$. Observe that $A_{t,s}\to A^+(t)$. We get

$$(\partial^+\mathbf{y}(t))\cdot u_t = (A^+(t) - \mathbf{y}(t))\cdot v_t = -g^+(t)$$

as $g^+(t) = (\mathbf{y}(t) - A^+(t))\cdot v_t$. A similar argument can be applied to show $(\partial^+\mathbf{y}(t))\cdot v_t = h^+(t)$ and thus the first equation of the theorem. The right derivative of $\mathbf{x}_K(t)$ comes from $\mathbf{x}_K(t) = \mathbf{y}_K(t) - u_t - v_t$. A symmetric argument calculates the left derivative of the outer and inner corner. $\square$

Definition 4.7. For any cap $K\in\mathcal{K}_{\pi/2}$, define $i_K : [0,\pi)\to\mathbb{R}$ as $i_K(t) = h_K^+(t) - 1$ and $i_K(t+\pi/2) = g_K^+(t) - 1$ for every $t\in[0,\pi/2)$. Define ι_K as the measure on $[0,\pi]$ derived from the density function i_K. That is, $\iota_K(dt) = i_K(t)dt$.

Definition 4.7 is motivated by the following lemma.

Lemma 4.17. *Let $t_1, t_2\in[0,\pi/2]$ such that $t_1 < t_2$ be arbitrary. Let $K_1, K_2\in\mathcal{K}_{\pi/2}$ be arbitrary. Then the following holds.*

$$\int_{t_1}^{t_2}\mathbf{x}_{K_1}(t)\times d\mathbf{x}_{K_2}(t) = \langle p_{K_1} - 1, \iota_{K_2}\rangle_{[t_1,t_2]\cup[t_1+\pi/2,t_2+\pi/2]}$$

Proof. By Theorem 4.16, the derivative of $\mathbf{x}_{K_2}(t)$ with respect to t exists almost everywhere and is the following.

$$\mathbf{x}'_{K_2}(t) = -(g^+_{K_2}(t) - 1)u_t + (h^+_{K_2}(t) - 1)v_t$$

Meanwhile, we have the following.

$$\mathbf{x}_{K_1}(t) = (p_{K_1}(t) - 1)u_t + (p_{K_1}(t+\pi/2) - 1)v_t$$

So the cross-product $\mathbf{x}_{K_1}(t)\times\mathbf{x}'_{K_2}(t)$ is equal to the following almost everywhere.

$$(h^+_{K_2}(t) - 1)(p_{K_1}(t) - 1) + (g^+_{K_2}(t) - 1)(p_{K_1}(t+\pi/2) - 1)$$

Now the left-hand side is equal to

$$\int_{t_1}^{t_2}(h^+_{K_2}(t) - 1)(p_{K_1}(t) - 1) + (g^+_{K_2}(t) - 1)(p_{K_1}(t+\pi/2) - 1)\,dt$$

and by Definition 4.7 this integral is equal to $\langle p_{K_1} - 1, \iota_{K_2}\rangle_{[t_1,t_2]\cup[t_1+\pi/2,t_2+\pi/2]}$. $\square$

Theorem 4.18. *Let K_1 and K_2 be two caps in $\mathcal{K}_{\pi/2}$. Then we have the following.*

$$D\mathcal{A}_1(K_1;K_2) = \left.\frac{d}{d\lambda}\right|_{\lambda=0}\mathcal{A}_1((1-\lambda)K_1 + \lambda K_2) = \langle p_{K_2} - p_{K_1}, \beta_{K_1} - \iota_{K_1}\rangle_{[0,\pi]}$$

Proof. We start from $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K)$. By applying Theorem 4.14 to the first term and applying **??** to the second term, we get the following.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{A}_1(K) = \langle p_{K_2} - p_{K_1}, \beta_{K_1} \rangle_{[0,\pi]} - \int_0^{\pi/2} (\mathbf{x}_{K_2}(t) - \mathbf{x}_{K_1}(t)) \times d\mathbf{x}_{K_1}(t)$$

Apply Lemma 4.17 to the second term twice. □

We examine the intuitive meaning of Theorem 4.18 in depth before moving on to the rigorous details of maximizing $\mathcal{A}_1(K)$ with respect to K. In Theorem 2 (ODE3) of [11], Romik essentially computes the following calculus of variation. Assume for a moment that S is a monotone sofa of rotation angle $\pi/2$ with cap K, such that the niche $\mathcal{N}(K)$ is exactly the region bounded by the curve $\mathbf{x}_K(t)$. Assume that our cap K has differentiable $A_K(t) = A_K^{\pm}(t)$, $C_K(t) = C_K^{\pm}(t)$ and $\mathbf{x}_K(t)$ for every $t \in (0, \pi/2)$. Take an arbitrary angle t. Fix small positive δ and ϵ. We pertube S to a new sofa S' as the following.

The monotone sofa $S = H \cap \bigcap_{0 \leq u \leq \pi/2} L_K(u)$ is the intersection of rotating hallways $L_K(t)$. For every $t' \in [0, \pi/2]$, define a new family of rotating hallways $L'(t')$. For every $t' \in [t, t+\delta]$, let $L'(t') = L_S(t') + \epsilon u_t$, and for every other t', let $L'(t') = L_S(t')$. That is, we move $L_S(t')$ in the direction of ϵu_t for only $t' \in [t, t+\delta]$. Define $S' = H \cap \bigcap_{0 \leq u \leq \pi/2} L'(u)$, then S' is a slight perturbation of S. By Theorem 2.1, S' is also a moving sofa modulo that S' is connected. Comparing S and S', some area is gained near $A_K(t)$ and some area is lost near $\mathbf{x}_K(t)$. The gain near $A_K(t)$ is approximately $\epsilon\delta \left\| A_K'(t) \right\|$ as the shape of the gain is approximately a rectangle of sides $\delta A_K'(t)$ and ϵu_t. The loss near $\mathbf{x}_K(t)$ is $\epsilon\delta \mathbf{x}'(t) \cdot v_t$ as the shape of the loss is approximately a parallelogram of sides $\delta \mathbf{x}_K(t)$ and ϵu_t.

Now observe that this total gain of $\epsilon\delta \left\| A_K'(t) \right\| - \epsilon\delta \mathbf{x}'(t) \cdot v_t$ is captured in Theorem 4.18. The perturbation from K to K' can be described as $p_{K'} = p_K + \epsilon 1_{[t,t+\delta]}$ in terms of their support functions. Correspondingly, the value $\langle p_{K'} - p_K, \beta_{K_1} \rangle_{[0,\pi]} = \epsilon \beta_{K_1}((t, t+\delta])$ is approximately $\epsilon\delta \left\| A_K'(t) \right\|$, which is equal to the gain near $A_K(t)$. The value $\langle p_{K'} - p_K, \iota_{K_1} \rangle_{[0,\pi]} = \epsilon \iota_{K_1}((t, t+\delta])$, by Theorem 4.16 and Definition 4.7, is approximately $\epsilon\delta \mathbf{x}_K'(t) \cdot v_t$ which is equal to the loss near $\mathbf{x}_K(t)$. A similar observation can be made when we pertube for every $t' \in [t, t+\delta]$, $L'(t') = L_S(t') + \epsilon v_t$.

To summarize, $(p_{K'} - p_K)(t)$ and $(p_{K'} - p_K)(t + \pi/2)$ measures the movement of $\mathbf{x}_K(t)$ along the direction u_t and v_t respectively. Then $\beta_K(t)$ and $\beta_K(t + \pi/2)$ respectively measures the differential side lengths of the boundary of K near $A_K(t)$ and $C_K(t)$ respectively. Likewise, $\iota_K(t)$ and $\iota_K(t + \pi/2)$ measures the component of $\mathbf{x}'(t)$ in direction of v_t and $-u_t$ respectively.

4.6 Maximizer

We now solve for the maximizer $K = K_{\max,1}$ of our concave quadratic upper bound $\mathcal{A}_1(K)$. We do this *a priori* by solving for the cap K where $\beta_K = \iota_K$ on the set $[0, \pi] \setminus \{\pi/2\}$. Once the equation $\beta_K = \iota_K$ is solved, then for every other cap K' we have $D\mathcal{A}_1(K;K') = 0$ by Theorem 4.18 as we have $\beta_K = \iota_K$ on $[0,\pi] \setminus \{\pi/2\}$ and $p_{K'}(\pi/2) = p_K(\pi/2) = 1$. Then by Theorem 4.2 the cap K attains the maximum value of $\mathcal{A}_1$.

We need some preparation to solve for the equation $\beta_K = \iota_K$ on $[0,\pi]\setminus\{\pi/2\}$. From now on, let $K \in \mathcal{K}_{\pi/2}$ be an arbitrary cap. We will use the arm lengths $g_K^{\pm}(t)$ and $h_K^{\pm}(t)$ as our main tool for caculation.

From Theorem 4.5 with $a = t$ and $b = t + \pi/2$, we have the following.

Lemma 4.19. *For every $t \in [0,\pi/2]$ we have the followings.*

$$g_K^+(t) = \int_{(t,t+\pi/2]} \cos(u-t)\,\beta_K(du) \qquad g_K^-(t) = \int_{[t,t+\pi/2]} \cos(u-t)\,\beta_K(du)$$

$$h_K^+(t) = \int_{[t,t+\pi/2]} \sin(u-t)\,\beta_K(du) \qquad h_K^-(t) = \int_{[t,t+\pi/2)} \sin(u-t)\,\beta_K(du)$$

For most $t \in (0,\pi/2)$, the values $g_K^+(t)$ and $g_K^-(t)$ (resp. $h_K^+(t)$ and $h_K^-(t)$) are equal and we will denote the common value by dropping their signs.

Definition 4.8. For any angle $t \in (0,\pi/2)$, if $g_K^+(t) = g_K^-(t)$ (resp. $h_K^+(t) = h_K^-(t)$) then we simply denote the matching value as $g_K(t)$ (resp. $h_K(t)$). Note that the condition holds if and only if $\beta_K(\{t\}) = 0$ (resp. $\beta_K(\{t+\pi/2\})$) by Lemma 4.19.

Now we calculate the derivatives of $g_K(t)$ and $h_K(t)$.

Theorem 4.20. *Assume that there is a open interval U in $(0,\pi/2)$ and a continuous function $f : U \cup (U + \pi/2) \to \mathbb{R}$ such that the measure β_K has density function f on $U \cup (U+\pi/2)$. That is, we have $\beta_K(X) = \int_X f(x)\,dx$ for every measurable $X \subseteq U \cup (U+\pi/2)$. Then we have $g_K'(t) = -f(t) + h_K(t)$ and $h_K'(t) = f(t_0 + \pi/2) - g_K(t)$ for every $t \in U$.*

Proof. We have the followings for every $t \in U$ by Lemma 4.19.

$$g_K(t) = \int_t^{t+\pi/2} \cos(u-t)\,\beta_K(du)$$

$$h_K(t) = \int_t^{t+\pi/2} \sin(u-t)\,\beta_K(du)$$

Differentiate those at $t \in U$ using Leibniz integral rule to get the following.

$$g_K'(t) = -f(t) + \int_t^{t+\pi/2} \sin(u-t)\,\beta_K(du) = -f(t) + h_K(t)$$

$$h_K'(t) = f(t+\pi/2) - \int_t^{t+\pi/2} \cos(u-t)\,\beta_K(du) = f(t+\pi/2) - g_{K_1}(t)$$

□

The following theorem solves the equation $\beta_K = \iota_K$ on any open interval $(t_1, t_2) \cup (t_1 + \pi/2, t_2 + \pi/2)$. Note that this equation is a mesure-theoretic version of (ODE3) in Theorem 2 of [11] generalized to arbitrary caps.

Theorem 4.21. *Let $K \in \mathcal{K}_{\pi/2}$ be any cap. Let $0 \le t_1 < t_2 < \pi/2$ be two arbitrary angles. Then the followings are equivalent.*

- *We have $\beta_K = \iota_K$ on the set $(t_1, t_2) \cup (t_1 + \pi/2, t_2 + \pi/2)$*
- *We have $\beta_K(dt) = (a_0 - t)dt$ and $\beta_K(dt + \pi/2) = (c_0 + t)dt$ on the interval $t \in (t_1, t_2)$ where $a_0 = h_K^+(t_1) + t_1 - 1$ and $c_0 = g_K^+(t_1) - t_1 - 1$.*

Moreover, for such K we have $g_K(t) = g_K^+(t_1) + (t - t_1)$ and $h_K(t) = h_K^+(t_1) - (t - t_1)$ for $t \in (t_1, t_2)$, and $g_K^-(t_2) - g_K^+(t_1) = t_2 - t_1$ and $h_K^-(t_2) - h_K^+(t_1) = -(t_2 - t_1)$ consequently.

Proof. Assume that $\beta_K = \iota_K$ on the set $X = (t_1, t_2) \cup (t_1 + \pi/2, t_2 + \pi/2)$. Since the measure of β_K is zero for every point of X, by Definition 4.8 we have $g_K(t) = g_K^{\pm}(t)$ and $h_K(t) = h_K^{\pm}(t)$ for every $t \in (t_1, t_2)$. Also, by applying dominating convergence theorem to Lemma 4.19, the functions $g_K(t)$ and $h_K(t)$ are continuous with respect to every $t \in (t_1, t_2)$. Now we can apply Theorem 4.20 to K as $\beta_K = \iota_K$ has the continuous density function i_K on X. For every $t \in (0, \pi/2)$, we have the followings by Theorem 4.20 and Definition 4.7.

$$g_K'(t) = -i_K(t) + h_K(t) = 1 \tag{10}$$

$$h_K'(t) = i_K(t + \pi/2) - g_K(t) = -1 \tag{11}$$

As $g_K(t) \to g_K^+(t_1)$ as $t \to t_1^+$, we have $g_K(t) = g_K^+(t_1) + (t - t_1)$ for every $t \in (t_1, t_2)$. Likewise, as $h_K(t) \to h_K^+(t_1)$ as $t \to t_1^+$, we have $h_K(t) = h_K^+(t_1) - (t - t_1)$ for every $t \in (t_1, t_2)$. By Definition 4.7, we have $i_K(t) = h_K(t) - 1 = h_K^+(t_1) - (t - t_1) - 1$ and $i_K(t + \pi/2) = g_K(t) - 1 = g_K^+(t_1) + (t - t_1) - 1$. As $\beta_K(dt) = i_K(t)dt$ on X, we have $\beta_K(dt) = (a_0 - t)dt$ and $\beta_K(dt + \pi/2) = (c_0 + t)dt$ where $a_0 = h_K^+(t_1) + t_1 - 1$ and $c_0 = g_K^+(t_1) - t_1 - 1$.

Now let K be any cap satisfying $\beta_K(dt) = (a_0 - t)dt$ and $\beta_K(dt + \pi/2) = (c_0 + t)dt$ on the interval $t \in (t_1, t_2)$ where $a_0 = h_K^+(t_1) + t_1 - 1$ and $c_0 = g_K^+(t_1) - t_1 - 1$. Then by Theorem 4.20 we have the followings.

$$g_K'(t) = -(a_0 - t) + h_K(t) \tag{12}$$

$$h_K'(t) = (c_0 + t) - g_K(t) \tag{13}$$

We have $g_K(t) \to g_K^+(t_1)$ and $g_K'(t) = -(a_0 - t) + h_K(t) \to -(h_K^+(t_1) - 1) + h_K(t_1) = 1$ as $t \to t_1^+$. Likewise, $h_K(t) \to h_K^+(t_1)$ and $h_K'(t) \to -1$ as $t \to t_1^+$. So the unique solutions $g_K(t)$ and $h_K(t)$ satisfying the ODEs above are $g_K(t) = g_K^+(t_1) + (t - t_1)$ and $h_K(t) = h_K^+(t_1) - (t - t_1)$ for $t \in (t_1, t_2)$, completing the proof. $\square$

With Theorem 4.21, we can now solve for the equation $\beta_K = \iota_K$ on $[0, \pi] \setminus \{\pi/2\}$ with $t_1 = 0$ and $t_2 = \pi/2$. Let K be a solution, then it should be that $g_K(0) = 1$ and $h_K(\pi/2) = 1$ as $\beta_K(\{0\}) = \beta_K(\{\pi\}) = 0$. So by Theorem 4.21 we should have $\beta_K(dt) = (\pi/2-t)dt$ and $\beta_K(dt+\pi/2) = tdt$ on the set $t \in (0, \pi/2)$. We also have $g_K^-(\pi/2) = 1 + \pi/2$. Another calculation by Lemma 4.19 shows $g_K^+(\pi/2) = \int_0^{\pi/2} t \cos t \, dt = \pi/2 - 1$. So we also have $\beta_K(\{\pi/2\}) = 2$ as well (note that on the other hand we have $\iota_K(\{\pi/2\}) = 0$). So the solution K should be the following cap.

Definition 4.9. Define the cap $K = K_{\max,1} \in \mathcal{K}_{\pi/2}$ as the cap with following boundary measure β_K and $p_K(0) = 1$.

- $\beta_K(\{\pi/2\}) = 2$
- $\beta_K(dt) = (\pi/2 - t)dt$ on $t \in [0, \pi/2)$ and $\beta_K(dt) = (t - \pi/2)dt$ on $t \in (\pi/2, \pi]$.

To justify the existence of such $K = K_{\max,1}$ we can use Theorem 4.7. By Theorem 4.7, we only need to check $\int_0^{\pi/2} \cos(t)(\pi/2 - t) \, dt = 1$ which is true by calculation. So there is $K_{\max,1}$ satisfying Definition 4.9.

By Theorem 4.21 we have $\beta_K = \iota_K$ on $[0, \pi] \setminus \{\pi/2\}$ for $K = K_{\max,1}$ indeed, and so by Theorem 4.2 we get the following.

Theorem 4.22. *The cap $K_{\max,1}$ maximizes $\mathcal{A}_1 : \mathcal{K}_{\pi/2} \to \mathbb{R}$.*

The maximum value of $\mathcal{A}_1$ is very close to the area $\mu_G = 2.2195\dots$ of Gerver's sofa.

Theorem 4.23. *The maximum value $\mathcal{A}_1(K_{\max,1})$ of $\mathcal{A}_1$ is $1+\pi^2/8 = 2.2337\dots$.*

Proof. (TODO: plug in $\omega = \pi/2$ for this general calculation with arbitrary rotation angle) If $\omega = \pi/2$, let $p_\omega = (0, 1)$. Let $\tau = (\pi/2 - \omega)/2$. Then we have $p_\omega = (\tan(\tau), 1)$ for any angle $\omega \leq \pi/2$. Note that K_1 has mirror symmetry along the line segment connecting $(0, 0)$ and p_ω.

Let us compute the value of $p_{K_1}(t)$ for $t \in [0, \omega]$.

$$
\begin{aligned}
p_{K_1}(t) - p_\omega \cdot u_t = (A_{K_1}^-(t) - p_\omega) \cdot u_t &= \\
&= \sin(\omega - t) + \int_{t_0 \in [t, \omega]} (\omega - t_0) \sin(t_0 - t) \, dt_0 \\
&= \omega - t
\end{aligned}
$$

So $p_{K_1}(t) = \omega - t + p_\omega \cdot u_t$ on $t \in [0, \omega]$. By symmetry, $p_{K_1}(t) = t - \pi/2 + p_\omega \cdot u_t$ on $t \in [\pi/2, \omega + \pi/2]$. Now use the mirror symmetry to calculate half the area of K_1.

$$
\begin{aligned}
\frac{1}{2} \int_{t \in [0, \omega]} p_{K_1}(t) \, \beta(dt) &= \frac{1}{2} + \frac{1}{2} \int_{t \in [0, \omega]} (\omega - t + p_\omega \cdot u_t)(\omega - t) \, dt \\
&= \frac{1}{2} + \frac{1}{2} \left(\omega^3/3 + p_\omega \cdot (1 - \cos(\omega), \omega - \sin(\omega)) \right)
\end{aligned}
$$

So we have the following.

$$
\begin{aligned}
|K_1| &= 1 + \omega^3/3 + p_\omega \cdot (1 - \cos\omega, \omega - \sin\omega) \\
&= \omega^3/3 + \omega + \cot(\pi/4 + \omega/2)
\end{aligned}
$$

Next, we compute the area of the niche $I(\mathbf{x}_{K_1})$. We have

$$
\mathbf{x}_{K_1}(t) = (p_{K_1}(t) - 1)u_t + (p_{K_1}(t + \pi/2) - 1)v_t
$$

and

$$
\mathbf{x}'_{K_1} = -(g^+_{K_1}(t) - 1) \cdot u_t + (h^+_{K_1}(t) - 1) \cdot v_t = -t \cdot u_t + (\omega - t) \cdot v_t
$$

so

$$
\begin{aligned}
I(\mathbf{x}_{K_1}) &= \frac{1}{2}\int_0^\omega (p_{K_1}(t) - 1)(\omega - t) + (p_{K_1}(t + \pi/2) - 1)t\,dt \\
&= \frac{1}{2}\int_0^\omega (\omega - t + p_\omega \cdot u_t - 1)(\omega - t)\,dt + \frac{1}{2}\int_0^\omega (t + p_\omega \cdot v_t - 1)t\,dt \\
&= \omega^3/3 - \omega^2/2 + (-1 + \omega + \cot(\pi/4 + \omega/2))
\end{aligned}
$$

Finally, we compute $\mathcal{A}_1(K_1) = |K_1| - I(\mathbf{x}_{K_1}) = 1 + \omega^2/2$. $\square$

5 Gerver's Conjecture

5.1 Overview (WIP)

Assuming the injectivity hypothesis, we now prove Gerver's conjecture. By utilizing the first upper bound $\mathcal{A}_1$, we first show that any cap $K \in \mathcal{K}_{\pi/2}$ that maximizes the sofa area functional $\mathcal{A}(K)$ should satisfy certain properties. Then, we use those properties to construct a much refined second upper bound $\mathcal{A}_2$. By maximizing $\mathcal{A}_2$, we will see that the maximizer of $\mathcal{A}_2$ is the Gerver's sofa S_G, and that the value of $\mathcal{A}_2$ at S_G matches with the actual area $|S_G|$, completing the proof of Gerver's conjecture.

5.1.1 Properties of maximizer K of $\mathcal{A}(K)$

Let $K \in \mathcal{K}_{\pi/2}$ be any well-behaved cap that is the maximizer of the sofa area functional $\mathcal{A}(K)$. Then in particular, the upper bound $\mathcal{A}_1(K)$ of $\mathcal{A}(K)$ should be at least the area $\mu_G = 2.2195\ldots$ of Gerver's sofa. From $\mathcal{A}_1(K) \geq \mu_G$, we will show the following.

Theorem 5.1. *For any $t < \pi/4$, the point $\mathbf{x}_K(t)$ is always strictly on the right side of $\mathbf{x}_K(\pi/4)$.*

Assume the contrary, then it should be that $f(K) = (\mathbf{x}_K(t) - \mathbf{x}_K(\pi/4)) \cdot u_0 \leq 0$ for some $t < \pi/4$ where f is a linear functional on K. We maximize $\mathcal{A}_1(K)$ on the convex subspace $f(K) \leq 0$, and show that the maximum is strictly less than μ_G, showing that it is impossible for K to maximize $\mathcal{A}(K)$.

(Note as of Nov. 4 2023: The complete proof for the optimization problem described above is not done. But a numerical approximation of the problem has been computed in Mathematica, and it strongly suggests a 'pre-sofa' solution consisting of two stages of movement (as Gerver's sofa has five stages of *contact points* described by Romik). Explicitly writing down the constants and working out complicated symbols is the part that will take the most time, but once that is done, the theoretical proof more or less follows the tools and derivation we did for maximizing of $\mathcal{A}_1$.)

By using $\mathcal{A}_1$ similarly, we also show the following.

Theorem 5.2. *For any $t > \pi/4$, the point $\mathbf{x}_K(t)$ is always strictly on the left side of $\mathbf{x}_K(\pi/4)$.*

(Note as of Nov. 4 2023: The proof is not done here similarly. We do have a numerical approximation solution done in Mathematica suggesting the two stages of contact points)

With Theorem 5.1, we can now define the convex curve $\mathbf{b}_K$ which is the envelope of the lines $b_K(t)$ for $t \in [\pi/2, \pi/4]$. Define $\mathbf{b}_K$ as the curve that starts with $\mathbf{x}_K(\pi/4)$ and ends with the point $A_K^-(0)$, with tangent lines $b_K(t)$ for $t \in [\pi/2, \pi/4]$. The curve $\mathbf{d}_K$ is the curve that starts with the point $C_K^+(\pi/2)$ and ends with the point $\mathbf{x}_K(\pi/4)$. Theorem 5.1 ensures that the region under both $\mathbf{b}_K$ and $\mathbf{d}_K$ are subsets of the niche $\mathcal{N}(K)$. That is, $\mathbf{b}_K$ and $\mathbf{d}_K$ corresponds to the extensions of the tails of the Gerver's sofa.

Next, we show the following, again by showing that for any possible counterexample K we have the maximum value of $\mathcal{A}_1(K)$ strictly less than μ_G by solving the convex optimization problem on $\mathcal{A}_1$.

Theorem 5.3. *For any $t \in [\pi/4, \pi/2]$, we have $\mathbf{x}_K(\psi_m)$ always strictly above the line $b_K(t)$ for $\psi_m = \pi/6$. By symmetry, for any $t \in [0, \pi/4]$, we have $\mathbf{x}_K(\pi/2 - \psi_m)$ strictly above the line $d_K(t)$.*

(Note as of Nov. 4 2023: The proof is not done here similarly.)

5.1.2 Construction of $\mathcal{A}_2$

With Theorem 5.3, observe that $\mathbf{x}_K(\psi_m)$ is strictly above the curve $\mathbf{b}_K$ and $\mathbf{x}_K(0)$ is on the x-axis. So by intermediate value theorem, there is a value ψ_R of $t \in [0, \psi_M]$ such that $\mathbf{x}_K(\psi_R)$ is exactly on the curve $\mathbf{b}_K$. Take the convex curve segment $\mathbf{r}$ of $\mathbf{b}_K$ that connects $\mathbf{x}_K(\psi_R)$ and $A_K^-(0)$. Because $\mathbf{r}$ is a part of the curve $\mathbf{b}_K$, we should have $p_K(t) - p_{\mathbf{r}}(t) \leq 1$ for every $t \in [0, \pi/4]$. Likewise, there is a maximum value $\psi_L \in [0, \psi_M]$ such that $\mathbf{x}_K(\pi/2 - \psi_L)$ is exactly on the curve $\mathbf{d}_K$. Take the convex curve segment $\mathbf{l}$ of $\mathbf{d}_K$ that connects $C_K^+(\pi/2)$ to $\mathbf{x}_K(\pi/2 - \psi_L)$. Because $\mathbf{l}$ is a part of the curve $\mathbf{d}_K$, we should have $p_K(t + \pi/2) - p_{\mathbf{l}}(t) \leq 1$ for every $t \in [\pi/4, \pi/2]$.

Now the tuple $(K, \mathbf{l}, \mathbf{r})$ is a member of the following space of caps and tails.

Definition 5.1. Let $\psi_R, \psi_L \in [0, \psi_{\max}]$ be arbitrary fixed constants. Define the space $\mathcal{K}_{\psi_R,\psi_L}$ as the tuple $(K, \mathbf{r}, \mathbf{l})$ of a cap K with angle set $[0, \pi/2]$, convex curve $\mathbf{r}$ with normal angle $[-3\pi/2, -\pi/2]$ from $\mathbf{x}_K(\psi_R)$ to $A_K^-(0)$, and convex curve $\mathbf{l}$ with normal angle $[-\pi/2, -\pi/4]$ from $C_K^+(\pi/2)$ to $\mathbf{x}_K(\pi/2 - \psi_L)$, such that they satisfy the following constraints.

- $p_K(t) - p_{\mathbf{r}}(t) \leq 1$ for every $t \in [0, \pi/4]$.
- $p_K(t + \pi/2) - p_{\mathbf{l}}(t) \leq 1$ for every $t \in [\pi/4, \pi/2]$.

One key observation is that, once we fix the constants ψ_R and ψ_L, the space $\mathcal{K}_{\psi_R,\psi_L}$ becomes a convex space. Moreover, the area of the region bounded by $\mathbf{r}$, $\mathbf{x}_K|_{[\psi_R,\pi/2-\psi_L]}$ and $\mathbf{l}$ consecutively is a quadratic functional on the space $\mathcal{K}_{\psi_R,\psi_L}$. So we use this to define the quadratic upper bound $\mathcal{A}_2$.

Definition 5.2. Define $\mathcal{A}_2 : \mathcal{K}_{\psi_R,\psi_L} \to \mathbb{R}$ as the functional

$$\mathcal{A}_2(K, \mathbf{r}, \mathbf{l}) = |K| - \mathcal{I}(\mathbf{l}) - \mathcal{I}(\mathbf{x}_K) - \mathcal{I}(\mathbf{r})$$

Theorem 5.4. *$\mathcal{A}_2$ is concave.*

(Note as of Nov. 4 2023: I can do this by using Mamikon's theorem, and can write it in at most a week including the preparations required)

5.1.3 Maximization of $\mathcal{A}_2$

Observe that the area under the concatenation of curves $\mathbf{l}$, $\mathbf{x}_K$ and $\mathbf{r}$ is always a subset of $\mathcal{N}(K)$. So for our K, we have $\mathcal{A}_2(K, \mathbf{r}, \mathbf{l}) \geq \mathcal{A}(K)$ by Lemma 4.8. So $\mathcal{A}_2$ is an upper bound of our area functional $\mathcal{A}$. Also note that for the cap K_G of Gerver's sofa and its tails $\mathbf{r}_G$ and $\mathbf{l}_G$, we have $\mathcal{A}_2(K, \mathbf{r}_G, \mathbf{l}_G) = \mathcal{A}(K)$. This combined with the following theorem proves Gerver's conjecture.

Theorem 5.5. *For any $0 \leq \psi_R, \psi_L \leq \psi_{\max}$, the maximum value of $\mathcal{A}_2^{\psi_R,\psi_L}(K, \mathbf{r}, \mathbf{l})$ is at most the area μ_G of Gerver's sofa.*

The maximization of $\mathcal{A}_2$ in Theorem 5.5 is where the difficulty of the moving sofa problem is most concentrated as optimization problem.

(Note as of Nov. 4 2023: Of course the proof is not done. The maximization of $\mathcal{A}_2$ is where the difficulty of the moving sofa problem is most concentrated as optimization problem, while the main theme of all previous parts are all about 'looking at the problem in the right way'.

Numerical calculation suggests me a maximizer consisting of 9 different stages of movement with different set of contact points. If we work the nine different ODEs out similarly as Romik did in his derivation of Gerver's sofa, then the rest of the argument more or less can be done using same methods for maximizing $\mathcal{A}_1$.

The real time-consuming difficulty is actually formulating and solving the nine different stages analytically. We will have something like 10 different real variables including ψ_L and ψ_R, and 8 different equations involving those variables. Note also that we can't also exploit symmetry like Gerver or Romik in their derivation of S_G. Numerical calculation suggests that the other 8 variables are functions with respect to ψ_L and ψ_R, but actually showing that it's true is a different problem. The equations and value of $\mathcal{A}_2$ is a extremely complicated analytic expression of the area involving the 10 variables, which I think will definitely have to rely on CAS like Mathematica to work out. Then we also have to show that $\mathcal{A}_2$ maximizes exactly when ψ_L and ψ_R match the value of ψ of Gerver's sofa, which seems to be true numerically.

So, after a couple months of work, I expect we will be able to say the following at least:

"Assuming the injectivity hypothesis, we reduce the moving sofa problem to a set of decision/optimization problems involving a finite number of real variables and equations. The equations used in the decision problem is extremely complicated, but here are numerical evidences of why those problems should be true")

A Convex Bodies

A.1 Geometric Results

We will use the following results for the sofa problem. The proof of these results are mostly independent to the main discussion on sofas, so we delegate their proofs in an appendix.

A.1.1 Convex Bodies

We use the *surface area measure* β_K of a convex body K [2] [13]. Because we only work with two-dimensional convex bodies, we call such measure the *boundary measure* of K.

If every tangent line $l_K(t)$ of a convex body K touches K at the single point $v_K(t) = v_K^{\pm}(t)$, and the vertex $v_K(t)$ is a smooth function with respect to t, the measure β_K is the measure on S^1 with distribution function $||v_K'(t)||$. That is, $\beta(dt) = ||v_K'(t)||dt$. So essentially, β_K measures the differential side lengths of K. β_K generalizes naturally to K with non-smooth boundary. For example, if K is the unit square $[0, 1]^2$ then β_K is concentrated on four points $t = 0, \pi/2, \pi, 3\pi/2$ each with a measure 1, measuring the exact lengths of each edge with a given normal angle.

A lot of information of a convex body K, including β_K, is linear with respect to K.

Definition A.1. Let V be a vector space and f be a map from the collection of all convex bodies to V. If for any convex bodies K_1, K_2 and $a, b \geq 0$ we have

$f(aK_1 + bK_2) = af(K_1) + bf(K_2)$, then we say that f is *linear* on the convex bodies. If $V = \mathbb{R}$ call such f a *linear functional* on convex bodies.

Theorem A.1. *Support function p_K, boundary measure β_K, vertices v_K^+ and v_K^- are all linear with respect to convex body K. Also, for fixed and different $t_1, t_2 \in S^1$ such that $t_2 \neq t_1 + \pi$, the intersection point $l_K(t_1) \cap l_K(t_2)$ is linear respect to K.*

We define the curve parametrization of the boundary of any convex body K with nonempty interior.

Theorem A.2. *For any convex body K with nonempty interior, the topological boundary ∂K is both (i) a Jordan curve and (ii) the union $\cup_{t \in S^1} e_K(t)$ of all the edges of K. Write B_K for the length of ∂K. For any angle $t \in \mathbb{R}$ there is a unique positively oriented arc-length parametrization $\mathbf{x}_{K,t} : [0, B_K]$ of ∂K that starts and ends with $v_K^+(t)$.*

Theorem A.3. *Let K be any convex body K with nonempty interior. For every angles $t_1, t_2 \in \mathbb{R}$ such that $t_2 \in [t_1, t_1 + 2\pi]$, we can assign an arc-length curve parametrization $\mathbf{x}_{t_1,t_2}$ such that it satisfies the followings.*

- *$\mathbf{x}_{K,t_1,t_2}$ is a Jordan arc or curve from $v_K^+(t_1)$ to $v_K^+(t_2)$ parametrizing the set $\{v_K^+(t_1)\} \bigcup \cup_{t \in (t_1,t_2]} e_K(t)$.*
- *$\mathbf{x}_{K,t,t+2\pi}$ is $\mathbf{x}_{K,t}$ for any $t \in \mathbb{R}$.*
- *For any t_0, t_1, t_2 such that $t_0 \leq t_1 \leq t_2 \leq t_0 + 2\pi$, $\mathbf{x}_{K,t_0,t_2}$ is the join of $\mathbf{x}_{K,t_0,t_1}$ and $\mathbf{x}_{K,t_1,t_2}$.*

Theorem A.4. *Moreover, for every t_1 and $t_2 \in [t_1, t_1 + 2\pi)$ we can assign an arc-length curve parametrization $\mathbf{x}_{K,t_1-,t_2}$ such that it satisfies the followings.*

- *$\mathbf{x}_{K,t_1-,t_2}$ is a Jordan arc or curve from $v_K^-(t_1)$ to $v_K^+(t_2)$ parametrizing the set $\bigcup_{t \in [t_1,t_2]} e_K(t)$.*
- *$\mathbf{x}_{K,t_1-,t_2}$ is the join of $e_K(t_1)$ and $\mathbf{x}_{K,t_1,t_2}$.*
- *For any t_0, t_1, t_2 such that $t_0 \leq t_1 \leq t_2 < t_0 + 2\pi$, $\mathbf{x}_{K,t_0-,t_2}$ is the join of $\mathbf{x}_{K,t_0-,t_1}$ and $\mathbf{x}_{K,t_1,t_2}$.*

This theorem connects the boundary measure of K to the area of K.

Theorem A.5. *For any convex body K, the boundary measure satisfies the* area formula*:*

$$|K| = \frac{1}{2} \int_{S^1} p_K(t)\, \beta_K(dt)$$

Moreover, we have the following formulas for the area functional on boundary segments. For all $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1 \leq t_0 + 2\pi$, we have the followings.

$$I(\mathbf{x}_{K,t_0,t_1}) = \frac{1}{2} \int_{(t_0,t_1]} p_K(t)\, \beta_K(dt)$$

If $t_1 < t_0 + 2\pi$,

$$I\left(\mathbf{x}_{K,t_0-,t_1}\right) = \frac{1}{2}\int_{[t_0,t_1]} p_K(t)\,\beta_K(dt).$$

The following theorem is known as the Gauss-Minkowski theorem [8]. It gives a bijection between any convex body K and its boundary measure β_K.

Theorem A.6. *(Gauss-Minkowski) Let p_0 be an arbitrary point on $\mathbb{R}^2$ and $t_0 \in S^1$ be an arbitrary angle. Then any convex body K with the vertex $v_K^+(t_0) = p_0$ corresponds bijectively to an arbitrary measure β on S^1 such that $\int_{S^1} v_t\,\beta(dt) = 0$ by $\beta = \beta_K$. Moreover, any convex body K' such that $\beta_{K'} = \beta$ is a translation of K, so for any β with $\int_{S^1} v_t\,\beta(dt) = 0$ there is a unique convex body K with $\beta_K = \beta$ up to translations.*

The following theorems use the angular support of a convex body K.

Definition A.2. Define the *angular support* of a convex body K as the support of the boundary measure β_K on S^1.

Theorem A.7. *Let K be a convex body, and let (t_1, t_2) be an open interval of S^1 of length $< \pi$. The followings are equivalent.*

1. *(t_1, t_2) is disjoint from the angular support of K*
2. *$v_K^+(t_1) = v_K^-(t_2)$*
3. *Both $v_K^+(t_1)$ and $v_K^-(t_2)$ are equal to a point p. Also, $p = v_K^\pm(t)$ for every $t \in (t_1, t_2)$ and $p = l_K(a) \cap l_K(b)$ for every $a, b \in [t_1, t_2]$ where $a \neq b$.*

Proof. (of Theorem A.7)

(1 $\Leftrightarrow$ 2) From $\int_{t\in(t_1,t_2)} v_t\,\beta(dt) = v_K^-(t_2) - v_K^+(t_1)$ we have (1 $\Rightarrow$ 2). Conversely, if β is not zero on (t_1, t_2) then the integral $\int_{t\in(t_1,t_2)} v_t\,\beta(dt)$ is nonzero when measured in the direction of u_{t_1}.

(3 $\Rightarrow$ 2) is trivial.

(1 $\Rightarrow$ 3) Letting $p = v_K^+(t_1) = v_K^-(t_2)$, we also have $p = v_K^\pm(t)$ for every $t \in (t_1, t_2)$ as well, as β is zero on the interval (t_1, t_2). So p is on the line $l_K(t)$ for every $t \in [t_1, t_2]$, and (3) follows. $\square$

Theorem A.8. *Let Π be any closed subset of S^1 such that $S^1 \setminus \Pi$ is the disjoint union of open intervals of length $< \pi$. Then for any convex body K, $K = \bigcap_{t\in\Pi} H_K(t)$ holds if and only if the angular support of K is contained in Π.*

Proof. (of Theorem A.8) We have $K = \bigcap_{t\in S^1} H_K(t) \subseteq \bigcap_{t\in\Pi} H_K(t)$ always. The equality holds if and only if for every t not in Π, $H_K(t)$ contains the set $\bigcap_{t\in\Pi} H_K(t)$. Let (t_1, t_2) be any connected component of $S^1 \setminus \Pi$. Then the interval has length $< \pi$ by assumption. Now take any $t \in (t_1, t_2)$. Observe that $\bigcap_{t\in\Pi} H_K(t) \subseteq H_K(t)$ if and only if $p = l_K(t_1) \cap l_K(t_2) \in H_K(t)$, as p is the point of $\bigcap_{t\in\Pi} H_K(t)$ farthest in the direction of u_t. Then $p \in H_K(t)$ holds for every $t \in (t_1, t_2)$ if and only if (t_1, t_2) is disjoint from the angular support of K by Theorem A.7. This completes the proof. $\square$

The following follows immediately from Definition A.2.

Theorem A.9. *Let Π be any closed subset of S^1 such that $S^1 \setminus \Pi$ is the disjoint union of open intervals of length $< \pi$. The correspondence in Theorem A.6 maps the convex bodies K with angular support in Π bijectively to arbitrary (nonnegative) measure β on S^1 with support in Π such that $\int_{S^1} v_t \, \beta(dt) = 0$.*

A.1.2 Convex Curve

A *convex curve* $\mathbf{x} : [a,b] \to \mathbb{R}^2$ with normal angles in a closed interval $[t_1, t_2]$ of S^1. It is a connected subset of the boundary

Define its boundary measure $\beta_{\mathbf{x}}$ on the interval $I = [t_1, t_2]$. Then the following holds.

$I(\mathbf{x}) = \langle p_{\mathbf{x}}, \beta_{\mathbf{x}} \rangle_I$

So we can do convex-linear interpolation between two different curves.

A.1.3 Jordan Curve

Recall the following two standard theorems.

Theorem A.10. *(Jordan curve theorem. Theorem 8-40 of [1]) Any Jordan curve J divides the complement $\mathbb{R}^2 \setminus J$ into two connected components: an unbounded region $\mathcal{U}(J)$, and a bounded region $\mathcal{B}(J)$. Their boundaries are both J.*

The following theorem is a special case of Green's theorem.

Theorem A.11. *(Theorem 10-43 of [1] with $P = -y/2$ and $Q = x/2$) If J is a Jordan curve with a positively oriented, rectifiable parametrization $\mathbf{x} : [a,b] \to \mathbb{R}^2$, then the area of the bounded region $\mathcal{B}(J)$ can be computed as the following.*

$$|\mathcal{B}(J)| = I(\mathbf{x})$$

Any Jordan curve is either positively or negatively oriented. Although we won't provide the precise definition of the orientation of a Jordan curve (e.g. [1]), we use the following lemma to determine the orientation of a Jordan curve.

Lemma A.12. *Let p and q be two different points of $\mathbb{R}^2$. Define the closed half-planes H_0 and H_1 as the closed half-planes separated by the line l connecting p and q, so that for any point x in the interior of H_0 (resp. H_1) the points x, p, q are in clockwise (resp. counterclockwise) order. If a Jordan curve J consists of the join of two arcs Γ_0 and Γ_1, where Γ_0 connects p to q inside H_0, and Γ_1 connects q to p inside H_1, then J is positively oriented.*

Proof. (sketch) We first show that it is safe to assume the case where J only intersects l at two points p and q. Observe that H_i has a deformation retract to some subset $S_i \subseteq H_i$ with $S_i \cap l = \{p, q\}$ (push the three segments of $l \setminus \{p, q\}$ towards the interior of H_i). Using the retracts, we may continuously deform the arcs Γ_0 and Γ_1 inside S_0 and S_1 respectively without chainging the orientation of J. Now take any point r inside the segment connecting p and q. Continuously move a point x inside J in the orientation of J starting with $x = p$. As x moves along Γ_0 from p to q the argument of x with respect to r increases by π. And as x moves along Γ_1 the argument of x with respect to r again increases by π. So the total increase in the argument of $x \in J$ is 2π and J is positively oriented. □

A.1.4 Calculus of Variations

The following calculations are used as subroutines for the calculus of variation on sofas.

Theorem A.13. *Let $\mathbf{x}_1, \mathbf{x}_2 : [a, b] \to \mathbb{R}^2$ be two rectifiable curves. Let $\mathbf{x} = (1 - \lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2$ be the interpolation between $\mathbf{x}_1$ and $\mathbf{x}_2$ where $\lambda \in [0, 1]$. Then the following holds.*

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} I(\mathbf{x}) = I(\mathbf{x}_1(b), \mathbf{x}_2(b)) - I(\mathbf{x}_1(a), \mathbf{x}_1(a)) + \int_a^b (\mathbf{x}_2(t) - \mathbf{x}_1(t)) \times d\mathbf{x}_1(t)$$

Proof. Direct calculation gives

$$\begin{aligned} I(\mathbf{x}) &= \frac{1}{2}\int_a^b \mathbf{x}(t) \times d\mathbf{x}(t) \\ &= \frac{1}{2}\int_a^b (\mathbf{x}_1(t) + \lambda(\mathbf{x}_2(t) - \mathbf{x}_1(t))) \times (d\mathbf{x}_1(t) + \lambda d(\mathbf{x}_2(t) - \mathbf{x}_1(t))) \\ &= \frac{1}{2}\lambda\left(\int_a^b (\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1 + \mathbf{x}_1 \times d(\mathbf{x}_2 - \mathbf{x}_1)\right) + \dots \end{aligned}$$

where only the linear term of λ is displayed in the last expression. So in particular we have the following derivative.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} I(\mathbf{x}) = \frac{1}{2}\int_a^b (\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1 + \mathbf{x}_1 \times d(\mathbf{x}_2 - \mathbf{x}_1)$$

Do integration by parts (**??**) on one integral.

$$\begin{aligned} \int_a^b \mathbf{x}_1 \times d(\mathbf{x}_2 - \mathbf{x}_1) &= \mathbf{x}_1(t) \times (\mathbf{x}_2(t) - \mathbf{x}_1(t))|_{t=a}^b + \int_a^b (\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1 \\ &= \mathbf{x}_1(b) \times \mathbf{x}_2(b) - \mathbf{x}_1(a) \times \mathbf{x}_2(a) + \int_a^b (\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1 \end{aligned}$$

Substituting this back, we get

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} I(\mathbf{x}) = \frac{\mathbf{x}_1(b) \times \mathbf{x}_2(b) - \mathbf{x}_1(a) \times \mathbf{x}_2(a)}{2} + \int_a^b (\mathbf{x}_2 - \mathbf{x}_1) \times d\mathbf{x}_1$$

that matches the claimed equation. □

A.2 Convex Bodies

Recall that a convex body K is a nonempty compact subset of $\mathbb{R}^2$ which is an intersection of half-planes (Definition 2.8). We defined its support function (Definition 2.5) p_K, tangent lines (Definition 2.7) $l_K(t)$, half-planes (Definition 2.8) $H_K(t)$, vertices $v_K^{\pm}(t)$, and edges $e_K(t)$ (Definition 2.11).

Theorem A.14. *Support function p_K, boundary measure β_K, vertices v_K^+ and v_K^- are all linear with respect to convex body K. Also, for fixed and different angles $t_1, t_2 \in S^1$ such that $t_2 \neq t_1 + \pi$, the intersection point $l_K(t_1) \cap l_K(t_2)$ is linear respect to K.*

The support function p_K is Lipschitz so continuous.

Theorem A.15. *For any convex body K, its support function p_K is Lipschitz. Moreover, if K is inside the closed ball of radius R centered at the origin, then p_K is R-Lipschitz.*

Proof. For every point $z \in K$, define $f_z : S^1 \to \mathbb{R}$ as the function $t \mapsto z \cdot u_t$. Observe that f_z is a sinusoidal function with an amplitude of $\leq R$, so is R-Lipschitz. The support function p_K is the supremum of f_z for all $z \in K$, so is itself R-Lipschitz as well. $\square$

Next, we prove two technical lemmas to proceed. The following lemma is the hard/technical part of computing limits.

Lemma A.16. *Let K be a convex body and t be an arbitrary angle. We have the following right limits all converging to $v_K^+(t)$. In particular, the vertex $v_K^+(t)$ is a right-continuous function on $t \in S^1$.*

$$\lim_{u \to t^+} v_K^+(u) = \lim_{u \to t^+} v_K^-(u) = \lim_{u \to t^+} l_K(u) \cap l_K(t) = v_K^+(t)$$

Similarly, we have the following left limits.

$$\lim_{u \to t^-} v_K^+(u) = \lim_{u \to t^-} v_K^-(u) = \lim_{u \to t^-} l_K(u) \cap l_K(t) = v_K^-(t)$$

Proof. We only compute the right limits. Left limits can be shown using a symmetric argument.

Let $\epsilon > 0$ be arbitrary. Let $p = v_K^+(t) + \epsilon v_t$. Then by the definition of $v_K^+(t)$ the point p is not in K. As $\mathbb{R}^2 \setminus K$ is open, any sufficiently small open neighborhood of p is disjoint from K, so we can take some positive $\epsilon' < \epsilon$ such that the closed line segment connecting p and $q = p - \epsilon' u_t$ is disjoint from K as well. Consider the line l that passes through both q and $v_K^+(t)$. By the definition of $q = v_K^+(t) - \epsilon' u_t + \epsilon v_t$, we have $l = l(t', h')$ for some $t' \in (t, t + \pi/2)$ and $h' \in \mathbb{R}$. Define the closed right-angled triangle formed by vertices $v_K^+(t)$, p, and q as T. Then T is on the opposite sitde of $H(t', h')$ along the line l.

Define $H' = H(t' + \pi, -h')$ as the half-plane opposite to the half-plane $H(t', h')$. We show that $K \cap H' \subseteq T$. Assume by contradiction that there is

$r \in K \cap H'$ not in T. As $r \in K$, r should be in the tangential half-plane $H_K(t)$. As $r \in H' \cap H_K(t)$ and $r \notin T$, the line segment connecting r and $v_K^+(t)$ should cross the line segment connecting p and q at some point s. As $r, v_K^+(t) \in K$ we also have $s \in K$ by convexity. But the line segment connecting p and q is disjoint from K by how we took q, so we get contradiction. Thus we have $K \cap H' \subseteq T$.

We show that for any angle $t_0 \in (t, t')$, the edge $e_K(t_0)$ should lie inside the triangular region T. Observe that for all points $z \in K$ which is also in the fan $H_K(t) \cap H(t', h')$, the vertex $z = v_K^+(t)$ attains the maximum of $z \cdot u_{t_0}$. All other points $z \in K$ outside the fan $H_K(t) \cap H(t', h')$ is in T because $K \cap H' \subseteq T$. So the maximizer $z \in K$ of $z \cdot u_{t_0}$ lies in T either way, and so $e_K(t_0) \subseteq T$. Observe that the triangle T contains $v_K^+(t)$ and has diameter $< 2\epsilon$ because two perpendicular sides of T containing p has length $\leq \epsilon$. So both endpoints $v_K^+(u)$ and $v_K^-(u)$ are distance $< 2\epsilon$ away from $v_K^+(t)$. This completes the epsilon-delta argument for $\lim_{u \to t^+} v_K^+(u) = \lim_{u \to t^+} v_K^-(u) = v_K^+(t)$. Also, note that $l_K(u) \cap l_K(t)$ is inside T as well, so we also show $\lim_{u \to t^+} l_K(u) \cap l_K(t) = v_K^+(t)$. $\square$

We prepare another technical lemma on the limit of the intersection of two tangent lines.

...that the parametrization is absolutely continuous is a bit nontrivial. A computation shows that

Lemma A.17. *Let $t, t' \in S^1$ be arbitrary with $t' = t + \theta$ for $\theta \in (-\pi, \pi) \setminus \{0\}$. Then the intersection $l_K(t) \cap l_K(t')$ is a point and it is exactly the following.*

$$l_K(t) \cap l_K(t') = p_K(t) u_t + \alpha_K(t, \theta) v_t$$

Here, the value $\alpha_K(t, \theta)$ is the following.

$$\alpha_K(t, \theta) = \frac{p_K(t + \theta) - p_K(t) \cos \theta}{\sin \theta}$$

As a function of θ, $\alpha_K(t, \theta)$ is continuous and monotonically increasing on the intervals $(-\pi, 0)$ and $(0, \pi)$.

Proof. Because the point $p = l_K(t) \cap l_K(t')$ is on the line $l_K(t)$, we have $p = p_K(t) u_t + \alpha v_t$ for some constant $\alpha \in \mathbb{R}$. We use $p \cdot u_{t'} = p_K(t')$ to derive the unique value $\alpha = \alpha_K(t, \theta)$.

$$\begin{aligned} p_K(t') &= p_K(t)(u_t \cdot u_{t'}) + \alpha(v_t \cdot u_{t'}) \\ &= p_K(t) \cos \theta + \alpha \sin \theta \end{aligned}$$

This gives $\alpha = p_C(t') \csc \theta - p_C(t) \cot \theta$ as claimed and completes the calculation. The value α is continuous on $(-\pi, \pi) \setminus \{0\}$ by the formula.

We show that $\alpha_K(t, \theta)$ is monotonically increasing on the interval $\theta \in (0, \pi)$. A similar argument works for the interval $\theta \in (-\pi, 0)$ as well.

Take two values $\theta_1 < \theta_2$ in $(0, \pi)$. Let $\alpha_i = \alpha_K(t, \theta_i)$ and $v_i = l_K(t) \cap l_K(t + \theta_i) = p_K(t) u_t + \alpha_i v_t$ for $i = 1, 2$. Observe that K is contained in the intersection

C of $H_K(t)$ and $H_K(t+\theta_2)$. The intersection C is a cone with vertex v_2, and by geometric considerations the vertex v_2 is furthest in the direction of $u_{t+\theta_1}$ among all the points in C. So the interior of $H = H_K(t+\theta_1)$ does not contain v_2, but v_1 is on the boundary of H. This implies that v_2 is further than v_1 in the direction of v_t, so $\alpha_1 \leq \alpha_2$. □

Observe that the edge $e_K(t)$ is a segment of the line $l_K(t)$, so $l_K(t) \setminus e_K(t)$ is a union of two disjoint open half-lines. Using the previous technical lemmas, we can parametrize those half-lines by the intersection of two tangent lines.

Theorem A.18. *Take any convex body K and $t \in S^1$. Define $\mathbf{p} : [0, \pi) \to \mathbb{R}^2$ as $\mathbf{p}(\theta) = l_K(t-\theta) \cap l_K(t)$ for $\theta > 0$ and $\mathbf{p}(0) = v_K^-(t)$. Likewise, define $\mathbf{q} : [0, \pi) \to \mathbb{R}^2$ as $\mathbf{q}(\theta) = l_K(t+\theta) \cap l_K(t)$ for $0 < \theta$ and $\mathbf{q}(0) = v_K^+(t)$.*

If the width $w_K(t) = p_K(t+\pi) + p_K(t)$ of K measured in the direction of u_t is positive, then $\mathbf{p}$ (resp. $\mathbf{q}$) is an absolutely-continuous parametrization of the ray starting from $v_K^-(t)$ (resp. $v_K^+ t$) in the direction of $-v_t$ (resp. v_t).

If the width $w_K(t)$ is zero, then K is the line segment connecting $v_K^+(t)$ and $v_K^-(t)$, so $\mathbf{p}$ and $\mathbf{q}$ are constant functions $v_K^-(t)$ and $v_K^+(t)$ respectively.

Proof. As K is nonempty, the width $w_K(t)$ has to be nonnegative. If it is zero, then as K is in the line $H_K(t) \cap H_K(t+\pi)$ we get the conclusion. Assume that the width $w_K(t)$ is strictly positive. By Lemma A.16 and Lemma A.17, we only need to show that $\alpha_K(t,\theta) \to \pm\infty$ as $\theta \to \pm\pi$. Observe that as we take $\theta \to \pm\pi$, $p_K(t+\theta) - p_K(t)\cos\theta$ converges to the width $w_K(t)$, so $\alpha_K(t,\theta) \to \pm\infty$ by using the formula in Lemma A.17. □

Theorem A.19. *Let K be any convex body. Let $t_0 \in \mathbb{R}$ be any angle. On the interval $t \in [t_0, t_0+\pi]$, the value $v_K^+(t) \cdot v_t$ is monotonically decreasing. On the interval $t \in [t_0 - \pi, t_0]$, the value $v_K^+(t) \cdot v_t$ is monotonically increasing.*

Proof. Take two arbitrary values $t_1 < t_2$ in the interval $[t_0, t_0+\pi]$. By Theorem A.18, the points $v_K^+(t_1), l_K(t_1) \cap l_K(t_2), v_K^-(t_2), v_K^+(t_2)$ goes further in the direction of $-v_t$ in the increasing order. This shows that the value $v_K^+(t) \cdot v_t$ is monotonically decreasing on the interval $t \in [t_0, t_0+\pi]$. A symmetric argument proves the other claim. □

Theorem A.20. *On any bounded interval $t \in I$ of $\mathbb{R}$, $v_K^+(t)$ is of bounded variation.*

Proof. The x and y-coordinates of $v_K^+(t)$ either monotonically increases or decreases on each of the intervals $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$ by Theorem A.19. So the coordinates are of bounded variation on each interval. □

We now calculate the left/right differentiation of $p_K(t)$.

Theorem A.21. *For any convex body K, p_K is both left and right differentiable. Moreover, for any angle $t \in S^1$ the following equalities hold.*

$$v_K^+(t) = p_K(t)u_t + \partial^+ p_K(t) v_t$$

$$v_K^-(t) = p_K(t)u_t + \partial_- p_K(t) v_t.$$

Proof. We show the first equality. The last limit of Lemma A.17 and Lemma A.17 shows that we have the following.

$$v_K^+(t) = p_K(t)u_t + \lim_{\theta \to 0^+} \alpha_K(t,\theta)v_t$$

Computing the limit,

$$\begin{aligned}
\lim_{\theta \to 0^+} \alpha_K(t,\theta) &= \lim_{\theta \to 0^+} \frac{p_K(t+\theta) - p_K(t)\cos\theta}{\sin\theta} \\
&= \lim_{\theta \to 0^+} \left(\frac{p_K(t+\theta) - p_K(t)}{\theta} + p_K(t)\frac{1-\cos\theta}{\theta} \right) \cdot \frac{\theta}{\sin\theta} \\
&= \lim_{\theta \to 0^+} \frac{p_K(t+\theta) - p_K(t)}{\theta} \\
&= \partial^+ p_K(t)
\end{aligned}$$

as $(1-\cos\theta)/\theta \to 0$ and $\theta/\sin\theta \to 1$ as $\theta \to 0$. The second inequality can be proved in a similar way. □

Corollary A.22. *For any convex body K, $\partial^+ p_K$ is right-continuous.*

Proof. By Lemma A.16 and Theorem A.21 we have $p_K(t)u_t + \partial^+ p_K(t)v_t$ right-continuous. $p_K(t)$ is continuous by Theorem A.15. So subtracting and taking the dot product with continuous v_t we get that $\partial^+ p_K$ is right-continuous. □

Corollary A.23. *For any convex body K, $\partial^+ p_K$ is bounded-variation.*

Proof. Use that $v_K^+(t) = p_K(t)u_t + \partial^+ p_K(t)v_t$ is bounded variation (Theorem A.20) and that p_K is Lipschitz (Theorem A.15). □

Definition A.3. For any convex body K, define its *boundary measure* β_K on S^1 as the signed measure $\beta_K(dt) = p_K(t) \cdot dt + d\left(\partial^+ p_K(t)\right)$ on S^1. Here, dt is the Lebesgue measure on S^1.

Proof. Note that p_K is bounded and continuous (Theorem A.15) on S^1 so we can construct the measure $p_K(t) \cdot dt$ on S^1. We use that $\partial^+ p_K(t)$ is both right-continuous (Corollary A.22) and bounded-variation (Corollary A.23) to construct the Lebesgue-Stieltjes measure (**??**). □

This measure is called as the *surface area measure* for general n-dimensional convex bodies. Now we prove **??**.

Proof. (of **??**) First, note that v_K^+ is both right-continuous and locally bounded-variation so it makes sense to talk about the Lebesgue-Stieltjes measure $dv_K^+(t)$. If β_K is defined as Definition A.3, the following chain of equalities justify $dv_K^+(t) = v_t\beta_K(dt)$.

$$
\begin{aligned}
v_t\beta_K(dt) &= v_t p_K(t)dt + v_t d\left(\partial^+ p_K(t)\right)\\
&= v_t p_K(t)dt + d\left(v_t\partial^+ p_K(t)\right) - (-u_t)\partial^+ p_K(t)dt\\
&= v_t p_K(t)dt + d\left(v_t\partial^+ p_K(t)\right) + d\left(u_t p_K(t)\right) - v_t p_K(t)dt\\
&= d\left(v_t\partial^+ p_K(t) + u_t p_K(t)\right) = dv_K^+(t)
\end{aligned}
$$

The first equality ues Definition A.3. The second and third equality uses **??**. The rest uses linearity and definition. The equality

$$\int_{(a,b]} v_t\,\beta(dt) = v_K^+(b) - v_K^+(a)$$

comes from the definition of $dv_K^+(t)$. The inequality

$$\int_{[a,b]} v_t\,\beta(dt) = v_K^+(b) - v_K^-(a)$$

comes from taking the left limit of a in the first equality and using Lemma A.16 on the right side.

We still haven't showed two things. First is that β_K is non-negative so is a measure which is not signed. Second is that this β_K defined using Definition A.3 is the only measure that satisfies the vertex formulas.

We first show that β_K defined using Definition A.3 is non-negative. We only need to show that β_K is non-negative near any $t \in \mathbb{R}$. Take the interval $I = (t - \pi/4, t + \pi/4)$. Consider the function $f(u) = \cos(u - t)$ on I. Both f and $1/f$ are bounded continous functions on I. Also for any $a, b \in I$ such that $a < b$, $\int_{(a,b]} f(u)\beta_K(du) = \left(v_K^+(b) - v_K^+(a)\right)\cdot v_t$ is nonnegative by Theorem A.19. So $f \cdot \beta_K$ is nonnegative on I, and $(1/f)\cdot f \cdot \beta_K = \beta_K$ is also nonnegative on I.

Finally, we observe that the β_K satisfying the vertex formula $dv_K^+(t) = v_t\beta_K(dt)$ is unique because $v_t \cdot dv_K^+(t) = \beta_K(dt)$ determines β_K uniquely. $\square$

We end by showing that multiple values attached to a convex body K is linear in terms of K (Definition A.1).

Proof. (of Theorem A.1) The support function p_K is linear in K. If $t_1, t_2 \in S^1$ are two different angles, the intersection $p = l_K(t_1) \cap l_K(t_2)$ is linear in K because p is the unique point satisfying $p \cdot u_{t_1} = p_K(t_1)$ and $p \cdot u_{t_2} = p_K(t_2)$. The vertices v_K^+ and v_K^- are linear in K because they are the limits of intersection points as in Definition A.1. The boundary measure β_K is linear in K because $v_t \cdot dv_K^+(t) = \beta_K(dt)$. $\square$

A.3 Boundary of Convex Bodies

In this section, fix an arbitrary convex body K and the starting angle $t_0 \in \mathbb{R}$.

Definition A.4. Define $B_K = \beta_K((0, 2\pi])$ as the *perimeter* of K. As β_K is a measure on S^1 we also observe that $B_K = \beta_K((t_0, t_0 + 2\pi])$ for any $t_0 \in \mathbb{R}$ as well.

Why is B_K the actual perimeter of K will be justified soon. We would like to construct an arc-length parametrization $\mathbf{x}_{K,t_0} : [0, B_K] \to \mathbb{R}^2$ of the boundary ∂K starting with the point $v_K^+(t_0)$. Take an arbitrary point p' on the boundary ∂K. Let $s' \in [0, B_K]$ be the arc length s' between p' and $v_K^+(t_0)$ on ∂K, so that $p' = \mathbf{x}_{K,t_0}(s')$. As the point p' is on the boundary of K, p' is inside a tangent line $l_K(t')$ for some angle $t' \in (t_0, t_0 + 2\pi]$. Note that the arc length s' and the tangent line angle t' are two different variables attached to $p' \in \partial K$. The relation between s' and t' can't be described as a function from one to another: a single value of s' (resp. t') can have multiple possible values of t' (resp. s'). However, the values s' and t' form a generalized inverse of each other as described in [3] and for each s' (resp. t') it makes sense to find *some* corresponding value of t' (resp. s'). The function s_{K,t_0} maps parameter t' to a possible value of the parameter s', and t_{K,t_0} maps parameter s' to t' in a similar way. Our definition of t_{K,t_0} corresponds to the minimum inverse $s^{\wedge}_{K,t_0}$ of s_{K,t_0} in [3].

Definition A.5. Define monotonically increasing functions: - $s_{K,t_0} : [t_0, t_0 + 2\pi] \to [0, B_K]$ as $s_{K,t_0}(t) = \beta_K((t_0, t])$ - and $t_{K,t_0} : [0, B_K] \to [t_0, t_0 + 2\pi]$ as $t_{K,t_0}(s) = \min\{t \geq t_0 : \beta_K((t_0, t]) \geq s\}$.

Proof. It is easy to check that s_{K,t_0} is monotonically increasing. To see the well-definedness of t_{K,t_0} as a function, we only need to observe that for any $s \in [0, B_K]$ the minimum value t of the set $X = \{t \in \mathbb{R} : \beta((a, t]) \geq s\}$ is well-defined. As $\beta((a, t])$ is monotonically increasing, zero at $t = a$, and B_K at $t = a + 2\pi$ the set X is nonempty. The infimum of X is contained in X because $\beta_K((t_0, t])$ is right-continuous with respect to t. That t_{K,t_0} is monotonically increasing follows immediately from the definition of t_{K,t_0}. $\square$

Definition A.6. Define $\mathbf{x}_{K,t_0} : [0, B_K] \to \mathbb{R}^2$ as the following absolutely continuous (and thus rectifiable) function with initial condition $\mathbf{x}_{K,t_0}(0) = v_K^+(t_0)$ and the derivative $\mathbf{x}'_{K,t_0}(s') = u_{t_{K,t_0}(s')}$ almost everywhere.

$$\mathbf{x}_{K,t_0}(s) := v_K^+(t_0) + \int_{s' \in (0,s]} u_{t_{K,t_0}(s')}\, ds'$$

As t_{K,t_0} is a monotone function which is well-defined on $[0, B_K]$, it is a Borel function and the integral is well-defined. By a criterion of absolute continuity, $\mathbf{x}_{K,t_0}$ is absolutely continous with the derivative $\mathbf{x}'_{K,t_0}(s') = u_{t_{K,t_0}(s')}$ almost everwhere. Length of an absolutely continuous curve $\mathbf{x}$ in $\mathbb{R}^2$ is the integral of $||\mathbf{x}'(t)||$ [6]. So we have the following.

Theorem A.24. *The function $\mathbf{x}_{K,t_0} : [0, B_K] \to \mathbb{R}^2$ is an arc-length parametrization.*

The following is a technical computation.

Lemma A.25. *The followings hold.*

- *For any* $t_1 \in (t_0, t_0{+}2\pi]$*, we have* $t^{-1}_{K,t_0}([t_0, t_1]) = [0, s_{K,t_0}(t_1)] = [0, \beta_K((t_0, t_1])]$.
- *Moreover, the set* $t^{-1}_{K,t_0}(\{t_1\})$ *is either* $[s_{K,t_0}(t_1-), s_{K,t_0}(t_1)]$ *or* $(s_{K,t_0}(t_1-), s_{K,t_0}(t_1)]$.

Proof. The first statement comes from manipulating the definitions as the following.

$$
\begin{aligned}
t^{-1}_{K,t_0}([t_0, t_1]) &= \{s \in [0, B_K] : \min\{t \geq t_0 : \beta((t_0, t]) \geq s\} \in [t_0, t_1]\} \\
&= \{s \in [0, B_K] : \beta((t_0, t_1]) \geq s\} \\
&= [0, \beta_K((t_0, t_1])] = [0, s_{K,t_0}(t_1)]
\end{aligned}
$$

Now send $t \to t_1^-$ in the equality $t^{-1}_{K,t_0}([t_0, t]) = [0, s_{K,t_0}(t)]$ to get that $t^{-1}_{K,t_0}([t_0, t_1)) = \bigcup_{t<t_1}[0, s_{K,t_0}(t)]$ is either $[0, s_{K,t_0}(t_1-))$ or $[0, s_{K,t_0}(t_1-)]$. Then use $t^{-1}_{K,t_0}(\{t_1\}) = t^{-1}_{K,t_0}([t_0, t_1]) \setminus t^{-1}_{K,t_0}([t_0, t_1))$ to get the second statement of the lemma. ◻

Theorem A.26. $\mathbf{x}_{K,t_0}(s_{K,t_0}(t)) = v_K^+(t)$ *for all* $t \in [t_0, t_0{+}2\pi]$ *and* $\mathbf{x}_{K,t_0}(s_{K,t_0}(t-)) = v_K^-(t)$ *for all* $t \in (t_0, t_0{+}2\pi]$*. Moreover, for all* $t \in (t_0, t_0{+}2\pi]$ *the function* $\mathbf{x}_{K,t_0}$ *restricted to the interval* $[s_{K,t_0}(t-), s_{K,t_0}(t)]$ *is the arc-length parametrization of the edge* $e_K(t)$ *from vertex* $v_K^-(t)$ *to* $v_K^+(t)$*.*

Proof. The first statement of Lemma A.25 shows that the measure β_K on $(t_0, t_0 + 2\pi]$ is the pushforward of the Lebesgue measure on $(0, B_K]$ with respect to the map $t_{K,t_0} : (0, B_K] \to (t_0, t_0 + 2\pi]$. With this, we have the following calculation.

$$
\begin{aligned}
\mathbf{x}_{K,t_0}(s_{K,t_0}(t)) &= v_K^+(t_0) + \int_{s' \in (0, s_{K,t_0}(t)]} u_{t_{K,t_0}(s')}\, ds' \\
&= v_K^+(t_0) + \int_{s' \in t^{-1}_{K,t_0}([t_0,t])} u_{t_{K,t_0}(s')}\, ds' \\
&= v_K^+(t_0) + \int_{t \in (t_0, t]} u_t\, \beta(dt) = v_K^+(t)
\end{aligned}
$$

For the proof of $\mathbf{x}_{K,t_0}(s_{K,t_0}(t-)) = v_K^-(t)$, send $t' \to t^-$ in the equation $\mathbf{x}_{K,t_0}(s_{K,t_0}(t')) = v_K^+(t')$ and use Lemma A.16. From the second statement of Lemma A.25, we get that on the interval $s' \in (s_{K,t_0}(t-), s_{K,t_0}(t)]$ the value $t_{K,t_0}(s')$ is equal to t. So the derivative of $\mathbf{x}_{K,t_0}(s')$ restricted to the interval $[s_{K,t_0}(t-), s_{K,t_0}(t)]$ is almost everywhere equal to u_t, and $\mathbf{x}_{K,t_0}$ is the arc-length parametrization of the edge $e_K(t)$ from vertex $v_K^-(t)$ to $v_K^+(t)$ on the interval $[s_{K,t_0}(t-), s_{K,t_0}(t)]$. ◻

Definition A.7. Let K be any convex body. For any $t_0, t_1 \in \mathbb{R}$ such that $t_1 \in [t_0, t_0 + 2\pi]$, define $\mathbf{x}_{K,t_0,t_1}$ as the curve $\mathbf{x}_{K,t_0}(s)$ restricted on the interval $s \in [0, s_{K,t_0}(t_1)]$.

Observe that $\mathbf{x}_{K,t,t+2\pi}$ is $\mathbf{x}_{K,t}$ because $s_{K,t}(t + 2\pi) = \beta((t, t + 2\pi]) = B_K$.

Theorem A.27. *$\mathbf{x}_{K,t_0,t_1}$ is the arc-length parametrization from $v_K^+(t_0)$ to $v_K^+(t_1)$ of the set $\left\{v_K^+(t_0)\right\} \cup \bigcup_{t\in(t_0,t_1]} e_K(t)$. Moreover, if we define $\mathbf{x}_{K,t_0,t_1-}$ as the curve $\mathbf{x}_{K,t_0}(s)$ resticted on the interval $s \in [0, s_{K,t_0}(t_1-)]$, then $\mathbf{x}_{K,t_0,t_1}$ is the join of $\mathbf{x}_{K,t_0,t_1-}$ and the edge $e_K(t)$ directed from $v_K^-(t)$ to $v_K^+(t)$.*

Proof. The interval $[0, s_{K,t_0}(t_1)]$ is equal to the inverse image $t_{K,t_0}^{-1}([t_0, t_1])$, and so is the disjoint union of the singleton $t_{K,t_0}^{-1}(\{t_0\}) = \{0\}$ and the intervals $t_{K,t_0}^{-1}(\{t\})$ whose closure is $[s_{K,t_0}(t-), s_{K,t_0}(t)]$ for all $t \in (t_0, t_1]$. Under the map $\mathbf{x}_{K,t_0}$, the singleton $\{0\}$ maps to $\left\{v_K^+(t_0)\right\}$ and the set $[s_{K,t_0}(t-), s_{K,t_0}(t)]$ maps to $e_K(t)$ for all $t \in (t_0, t_1]$ by Theorem A.26. This proves that the image of the interval $[0, s_{K,t_0}(t_1)]$ under the map $\mathbf{x}_{K,t_0}$ is the set $\left\{v_K^+(t_0)\right\} \cup \bigcup_{t\in(t_0,t_1]} e_K(t)$. That $\mathbf{x}_{K,t_0,t_1}$ is the join of $\mathbf{x}_{K,t_0,t_1-}$ and the edge $e_K(t)$ directed from $v_K^-(t)$ to $v_K^+(t)$ is a direct consequence of Theorem A.26. □

Theorem A.28. *Let K be any convex body. For any $t_0, t_1, t_2 \in \mathbb{R}$ such that $t_0 \leq t_1 \leq t_2 \leq t_0 + 2\pi$, the curve $\mathbf{x}_{K,t_0,t_2}$ is the join of the curve $\mathbf{x}_{K,t_0,t_1}$ and $\mathbf{x}_{K,t_1,t_2}$.*

Proof. The curve $\mathbf{x}_{K,t_0,t_1}$ is an initial part of the curve $\mathbf{x}_{K,t_0,t_2}$. So it remains to show that $\mathbf{x}_{K,t_0}$ restricted to the interval $[s_{K,t_0}(t_1), s_{K,t_0}(t_2)]$ is $\mathbf{x}_{K,t_1}$ restricted to $[0, s_{K,t_1}(t_2)]$. Observe that the interval lengths match by Definition A.5. The initial point of the two curves is equal to $v_K^+(t_1)$ by Theorem A.26. We show that their derivatives match, that $\mathbf{x}'_{K,t_0}(t + s_{K,t_0}(t_1)) = \mathbf{x}'_{K,t_1}(t)$ for all $t \in [0, s_{K,t_1}(t_2)]$. By the definition of $s_{K,t_0}(t) = \beta_K((t_0, t])$ and Definition A.6, it remains to show that $t_{K,t_0}(t+s_{K,t_0}(t_1)) = t_{K,t_1}(t)$. This can be checked using the definition of t_{K,t_0} and t_{K,t_1}. □

The boundary ∂K is the union of all the edges.

Theorem A.29. *Let K be any convex body. Then the topological boundary ∂K of K as a subset of $\mathbb{R}^2$ is the union of all edges $\cup_{t\in S^1} e_K(t)$.*

Proof. Let $E = \cup_{t\in S^1} e_K(t)$. $E \subseteq \partial K$ comes from $E \subseteq K$ and that any point in E is on some tangent line of K so its neighborhood contains a point outside K. Now take any point $p \in \partial K$. As K is closed we have $p \in K$. So $p \cdot u_t \leq p_K(t)$ for any $t \in S^1$. Assume that the equality does not hold for any $t \in S^1$. Then by compactness of S^1 and continuity of p_K there is some $\epsilon > 0$ such that $\epsilon \leq p_K(t) - p \cdot u_t$ for any t. This implies that the ball of radius ϵ centered at p is contained in K. This contradicts $p \in \partial K$. So it should be that there is some $t \in S^1$ such that $p \cdot u_t = p_K(t)$. That is, $p \in e_K(t)$. □

Theorem A.30. *Let K be any convex body with nonempty interior. For any $t \in \mathbb{R}$, the curve $\mathbf{x}_{K,t} : [0, B_K] \to \mathbb{R}$ is a positively oriented arc-length parametrization of the boundary ∂K as a Jordan curve that starts and ends with the point $v_K^+(t)$.*

Proof. By Theorem A.28 the curve $\mathbf{x}_{K,t} = \mathbf{x}_{K,t,t+2\pi}$ is the join of two curves $\mathbf{x}_{K,t,t+\pi}$ and $\mathbf{x}_{K,t+\pi,t+2\pi}$ connecting $p = v_K^+(t)$ and $q = v_K^+(t+\pi)$ and vice versa. As K has nonempty interior, the width of K measured in the direction of u_t is strictly positive, and the point p is strictly further than the point q in the direction of u_t.

We first show that the curve $\mathbf{x}_{K,t,t+\pi}$ is a Jordan arc from p to q. The curve $\mathbf{x}_{K,t,t+\pi}$ is the join of the curve $\mathbf{x}_{K,t,t+\pi-}$ and $e_K(t+\pi)$ by Theorem A.27. Also, the derivative $\mathbf{x}'_{K,t,t+\pi-}(s')$ for an arbitary $s' \in (0, s_{K,t_0}(t_1+\pi-))$ is always a unit vector $v_{t'}$ where $t' = t_{K,t}(s') \in (t, t+\pi)$ so the curve $\mathbf{x}_{K,t,t+\pi-}$ is strictly monotonically decrasing in the direction of u_t. This with the fact that $e_K(t+\pi)$ is parallel to u_t shows that the curve $\mathbf{x}_{K,t,t+\pi}$ is injective and thus a Jordan arc. A similar argument shows that $\mathbf{x}_{K,t+\pi,t+2\pi}$ is also a Jordan arc.

Define the closed half-planes H_0 and H_1 as the half-planes divided by the line l connecting p and q, so that for any point x in the interior of H_0 (resp. H_1) the points x, p, q are in clockwise (resp. counterclockwise) order. We first observe that $\mathbf{x}_{K,t,t+\pi}$ is in H_0: as the image of $\mathbf{x}_{K,t,t+\pi}$ is the point p and the union of edges $e_K(t_1)$ for all $t_1 \in (t, t+\pi]$ by Theorem A.27, and as $p, q \in K$ and $t_1 \in (t, t+\pi]$ the edges $e_K(t_1)$ should lie in H_0. A similar argument shows that $\mathbf{x}_{K,t+\pi,t+2\pi}$ is in H_1.

If K contains the point r which is in the interior of H_0, then by convexity of K the triangle pqr is contained in K. Note again that the image of $\mathbf{x}_{K,t,t+\pi}$ is the point p and the union of edges $e_K(t_1)$ for all $t_1 \in (t, t+\pi]$ by Theorem A.27. Each edge $e_K(t_1)$ can only intersect with the line l at the points p or q because the triangle pqr is inside K. So if K contains some point in the interior of H_0, then the curve $\mathbf{x}_{K,t,t+\pi}$ connects p and q only using the points in the interior of H_0. Similarly, if K contains some point in the interior of H_1, then the curve $\mathbf{x}_{K,t+\pi,t+2\pi}$ connects q and p only using the points in the interior of H_1. Because K has nonempty interior, K contains a point in the interior of either H_0 or H_1. So in either case the Jordan arcs $\mathbf{x}_{K,t,t+\pi}$ and $\mathbf{x}_{K,t+\pi,t+2\pi}$ overlap only at the endpoints. This completes the proof that the curve $\mathbf{x}_{K,t}$ is a Jordan curve. That $\mathbf{x}_{K,t}$ is an arc-length parametrization is Theorem A.24. That $\mathbf{x}_{K,t}$ is positively oriented is a consequence of Lemma A.12. Finally, use Theorem A.27 and Theorem A.29 to conclude that $\mathbf{x}_{K,t}$ parametrizes the boundary ∂K. $\square$

Now we are ready to prove the results we claimed on the boundary of K.

Proof. (of Theorem A.2) The theorem is a combination of Theorem A.29 and Theorem A.30. $\square$

Proof. (of Theorem A.3) This theorem is a combination of **??** and Theorem A.28. To show that the function $\mathbf{x}_{K,v_K^-(t_1),v_K^+(t_2)}$ parametrizes the set $\bigcup_{t \in [t_1,t_2]} e_K(t)$, take $t_0 = t_2 - 2\pi$. Then $\mathbf{x}_{t_0,t_2}$ parametrizes ∂K, and is a join of $\mathbf{x}_{t_0,t_1-}$, the edge $e_K(t_1)$, and $\mathbf{x}_{t_1,t_2}$ by Theorem A.28. □

Proof. (of Theorem A.4) Define $\mathbf{x}_{K,t_1-,t_2}$ as the join of $e_K(t_1)$ and $\mathbf{x}_{K,t_1,t_2}$, where $e_K(t_1)$ is arc-length parametrized from $v_K^-(t_1)$ to $v_K^+(t_1)$. So it parametrizes $\bigcup_{t\in[t_1,t_2]} e_K(t)$ from $v_K^-(t_1)$ to $v_K^+(t_2)$ by definition.

Let $t_0 = t_2 - 2\pi$. Then $\mathbf{x}_{t_0,t_2}$ parametrizes ∂K, and is a join of $\mathbf{x}_{t_0,t_1-}$, the edge $e_K(t_1)$, and $\mathbf{x}_{t_1,t_2}$ by Theorem A.28 and Theorem A.27. That is, $\mathbf{x}_{t_0}$ is the join of $\mathbf{x}_{t_0,t_1-}$ and $\mathbf{x}_{t_1-,t_2}$. This shows that $\mathbf{x}_{K,t_1-,t_2}$ is either a Jordan arc or curve. □

We compute the area functional of the boundary parametrizations.

Theorem A.31. *Let K be any convex body. For any $t_0, t_1 \in \mathbb{R}$ such that $t_1 \in (t_0, t_0 + 2\pi]$. The boundary segment $\mathbf{x}_{K,t_0,t_1}(s)$ of K from $v_K^+(t_0)$ to $v_K^+(t_1)$ has the following area functional.*

$$I\left(\mathbf{x}_{K,t_0,t_1}\right) = \frac{1}{2}\int_{t'\in(t_0,t_1]} p_K(t')\,\beta(dt') = \frac{1}{2}\int_{t'\in(t_0,t_1]} v_K^+(t') \times dv_K^+(t')$$

Here, the Lebesgue-Stieltjes measure $dv_K^+(t)$ is well-defined from Theorem A.20 and Lemma A.16.

Proof. Take any $s' \in (0, s_{K,t_0}(t_1)]$. Let $t' = t_{K,t_0}(s')$ and observe that $t' \in (t_0, t_1]$ and $s' \in t_{K,t_0}^{-1}(\{s'\})$. Then as $\mathbf{x}_{K,t_0}(s') \in e_K(t')$ by Theorem A.26, we have $\mathbf{x}_{K,t_0}(s') \times v_{t'} = p_K(t')$. So we have the following.

$$\begin{aligned} I\left(\mathbf{x}_{K,t_0,t_1}\right) &= \frac{1}{2}\int_{s'\in(0,s_{K,t_0}(t_1)]} \mathbf{x}_{K,t_0}(s') \times u_{t_{K,t_0}(s')}\,ds' \\ &= \frac{1}{2}\int_{s'\in t_{K,t_0}^{-1}((t_0,t_1])} p_K(t_{K,t_0}(s'))\,ds' \\ &= \frac{1}{2}\int_{t'\in(t_0,t_1]} p_K(t')\,\beta(dt') \end{aligned}$$

The last equality uses that the measure β_K is the pushforward of the Lebesgue measure with respect to the map $t_{K,t_0} : (0, B_K] \to (t_0, t_0 + 2\pi]$ (Lemma A.25). This proves the first equality stated in the theorem. To show the second equality in the statement, check $v_K(t') \times dv_K^+(t') = v_K^+(t') \times v_{t'}\beta_K(dt') = p_K(t')\beta(dt')$. □

Proof. (of Theorem A.5) The second equation is exactly Theorem A.31. By Green's theorem (Theorem A.11) and Theorem A.30 we have $|K| = I(\mathbf{x}_{K,0})$. This with the second equation proves the first equation.

The last equation comes from the second equality and $I(v_K^-(t_0), v_K^+(t_0)) = p_K(t_0)\beta_K(\{t_0\})/2$. Observe $v_K^+(t_0) = v_K^-(t_0) + v_{t_0}\beta_K(\{t_0\})$ from **??** on the interval $[t_0, t_0]$. □

A.4 Gauss-Minkowski Theorem

Lemma A.32. *All extreme points of any convex body K are the points $v_K^-(t)$ and $v_K^+(t)$ for arbitrary $t \in S^1$.*

Proof. First, we show that any extreme point is of form $v_K^-(t)$ or $v_K^+(t)$. Any extreme point p should be on the topological boundary of K, and by Theorem A.29 p is in $e_K(t)$ for some t. Now p should be one of the endpoints $v_K^-(t)$ or $v_K^+(t)$ of $e_K(t)$, or otherwise p can be represented as strict convex combination of the endpoints and thus not a extreme point. Next, we show that the endpoints $v_K^-(t)$ or $v_K^+(t)$ are extreme points of K. $\square$

This is an immediate corollary of Lemma A.32 and the Krein–Milman theorem.

Corollary A.33. *For any convex body K, the convex hull of the points $v_K^-(t)$ and $v_K^+(t)$ for all $t \in S^1$ is K.*

Now we prove the Gauss-Minkowski Theorem.

Proof. (of Theorem A.6) For any convex body K such that $v_K^+(t_0) = p_0$, we have $\int_{S^1} v_t \, \beta_K(dt) = v_K^+(t + 2\pi) - v_K^+(t) = 0$ by **??**. On the other hand, let β be an arbitrary measure on S^1 such that $\int_{S^1} v_t \, \beta(dt) = 0$. If K is any convex body such that $\beta_K = \beta$ and $v_K^+(t_0) = p_0$, then by Corollary A.33 it should be that K is the convex hull of points $v_K^+(t) = p_0 + \int_{u \in (t_0, t]} v_u \, \beta(du)$ and $v_K^-(t) = \int_{u \in (t_0, t)} v_u \, \beta(du)$ completely determined by β. Thus, the correspondence $K \mapsto \beta_K$ is well-defined and injective.

In the other direction, let's take arbitrary measure β on S^1 such that $\int_{S^1} v_t \, \beta(dt) = 0$. Our only possible choice of K is the convex hull of points $w^+(t) = p_0 + \int_{u \in (t_0, t]} v_h \, \beta(du)$ and $w^-(t) = p_0 + \int_{u \in (t_0, t)} v_h \, \beta(du)$ for all $t \in (t_0, t_0 + 2\pi]$ like we have just observed. Let K be defined that way. We first show that we indeed have $w^+(t) = v_K^+(t)$ and $w^-(t) = v_K^-(t)$ for all t respectively. For any $t' \in (t, t + \pi]$, we have

$$(w^+(t') - w^+(t)) \cdot u_t = \int_{u \in (t, t']} v_u \cdot u_t \, \beta(du) \leq 0$$

and similarly, for any $t' \in (t - \pi, t]$ we have

$$(w^+(t) - w^+(t')) \cdot u_t = \int_{u \in (t', t]} v_u \cdot u_t \, \beta(du) \geq 0$$

so fixing t and changing t', $w^+(t') \cdot u_t$ attains the largest value when $t' = t$. Similarly, we can show that $w^+(t) \cdot u_t$ is an upper bound of any $w^-(t') \cdot u_t$ for $t' \in S^1$. This implies that $w^+(t)$ is in the edge $e_K(t)$ by the convex hull definition of K. Moreover, if $w^+(t')$ is on the line $l_K(t)$ then $w^+(t) \cdot u_t = w^+(t') \cdot u_t$ holds and by the formulas above the measure β between t and t' should be zero, so it implies $w^+(t') = w^+(t)$. Similarly, if $w^-(t')$ is on the line $l_K(t)$ then

$w^-(t') = w^-(t)$. This implies that $e_K(t)$ is the line segment connecting $w^-(t)$ and $w^+(t) = w^-(t) + \beta(\{t\})v_t$, and so $w^+(t) = v_K^+(t)$ and $w^-(t) = v_K^-(t)$.

Now observe that $v_t\beta(dt) = dw^+(t) = dv_K^+(t) = v_t\beta_K(dt)$, so $\beta = \beta_K$. As we have successfully constructed the inverse of the injective map $K \mapsto \beta_K$, this shows that the correspondence is bijective and finishes the proof. □

A.5 Mamikon's Theorem

Proof. (of Theorem 4.11) For this proof only, let $\mathbf{x} = v_K^+$ be the alias of $v_K^+ : [t_0, t_1] \to \mathbb{R}^2$. First, we verify the differential version of the theorem by the following calculation. Here $t \in (t_0, t_1]$ and the following is an equality of measures on $(t_0, t_1]$. Note that $\mathbf{y}$ is continuous by definition and $\mathbf{x}$ is right-continuous (Lemma A.16), so in particular $\mathbf{y} \times \mathbf{x}$ is also right-continuous, so $d(\mathbf{y} \times \mathbf{x})$ makes sense as a Lebesgue-Stieltjes measure.

$$\begin{aligned}
&\mathbf{y}(t) \times d\mathbf{y}(t) - \mathbf{x}(t) \times d\mathbf{x}(t) + d\left(\mathbf{y}(t) \times \mathbf{x}(t)\right)\\
&= \mathbf{y}(t) \times d\mathbf{y}(t) - \mathbf{x}(t) \times d\mathbf{x}(t) + \left(d\mathbf{y}(t) \times \mathbf{x}(t) + \mathbf{y}(t) \times d\mathbf{x}(t)\right)\\
&= \left(\mathbf{y}(t) - \mathbf{x}(t)\right) \times d\left(\mathbf{y}(t) + \mathbf{x}(t)\right)\\
&= \left(\mathbf{y}(t) - \mathbf{x}(t)\right) \times d\left(\mathbf{y}(t) - \mathbf{x}(t)\right)\\
&= f(t)u_t \times d\left(f(t)u_t\right) = f(t)u_t \times \left(u_t df(t) + f(t)v_t dt\right) = f(t)^2 dt
\end{aligned}$$

The first equality uses the product rule of differentials (**??**). The second equality is an rearrangement using linearity (note that $d\mathbf{y}(t) \times \mathbf{x}(t) = -\mathbf{x}(t) \times d\mathbf{y}(t)$ by antisymmetry of $\times$). As we have $d\mathbf{x}(t) = \beta(dt)v_t$ by **??**, and $\mathbf{y}(t) - \mathbf{x}(t) = f(t)v_t$, they are parallel and we get $(\mathbf{y}(t) - \mathbf{x}(t)) \times d\mathbf{x}(t) = 0$ which is used in the third equality. The last chain of equalities are calculus.

When we integrate the differential formula on the interval $(t_0, t_1]$, $\mathbf{y}(t) \times d\mathbf{y}(t)$ becomes $2I(\mathbf{y})$ by definition (Definition 4.3), $\mathbf{x}(t) \times d\mathbf{x}(t)$ becomes $2I(\mathbf{x})$ by Theorem A.31, and $d(\mathbf{y}(t) \times \mathbf{x}(t))$ becomes the difference $2I\left(\mathbf{y}(t_1), v_K^+(t_1)\right) - 2I\left(\mathbf{y}(t_0), v_K^+(t_0)\right)$. So the integral matches twice the left-hand side of the claimed equality. We conclude the proof by dividing by two.

For the variant, use that $\mathbf{x}_{K,t_0-,t_1}$ is the join of $e_K(t_0)$ and $\mathbf{x}_{K,t_0,t_1}$, and that $I(e_K(t_0)) = I(v_K^-(t_0), v_K^+(t_0))$. □

Proof. (of Theorem 4.12) Define $\mathbf{y} : [t_0, t_1] \to \mathbb{R}^2$ as $\mathbf{y}(t) = l_K(t) \cap l_K(t_1)$ for every $t < t_1$ and $\mathbf{y}(t_1) = v_K^-(t_1)$, then $\mathbf{y}$ is continuous by Theorem A.18 and parametrizes the line segment connecting r and $v_K^-(t_1)$. So $f(t)$ is integrable. Apply Theorem 4.11 to the curves $\mathbf{x}_{K,t_0,t_1}$ and $\mathbf{y}$, and use $I(\mathbf{y}) = I(r, \mathbf{y}(t_1)) = I(r, v_K^+(t_1)) - I(e_K(t_1))$ to obtain the claimed equality. □

References

[1] Tom M Apostol. “Mathematical Analysis A Modern Approach To Advanced Calculus”. In: ().

[2] Herbert Busemann. *Convex Surfaces*. New York, Interscience Publishers, 1958.

[3] Arnaud de La Fortelle. "A Study on Generalized Inverses and Increasing Functions Part I: Generalized Inverses". In: ().

[4] Joseph L. Gerver. "On Moving a Sofa around a Corner". In: *Geometriae Dedicata* 42.3 (June 1992), pp. 267–283. ISSN: 0046-5755, 1572-9168. DOI: 10.1007/BF02414066. URL: http://link.springer.com/10.1007/BF02414066 (visited on 05/16/2023).

[5] Philip Gibbs. "A Computational Study of Sofas and Cars". In: ().

[6] Frank Jones. *Lebesgue Integration on Euclidean Space*. Jones & Bartlett Learning, 2001.

[7] Yoav Kallus and Dan Romik. "Improved Upper Bounds in the Moving Sofa Problem". In: *Advances in Mathematics* 340 (Dec. 2018), pp. 960–982. ISSN: 00018708. DOI: 10.1016/j.aim.2018.10.022. arXiv: 1706.06630 [math]. URL: http://arxiv.org/abs/1706.06630 (visited on 05/16/2023).

[8] Jean-François Marckert and David Renault. "Compact Convex Sets of the Plane and Probability Theory". In: *ESAIM: Probability and Statistics* 18 (2014), pp. 854–880.

[9] Mamikon Mnatsakanian. "Annular Rings of Equal Area". In: *Math Horizons* 5.2 (Nov. 1997), pp. 5–8. ISSN: 1072-4117, 1947-6213. DOI: 10.1080/10724117.1997.11975029. URL: https://www.tandfonline.com/doi/full/10.1080/10724117.1997.11975029 (visited on 06/16/2023).

[10] nLab authors. *Convex Space*. May 2023. URL: https://ncatlab.org/nlab/show/convex+space.

[11] Dan Romik. "Differential Equations and Exact Solutions in the Moving Sofa Problem". In: *Experimental Mathematics* 27.3 (July 3, 2018), pp. 316–330. ISSN: 1058-6458, 1944-950X. DOI: 10.1080/10586458.2016.1270858. URL: https://www.tandfonline.com/doi/full/10.1080/10586458.2016.1270858 (visited on 05/16/2023).

[12] Rolf Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. 2nd ed. Encyclopedia of Mathematics and Its Applications. Cambridge: Cambridge University Press, 2013. DOI: 10.1017/CBO9781139003858.

[13] Rolf Schneider. "Convex Surfaces, Curvature and Surface Area Measures". In: *Handbook of Convex Geometry*. Elsevier, 1993, pp. 273–299. ISBN: 978-0-444-89596-7. DOI: 10.1016/B978-0-444-89596-7.50013-4. URL: https://linkinghub.elsevier.com/retrieve/pii/B9780444895967500134 (visited on 06/16/2023).