On moving sofas I: A new area upper bound via calculus of variation on convex bodies

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April 4, 2024

Contents

Abstract

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1 Introduction

1.0.1 Moving Sofa Problem

Moving a large couch through a narrow hallway requires not only strength but also well-planned pivoting. The *moving sofa problem*, first published by Leo Moser in 1966, is a two-dimensional idealization of such a situation:

> What is the largest area $\mu_{\max}$ of a connected shape that can move around the right-angled corner of a hallway with unit width?

More precisely, define the hallway L as the union $L = L_H \cup L_V$ of sets $L_H = (-\infty, 1] \times [0, 1]$ and $L_V = [0, 1] \times (-\infty, 1]$ representing the horizontal and vertical side of L respectively. A *moving sofa* S may be defined as a connected subset of L_H that can be moved inside L by a continuous rigid motion to a subset of L_V. It is known that there exists a moving sofa attaining the maximum area $\mu_{\max}$ [4], but the precise value of $\mu_{\max}$ remains unknown despite decades of attempts.

The best bounds currently known on $\mu_{\max}$ are summarized as

$$2.2195\dots \leq \mu_{\max} \leq 2.37. \tag{1}$$

The lower bound $2.2195\dots \leq \mu_{\max}$ comes from Gerver's sofa S_G of area $\mu_G := 2.2195\dots$ constructed in 1994 [4] (see Figure 1). Gerver conjectured that his sofa attains the maximum possible area, so that $\mu_G = \mu_{\max}$ (Conjecture 1.1). Approximate solutions found by computer experiments are consistent with his Conjecture 1.1.[1] On the other hand, the upper bound $\mu_{\max} \leq 2.37$ was proved by Kallus and Romik using computer assistance [7].

Conjecture 1.1. *(Gerver's conjecture) Gerver's sofa S_G attains the maximum area $\mu_{\max} = \mu_G$.*

1.0.2 Rotation Angle

The *rotation angle* ω of a moving sofa S is the clockwise angle that S rotates as it moves from L_H to L_V inside L. Gerver's sofa has the rotation angle $\omega = \pi/2$. On the other hand, the unit square $[0, 1]^2$ can move inside L with only translation, so it has the rotation angle $\omega = 0$. Let ω is the rotation angle of a moving sofa of maximum area. Gerver showed in [4] that we can assume $\pi/3 \leq \omega \leq \pi/2$. Kallus and Romik improved

[1] Wagner used Monte Carlo simulation to find an approximate solution (Figure 2 of [13]) that resembles Gerver's sofa in shape. More recent approximate solutions, as found by Gibbs [5] in 2014 and Batsch [2] in 2022, deviate in area from Gerver's sofa by small margins of 1.7×10^{-7} and 5.7×10^{-9} respectively.

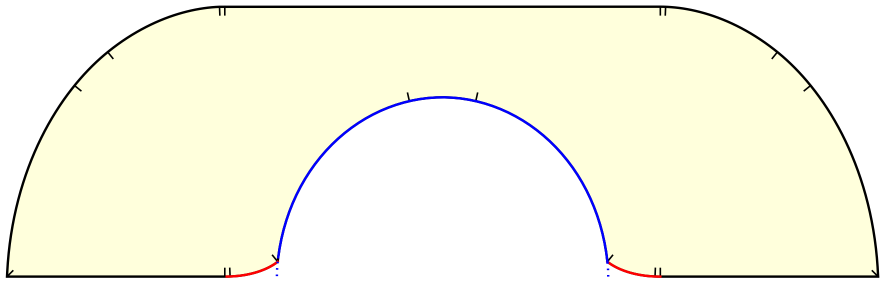

Figure 1: Gerver's sofa S_G. The ticks denote the endpoints of 18 analytic curves and segments constituting the boundary of S_G (see [11] for details). The lower portion of S_G is made of two small 'tails' (depicted red) and one large 'core' (depicted blue).

the lower bound of ω by showing that $\omega \geq \arcsin(84/85) = 81.203\ldots^\circ$ [7]. With this, we will only consider moving sofas with rotation angle $\omega \in (0, \pi/2]$, and it seems reasonable to conjecture that a maximum-area moving sofa has $\omega = \pi/2$.

Conjecture 1.2. *(Angle Hypothesis) There exists a maximum-area moving sofa with a movement of rotation angle $\omega = \pi/2$.*

Remark 1.1. A single shape S may admit different moving sofa movements in L with varying rotation angles ω. For any moving sofa S mentioned in this paper, we will always assume a fixed rotation angle ω attached to it. So any moving sofa S in this paper is technically a tuple of a shape and its fixed rotation angle. In this way, we can talk about *the* rotation angle of a moving sofa.

1.0.3 Monotone Sofas

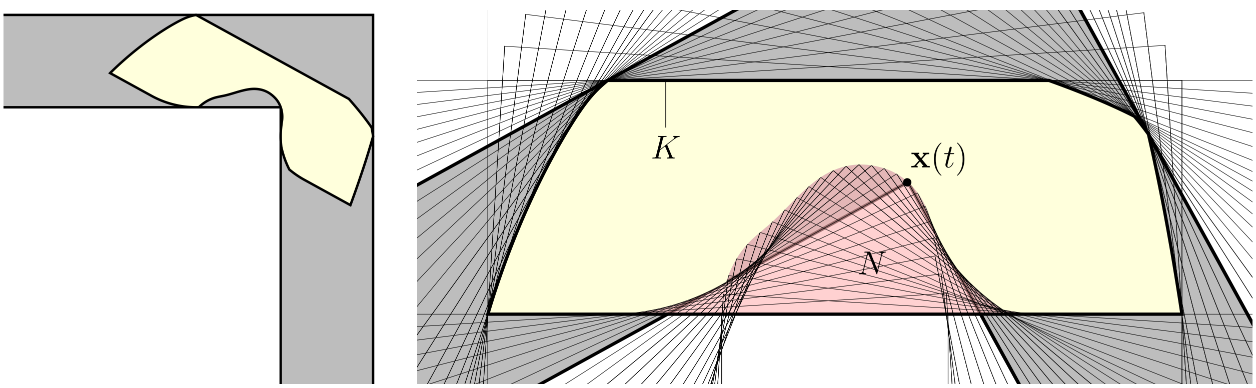

Figure 2: The movement of a monotone sofa S in perspective of the hallway (left) and the sofa (right). The monotone sofa S is equal to the cap K minus its niche N. The cap K is a convex body with the outer walls of $L_t = L_S(t)$ as tangent lines of K (and S). The niche N is the union of all triangular regions bounded by the inner walls of $L_t = L_S(t)$.

Both bounds in Equation (1) are based on an important observation of Gerver in [4]. Take any moving sofa S with rotation angle $\omega \in (0, \pi/2]$. He looked at the hallway L in the perspective of the moving sofa S, so that S is fixed and the hallway L moves around the sofa (see Figure 2). In this perspective, the sofa S is now a common subset of the rotated hallways and the two unit-width strips as the following.

1. For every angle $t \in [0, \omega]$, the hallway L_t, which is L rotated counterclockwise by t and translated so that the outer walls of L_t is in contact with S.

2. Horizontal strip $H = \mathbb{R} \times [0, 1]$.
3. A translation of V_ω, where V_ω is the vertical strip $V = [0, 1] \times \mathbb{R}$ rotated counterclockwise by ω along the origin.

To observe that $S \subseteq L_t$ for every $t \in [0, \omega]$, note that the sofa S is rotated initially by a clockwise angle of 0 and finally by ω during its movement. By the intermediate value theorem, there is a moment where a copy of S rotated clockwise by t inside L. Push that copy inside L towards the positive x and y directions, until it makes contact with the outer walls $x = 1$ and $y = 1$ of L. Now see this configuration in the perspective of S to conclude $S \subseteq L_t$. The sofa S is initialy in L_H and finally in L_H rotated clockwise by ω; see the configurations in perspective of S ot conclude that S is in H and a translation of V_ω.

Without loss of generality, we can translate the moving sofa S horizontally to further assume that S is contained in V_ω, not its translation.[2] If S attains the maximum area, we can also assume that S is simply the intersection of the sets H, V_ω, and L_t for all $t \in [0, \omega]$ containing S.[3] In this paper, we define such a moving sofa S as a *monotone sofa* (Definition 3.9; see Figure 2). Gerver's main observation in [4] summarizes to the following theorem.

Theorem 1.3. *(Gerver) There exists a monotone sofa of maximum area with rotation angle $\omega \in (0, \pi/2]$.*

1.0.4 Main Theorem

Let S be a monotone sofa with rotation angle $\omega \in (0, \pi/2]$, so that S is the intersection of H, V_ω, and the hallways L_t rotating counterclockwise by an angle of $t \in [0, \omega]$. For each $t \in [0, \omega]$, the coordinate of the inner corner $\mathbf{x}_S(t)$ of L_t determines the location of L_t. So the trajectory $\mathbf{x}_S : [0, \omega] \to \mathbb{R}^2$ of the inner corner, defined as the *rotation path* of S in [11], determines the monotone sofa S completely. In particular, Romik derived Gerver's sofa S_G by solving for the area optimality as a set of ordinary differential equations on $\mathbf{x}_S$ [11].[4]

In this paper, we introduce the following condition on monotone sofas.

Definition 1.1. A monotone sofa S with rotation angle $\omega \in (0, \pi/2]$ satisfies the *injectivity condition*, if its rotation path $\mathbf{x}_S : [0, \omega] \to \mathbb{R}^2$ is injective and never below the bottom line $y = 0$ of H nor the bottom line $x \cos \omega + y \sin \omega = 0$ of $R_\omega(V)$.

In particular, Gerver's sofa S_G is a monotone satisfying the injectivity condition. We establish the following upper bound on monotone sofas.

Theorem 1.4. *The area of any monotone sofa S satisfying the injectivity condition is at most $1 + \omega^2/2$.*

So the upper bound of Theorem 1.4 is effective on a domain containing the conjectured maximum S_G. The upper bound $1 + \omega^2/2$ of Theorem 1.4 maximizes at $\omega = \pi/2$ with the value $1 + \pi^2/8 = 2.2337\ldots$. This is much closer to the lower bound $2.2195\ldots$ of Gerver than the currently best upper bound of 2.37 of Kallus and Romik (Equation (1)).

We conjecture that a monotone sofa of maximum area should satisfy the premise of Theorem 1.4.

Conjecture 1.5. *(Injectivity Hypothesis) There exists a monotone sofa S of maximum area with rotation angle $\omega \in (0, \pi/2]$, satisfying the injectivity hypothesis.*

So the upper bound in Theorem 1.4 can be made unconditional if the injectivity hypotheses are true.

Our main idea for proving Theorem 1.4 is to overestimate the area of a monotone sofa S. For simpler explanation, assume the rotation angle $\omega = \pi/2$ of S. Observe that the lower boundary of Gerver's sofa S_G consists of two 'tails' and one 'core' (see Figure 1). The core is the trajectory of the inner corner $\mathbf{x}_{S_G}(t)$ for $t \in [\varphi, \pi/2 - \varphi]$ with $\varphi = 0.0392\ldots$ [11] and forms the majority of the lower boundary. The region below

[2] Technically, translating the moving sofa S may invalidate the initial condition $S \subseteq L_H$. To overcome this, we relax the definition of a moving sofa S in Definition 2.2 so that only a *translation* of S is required to be movable from L_H to L_V inside L.

[3] This uses an implicit assumption that the intersection of H, V_ω and L_t for all $t \in [0, \omega]$ should be connected. Indeed, Gerver in [4] assumed this in his proof of Theorem 1.3 without any proof. We fill this gap in Theorem 3.7.

[4] Gerver in [4] had to assume five specific stages of sofa movement to derive his sofa S_G. Romik's derivation of Gerver's sofa S_G in [11] also relies on this assumption, so their derivations do not constitute a full proof of Conjecture 1.1.

the two tails, trimmed out by the inner walls of L_t, constitutes only 0.28% of the area $|S_G| = 2.2195\dots$ of the whole sofa. Inspired by this, we define the overestimation $\mathcal{A}_1$ of the area $\mathcal{A}$ of a monotone sofa S with rotation angle $\omega = \pi/2$ as: the area of the convex hull K of S (that we call the *cap* of S), subtracted by the region enclosed by $\mathbf{x}_S : [0, \omega] \to \mathbb{R}^2$ and the line $y = 0$.[5]

Maximizing the area $\mathcal{A}$ of a monotone sofa S hard, because there is no managable formula of $\mathcal{A}$ that works for all possible shapes of S. On the other hand, the overestimation $\mathcal{A}_1$ has a definite formula with respect to the cap K of S. Moreover, it turns out that $\mathcal{A}_1$ is a *concave quadratic functional* on a convex domain $\mathcal{K}_\omega$ of convex bodies K. So the maximization of $\mathcal{A}_1$ becomes a single quadratic programming on the domain $\mathcal{K}_\omega$. We use the Brunn-Minkowski theory [12] on convex bodies to establish that $\mathcal{A}_1$ is quadratic on $\mathcal{K}_\omega$. Then we use Mamikon's theorem [9], a theorem in classical geometry, to prove that $\mathcal{A}_1$ concave on $\mathcal{K}_\omega$. Then we introduce a calculus of variation based on the Brunn-Minkowski theory to find a local optimum S_1 of $\mathcal{A}_1$, which is a global optimum since $\mathcal{A}_1$ is concave.

For the rotation angle $\omega = \pi/2$, the maximizer of $\mathcal{A}_1$ gives an unmovable sofa S_1 of area $1 + \pi^2/8 = 2.2337\dots$ very close to the area of Gerver's sofa S_G (see Figure 3). The shape of S_1 is very close to S_G, and cutting away the region under the red curves from S_1 gives a valid moving sofa of area approximately $2.2009\dots$, which is again very close to the area of S_G.

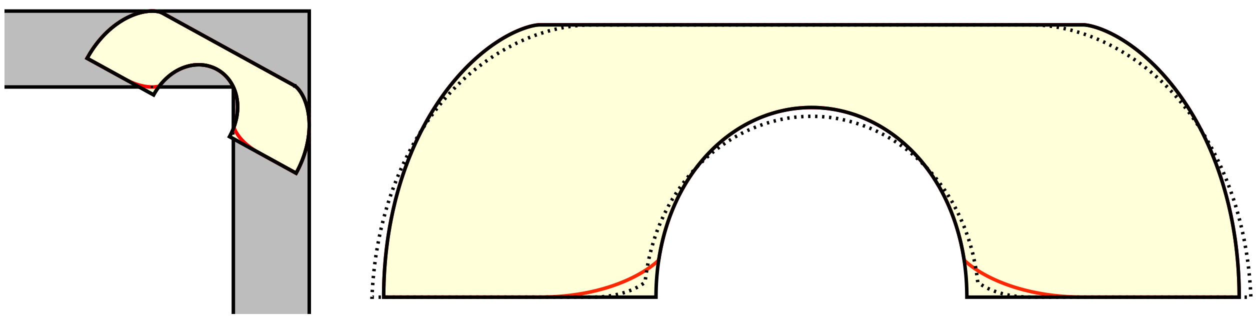

Figure 3: The maximizing shape S_1 of $\mathcal{A}_1$ of area $1 + \pi^2/8 = 2.2337\dots$. The regions below two tails (red curves) stick out of the hallway L during the movement of S_1 in L (left). The shape S_1 is very similar to Gerver's sofa S_G whose boundary is drawn in dotted lines (right).

1.0.5 Outline

Section 2 contains basic definitions that will be used thoroughout this paper.

Section 3 proves Theorem 1.3 that there exists a monotone sofa with the maximum area $\mu_{\max}$. Using this, Section 4 reduces the moving sofa problem to the maximization of the *sofa area functional* $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ on the space of all possible caps $\mathcal{K}_\omega$. Section 5 proves the main Theorem 4.3 by establishing the upper bound $1 + \omega^2/2$ of the sofa area $\mathcal{A}$ using the overestimation $\mathcal{A}_1$.

Appendix A proves numerous properties of an arbitrary planar convex body K that we will use thoroughly in this paper. So the logical ordering of this paper is Section 2, followed by Appendix A, then the sections starting Section 3. A logically inclined reader may read in this ordering to verify the correctness of all arguments. Readers who are interested in the overall idea, on the other hand, may start by reading the sections in order and refer to the appendix when needed.

2 Notations and conventions

We set up basic notations, definitions and conventions that will be used thoroughout the rest of the document.

[5]For arbitrary rotation angle $\omega \in (0, \pi/2]$ of monotone sofa S, the cap K of S is the convex hull of $S \cup \{(0,0)\}$. The overestimation $\mathcal{A}_1$ of the area of S is $|K|$ minus the area enclosed by rotation path $\mathbf{x}_S$, the bottom line $y = 0$ of H, and the bottom line $x \cos\omega + y \sin\omega = 0$ of V_ω.

2.0.1 Moving Sofa

Definition 2.1. The *hallway* $L = L_H \cup L_V$ is the union of sets $L_H = (-\infty, 1] \times [0, 1]$ and $L_V = [0, 1] \times (-\infty, 1]$, each representing the horizontal and vertical side of L respectively.

In the introduction, we gave a definition of a moving sofa S as a subset of L_H. However, the condition that S should be confined in L_H is a bit restrictive for our future use. So we will also call any translation of such $S \subseteq L_H$ a *moving sofa* as well without loss of generality.

Definition 2.2. A *moving sofa* S is a connected, nonempty and compact subset of $\mathbb{R}^2$, such that a translation of S is a subset of L_H that admits a continuous rigid motion inside L from L_H to L_V.

It is safe assume that a moving sofa is always closed, since for any subset of L its closure is also contained in L. We also define the rotation angle ω of a moving sofa S.

Definition 2.3. Say that a moving sofa S have the *rotation angle* $\omega \in (0, \pi/2]$ if the continuous rigid motion of a translate of S from L_H to L_V inside L rotates the body clockwise by ω in its full movement.

With the result of [7] that $\omega \in [81.203\ldots^\circ, 90^\circ]$ for a maximum-area moving sofa, we will always assume that a moving sofa have rotation angle $\omega \in (0, \pi/2]$.

2.0.2 Basic Notations

Denote the area (Borel measure) of a measurable $X \subseteq \mathbb{R}^2$ as $|X|$. For any subset X of $\mathbb{R}^2$, denote the topological closure, boundary, and interior as $\overline{X}$, ∂X, and X° respectively. For a subset X of $\mathbb{R}^2$ and a vector v in $\mathbb{R}^2$, define the set $X + v = \{x + v : x \in X\}$. For any two subsets X, Y of $\mathbb{R}^2$, the set $X + Y = \{x + y : x \in X, y \in Y\}$ denotes the Minkowski sum of X and Y.

We use the convention $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For any function f on S^1 and any $t \in \mathbb{R}$, the notation $f(t)$ denotes the value $f(t + 2\pi\mathbb{Z})$. That is, a real value coerces to a value in S^1 when used as an argument of a function that takes a value in S^1. We will often denote an interval of S^1 by its lift under the canonical map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} = S^1$. Specifically, for any $t_1 \in \mathbb{R}$ and $t_2 \in (t_1, t_1 + 2\pi]$, the intervals $(t_1, t_2]$ and $[t_1, t_2)$ of $\mathbb{R}$ are used to denote the corresponding intervals of S^1 mapped under $\mathbb{R} \to S^1$. Likewise, for any $t_1 \in \mathbb{R}$ and $t_2 \in [t_1, t_1 + 2\pi)$, the interval $[t_1, t_2]$ of $\mathbb{R}$ is used to denote the corresponding interval of S^1 mapped under $\mathbb{R} \to S^1$.

For any function $f : \mathbb{R} \to \mathbb{R}$ or $f : S^1 \to \mathbb{R}$, $f(t-)$ denotes the left limit of f at t and $f(t+)$ denotes the right limit of f at t. For any function $f : X \to \mathbb{R}$ defined on some open subset X of either $\mathbb{R}$ and S^1, and $t \in X$, define $\partial^+ f(t)$ and $\partial^- f(t)$ as the right and left differentiation of f at t if they exists.

We denote the integral of a measurable function f with respect to a measure μ on a set X as either $\int_{x \in X} f(x)\, \mu(dx)$ or $\langle f, \mu \rangle_X$. The latter notation is used especially when we want to emphasize that the integral is bi-linear with respect to both f and μ.

2.0.3 Convex Body

Definition 2.4. In this paper, a *shape* S is a nonempty and compact subset of $\mathbb{R}^2$.

Definition 2.5. For any angle t in S^1 or $\mathbb{R}$, define the unit vectors $u_t = (\cos t, \sin t)$ and $v_t = (-\sin t, \cos t)$.

Any line on $\mathbb{R}^2$ can be described by the angle t of its normal vector u_t and its (signed) distance from the origin.

Definition 2.6. For any angle t in S^1, and a value $h \in \mathbb{R}$, define the line $l(t, h)$ with the *normal angle* t and the distance h from the origin as the following.

$$l(t, h) = \left\{ p \in \mathbb{R}^2 : p \cdot u_t = h \right\}$$

A line on $\mathbb{R}^2$ divides the plane into two half-planes. Following Definition 2.6, we also give a name to one of the half-planes in the direction of $-u_t$.

Definition 2.7. For any angle t in S^1, and a value $h \in \mathbb{R}$, define the closed *half-plane* $H(t,h)$ with the boundary $l(t,h)$ as the following. We say that the closed half-plane $H(t,h)$ has the *normal angle* t.

$$H(t,h) = \left\{p \in \mathbb{R}^2 : p \cdot u_t \leq h\right\}$$

Fix a shape S and angle $t \in S^1$. Take a sufficiently large $h \in \mathbb{R}$ so that $H(t,h) \supseteq S$. As we decrease h continuously, the line $l(t,h)$ will get close to S until it makes contact with S for the first time. We define the value of h, tangent line $l(t,h)$, tangent half-plane $H(t,h)$ as the following Definition 2.8, Definition 2.9 and Definition 2.10.

Definition 2.8. For any shape S, define its *support function* $p_S : S^1 \to \mathbb{R}$ as the value $p_S(t) = \sup\{p \cdot u_t : p \in S\}$.

Definition 2.9. For any shape S and angle $t \in S^1$, define the *tangent line* $l_S(t)$ of S with *normal angle* t as the line $l_S(t) = l(t, p_S(t))$.

Definition 2.10. For any shape S and angle $t \in S^1$, define the *tangent half-plane* $H_S(t)$ of S with *normal angle* t as the line $H_S(t) = H(t, p_S(t))$.

Observe that the support function $p_S(t)$ measures the signed distance from the origin $(0,0)$ to the tangent line $l_S(t)$ of S with the normal vector u_t directing outwards from S. Support function and tangent lines of S are usually studied when S is a convex body (e.g. p45 of [12]), but in this paper we generalized the notion to arbitrary shape S.

The notion of *width* along a direction is also studied for convex bodies (e.g. p49 of [12]). We generalize this notion to arbitrary shapes.

Definition 2.11. For any shape S and angle t in S^1 or $\mathbb{R}$, the *width* of S along the direction of unit vector u_t is $p_S(t) + p_S(t+\pi)$.

Geometrically, the width of S along u_t measures the distance between the parallel tangent lines $l_S(t)$ and $l_S(t+\pi)$ of S.

We adopt the following definition of convex bodies (p8 of [12]).

Definition 2.12. A *convex body* K is a nonempty, compact, and convex subset of $\mathbb{R}^2$.

Many authors often also include the condition that K° is nonempty, but we allow K° to be empty (that is, K can be a closed line segment or a point).

In this paper, we use the following terminologies of *vertices* and *edges* of a planar convex body K.

Definition 2.13. For any convex body K and $t \in S^1$, define the *edge* $e_K(t)$ of K as the intersection of K with the tangent line $l_K(t)$.

Definition 2.14. For any convex body K and $t \in S^1$, let $v_K^+(t)$ and $v_K^-(t)$ be the endpoints of the edge $e_K(t)$ such that $v_K^+(t)$ is positioned farthest in the direction of v_t and $v_K^-(t)$ is positioned farthest in the opposite direction of v_t. We call $v_K^\pm(t)$ the *vertices* of K.

It is possible that the edge $e_K(t)$ can be a single point. In such case, the tangent line $l_K(t)$ touches K at the single point $v_K^+(t) = v_K^-(t)$. In fact, this holds for every $t \in S^1$ except for a countable number of values.

2.0.4 Parts of hallway L

We give names to the different parts of the hallway L for future reference.

Definition 2.15. Let $\mathbf{x} = (0,0)$ and $\mathbf{y} = (1,1)$ be the inner and outer corner of L respectively.

Definition 2.16. Let a and c be the lines $x = 1$ and $y = 1$ representing the *outer walls* of L passing through $\mathbf{y}$. Let b and d be the half-lines $(-\infty, 0] \times \{0\}$ and $\{0\} \times (-\infty, 0]$ representing the *inner walls* of L emanating from the inner corner $\mathbf{x}$.

Definition 2.17. Let $Q^+ = (-\infty, 1]^2$ be the closed quarter-plane bounded by outer walls a and c. Let $Q^- = (-\infty, 0)^2$ be the open quarter-plane bounded by inner walls b and d, so that $L = Q^+ \setminus Q^-$.

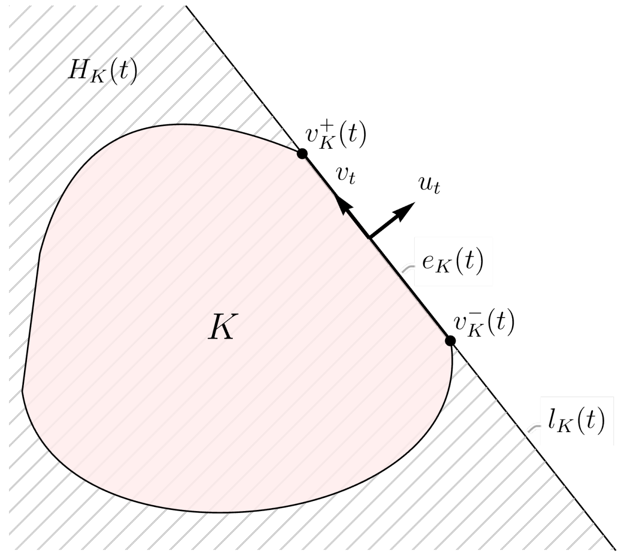

Figure 4: A convex body K with its edge, vertices, tangent line, and half-plane.

3 Monotone sofas

In this section, we rigorously define monotone sofas in Definition 3.9. Then we establish a process called *monotonization* that can take any moving sofa S with rotation angle $\omega \in d$ to a larger monotone sofa $\mathcal{M}(S)$. Then Theorem 1.3 is a consequence of monotonization. Showing that $\mathcal{M}(S)$ is connected is a key step in establishing the process.

We prove a structural theorem on every monotone sofa S (Theorem 3.12). We show that S is equal to $K \setminus \mathcal{N}(K)$, where $K = \mathcal{C}(S)$ is a convex set called the *cap of* S, and $\mathcal{N}(K)$ is a subset of K called the *niche* determined by the cap K (Theorem 3.11 Definition 3.16; see Figure 2). Then we show $\mathcal{N}(K) \subseteq K$, so that the area $|S| = |K| - |\mathcal{N}(K)|$ of S can be understood in terms of cap K and niche $\mathcal{N}(K)$ separately.

3.1 Tangent Hallway

3.1.1 Tangent hallway

We start by defining the *tangent hallways* for a general shape S (that is, any nonempty compact subset S of $\mathbb{R}^2$ by Definition 2.4).

Definition 3.1. Define $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ as the rotation map of $\mathbb{R}^2$ along the origin by a counterclockwise angle of $\theta \in S^1$.

Definition 3.2. For any shape S and angle $t \in S^1$, define the *tangent hallway* $L_S(t)$ of S with angle t as the following.

$$L_S(t) = R_t(L) + (p_S(t) - 1)u_t + (p_S(t + \pi/2) - 1)v_t$$

The equation of $L_S(t)$ in Proposition 3.1 is motivated by the following defining property of $L_S(t)$.

Proposition 3.1. *For any shape S and angle $t \in S^1$, the tangent hallway $L_S(t)$ is the unique rigid transformation of L rotated counterclockwise by t, such that the outer walls of $L_S(t)$ corresponding to the outer walls a and c of L are the tangent lines $l_S(t)$ and $l_S(t + \pi/2)$ of S respectively.*

Proof. Let c_1 and c_2 be arbitrary real values. Then $L' = R_t(L) + c_1 u_t + c_2 v_t$ is an arbitrary rigid transformation of L rotated counterclockwise by t. The outer walls of L' corresponding to the outer walls a and c of L (Definition 2.16) are then $l(t, c_1 + 1)$ and $l(t + \pi/2, c_2 + 1)$ respectively. They match with $l_S(t) = l(t, p_S(t))$ and $l_S(t + \pi/2) = l(t + \pi/2, p_S(t + \pi/2))$ if and only if $c_1 = p_S(t) - 1$ and $c_2 = p_S(t + \pi/2) - 1$. That is, if and only if $L' = L_S(t)$. ☐

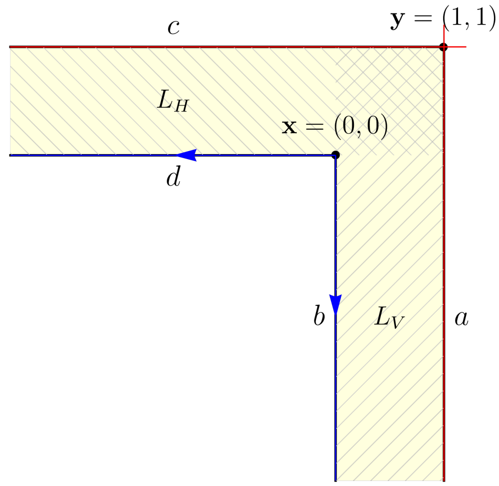

Figure 5: The standard hallway L and its parts.

Name the parts of tangent hallway $L_S(t)$ according to the parts of L (Definition 2.15, Definition 2.16, and Definition 2.17) for future use.

Definition 3.3. For any shape S and angle $t \in S^1$, define the rigid transformation $f_{S,t} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f_{S,t}(z) = R_t(z) + (p_S(t) - 1)u_t + (p_S(t + \pi/2) - 1)v_t$$

so that $f_{S,t}$ maps L to $L_S(t)$.

Definition 3.4. For any shape S and angle $t \in S^1$, let $\mathbf{x}_S(t), \mathbf{y}_S(t), a_S(t), b_S(t), c_S(t), d_S(t), Q_S^+(t), Q_S^-(t)$ be the parts of $L_S(t)$ corresponding to the parts $\mathbf{x}, \mathbf{y}, a, b, c, d, Q^+, Q^-$ of L respectively. That is, for any $? = \mathbf{x}, \mathbf{y}, a, b, c, d, Q^+, Q^-$, let $?_S(t) := f_{S,t}(?)$.

Proposition 3.2. *We have* $L_S(t) = Q_S^+(t) \setminus Q_S^-(t)$ *and* $Q_S^+(t) = H_S(t) \cap H_S(t + \pi/2)$*. Also we have the following representations of the parts purely in terms of the supporting function* p_S *of* S*.*

$$\begin{aligned}
\mathbf{x}_S(t) &= (p_S(t) - 1)u_t + (p_S(t + \pi/2) - 1)v_t \\
\mathbf{y}_S(t) &= p_S(t)u_t + p_S(t + \pi/2)v_t \\
a_S(t) &= l_S(t) = l(t, p_S(t)) \\
b_S(t) &\subseteq l(t, p_S(t) - 1) \\
c_S(t) &= l_S(t + \pi/2) = l(t + \pi/2, p_S(t + \pi/2)) \\
d_S(t) &\subseteq l(t + \pi/2, p_S(t + \pi/2) - 1)
\end{aligned}$$

Proof. The equality $L_S(t) = Q_S^+(t) \setminus Q_S^-(t)$ follows from mapping $L = Q^+ \setminus Q^-$ under $f_{S,t}$. The equality $Q_S^+(t) = H_S(t) \cap H_S(t + \pi/2)$ follows from Proposition 3.1 which states that the outer walls of $L_S(t)$ (and thus of $Q_S^+(t)$) are the tangent lines $l_S(t)$ and $l_S(t + \pi/2)$. The formulas for $a_S(t)$ and $c_S(t)$ comes from Proposition 3.1. The formulas for $b_S(t)$ and $d_S(t)$ comes from that the lines are respectively of distance 1 away from $a_S(t)$ and $c_S(t)$. The formulas for $\mathbf{x}_S(t)$ and $\mathbf{y}_S(t)$ are obtained by reading the coordinates of $\mathbf{x}_S(t) = a_S(t) \cap c_S(t)$ and $\mathbf{y}_S(t) = b_S(t) \cap d_S(t)$ with respect to the orthogonal unit vectors u_t and v_t. $\square$

3.1.2 Moving Hallway Problem

Any translation of a moving sofa S is also a moving sofa (Definition 2.2). From now on, we will always translate S in *standard position* (Definition 3.5) according to the following Proposition 3.3.

Definition 3.5. A moving sofa S with rotation angle $\omega \in (0, \pi/2]$ is in *standard position* if $p_S(\omega) = p_S(\pi/2) = 1$.

Proposition 3.3. *For any angle $\omega \in (0, \pi/2]$ and shape S, there is a translation S' of S such that $p_{S'}(\omega) = p_{S'}(\pi/2) = 1$ which is (i) unique if $\omega < \pi/2$, or (ii) unique up to horizontal translations if $\omega = \pi/2$.*

Proof. Since the support function $p_{S'}(t)$ measures the signed distance from origin to tangent line $l_{S'}(t)$ (see the remark above Definition 2.8), the translation S' of S satisfies the condition $p_{S'}(\omega) = p_{S'}(\pi/2) = 1$ if and only if the lines $l(\omega, 1)$ and $l(\pi/2, 1)$ are tangent to S' and S' is below the lines. Translate S below the lines $l(\omega, 1)$ and $l(\pi/2, 1)$ so that it makes contact with the two lines. If $\omega < \pi/2$, then the constraints determine the unique location of S'. If $\omega = \pi/2$, then the two lines are equal to the horizontal line $y = 1$, and S' can move freely horizontally as long as the line $y = 1$ makes contact with S' from above. $\square$

Assume any moving sofa S with rotation angle $\omega \in (0, \pi/2]$. By Proposition 3.3 any moving sofa can be put in standard position by translating it. Gerver also observed in [4] that S should be contained in the tangent hallways $L_S(t)$ for all $t \in [0, \omega]$.

Proposition 3.4. *For any shape S contained in a translation of $R_t(L)$ with an angle $t \in S^1$, the tangent hallway $L_S(t)$ with angle t contains S.*

Proof. Assume that an arbitrary shape S is contained in a translation L' of $R_t(L)$ for some angle t. Then while keeping S inside L', we can push L' towards S in the directions $-u_t$ and $-v_t$ until the outer walls of the final $L' = L_S(t)$ make contact with S. The pushed hallway $L_S(t)$ still contains S because the directions $-u_t$ and $-v_t$ of movement only push the inner walls of L' away from S. $\square$

We summarize the full details of Gerver's observation (line 18-22, p269; line 24-31, p270 of [4]) as Theorem 3.5.

Definition 3.6. Define the unit-width horizontal and vertical strips $H = \mathbb{R} \times [0, 1]$ and $V = [0, 1] \times \mathbb{R}$ respectively.

Definition 3.7. For any angle $\omega \in [0, \pi/2]$, define the *parallelogram* $P_\omega = H \cap R_\omega(V)$ with *rotation angle* ω.[6]

Theorem 3.5. *Let $\omega \in (0, \pi/2]$ be an arbitrary angle. For a connected shape S, the following conditions are equivalent.*

1. *S is a moving sofa with rotation angle ω.*
2. *S is contained in a translation of H and $R_\omega(V)$. Also, for every $t \in [0, \omega]$, S is contained in a translation of $R_t(L)$, the hallway rotated counterclockwise by an angle of t.*
3. *Let S' be any translation of S such that $p_{S'}(\omega) = p_{S'}(\pi/2) = 1$. Then (i) $S' \subseteq P_\omega$, (ii) $S' \subseteq L_{S'}(t)$ for all $t \in [0, \omega]$, and (iii) S' is a moving sofa with rotation angle ω in standard position.*

Proof. ($1 \Rightarrow 2$) Consider the movement of S inside the hallway L. For any angle $t \in [0, \omega]$, there is a moment where the sofa S is rotated clockwise by an angle of t inside L. Viewing this from the perspective of the sofa S, S is contained in some translation of L rotated *counterclockwise* by an arbitrary $t \in [0, \omega]$. Likewise, by looking at the initial (resp. final) position of S inside L_H (resp. L_V) from the perspective of S, the set S is contained in a translation of H and $R_\omega(V)$ respectively.

($2 \Rightarrow 3$) Take any S satisfying condition 2, and take its arbitrary translation S' satisfying $p_{S'}(\omega) = p_{S'}(\pi/2) = 1$ using Proposition 3.3. Then the translate S' of S also satisfies condition 2. So without loss of generality, we can simply assume $S = S'$.

Since S is contained in a translation of H and $R_\omega(V)$, the width of S along the direction of u_ω and v_0 (Definition 2.11) are at most 1. So $p_S(\omega) = p_S(\pi/2) = 1$ implies (i) $S \subseteq P_\omega$. Proposition 3.4 implies (ii) $S \subseteq L_S(t)$. It remains to show that S' is a moving sofa to verify (iii).

[6] If $\omega = \pi/2$, then the set $P_{\pi/2} = H$ is technically not a parallelogram. We will however call it as the parallelogram with rotation angle $\pi/2$ with a slight abuse of notation.

Because the support function $p_S(t)$ of S is continuous, the tangent hallway $L_S(t)$ moves continuously with respect to t by Definition 3.2. For every $t \in [0, \omega]$, let g_t be the unique rigid transformation that maps $L_S(t)$ to L. Then the rigid transformation $S_t := g_t(S)$ of S also changes continuously with respect to t. Mapping $S \subseteq L_S(t)$ under g_t we have $S_t \subseteq L$. Because $L_S(0)$ is a translation of L by $t = 0$ in Definition 3.2, g_0 is a translation and S_0 is a translation of S. It remains to show that $S_0 \subseteq H$ and $S_\omega \subseteq V$. Because $p_S(\pi/2) = 1$, $L_S(0)$ is a translation of L along the direction u_0, and the map g_0 is also a translation along the direction u_0. Because $S \subseteq H$, we also have $S_0 = g_0(S) \subseteq H$. Likewise, since $p_S(\omega) = 1$ the hallway $L_S(\omega)$ is a translation of L along the direction v_ω. So the map g_ω is the composition of a translation along the direction v_ω and $R_{-\omega}$. Because $S \subseteq R_\omega(V)$, we also have $S_\omega = g_\omega(S) \subseteq g_\omega(R_\omega(V)) = V$.

($3 \Rightarrow 1$) By Proposition 3.3, any connected shape S have a translation S' that satisfies the premise $p_{S'}(\omega) = p_{S'}(\pi/2) = 1$ of condition 3. So condition 3 is not vacuously true and S' is a moving sofa. Now its translation S is a moving sofa as well. □

3.2 Monotonization

Here, we rigorously define the notion of monotone sofas and establish Theorem 1.3. First, define the *monotonization* of any moving sofa S in standard position as the following set.

Definition 3.8. Let S be any moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. The *monotonization* of S is the intersection

$$\mathcal{M}(S) = P_\omega \cap \bigcap_{0 \leq t \leq \omega} L_S(t).$$

Condition 3 of Theorem 3.5 implies that the set $\mathcal{M}(S)$ contains S.

Corollary 3.6. *$\mathcal{M}(S) \supseteq S$ for any moving sofa S in standard position.*

Gerver proved Theorem 1.3 in [4] that the monotonization $\mathcal{M}(S)$ is also a moving sofa. However, he implicitly assumed that $\mathcal{M}(S)$ is connected, which is essential for $\mathcal{M}(S)$ to be a moving sofa. We fill the gap in his argument by showing that $\mathcal{M}(S)$ is indeed connected.

Theorem 3.7. *Let S be a moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. Then the monotonization $\mathcal{M}(S)$ is connected.*

Once the connectedness of $\mathcal{M}(S)$ is established, we can show that monotonization enlarges any moving sofa to a larger moving sofa.

Theorem 3.8. *Let S be any moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. The monotonization $\mathcal{M}(S)$ of S is then a moving sofa with the same rotation angle ω in standard position containing S.*

Proof. By Theorem 3.7, the shape $\mathcal{M}(S)$ is connected. By Definition 3.8, the set $\mathcal{M}(S)$ is contained P_ω and $L_S(t)$ for all $t \in [0, \omega]$, so it satisfies the second condition of Theorem 3.5. So the set $\mathcal{M}(S)$ is a moving sofa with rotation angle ω. $\mathcal{M}(S)$ contains S by Corollary 3.6. From $S \subseteq \mathcal{M}(S) \subseteq P_\omega$ and

$$p_S(\omega) = p_{P_\omega}(\omega) = p_S(\pi/2) = p_{P_\omega}(\pi/2) = 1$$

we have $p_{\mathcal{M}(S)}(\omega) = p_{\mathcal{M}(S)}(\pi/2) = 1$. So $\mathcal{M}(S)$ is in standard position. □

Now define the resulting sofa of monotonization a *monotone sofa*.

Definition 3.9. A *monotone sofa* is a moving sofa in standard position, which is the monotonization of some moving sofa in standard position.

The existence of a monotone sofa attaining the maximum area (Theorem 1.3) is an immediate consequence of Theorem 3.8. The rest of this subsection is dedicated to proving Theorem 3.7.

3.2.1 Proof of Theorem 3.7

We prepare the following terminologies.

Definition 3.10. Let S be any moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. Define the set

$$\mathcal{C}(S) = P_\omega \cap \bigcap_{0 \leq t \leq \omega} Q_S^+(t).$$

Definition 3.11. Say that a set $X \subseteq \mathbb{R}^2$ is *closed in the direction of* vector $v \in \mathbb{R}^2$ if, for any $x \in X$ and $\lambda \geq 0$, we have $x + \lambda v \in X$.

Definition 3.12. Any line l of $\mathbb{R}^2$ divides the plane into two half-planes. If l is not parallel to the y-axis, call the *left side* (resp. *right side*) of l as the closed half-plane with boundary l containing the point $-Nu_0$ (resp. Nu_0) for a sufficiently large N. If a point p is on the left (resp. right) side of l and not on the boundary l, we say that p is *strictly on the left* (resp. *right*) *side* of l.

We also prepare a lemma.

Lemma 3.9. *Let S be any moving sofa with rotation angle $\omega \in [0, \pi/2]$ in standard position. Then the support functions p_S, $p_{\mathcal{M}(S)}$, and $p_{\mathcal{C}(S)}$ of S, $\mathcal{M}(S)$ and $\mathcal{C}(S)$ agree on the set $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$.*

Proof. We have $S \subseteq \mathcal{M}(S) \subseteq \mathcal{C}(S)$ by Corollary 3.6 and the definitions of $\mathcal{M}(S)$ and $\mathcal{C}(S)$. So it remains to show that $p_{\mathcal{C}(S)}(t) \leq p_S(t)$ on every t in the set $J := [0, \omega] \cup [\pi/2, \pi/2 + \omega]$. Define $S' = \bigcap_{t \in J} H_S(t)$, then by convexity of S we have $p_{S'}(t) = p_S(t)$ on every $t \in J$. By the definition of $\mathcal{C}(S)$ we have $\mathcal{C}(S) \subseteq S'$, and so $p_{\mathcal{C}(S)}(t) \leq p_{S'}(t) = p_S(t)$ for all $t \in J$. $\square$

An important consequence of Lemma 3.9 is that a moving sofa and its cap shares the same tangent hallway.

Proposition 3.10. *For any moving sofa S with rotation angle $\omega \in [0, \pi/2]$ in standard position, the tangent hallway $L_S(t)$ of S and the tangent hallway $L_K(t)$ of cap $K = \mathcal{C}(S)$ are equal for every $t \in [0, \omega]$.*

Proof. The tangent hallways $L_X(t)$ of $X = S, K$ depend solely on the values of the support function p_X of X on $J := [0, \omega] \cup [\pi/2, \pi/2 + \omega]$ by the equation of $L_X(t)$ in Definition 3.2. The support functions of S and K match on the set J by Lemma 3.9, so the result follows. $\square$

We are now ready to prove that $\mathcal{M}(S)$ is connected.

Proof. (of Theorem 3.7) Define the set $X = \bigcup_{0 \leq t \leq \omega} Q_S^-(t)$. By plugging the equation $L_S(t) = Q_S^+(t) \setminus Q_S^-(t)$ to Definition 3.8, we have $\mathcal{M}(S) = \mathcal{C}(S) \setminus X$. Observe that $\mathcal{C}(S)$ is a convex body containing S. Also observe that $X = \bigcup_{t \in [0, \omega]} Q_S^-(t)$ is closed in the direction of $-u_\theta$.

Fix an arbitrary point p in $\mathcal{M}(S)$. Take the line l_θ passing p in the direction of u_θ for an arbitrary angle $\theta \in [\omega, \pi/2]$. The set $s_\theta = l_\theta \cap \mathcal{M}(S)$ is a nonempty line segment, because s_θ is the line segment $l_\theta \cap \mathcal{C}(S)$ minus the half-line $l_\theta \setminus X$. Now if the line l_θ meets S for any $\theta \in [\omega, \pi/2]$, then p is connected to S inside the line segment s_θ of $\mathcal{M}(S)$ and the proof is done. Our goal now is to prove that there is some $\theta \in [\omega, \pi/2]$ such that l_θ meets S.

Assume by contradiction that for every $\theta \in [\omega, \pi/2]$ the line l_θ is disjoint from S. By Lemma 3.9 the support function of $\mathcal{M}(S)$ agrees with that of S on the set $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$. Because $p \in \mathcal{M}(S)$, the line $l_{\pi/2}$ is either equal to $l_{\mathcal{M}(S)}(0) = l_S(0)$ or strictly on the left side of $l_S(0)$. If $l_{\pi/2} = l_S(0)$ then $l_{\pi/2}$ contains some point of S, contradicting our assumption. So the line $l_{\pi/2}$ is strictly on the left side of $l_S(0)$, and there is a point of S strictly on the right side of $l_{\pi/2}$. Likewise, as $p \in \mathcal{M}(S)$, the line l_ω that passes through p is either equal to $l_{\mathcal{M}(S)}(\omega + \pi/2) = l_S(\omega + \pi/2)$ or strictly on the right side of $l_S(\omega + \pi/2)$. The line l_ω cannot be equal to $l_S(\omega + \pi/2)$ because we assumed that l_ω is disjoint from S. So the line l_ω is strictly on the right side of $l_S(\omega + \pi/2)$, and there is a point of S strictly on the left side of l_ω.

Because the line l_θ is disjoint from S for any $\theta \in [\omega, \pi/2]$, the set S is inside the set $Y = \mathbb{R}^2 \setminus \bigcup_{\theta \in [\omega, \pi/2]} l_\theta$. Note that Y has exactly two connected components Y_L and Y_R on the left and right side of the lines l_θ respectively. As there is a point of S strictly on the right side of $l_{\pi/2}$, the set $S \cap Y_R$ is nonempty. As there is also a point of S strictly on the left side of l_ω, the set $S \cap Y_L$ is also nonempty. We get contradiction as S should be a connected subset of Y. $\square$

3.3 Structure of a monotone sofa

Here, we show that any monotone sofa S is always equal to a *cap* K minus its *niche* $\mathcal{N}(K)$ (Theorem 3.14; see Figure 2). The key idea is in Equation (2), which an eager reader may peek now before reading the details of this section.

Define a *cap* as a convex body satisfying certain properties.

Definition 3.13. For any $\omega \in (0, \pi/2]$, define the set $J_\omega = [0, \omega] \cup [\pi/2, \omega + \pi/2]$.

Definition 3.14. A *cap* K with *rotation angle* $\omega \in (0, \pi/2]$ is a convex body such that the followings hold.

1. $p_K(\omega) = p_K(\pi/2) = 1$ and $p_K(\pi + \omega) = p_K(3\pi/2) = 0$.
2. K is an intersection of closed half-planes with normal angles (Definition 2.7) in $J_\omega \cup \{\pi + \omega, 3\pi/2\}$.

Geometrically, the first condition of Definition 3.14 states that K is contained in the parallelogram P_ω and touches all sides of P_ω. By Theorem A.49, the second condition of Definition 3.14 is equivalent to saying that the *normal angles* $\mathbf{n}(K)$ of K defined in Definition A.13 is contained in the set $J_\omega \cup \{\pi + \omega, 3\pi/2\}$; see Appendix A.5 for a quick overview of $\mathbf{n}(K)$.

We will show that the set $\mathcal{C}(S)$ in Definition 3.10 is a cap with rotation angle ω. This justifies calling $\mathcal{C}(S)$ *the cap of* S associated to S. The proof is technical, so we delegate it at the end of this subsection.

Theorem 3.11. *The set $\mathcal{C}(S)$ in Definition 3.10 is a cap with rotation angle ω as in Definition 3.14. With this, call $\mathcal{C}(S)$ the* cap *of the moving sofa S.*

Define the *niche* $\mathcal{N}(K)$ associated to any cap K.

Definition 3.15. For any angle $\omega \in [0, \pi/2]$, define the *fan* $F_\omega = H(\pi + \omega, 0) \cap H(3\pi/2, 0)$ with angle ω as the convex cone pointed at O bounded by the bottom edges $l(\omega, 0)$ and $l(3\pi/2, 0)$ of the parallelogram P_ω.

Definition 3.16. Let K be any cap with rotation angle $\omega \in [0, \pi/2]$ and angle set $\Theta \subseteq [0, \omega]$. Define the *niche* of K as the following.

$$\mathcal{N}(K) = F_\omega \cap \bigcup_{t \in \Theta} Q_K^-(t)$$

Now we establish the structure of any monotonization of a sofa.

Theorem 3.12. *Let S be a moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. The monotonization $\mathcal{M}(S)$ of S is equal to $K \setminus \mathcal{N}(K)$, where $K = \mathcal{C}(S)$ is the cap of sofa S and $\mathcal{N}(K)$ is the niche of the cap K.*

Proof. Fix an arbitrary moving sofa S with rotation angle $\omega \in (0, \pi/2]$ in a standard position. Let $K = \mathcal{C}(S)$ be the cap of S. By breaking down each $L_S(t)$ into $Q_S^+(t) \setminus Q_S^-(t)$, the monotonization $\mathcal{M}(S)$ of S can be represented as the following subtraction of two sets.

$$\begin{aligned} \mathcal{M}(S) &= P_\omega \cap \bigcap_{0 \le t \le \omega} L_S(t) \\ &= \left(P_\omega \cap \bigcap_{0 \le t \le \omega} Q_S^+(t) \right) \setminus \left(F_\omega \cap \bigcup_{0 \le t \le \omega} Q_S^-(t) \right) \\ &= K \setminus \mathcal{N}(K) \end{aligned} \tag{2}$$

By Proposition 3.10 we have $Q_S^-(t) = Q_K^-(t)$. ☐

Remark 3.1. The equation Equation (2) can understood intuitively as the following (see Figure 2). The cap K is a convex body bounded from below by the boundary of fan F_ω, and bounded from above by the outer walls $a_S(t)$ and $c_S(t)$ of $L_S(t)$. Imagine the set K as a block of clay that rotates inside the hallway L in the clockwise angle of $t \in [0, \omega]$ while always touching the outer walls a and c of L. As K rotates inside L, the inner corner of L carves out the niche N which is the regions bounded by inner walls $b_S(t)$ and $d_S(t)$ of $L_S(t)$ from K. After the full movement of K, the final clay $K \setminus N$ is a moving sofa $\mathcal{M}(S)$.

A moving sofa S and its monotonization $\mathcal{M}(S)$ shares the same cap.

Proposition 3.13. *For any moving sofa S with rotation angle $\omega \in [0, \pi/2]$ in standard position, we have $\mathcal{C}(\mathcal{M}(S)) = \mathcal{C}(S)$.*

Proof. The set $\mathcal{C}(X)$ of $X = S, \mathcal{M}(S)$ depend only on the values of the support function p_X on $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$ by Definition 3.10 and the equation $Q_X^+(t) = H_X(t) \cap H_X(t + \pi/2)$ in Proposition 3.2. The support functions of S and $\mathcal{M}(S)$ match on $[0, \omega] \cup [\pi/2, \pi/2 + \omega]$ by Lemma 3.9, completing the proof. □

We will always use the intrinsic variant Theorem 3.14 of Theorem 3.12 to represent any monotone sofa S as its cap minus niche.

Theorem 3.14. *Let S be any monotone sofa with rotation angle $\omega \in (0, \pi/2]$. Then S is in standard position and $S = K \setminus \mathcal{N}(K)$, where $K := \mathcal{C}(S)$ is the cap of S with rotation angle ω, and $\mathcal{N}(K)$ is the niche of the cap K.*

Proof. Let $S = \mathcal{M}(S')$ be any monotone sofa, so that it is the monotonization of a moving sofa S' in standard position. Then $K := \mathcal{C}(S) = \mathcal{C}(S')$ by Proposition 3.13. Now apply Theorem 3.12 to $\mathcal{M}(S')$ to conclude that $S = \mathcal{M}(S') = \mathcal{C}(S') \setminus \mathcal{N}(\mathcal{C}(S')) = K \setminus \mathcal{N}(K)$. □

Note that this variant does not mention anything about monotonization. In particular, by Theorem 3.14 any monotone sofa S can be recovered from its cap $K = \mathcal{C}(S)$.

Monotization $S \mapsto \mathcal{M}(S)$ is a process that enlarges any moving sofa S by Theorem 3.8. Moreover, if S is already monotone (so that $S = \mathcal{M}(S')$ for some S'), then the monotonization fixes S.

Theorem 3.15. *For any monotone sofa S, we have $\mathcal{M}(S) = S$.*

Proof. Since S is monotone, $S = \mathcal{M}(S')$ for some other moving sofa S'. Now check

$$\mathcal{M}(S) = \mathcal{C}(S) \setminus \mathcal{N}(\mathcal{C}(S)) = \mathcal{C}(S') \setminus \mathcal{N}(\mathcal{C}(S')) = \mathcal{M}(S') = S$$

which holds from Theorem 3.12 and **??**. □

Thus, the monotonization $S \mapsto \mathcal{M}(S)$ can be said as a 'projection' from all moving sofas to monotone sofas, in the sense that $\mathcal{M}$ is a surjective map that fixes monotone sofas.

3.3.1 Proof of Theorem 3.11

If $\omega = \pi/2$, then the set P_ω is the horizontal strip H. If $\omega < \pi/2$, P_ω is a proper parallelogram with the following points as vertices.

Definition 3.17. Let $O = (0, 0)$ be the origin. For any angle $\omega \in (0, \pi/2]$, define the point $o_\omega = (\tan(\omega/2), 1)$.

Note that if $\omega < \pi/2$, then O is the lower-left corner of P_ω and $o_\omega = l(\omega, 1) \cap l(\pi/2, 1)$ is the upper-right corner of P_ω. Define the following subset of P_ω.

Definition 3.18. Let $\omega \in (0, \pi/2]$ be arbitrary. Define M_ω as the convex hull of the points $O, o_\omega, o_\omega - u_\omega, o_\omega - v_0$.

Geometrically, M_ω is a subset of P_ω enclosed by the perpendicular legs from o_ω to the bottom sides $l(\omega, 0)$ and $l(\pi/2, 0)$ of P_ω. We also introduce the following terminology.

Definition 3.19. Say that a point p_1 is *further than* (resp. *strictly further than*) the point p_2 *in the direction* of vector $v \in \mathbb{R}^2$ if $p_1 \cdot v \geq p_2 \cdot v$ (resp. $p_1 \cdot v > p_2 \cdot v$).

We show the following lemma.

Lemma 3.16. *If $\omega < \pi/2$, then the set $\mathcal{C}(S)$ in Definition 3.10 contains M_ω.*

Proof. Since $p_S(\omega) = p_S(\pi/2) = 1$, we can take points q and r of S so that q is on the line $l(\pi/2, 1)$ further than o_ω in the direction of $-u_0$, and r is on the line $l(\omega, 1)$ further than o_ω in the direction of $-v_\omega$. Take an arbitrary $t \in [0, \omega]$. Because $Q_S^+(t)$ is a right-angled convex cone with normal vectors u_t and v_t containing q and r, $Q_S^+(t)$ also contains o_ω. Because $Q_S^+(t)$ contains o_ω and is closed in the direction of $-u_t$ and $-v_t$ (Definition 3.11), $Q_S^+(t)$ contains M_ω as a subset. $\square$

We finish the proof of Theorem 3.11.

Proof. (of Theorem 3.11) Let S be any moving sofa with rotation angle $\omega \in (0, \pi/2]$ in standard position. Let $K = \mathcal{C}(S)$. That $S \subseteq K$ is an immediate consequence of the third condition of Theorem 3.5. We now show that K is a cap with rotation angle ω.

Assume the case $\omega < \pi/2$. Then by Definition 3.10 and Lemma 3.16 we have $M_\omega \subseteq K \subseteq P_\omega$, and the support function of M_ω and P_ω agree on the angles $\omega, \pi/2, \omega + \pi, 3\pi/2$. So the first condition of Definition 3.14 is satisfied. Now assume $\omega = \pi/2$. Since $S \subseteq K \subseteq H$ and S is in standard position we have $p_S(\pi/2) = p_K(\pi/2) = 1$. With $p_K(\pi/2) = 1$, take the point $z \in K$ on the line $y = 1$. Let $X := \bigcap_{t \in [0, \pi/2]} Q_S^+(t)$, then by the definition of K we have $K = H \cap X$. Since X is closed in the direction of $-v_0$, the point $z' = z - (0, 1)$ is also in X. So $z' \in H \cap X = K$ and z' is on the line $y = 0$. This implies that $p_K(3\pi/2) = 0$. So the first condition of Definition 3.14 is true.

The set P_ω is the intersection of four half-planes with normal angles $\omega, \pi/2, \pi + \omega, 3\pi/2$. The set $Q_S^+(t)$ is an intersection of two half-planes with normal angles t and $t + \pi/2$. Now the second condition of Definition 3.14 follows. $\square$

3.4 Cap contains niche

In this subsection, we will establish the following theorem.

Theorem 3.17. *For any monotone sofa S with cap $K = \mathcal{C}(S)$, the cap K contains the niche $\mathcal{N}(K)$.*

Note that $S = K \setminus \mathcal{N}(K)$ by Theorem 3.14. With Theorem 3.17, the area $|S| = |K| - |\mathcal{N}(K)|$ of a monotone sofa can be understood separately in terms of its cap and niche.

Remark 3.2. In spite of Theorem 3.17, a general cap K following Definition 3.14 may not always contain its niche $\mathcal{N}(K)$ in Definition 3.16; this happens if K is too wide. In this case, the cap K is never the cap $\mathcal{C}(S)$ associated to a particular moving sofa S as in Theorem 3.11. Theorem 3.24 identifies the exact condition of K where $\mathcal{N}(K) \subseteq K$.

3.4.1 Geometric definitions on cap and niche

We need a handful of geometric definitions to prove Theorem 3.17. Define the *vertices* of a cap K.

Definition 3.20. Let K be a cap with rotation angle ω. For any $t \in [0, \omega]$, define the vertices $A_K^+(t) = v_K^+(t)$, $A_K^-(t) = v_K^-(t)$, $C_K^+(t) = v_K^+(t + \pi/2)$, and $C_K^-(t) = v_K^-(t + \pi/2)$ of K.

Note that the outer wall $a_K(t)$ (resp. $c_K(t)$) of $L_K(t)$ is in contact with the cap K at the vertices $A_K^+(t)$ and $A_K^-(t)$ (resp. $C_K^+(t)$ and $C_K^-(t)$) respectively. We also define the *upper boundary* of a cap K.

Definition 3.21. For any cap K with rotation angle ω, define the *upper boundary* δK of K as the set $\delta K = \bigcup_{t \in [0, \omega + \pi/2]} e_K(t)$.

For any cap K with rotation angle ω, the upper boundary δK is exactly the points of K making contact with the outer walls $a_K(t)$ and $c_K(t)$ of tangent hallways $L_K(t)$ for every $t \in [0, \omega]$. We collect some observations on δK.

Proposition 3.18. *Let K be a cap with rotation angle ω. The set $K \setminus \delta K$ is the interior of K in the subset topology of F_ω.*

Proof. Since K and F_ω are closed in $\mathbb{R}^2$, the set K is closed in the subset topology of F_ω. Let X be the boundary of K in the subset topology of F_ω, then we have $X \subseteq K$ because K is closed in the subset topology of F_ω. We will show that δK is equal to X, then it follows that the set $K \setminus \delta K$ is the interior of K in the subset topology of F_ω.

We show $\delta K \subseteq X$ and $X \subseteq \delta K$ respectively. Take any point z of δK. Then $z \in e_K(t)$ for some $t \in [0, \omega + \pi/2]$. Since K is a planar convex body, for any $\epsilon > 0$ the point $z' = z + \epsilon u_t$ is not in K. Since the set F_ω is closed in the direction of u_t (Definition 3.11), the point z' is also in F_ω. Thus we have a point z' in the neighborhood of z which is outside K, and δK is a subset of X.

On the other hand, take any point z of X and assume by contradiction that $z \in K \setminus \delta K$. Then for every $t \in [0, \omega + \pi/2]$ we have $z \notin e_K(t)$ so that $z \cdot u_t < p_K(t)$. Since p_K is continuous, the value $p_K(t) - z \cdot u_t$ has a global lower bound $\epsilon > 0$ on the compact interval $[0, \omega + \pi/2]$. So an open ball U of radius ϵ centered at z is contained in the half-space $H_K(t)$ for all $t \in [0, \omega + \pi/2]$. Now $U \cap F_\omega \subseteq K$ and so $z \notin X$, leading to contradiction. ☐

Geometrically, the upper boundary δK is an arc from $A^-_K(0)$ to $C^+_K(\omega)$ in the counterclockwise direction along the boundary ∂K of K. This is rigorously justified by the following corollary of Corollary A.42. For full details, see the introduction of Appendix A.4.

Corollary 3.19. *Let K be a cap with rotation angle ω. The upper boundary δK admits an absolutely-continuous, arc-length parametrization $\mathbf{b}_K^{0-,\pi/2+\omega}$ (Definition A.12) from $A^-_K(0)$ to $C^+_K(\omega)$ in the counter-clockwise direction along ∂K.*

We also give name to the convex polygons $F_\omega \cap Q^-_K(t)$ whose union over all $t \in [0, \omega]$ constitutes the niche $\mathcal{N}(K)$.

Definition 3.22. For any cap K with rotation angle ω, define $T_K(t) = F_\omega \cap Q^-_K(t)$ as the *wedge* of K with angle $t \in [0, \omega]$.

Proposition 3.20. *For any cap K with rotation angle ω, we have $\mathcal{N}(K) = \cup_{t \in [0,\omega]} T_K(t)$.*

Proof. Immediate from Definition 3.16. ☐

We give names to the parts of the wedge $T_K(t)$.

Definition 3.23. For any cap K with rotation angle ω and $t \in (0, \omega)$, define $W_K(t)$ as the intersection of lines $b_K(t)$ and $l(\pi, 0)$. Define $w_K(t) = (A^-_K(0) - W_K(t)) \cdot u_0$ as the signed distance from point $W_K(t)$ and the vertex $A^-_K(0)$ along the line $l(\pi, 0)$ in the direction of u_0.

Likewise, define $Z_K(t)$ as the intersection of lines $d_K(t)$ and $l(\omega, 0)$. Define $z_K(t) = (C^+_K(\omega) - Z_K(t)) \cdot v_\omega$ as the signed length between $Z_K(t)$ and the vertex $C^+_K(\omega)$ along the line $l(\omega, 0)$ in the direction of v_ω.

Note that if $T_K(t)$ contains the origin O, then the points $W_K(t)$ and $Z_K(t)$ are the leftmost and rightmost point of $\overline{T_K(t)}$ respectively.

3.4.2 Controlling the wedge inside cap

To show $\mathcal{N}(K) \subseteq K$ we need to control each wedge $T_K(t)$ inside K. First we show that $w_K(t), z_K(t) \geq 0$; this controls the endpoints $W_K(t)$ and $Z_K(t)$ of $T_K(t)$ inside K.

Lemma 3.21. *Let K be any cap with rotation angle ω. For any angle $t \in (0, \omega)$, we have $w_K(t), z_K(t) \geq 0$.*

Proof. To show that $w_K(t) \geq 0$, we need to show that the point $A^-_K(0)$ is further than the point $W_K(t)$ in the direction of u_0 (see Definition 3.19 for the terminology). The point $q := a_K(t) \cap l(\pi/2, 1)$ is further than $W_K(t) = b_K(t) \cap l(\pi/2, 0)$ in the direction of u_0, because the lines $a_K(t)$ and $b_K(t)$ form the boundary of a unit-width vertical strip rotated counterclockwise by t. The point $A^-_K(t)$ is further than $q = a_K(t) \cap a_K(\pi/2)$ in the direction of u_0 because K is a convex body. Finally, the point $A^-_K(0)$ is further than $A^-_K(t)$ in the direction of u_0 because K is a convex body. Summing up, the points $W_K(t), q, A^-_K(t), A^-_K(0)$ are aligned in the direction of u_0, completing the proof. A symmetric argument in the direction of v_ω proves $z_K(t) \geq 0$. ☐

Corollary 3.22. *Let K be any cap with rotation angle ω. Then $A^-_K(0), C^+_K(\omega) \in K \setminus \mathcal{N}(K)$.*

Proof. We only need to show that $A_K^-(0), C_K^+(\omega)$ are not in $\mathcal{N}(K)$. That is, for any $t \in (0, \omega)$ neither points are in $T_K(t)$. Since $w_K(t) \geq 0$ by Lemma 3.21, the point $A_K^-(0)$ is on the right side of the boundary $b_K(t)$ of $T_K(t)$. So $A_K^-(0) \notin T_K(t)$. Similarly, $z_K(t) \geq 0$ implies $C_K^+(\omega) \notin T_K(t)$. $\square$

We then show that if the corner $\mathbf{x}_K(t)$ is inside K, then the whole wedge $T_K(t)$ is always inside K.

Lemma 3.23. *Fix any cap K with rotation angle $\omega \in [0, \pi/2]$ and an angle $t \in (0, \omega)$. If the inner corner $\mathbf{x}_K(t)$ is in K, then the wedge $T_K(t)$ is a subset of K.*

Proof. Assume that $\mathbf{x}_K(t) \in K$.

If $\omega = \pi/2$, then by $\mathbf{x}_K(t) \in K$, the wedge $T_K(t)$ is the triangle with vertices $W_K(t)$, $\mathbf{x}_K(t)$, and $Z_K(t)$ in counterclockwise order. Note also that $W_K(t)$ is further in the direction of u_0 than $Z_K(t)$. As $w_K(t), z_K(t) \geq 0$, this implies that all three vertices of $T_K(t)$ are in K.

If $\omega < \pi/2$, we divide the proof into four cases on whether the origin O lies strictly below the lines $b_K(t)$ and $d_K(t)$ or not respectively.

- If $(0,0)$ lies on or above both $b_K(t)$ and $d_K(t)$, then we get contradiction as the corner $\mathbf{x}_K(t)$ should be outside the interior F_ω° of fan F_ω, but $\mathbf{x}_K(t) \in K$.
- If $(0,0)$ lies on or above $b_K(t)$ but lies strictly below $d_K(t)$, then $T_K(t)$ is a triangle with vertices $\mathbf{x}_K(t)$, $Z_K(t)$ and the intersection $p = l(\omega, 0) \cap b_K(t)$. In this case, the point p is in the line segment connecting $Z_K(t)$ and $(0,0)$. Also, as $z_K(t) \geq 0$ (Lemma 3.21) the point $Z_K(t)$ lies in the segment connecting $C_K^+(\omega)$ and the origin $(0,0)$. So the points $\mathbf{x}_K(t), Z_K(t), p$ are in K and by convexity of K we have $T \subseteq K$.
- The case where $(0,0)$ lies strictly below $b_K(t)$ but lies on or above $d_K(t)$ can be handed by an argument symmetric to the previous case.
- If $(0,0)$ lies strictly below both $b_K(t)$ and $d_K(t)$, then $T_K(t)$ is a quadrilateral with vertices $\mathbf{x}_K(t)$, $Z_K(t)$, $W_K(t)$ and $(0,0)$. As $w_K(t) \geq 0$ (resp. $z_K(t) \geq 0$) by Lemma 3.21, the point $W_K(t)$ (resp. $Z_K(t)$) is in the line segment connecting $(0,0)$ and $A_K^-(0)$ (resp. $C_K^+(\omega)$). So all the vertices of $T_K(t)$ are in K, and $T_K(t)$ is in K by convexity.

$\square$

3.4.3 Equivalent conditions for $\mathcal{N}(K) \subseteq K$

Now we prove Theorem 3.17. In fact, we identify the exact condition where $\mathcal{N}(K) \subseteq K$ for a general cap K (Definition 3.14) not necessary associated to a moving sofa.

Theorem 3.24. *Let K be any cap with rotation angle ω. Then the followings are all equivalent.*

1. *$\mathcal{N}(K) \subseteq K$*
2. *$\mathcal{N}(K) \subseteq K \setminus \delta K$*
3. *For every $t \in [0, \omega]$, either $\mathbf{x}_K(t) \notin F_\omega^\circ$ or $\mathbf{x}_K(t) \in K$.*
4. *The set $S = K \setminus \mathcal{N}(K)$ is connected.*

Proof. The conditions 1 and 2 are equivalent because the niche $\mathcal{N}(K)$ is open in the subset topology of F_ω by Definition 3.16, and the set $K \setminus \delta K$ is the interior of K in the subset topology of F_ω by Proposition 3.18.

($1 \Rightarrow 3$) We will prove the contraposition and assume $\mathbf{x}_K(t) \in F_\omega^\circ \setminus K$. Then a neighborhood of $\mathbf{x}_K(t)$ is inside F_ω and disjoint from K, so a subset of $T_K(t)$ is outside K, showing $\mathcal{N}(K) \not\subseteq K \setminus \delta K$.

($3 \Rightarrow 1$) If $\mathbf{x}(t) \notin F_\omega^\circ$, then $T_K(t)$ is an empty set. If $\mathbf{x}(t) \in K$, then by Lemma 3.23 we have $T_K(t) \subseteq K$.

($2 \Rightarrow 4$) As δK is disjoint from $\mathcal{N}(K)$, we have $\delta K \subseteq S$. We show that S is connected. First, note that δK is connected by Corollary 3.19. Next, take any point $p \in S$. Take the half-line r starting from p in the upward direction v_0. Then r touches a point in δK as $p \in K$. Moreover, r is disjoint from $\mathcal{N}(K)$ as the set $\mathcal{N}(K) \cup (\mathbb{R}^2 \setminus F_\omega)$ is closed in the direction $-v_0$. Now $r \cap K$ is a line segment inside S that connects the arbitrary point $p \in S$ to a point in δK. So S is connected.

($4 \Rightarrow 3$) Assume by contradiction that $\mathbf{x}(t) \in F_\omega^\circ \setminus K$ for some $t \in [0, \omega]$. Then it should be that $t \neq 0$ or ω. We first show that the vertical line l passing through $\mathbf{x}(t)$ in the direction of v_0 is disjoint from S. The ray with initial point $\mathbf{x}(t)$ and direction v_0 is disjoint from K as the set $F_\omega^\circ \setminus K$ is closed in the direction of v_0. The ray with initial point $\mathbf{x}(t)$ and direction $-v_0$ is not in S because $\mathbf{x}(t)$ is the corner of $Q_K^-(t)$, and $Q_K^-(t)$ is closed in the direction of $-v_0$. So the vertical line l passing through $\mathbf{x}(t)$ does not overlap with S.

Now separate the horizontal strip H into two chunks by the vertical line l passing through $\mathbf{x}(t)$. As S is connected, S should lie either strictly on left or strictly on right of l. As $\mathbf{x}(t)$ lies strictly inside F_ω, the point $W_K(t)$ is strictly further than $\mathbf{x}(t)$ in the direction of u_0, and by Lemma 3.21 the point $A_K^-(0)$ is further than $W_K(t)$ in the direction of u_0. So the endpoint $A_K^-(0)$ of K lies strictly on the right side of l. Similarly, the point $Z_K(t)$ is strictly further than $\mathbf{x}_K(t)$ in the direction of $-u_0$, and by Lemma 3.21 the point $C_K^+(\omega)$ is further than $W_K(t)$ in the direction of $-u_0$. So the endpoint $C_K^+(\omega)$ of K lies strictly on the left side of l. As the endpoints $A_K^-(0)$ and $C_K^+(\omega)$ are in $K \setminus \mathcal{N}(K)$ by Corollary 3.22, and the line l separates the two points, the set $K \setminus \mathcal{N}(K)$ is disconnected. □

We now show that for any monotone sofas, the niche is always inside the cap.

Proof. (of Theorem 3.17) We have $S = K \setminus \mathcal{N}(K)$ by Theorem 3.14. In particular, $K \setminus \mathcal{N}(K)$ is a moving sofa so it is connected. Use that condition 4 implies condition 1 in Theorem 3.24 to complete the proof. □

4 Sofa area functional

We use the findings in Section 3 to reduce the moving sofa problem to the maximization of *sofa area functional* $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$. We first define the domain.

Definition 4.1. Define the *space of caps* $\mathcal{K}_\omega$ with the *rotation angle* $\omega \in (0, \pi/2]$ as the collection of all caps K with rotation angle ω.

Define the *sofa area functional* $\mathcal{A}(K)$ on the space of caps $\mathcal{K}_\omega$ as following.

Definition 4.2. For any angle $\omega \in (0, \pi/2]$, define the *sofa area functional* $\mathcal{A}_\omega : \mathcal{K}_\omega \to \mathbb{R}$ on the space of caps $\mathcal{K}_\omega$ as $\mathcal{A}_\omega(K) = |K| - |\mathcal{N}(K)|$.

The corollary of Theorem 3.17 is then:

Corollary 4.1. *If $K \in \mathcal{K}_\omega$ is the cap $\mathcal{C}(S)$ for a monotone sofa S with rotation angle ω, we have $\mathcal{A}(K) = |S|$.*

But not all $K \in \mathcal{K}_\omega$ is the cap $\mathcal{C}(S)$ of a monotone sofa S as we observed in Remark 3.2.

Definition 4.3. Denote $\mathcal{M}_\omega$ as the subset of $\mathcal{K}_\omega$ consisting of the caps $\mathcal{C}(S)$ (Definition 3.10) of arbitrary monotone sofa S.

$\mathcal{M}_\omega$ is a proper subset of $\mathcal{K}_\omega$ by Remark 3.2. The set of all monotone sofas S with rotation angle ω embeds to the subset $\mathcal{M}_\omega$ of $\mathcal{K}_\omega$ by taking the cap $S \mapsto \mathcal{C}(S)$ (Theorem 3.14). By Theorem 1.3 and Corollary 4.1, the moving sofa problem for a fixed rotation angle $\omega \in (0, \pi/2]$ is now equivalent to the maximization of the sofa area functional $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ on the subspace $\mathcal{M}_\omega$ of $\mathcal{K}_\omega$. We will, however, try to optimize the sofa area functional $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ over the whole $\mathcal{K}_\omega$, not the subspace $\mathcal{M}_\omega$ of $\mathcal{K}_\omega$. This is because the space $\mathcal{K}_\omega$ of all caps is easier to understand than the subspace $\mathcal{M}_\omega$.

By extending the domain of optimization from $\mathcal{M}_\omega$ to $\mathcal{K}_\omega$, we get a counterpart of every statement on maximum-area monotone sofas. To start, we have the following stronger variant of Conjecture 1.1.

Conjecture 4.2. *The cap $K = K_G$ of Gerver's sofa S_G attains the maximum value $\mathcal{A}(K)$ of the sofa area functional $\mathcal{A} : \mathcal{K}_\omega \to \mathbb{R}$ over all $\omega \in (0, \pi/2]$.*

That Conjecture 4.2 implies Conjecture 1.1 is a corollary of Theorem 3.14 and Theorem 3.17. While we don't prove Conjecture 4.2, we expect it to be true.

00. Preface () proves the following strenghtening of Theorem 1.4 along the line of extending the domain from $\mathcal{M}_\omega$ to $\mathcal{K}_\omega$.

Theorem 4.3. *For any cap $K \in \mathcal{K}_\omega$ with rotation angle $\omega \in (0, \pi/2]$, if the rotation path $\mathbf{x}_K : [0, \omega] \to \mathbb{R}^2$ of S is injective, and always on the fan F_ω, then we have $\mathcal{A}(K) \leq 1 + \omega^2/2$.*

Let S be any monotone sofa of rotation angle $\omega \in (0, \pi/2]$ with cap $K = \mathcal{C}(S)$. By Proposition 3.10 we have $\mathbf{x}_S = \mathbf{x}_K$. By this and Theorem 3.14 and Theorem 3.17, we have:

Corollary 4.4. *Theorem 4.3 implies Theorem 1.4.*

We finish this section by mentioning the counterparts of angle and injectivity hypotheses (Conjecture 1.2 and Conjecture 1.5).

Conjecture 4.5. *The supremum of $\mathcal{A}_\omega : \mathcal{K}_\omega \to \mathbb{R}$ maximizes at $\omega = \pi/2$.*

Conjecture 4.6. *There exists a maximizer K of $\mathcal{A}_{\pi/2}$ such that the path $\mathbf{x}_K : [0, \pi/2] \to \mathbb{R}^2$ is injective and always on or above the line $y = 0$.*

Perhaps surprisingly, neither Conjecture 4.5 nor Conjecture 4.6 does not imply the angle nor injectivity hypotheses immediately. This is because a maximizer of $\mathcal{A}_\omega$ is not necessarily a cap of a monotone sofa (Remark 3.2). However, observe that Theorem 4.3 with Conjecture 4.6 implies the upper bound $1 + \pi^2/8$ of sofa area with $\omega = \pi/2$ unconditionally. And by assuming Conjecture 4.5 too, the bound can also be made unconditional with respect to arbitrary ω. Proving Conjecture 4.5 and Conjecture 4.6 will be the main goal of subsequent works.

5 Upper bound A1

We prove the main Theorem 4.3 in this section. We do this by constructing an upper bound $\mathcal{A}_1$ of the sofa area functional $\mathcal{A}$ as described in the introduction.

Section 5.1 defines $\mathcal{A}_1$ and shows that $\mathcal{A}_1$ is indeed an upper bound of $\mathcal{A}$. Section 5.3 establishes that $\mathcal{A}_1$ is a quadratic functional on its domain $\mathcal{K}_\omega$. Section 5.4 proves that $\mathcal{A}_1$ is concave, so that a local optimum of $\mathcal{A}_1$ is also a global optimum of $\mathcal{A}_1$. Section 5.5 calculates the directional derivative of $\mathcal{A}_1$. Finally, Section 5.6 finds the maximizer $K = K_{\omega,1}$ of $\mathcal{A}_1(K)$ by solving for the condition where directional derivative is always zero.

5.1 Definition of A1

We first define the upper bound $\mathcal{A}_1$. Define the *curve area functional* on an arbitrary rectifiable curve $\mathbf{x}$.

Definition 5.1. For two points $(a, b), (c, d) \in \mathbb{R}^2$, denote their cross product as $(a, b) \times (c, d) = ad - bc \in \mathbb{R}$.

Definition 5.2. Let Γ be any curve equipped with a continuous parametrization $\mathbf{x} : [a, b] \to \mathbb{R}^2$ of bounded variation. With $\mathbf{x}(t) = (x(t), y(t))$, define the *curve area functional* $\mathcal{I}(\mathbf{x})$ of Γ as the following.

$$\mathcal{I}(\mathbf{x}) := \frac{1}{2} \int_a^b \mathbf{x}(t) \times d\mathbf{x}(t) := \frac{1}{2} \int_a^b x(t) dy(t) - y(t) dx(t)$$

Also, write $\mathcal{I}(p, q)$ for the area functional of the line segment connecting point p to q, so that we have $\mathcal{I}(p, q) = 1/2 \cdot (p \times q)$.

Note that $\mathbf{x}(t)$ is of bounded variation, so the Lebesgue-Stieltjes measure $d\mathbf{x}(t) = (dx(t), dy(t))$ of the coordinates $x(t)$ and $y(t)$ exists, and the integral in Definition 5.2 is well-defined. Note also that the value of $\mathcal{I}(\mathbf{x})$ does not change even if we take a different parametrization of the curve Γ. By Green's theorem we have the following.

Proposition 5.1. *If $\mathbf{x}$ is a Jordan curve oriented counterclockwise (resp. clockwise), $\mathcal{I}(\mathbf{x})$ is the exact area of the region enclosed by $\mathbf{x}$ (resp. the area with a negative sign).*

If $\mathbf{x}$ is not closed (that is, $\mathbf{x}(a) \neq \mathbf{x}(b)$), the sofa area functional $\mathcal{I}(\mathbf{x})$ measures the signed area of the region bounded by the curve $\mathbf{x}$, and then the line segments connecting $\mathbf{x}(a)$ and $\mathbf{x}(b)$ respectiely. We also have the following additivity of $\mathcal{I}$.

Proposition 5.2. *If γ is the concatenation of two curves α and β then $\mathcal{I}(\gamma) = \mathcal{I}(\alpha) + \mathcal{I}(\beta)$.*

For any $\omega \in (0, \pi/2]$ and cap $K \in \mathcal{K}_\omega$, we want the value of $\mathcal{A}_1(K)$ to be essentially the area of K minus the area of the region enclosed by $\mathbf{x}_K : [0, \omega]$. We want to express the area enclosed by $\mathbf{x}_K$ as $\mathcal{I}(\mathbf{x}_K)$.

Proposition 5.3. *For any $\omega \in (0, \pi/2]$ and cap $K \in \mathcal{K}_\omega$, the inner corner $\mathbf{x}_K : [0, \omega] \to \mathbb{R}$ is Lipschitz.*

Proof. The support function p_K of K is Lipschitz, so

$$\mathbf{x}_K(t) = (p_K(t) - 1)u_t + (p_K(t + \pi/2) - 1)v_t$$

is also Lipschitz. □

Thus $\mathbf{x}_K$ is continuous and of bounded variation, and so the value $\mathcal{I}(\mathbf{x}_K)$ is well-defined. With this, define the functional $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$ as the following.

Definition 5.3. For any angle $\omega \in (0, \pi/2]$ and cap K in $\mathcal{K}_\omega$, define $\mathcal{A}_{1,\omega}(K) = |K| - \mathcal{I}(\mathbf{x}_K)$. If the rotation angle ω is clear from the context, denote $\mathcal{A}_{1,\omega}$ as simply $\mathcal{A}_1$.

We now show that $\mathcal{A}_1(K)$ is an upper bound of the area functional $\mathcal{A}(K)$ if $\mathbf{x}_K$ is injective and in the fan F_ω. Our key observation is the following.

Lemma 5.4. *Let $\omega \in (0, \pi/2]$ and $K \in \mathcal{K}_\omega$ be arbitrary. Let $\mathbf{z} : [t_0, t_1] \to \mathbb{R}^2$ be any open simple curve (that is, a curve with $t_0 < t_1$ and injective parametrization $\mathbf{z}$) inside the set $F_\omega \cap \bigcup_{0 \le t \le \omega} \overline{Q_K^-(t)}$, such that the starting point $\mathbf{z}(t_0)$ is on the boundary $l(\pi/2, 0) \cap F_\omega$ of F_ω, and the endpoint $\mathbf{z}(t_1)$ is on the boundary $l(\omega, 0) \cap F_\omega$ of F_ω. Then we have $\mathcal{I}(\mathbf{z}) \le |\mathcal{N}(K)|$.*

Proof. Define $\mathbf{b}$ as the curve from $\mathbf{z}(t_1)$ to $\mathbf{z}(t_0)$ along the boundary ∂F_ω of fan F_ω (so $\mathbf{b}$ is either a segment or the concatenation of two segments). Since $\mathbf{z}$ is injective, we have $\mathbf{z}(t_0) \neq \mathbf{z}(t_1)$ so $\mathbf{b}$ is also an open simple curve. For every $\epsilon > 0$, define the closed curve Γ_ϵ as the concatenation of the following curves in order: the curve $\mathbf{z}(t)$, the vertical segment from $\mathbf{z}(t_1)$ to $\mathbf{z}(t_1) - (0, \epsilon)$, the curve $\mathbf{b} - (0, \epsilon)$ shifted downwards by ϵ, and then the vertical segment from $\mathbf{z}(t_0) - (0, \epsilon)$ to $\mathbf{z}(t_0)$. The curve Γ_ϵ is a Jordan curve because $\mathbf{z}$ is an open simple curve inside F_ω. By Jordan curve theorem, the curve Γ_ϵ encloses an open set $\mathcal{N}_\epsilon$. Define $\mathcal{N}_0$ as the intersection $F_\omega \cap \mathcal{N}_\epsilon$, then $\mathcal{N}_0$ is independent of the choice of $\epsilon > 0$; for any $\epsilon_1 > \epsilon_2 > 0$, there is a continuous deformation of $\mathbb{R}^2$ that fixes F_ω and shrinks $\mathbb{R}^2 \setminus F_\omega$ vertically so that it shrinks Γ_{ϵ_1} to Γ_{ϵ_2}. Moreover, $\mathcal{N}_\epsilon$ is the disjoint union of $\mathcal{N}_0$ and the fixed region below ∂F_ω of area $|\mathbf{z}(t_1) - \mathbf{z}(t_0)|\, \epsilon$.

We have $|\mathcal{N}_\epsilon| = |\mathcal{I}(\Gamma_\epsilon)|$ by Green's theorem on Γ_ϵ regardless of the orientation of Γ_ϵ. By sending $\epsilon \to 0$, we have $|\mathcal{N}_0| = |\mathcal{I}(\mathbf{z})|$. We now show $\mathcal{N}_0 \subseteq \mathcal{N}(K)$ which finishes the proof. Take any $p \in \mathcal{N}_0$. Take the ray r emanating from p in the direction v_0, then it should cross some point $q \neq p$ in the curve $\mathbf{z}$. As $\mathbf{z}$ is inside the set $F_\omega \cap \bigcup_{0 \le t \le \omega} \overline{Q_K^-(t)}$, the point q is contained in $F_\omega \cap \overline{Q_K^-(t)}$ for some $0 \le t \le \omega$. We have $t \neq 0, \omega$ because q is strictly above the boundary of F_ω, and for $t = 0, \pi/2$ the set $Q_K^-(t)$ is either on or below ∂F_ω. Because the point p is in F_ω and strictly below the point q, it should be that p is contained in $F_{\pi/2} \cap Q_K^-(t)$. So the point p is in the niche $\mathcal{N}(K)$, and we have $\mathcal{N}_0 \subseteq \mathcal{N}(K)$. □

We can freely choose the curve $\mathbf{z}$ inside the set $F_{\pi/2} \cap \bigcup_{0 \le t \le \pi/2} \overline{Q_K^-(t)}$. In this paper, we simply choose $\mathbf{z} = \mathbf{x}_K$ and get the following.

Theorem 5.5. *For any $\omega \in (0, \pi/2]$ and $K \in \mathcal{K}_\omega$, if the curve $\mathbf{x}_K : [0, \omega] \to \mathbb{R}^2$ is injective and in F_ω, we have $\mathcal{A}(K) \le \mathcal{A}_1(K)$.*

Proof. Since $\mathbf{x}_K(t) \in \overline{Q_K^-(t)}$ for all $t \in [0, \omega]$ and we assumed that $\mathbf{x}_K(t) \in F_\omega$, the curve $z := \mathbf{x}_K$ is an open simple curve inside $F_\omega \cap \bigcup_{0 \le t \le \omega} \overline{Q_K^-(t)}$. Also, by $p_K(\omega) = p_K(\pi/2) = 1$ we have $\mathbf{x}_K(0) \in l(\pi/2, 0)$ and $\mathbf{x}_K(\omega) \in l(\omega, 0)$. So the curve $\mathbf{z} := \mathbf{x}_K$ satisfies the condition of Lemma 5.4, and we have $\mathcal{I}(\mathbf{x}_K) \le |\mathcal{N}(K)|$. So we have

$$\mathcal{A}(K) = |K| - |\mathcal{N}(K)| \le |K| - \mathcal{I}(\mathbf{x}_K) = \mathcal{A}_1(K)$$

which finishes the proof. □

5.1.1 Derivative of the inner corner

The formula (Definition 5.2) of $\mathcal{I}(\mathbf{x}_K)$ requires us to take derivative of the inner corner $\mathbf{x}_K$. We end this subsection by calculating the derivative of $\mathbf{x}_K$ explicitly, which will be used later. Define the *arm lengths* $g^{\pm}_K(t)$ and $h^{\pm}_K(t)$ of tangent hallways of a cap K as the following.

Definition 5.4. Let $K \in \mathcal{K}_\omega$ be arbitrary. For any $t \in [0, \omega]$, let $g^+_K(t)$ (resp. $g^-_K(t)$) be the unique real value such that $\mathbf{y}_K(t) = A^+_K(t) + g^+_K(t)v_t$ (resp. $\mathbf{y}(t) = A^-_K(t) + g^-_K(t)v_t$). Similarly, let $h^+_K(t)$ (resp. $h^-_K(t)$) be the unique real value such that $\mathbf{y}_K(t) = C^+_K(t) + h^+_K(t)u_t$ (resp. $\mathbf{y}(t) = C^-_K(t) + h^-_K(t)u_t$).

Observe that the point $\mathbf{y}_K(t)$ and the vertices $C^{\pm}_K(t)$ are on the tangent line $c_K(t)$ in the direction of u_t. Likewise $y_K(t)$ and $A^{\pm}_K(t)$ are on the tangent line $a_K(t)$ in the direction of v_t. So Definition 5.4 is well-defined. For the same reason, we also have the following equations.

Proposition 5.6. *Let $K \in \mathcal{K}_\omega$ and $t \in [0, \omega]$ be arbitrary. Then $g^{\pm}_K(t) = \left(C^{\pm}_K(t) - A^{\pm}_K(t)\right) \cdot v_t$ and $h^{\pm}_K(t) = (A^{\pm}_K(t) - C^{\pm}_K(t)) \cdot u_t$.*

The derivative of $\mathbf{x}_K$ can be expressed in terms of the arm lengths of K.

Theorem 5.7. *For any cap $K \in \mathcal{K}_\omega$, the right derivative of the outer and inner corner $\mathbf{y}_K(t)$ exists for any $0 \leq t < \omega$ and is equal to the following.*

$$\partial^+\mathbf{y}_K(t) = -g^+_K(t)u_t + h^+_K(t)v_t \qquad \partial^+\mathbf{x}_K(t) = -(g^+_K(t) - 1)u_t + (h^+_K(t) - 1)v_t$$

Likewise, the left derivative of $\mathbf{y}_K$ and $\mathbf{x}_K$ exists for all $0 < t \leq \omega$ and is equal to the following.

$$\partial^-\mathbf{y}_K(t) = -g^-_K(t)u_t + h^-_K(t)v_t \qquad \partial^+\mathbf{x}_K(t) = -(g^-_K(t) - 1)u_t + (h^-_K(t) - 1)v_t$$

Proof. Fix an arbitrary cap K and omit the subscript K in vertices $\mathbf{y}_K(t)$, $\mathbf{x}_K(t)$ and tangent lines $a_K(t)$. Take any $0 \leq t < \omega$ and set $s = t + \delta$ for sufficiently small and arbitrary $\delta > 0$. We evaluate $\partial^+\mathbf{y}(t) = \lim_{\delta \to 0^+}(\mathbf{y}(s) - \mathbf{y}(t))/\delta$. Define $A_{t,s} = a(t) \cap a(s)$. Since $A_{t,s}$ is on the lines $a(t)$ and $a(s)$, it satisfies both $A_{t,s} \cdot u_t = \mathbf{y}(t) \cdot u_t$ and $A_{t,s} \cdot u_s = \mathbf{y}(s) \cdot u_s$. Rewrite $u_s = \cos\delta \cdot u_t + \sin\delta \cdot v_t$ on the second equation and we have

$$A_{t,s}\cos\delta \cdot u_t + A_{t,s}\sin\delta \cdot v_t = \cos\delta(\mathbf{y}(s) \cdot u_t) + \sin\delta(\mathbf{y}(s) \cdot v_t).$$

Group by $\cos\delta$ and $\sin\delta$ and substitute $A_{t,s} \cdot u_t$ with $\mathbf{y}(t) \cdot u_t$, then

$$\cos\delta(\mathbf{y}(s) \cdot u_t - \mathbf{y}(t) \cdot u_t) = \sin\delta(A_{t,s}(s) \cdot v_t - \mathbf{y}(s) \cdot v_t).$$

Divide by δ and send $\delta \to 0^+$. We get the following limit as $A_{t,s} \to A^+(t)$ (Theorem A.4).

$$\partial^+(\mathbf{y}(t) \cdot u_t) = (A^+(t) - \mathbf{y}(t)) \cdot v_t = -g^+(t)$$

A similar argument can be applied to show $\partial^+(\mathbf{y}(t) \cdot v_t) = h^+(t)$ and thus the first equation of the theorem. The right derivative of $\mathbf{x}_K(t)$ comes from $\mathbf{x}_K(t) = \mathbf{y}_K(t) - u_t - v_t$. A symmetric argument calculates the left derivative of $\mathbf{y}_K$ and $\mathbf{x}_K$. $\square$

We prepare some observations on arm lengths that will be used later.

Remark 5.1. The resulting equation $\partial^+\mathbf{x}_K(t) = -(g^+_K(t) - 1)u_t + (h^+_K(t) - 1)v_t$ in Theorem 5.7 can be interpreted intuitively as the following. Imagine the hallway $L_K(t)$ where we increment t slightly by $\epsilon > 0$. If ϵ is very small, the wall $c_K(t)$ rotates around the pivot $C^+_K(t)$. As $\mathbf{x}_K(t)$ rotates differentially with the pivot $C^+_K(t)$ as center, the v_t component $h^+_K(t) - 1$ of the derivative $\partial^+\mathbf{x}_K(t)$ is the distance from the pivot $C^+_K(t)$ to $\mathbf{x}_K(t)$ measured in the direction of u_t. The u_t component $-(g^+_K(t) - 1)$ of $\partial^+\mathbf{x}_K(t)$ can be interpreted similarly as the distance from pivot $A^+_K(t)$ to $\mathbf{x}_K(t)$ along the direction v_t.

Except for a countable number of t, we don't need to differentiate $g_K^+(t)$ and $g_K^-(t)$ (and likewise for $h_K^\pm(t)$).

Definition 5.5. For any cap K of rotation angle $\omega \in (0, \pi/2]$ and any angle $t \in [0, \omega]$, if $g_K^+(t) = g_K^-(t)$ (resp. $h_K^+(t) = h_K^-(t)$) then simply denote the matching value as $g_K(t)$ (resp. $h_K(t)$).

Proposition 5.8. *Let K be any cap of rotation angle $\omega \in (0, \pi/2]$ and take any angle $t \in [0, \omega]$. The condition $g_K^+(t) = g_K^-(t)$ (resp. $h_K^+(t) = h_K^-(t)$) holds if and only if $\beta_K(\{t\}) = 0$ (resp. $\beta_K(\{t + \pi/2\}) = 0$) by Theorem A.21. So g_K and h_K are almost everywhere defined and integrable functions on $[0, \omega]$.*

By Proposition 5.6 and Theorem A.4 we have the following.

Corollary 5.9. *Let K be any cap of rotation angle $\omega \in (0, \pi/2]$ and take any angle $t \in [0, \omega]$. If $t > 0$, then $g_K^\pm(u) \to g_K^-(t)$ and $h_K^\pm(u) \to h_K^-(t)$ as $u \to t^-$. If $t < \omega$, then $g_K^\pm(u) \to g_K^+(t)$ and $h_K^\pm(u) \to h_K^+(t)$ as $u \to t^+$.*

5.2 Calculus of variation

Observe that the space of caps $\mathcal{K}_\omega$ is a *convex space* (see [10]) equipped with the barycentric combination operation $c_\lambda(-,-)$.

Definition 5.6. A *convex space* is a set $\mathcal{K}$ equipped with the *convex combination* operation $c_\lambda : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ for all $\lambda \in [0, 1]$, such that the followings hold.

1. $c_0(x, y) = x$
2. $c_\lambda(x, x) = x$ for all $\lambda \in [0, 1]$
3. $c_\lambda(x, y) = c_{1-\lambda}(y, x)$ for all $\lambda \in [0, 1]$
4. $c_\lambda(x, c_\mu(y, z)) = c_{\lambda\mu}(c_\tau(x, y), z)$ for all $\lambda, \mu, \tau \in [0, 1]$ such that $\lambda(1 - \mu) = (1 - \lambda\mu)\tau$.

Proposition 5.10. *The space of caps $\mathcal{K}_\Theta$ in Definition 4.1 is a convex space, equipped with the convex combination*

$$
\begin{aligned}
c_\lambda(K_1, K_2) &:= (1 - \lambda)K_1 + \lambda K_2 \\
&= \{(1 - \lambda)p_1 + \lambda p_2 : p_1 \in K_1, p_2 \in K_2\}
\end{aligned}
$$

given by the Minkowski sum of convex bodies.

Proof. The axiomatic equations in Definition 5.6 can be checked rudimentarily. □

We will prove Theorem 4.3 by optimizing $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$. Here, we prepare a minimal amount of language needed to execute the calculus of variation to a concave quadratic functional on a general convex space $\mathcal{K}$. Later, we will show that $\mathcal{A}_1$ is concave and quadratic on the cap space $\mathcal{K} = \mathcal{K}_\omega$, and then deploy the calculus of variation on $\mathcal{A}_1$.

Definition 5.7. A function $f : \mathcal{K} \to V$ from a convex space $\mathcal{K}$ to a convex space V is *convex-linear* if $f(c_\lambda(K_1, K_2)) = c_\lambda(f(K_1), f(K_2))$ for all $K_1, K_2 \in \mathcal{K}$ and $\lambda \in [0, 1]$. Call a functional $f : \mathcal{K} \to \mathbb{R}$ on $\mathcal{K}$ a *linear functional* on $\mathcal{K}$ if it is convex-linear.

Definition 5.8. For convex spaces $\mathcal{K}$ and V, call a function $g : \mathcal{K} \times \mathcal{K} \to V$ *convex-bilinear* if the maps $K \mapsto g(K_1, K)$ and $K \mapsto g(K, K_2)$ are convex-linear for any fixed $K_1, K_2 \in \mathcal{K}$.

Definition 5.9. Call $h : \mathcal{K} \to \mathbb{R}$ a *quadratic functional* on a convex space $\mathcal{K}$, if $h(K) = g(K, K)$ for some convex-bilinear function $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$.

Definition 5.10. For any quadratic functional $f : \mathcal{K} \to \mathbb{R}$ on a convex space $\mathcal{K}$ and $K, K' \in \mathcal{K}$, define:

$$
Df(K; K') = \left.\frac{d}{d\lambda}\right|_{\lambda=0} f(c_\lambda(K, K'))
$$

Definition 5.10 can be seen as a generalization of the Gateaux derivative to functionals on a general convex space. For any quadratic functional f and a fixed $K \in \mathcal{K}$, the value $Df(K; K')$ is well-defined and always a linear functional of K'.

Lemma 5.11. *Let f be a quadratic functional on a convex space $\mathcal{K}$, so that $f(K) = h(K, K)$ for a convex-bilinear map $h : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$. Then we have the following for any $K, K' \in \mathcal{K}$.*

$$Df(K; K') = h(K, K') + h(K', K) - 2h(K, K)$$

So in particular, the map $Df(K; -) : \mathcal{K} \to \mathbb{R}$ is always well-defined and a linear functional.

Proof. We have the following computation by bilinearlity of h.

$$\begin{aligned} f(c_\lambda(K, K')) &= h(c_\lambda(K, K'), c_\lambda(K, K')) \\ &= (1-\lambda)^2 h(K, K) + \lambda(1-\lambda)\left(h(K, K') + h(K', K)\right) + \lambda^2 h(K', K') \end{aligned} \tag{3}$$

Now take derivative at $\lambda = 0$. □

Definition 5.11. A functional $f : \mathcal{K} \to \mathbb{R}$ on a convex space $\mathcal{K}$ is *concave* (resp. *convex*) if $f(c_\lambda(K_1, K_2)) \geq (1-\lambda)f(K_1) + \lambda f(K_2)$ (resp. $f(c_\lambda(K_1, K_2)) \leq (1-\lambda)f(K_1) + \lambda f(K_2)$) for all $K_1, K_2 \in \mathcal{K}$ and $\lambda \in [0, 1]$.

To prove that K maximizes a concave quadratic functional $f(K)$ on $\mathcal{K}$, we only need to prove that $Df(K; -)$ is a nonpositive linear functional on $\mathcal{K}$.

Theorem 5.12. *For any concave quadratic functional f on a convex space $\mathcal{K}$, the value $K \in \mathcal{K}$ maximizes $f(K)$ if and only if the linear functional $Df(K; -)$ is nonpositive.*

Proof. Assume that K is the maximizer of $f(K)$. Then for any $K' \in \mathcal{K}$, the value $f(c_\lambda(K, K'))$ over all $\lambda \in [0, 1]$ is maximized at $\lambda = 0$. So taken derivative at $\lambda = 0$, we should have $Df(K; K') \leq 0$.

Now assume on the other hand that $K \in \mathcal{K}$ is chosen such that $Df(K; -)$ is always nonpositive. Take an arbitrary $K' \in \mathcal{K}$ and fix it. Observe that $f(c_\lambda(K, K'))$ is a polynomial $p(\lambda)$ of $\lambda \in [0, 1]$ by Equation (3). Because f is concave, the polynomial $p(\lambda)$ is also concave with respect to λ and the quadratic coefficient of $p(\lambda)$ is nonpositive. The linear coefficient of $p(\lambda)$ is $Df(K; K')$ which is nonpositive as well. So $p(\lambda)$ is monotonically decreasing with respect to λ and we have $f(K) \geq f(K')$ as desired. □

5.3 Boundary measure

We will observe that $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$ is a quadratic functional.

Theorem 5.13. *For any $\omega \in (0, \pi/2]$, the functional $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$ is quadratic.*

To establish Theorem 5.13, we will define the *boundary measure* β_K of $K \in \mathcal{K}_\omega$ and utilize it. Also, we will establish a correspondence between any cap $K \in \mathcal{K}_\omega$ and its boundary measure β_K (Theorem 5.19 and Theorem 5.20).

5.3.1 Convex-linear values of cap

We observe that a lot of values defined on the cap $K \in \mathcal{K}_\omega$ is convex-linear with respect to K. A reader interested in the details of proofs can read Appendix A.1 for the full details.

Theorem 5.14. *The following values are convex-linear with respect to $K \in \mathcal{K}_\omega$.*

- *Support function p_K*
- *Vertices $A^\pm_K(t)$ and $C^\pm_K(t)$ for a fixed $t \in [0, \omega]$*
- *The inner and outer corner $\mathbf{x}_K(t)$ and $\mathbf{y}_K(t)$ of the tangent hallway with any angle $t \in [0, \omega]$*
- *The points $W_K(t)$, $Z_K(t)$ and the values $w_K(t)$, $z_K(t)$ for a fixed $t \in (0, \omega)$*
- *The perpendicular leg lengths $g^\pm_K(t)$ and $h^\pm_K(t)$ for all $t \in [0, \omega]$*

Proof. Use Proposition A.2 for p_K, Corollary A.6 for $A_K^{\pm}(t)$ and $C_K^{\pm}(t)$, Lemma A.3 for $\mathbf{y}_K(t), W_K(t)$, and $Z_K(t)$. Use the equality $\mathbf{y}_K(t) = \mathbf{x}_K(t) + u_t + v_t$ for $\mathbf{x}_K(t)$, the equalities in Definition 3.23 for $w_K(t)$ and $z_K(t)$, and the equalities in Definition 5.4 for $g_K^{\pm}(t)$ and $h_K^{\pm}(t)$. ☐

Theorem 5.14 in particular establishes that the curve area functional $\mathcal{I}(\mathbf{x}_K)$ (Definition 5.2) is quadratic with respect to K.

Corollary 5.15. *The value $\mathcal{I}(\mathbf{x}_K)$ of a cap $K \in \mathcal{K}_\omega$ is quadratic with respect to K.*

5.3.2 Boundary Measure

We now define the *boundary measure* β_K of a cap K as the restriction of the *surface measure* σ_K of K (Definition A.3).

Definition 5.12. For any cap $K \in \mathcal{K}_\omega$ with rotation angle ω, define the *boundary measure* β_K of K as the surface area measure σ_K of K restricted to the set J_ω.

See Appendix A.3 for a thorough description on the surface area measure σ_K. The boundary measure β_K of cap K describes the information of length of the upper boundary δK. For example, let K be a cap made by attaching two quarter-circles of radius 1 at the vertical side of a box with width $4/\pi$ and height 1 (see). Then the boundary measure β_K is a measure on $J_{\pi/2} = [0, \pi]$ such that $\beta_K(\{\pi/2\}) = 1$ and β_K is equal to the Lebesgue measure ($\beta_K(dt) = dt$) on $[0, \pi] \setminus \{\pi/2\}$.

We collect properties of β_K.

Proposition 5.16. *The boundary measure β_K is convex-linear with respect to $K \in \mathcal{K}_\omega$.*

Proof. Immediate from Theorem A.17. ☐

Proposition 5.17. *For any cap $K \in \mathcal{K}_\omega$, we have*

$$|K| = \langle p_K, \beta_K \rangle_{J_\omega}.$$

Proof. By Theorem A.18 we have $|K| = \langle p_K, \sigma_K \rangle_{S^1}$. Apply Theorem A.49 to the second condition of Definition 3.14 to obtain that σ_K is supported on the set $J_\omega \cup \{\omega + \pi, 3\pi/2\}$. The first condition of Definition 3.14 gives $p_K(\omega + \pi) = p_K(3\pi/2) = 0$. From these, we have $|K| = \langle p_K, \sigma_K \rangle_{S^1} = \langle p_K, \beta_K \rangle_{J_\omega}$. ☐

Now the quadraticity of $|K|$ comes from convex-linearity of p_K (Proposition A.2) and β_K (Proposition 5.16) with respect to K.

Corollary 5.18. *The area $|K|$ of a cap $K \in \mathcal{K}_\omega$ is a quadratic functional on $\mathcal{K}_\omega$*

The quadraticity of $\mathcal{A}_1$ is now obtained.

Proof. (of Theorem 5.13) Immediate consequence of Corollary 5.15 and Corollary 5.18. ☐

Gauss-Minkowski theorem (Theorem A.50) states that any convex set K, up to translation, corresponds one-to-one to a measure σ on S^1 such that $\int_{S^1} u_t \, \sigma(dt) = 0$ by taking the surface area measure $\sigma = \sigma_K$. Using this correspondence, we can always construct a bijection between a cap $K \in \mathcal{K}_\omega$ and its boundary measure $\beta = \beta_K$.

Theorem 5.19. *For any cap $K \in \mathcal{K}_\omega$ with rotation angle ω, its boundary measure β_K satisfies the following equations.*

$$\int_{t \in [0,\omega]} \cos(t) \, \beta_K(dt) = 1 \qquad \int_{t \in [\pi/2, \omega+\pi/2]} \cos(\omega + \pi/2 - t) \, \beta_K(dt) = 1$$

Proof. By the second condition of Definition 3.14 and Theorem A.49, we have $\mathbf{n}(K) \subseteq J_\omega \cup \{\pi+\omega, 3\pi/2\}$. Now by Theorem A.48, that the interval $(-\pi/2, 0)$ of S^1 is disjoint from Π implies that the point $A_K^-(0)$ is on the line $l_K(3\pi/2)$ which is $y = 0$. Likewise, that the interval $(\omega, \pi/2)$ of S^1 is disjoint from Π implies that the point $A_K^+(\omega)$ is on the line $l_K(\pi/2)$ which is $y = 1$. By Corollary A.22 we have

$$\int_{t\in[0,\omega]} v_t \, \beta_K(dt) = A_K^+(\omega) - A_K^-(0)$$

and by taking the dot product with v_0, we have the first equality. The second equality can be proved similarly by measuring the displacement vector from $C_K^+(\omega)$ to $C_K^-(0)$ along the vector u_ω. □

Theorem 5.20. *Take arbitrary $\omega \in (0, \pi/2]$. Conversely to Theorem 5.19, let β be a measure on J_ω that satisfies the following equations.*

$$\int_{t\in[0,\omega]} \cos(t) \, \beta(dt) = 1 \qquad \int_{t\in[\pi/2,\omega+\pi/2]} \cos(\omega + \pi/2 - t) \, \beta(dt) = 1$$

Then there exists a cap $K \in \mathcal{K}_\omega$ such that $\beta_K = \beta$. Such K is unique if $\omega < \pi/2$, and unique up to horizontal translation if $\omega = \pi/2$.

Proof. We first show that there is a unique extension σ of β on the set $\Pi = J_\omega \cup \{\pi+\omega, 3\pi/2\}$ such that $\int_{t\in\Pi} v_t \, \sigma(dt) = 0$. The values of σ are determined on J_ω, and we need to find the values of $\sigma(\{\pi+\omega\})$ and $\sigma(\{3\pi/2\})$ that satisfies the equation $\int_{t\in\Pi} v_t \, \sigma(dt) = 0$.

If $\omega = \pi/2$, then by subtracting the two equations in the statement we have $\int_{t\in[0,\pi]} \cos(t) \, \beta(dt) = 0$. So the equation $\int_{t\in\Pi} v_t \, \sigma(dt) = 0$ becomes $\sigma(\{3\pi/2\}) = \int_{t\in[0,\pi]} \sin(t) \, \beta(dt)$ which immediately gives a unique solution σ.

Now assume $\omega < \pi/2$. Let $A := \int_{t\in[0,\omega]} \sin(t) \, \beta(dt) \geq 0$, then we have $\int_{t\in[0,\omega]} v_t \, \beta(dt) = -Au_0 + v_0$ by the first equality of the theorem statement. Likewise, if we let $B := \int_{t\in[\pi/2,\omega+\pi/2]} \sin(\omega + \pi/2 - t) \, \beta(dt) \geq 0$, then we have $\int_{t\in[\pi/2,\omega+\pi/2]} v_t \, \beta(dt) = Bv_\omega - u_\omega$ by the second equality of the theorem statement. Now the equation $\int_{t\in\Pi} v_t \, \sigma(dt) = 0$ we are solving for becomes

$$(-Au_0 + v_0) + (Bv_\omega - u_\omega) + \sigma(\{3\pi/2\}) \, u_0 - \sigma(\{\pi+\omega\}) \, v_\omega = 0$$

and $\sigma(\{\pi+\omega\}) = B + v_\omega \cdot p_\omega \geq 0$ and $\sigma(\{3\pi/2\}) = A + u_0 \cdot p_\omega \geq 0$ gives the unique solution of σ.

We now use Corollary A.51 on the measure σ extended on the set Π. There is a unique convex body K up to translation so that $\mathbf{n}(K) \subseteq \Pi$ (see Definition A.13) and $\sigma_K|_\Pi = \sigma$. Our goal is to translate K so that it is a cap with rotation angle ω. Since $\mathbf{n}(K) \subseteq \Pi$, the convex body K satisfies the second condition of cap in Definition 3.14. It remains to prove the first condition of Definition 3.14.

The width of K along the directions u_ω and v_0 are equal to 1, by applying the equations given in the theorem statement to Corollary A.24 with angles $t = \omega, \pi/2$. Now we translate K. If $\omega = \pi/2$, we only need $\beta_K(\pi/2) = 1$ for K to be a cap in $\mathcal{K}_{\pi/2}$, so such a cap K is unique up to horizontal translation. If $\omega < \pi/2$, we need both $\beta_K(\pi/2) = \beta_K(\omega) = 1$ so such a cap K exists uniquely among translations. □

5.4 Concavity of A1

We now show that $\mathcal{A}_1$ is concave. We use the following criterion to show the concavity of $\mathcal{A}_1$.

Lemma 5.21. *Let f be a quadratic functional on a convex space $\mathcal{K}$ with convex combination $c_\lambda(-,-)$ for all $\lambda \in [0,1]$. If there is a linear functional $g : \mathcal{K} \to \mathbb{R}$ and a convex-linear map $h : \mathcal{K} \to V$ to a real vector space V with inner product $\langle -,- \rangle_V$ such that $f(K) = g(K) - \langle h(K), h(K) \rangle_V$ for every $K \in \mathcal{K}$, then f is concave.*

Proof. Take arbitrary $K_1, K_2 \in \mathcal{K}$. Fixing K_1 and K_2, observe that $f(c_\lambda(K_1, K_2))$ is a quadratic polynomial of $\lambda \in [0,1]$ with the leading coefficient $-\|h(K_2) - h(K_1)\|_V^2$ of λ^2 by expanding the term $h(c_\lambda(K_1, K_2)) = h(K_1) + \lambda(h(K_2) - h(K_1))$ with respect to the inner product $\langle -,- \rangle_V$. This shows the concavity of f along the line segment connecting K_1 and K_2. Since K_1 and K_2 are chosen arbitrarily, this proves the concavity of f. □

We will use *Mamikon's theorem* [9] to express $\mathcal{A}_1$ in negative sum-of-squares. The theorem calculates the area of the region swept by tangent segments of a convex body. For the precise statement of Mamikon's theorem on a general planar convex body K, we refer to Theorem A.52 and Theorem A.53.

Theorem 5.22. *For every $\omega \in (0, \pi/2]$, the functional $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$ is concave.*

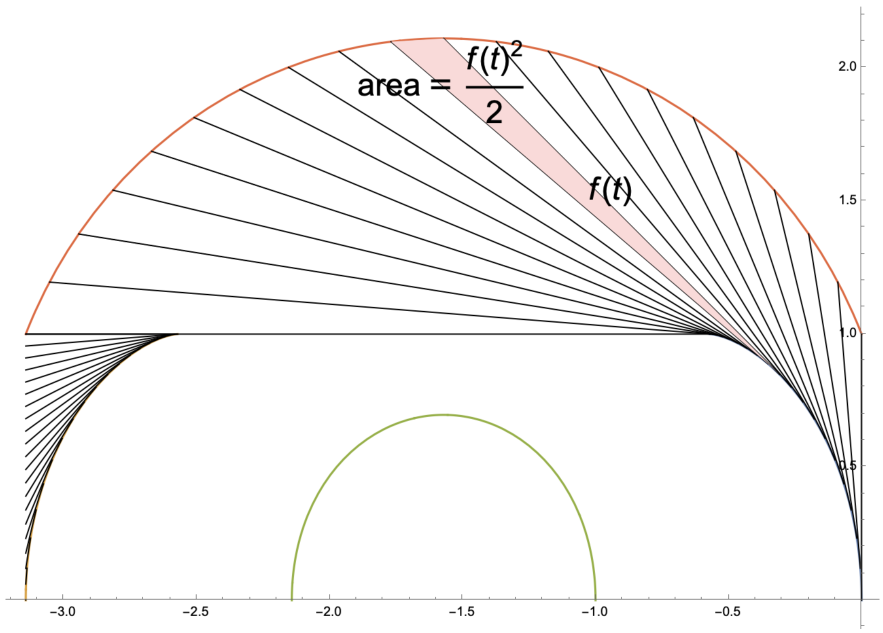

Figure 6: Expressing $\mathcal{A}_1$ as sum-of-squares via Mamikon's theorem.

Proof. We first express $|K|$ as a sum of curve area functionals. Note first that $|K| = \mathcal{I}(\partial K)$ where $\partial K = \mathbf{b}_K^{0,2\pi}$ is the boundary of K parametrized in the counterclockwise direction (Theorem A.29). Next, observe that the boundary ∂K is the concatenation of the upper boundary $\delta K = \mathbf{b}_K^{0-,\omega+\pi/2}$ (Corollary 3.19) and two line segments l_1 and l_2 each from $C_K^+(\omega)$ to O, and O to $A_K^-(\omega)$ respectively. So $\mathcal{I}(l_1) = \mathcal{I}(l_2) = 0$ and we have $|K| = \mathcal{I}(\delta K)$.

Next, define $\mathbf{b}_1$ as the path $\mathbf{b}_K^{0-,\omega}$ of Definition A.12. That is, $\mathbf{b}_1$ is the path from $A_K^-(0)$ to $A_K^+(\omega)$ along the upper boundary δK. Define $\mathbf{b}_2$ as the path $\mathbf{b}_K^{\omega,\omega+\pi/2}$ of Theorem A.25. That is, $\mathbf{b}_2$ is the path from $A_K^+(\omega)$ to $C_K^+(\omega)$ along the upper boundary δK. Then by Corollary A.44, the upper boundary δK is the concatenation of $\mathbf{b}_1$ and $\mathbf{b}_2$. So we have $|K| = \mathcal{I}(\delta K) = \mathcal{I}(\mathbf{b}_1) + \mathcal{I}(\mathbf{b}_2)$.

We now stitch the following instances of Mamikon's theorem. First, by applying Theorem A.53 to the curve $\mathbf{b}_1$ and the outer corner $\mathbf{y}_K(t)$ for $t \in [0, \omega]$, we have the following.

$$\begin{aligned}\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{b}_1) &= \frac{1}{2}\int_0^\omega h_K^+(t)^2\,dt - \mathcal{I}(\mathbf{y}_K(\omega), A_K^+(\omega)) + \mathcal{I}(\mathbf{y}_K(0), A_K^-(0)) \\ &= \frac{1}{2}\int_0^\omega h_K^+(t)^2\,dt - \mathcal{I}(\mathbf{y}_K(\omega), A_K^+(\omega)) - \frac{1}{2}p_K(0)\end{aligned}$$

For the second equation, we use the fact that the points $O, \mathbf{y}_K(0), A_K^-(0)$ form the vertices of a right-angled triangle of height 1 along the direction v_0 with base $p_K(0)$.

Second, by applying Theorem A.55 to the curve $\mathbf{b}_2$ we get the following. Note that the value $\tau_K(t, \omega+\pi/2)$ is defined in Definition A.14 and is convex-linear with respect to K by Corollary A.54.

$$
\begin{aligned}
-\mathcal{I}(\mathbf{b}_2) &= \frac{1}{2}\int_{\omega}^{\pi/2+\omega} \tau_K(t,\omega+\pi/2)^2\,dt - \mathcal{I}(\mathbf{y}_K(\omega), C_K^+(\omega)) + \mathcal{I}(\mathbf{y}_K(\omega), A_K^+(\omega)) \\
&= \frac{1}{2}\int_{\omega}^{\pi/2+\omega} \tau_K(t,\omega+\pi/2)^2\,dt - \frac{1}{2}p_K(\omega+\pi/2) + \mathcal{I}(\mathbf{y}_K(\omega), A_K^+(\omega))
\end{aligned}
$$

For the second equation, we use the fact that the points $O, \mathbf{y}_K(\omega), C_K^+(\omega)$ form the vertices of a right-angled triangle of height along the direction u_ω with base $p_K(\omega+\pi/2)$.

Add the two equations together and we get the following.

$$
\begin{aligned}
\mathcal{I}(\mathbf{y}_K) - |K| = \frac{1}{2}\int_0^{\omega} h_K^+(t)^2\,dt + \frac{1}{2}\int_{\omega}^{\pi/2+\omega} \tau_K(t,\omega+\pi/2)^2\,dt \\
- \frac{1}{2}\left(p_K(0) + p_K(\omega+\pi/2)\right)
\end{aligned}
\tag{4}
$$

Note that $\mathbf{y}_K(t) - \mathbf{x}_K(t) = u_t + v_t$ is constant with respect to K. So the expression $\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{x}_K)$ is convex-linear with respect to K. For more details, write $u_t + v_t$ as c_t, then we have

$$
\begin{aligned}
\mathcal{I}(\mathbf{y}_K) &= \frac{1}{2}\int_0^{\omega} \mathbf{y}_K(t) \times d\mathbf{y}_K(t) \\
&= \frac{1}{2}\int_0^{\omega} (\mathbf{x}_K(t) + c_t) \times d(\mathbf{x}_K(t) + c_t) \\
&= \mathcal{I}(\mathbf{x}_K) + \frac{1}{2}\left(\int_0^{\omega} c_t \times d\mathbf{x}_K(t) + \int_0^{\omega} \mathbf{x}_K(t) \times dc_t + \int_0^{\omega} c_t \times dc_t\right)
\end{aligned}
\tag{5}
$$

so $\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{x}_K)$ is convex-linear with respect to K by Theorem 5.14.

From Equation (4) and Equation (5) we have

$$
\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K) = \tag{6}
$$

$$
\frac{1}{2}\left(p_K(0) + p_K(\omega+\pi/2)\right) + \left(\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{x}_K)\right) \tag{7}
$$

$$
-\frac{1}{2}\int_0^{\omega} h_K^+(t)^2\,dt - \frac{1}{2}\int_{\omega}^{\pi/2+\omega} \tau_K(t,\omega+\pi/2)^2\,dt \tag{8}
$$

and since the terms h_K^+, τ_K, p_K and $\mathcal{I}(\mathbf{y}_K) - \mathcal{I}(\mathbf{x}_K)$ are convex-linear with respect to K, $\mathcal{A}_1$ is concave by Lemma 5.21. $\square$

5.5 Directional Derivative of A1

In this section, we calculate the directional derivative $D\mathcal{A}_1(K; -)$ of $\mathcal{A}_1$ (Definition 5.10) at any $K \in \mathcal{K}_\omega$. As $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K)$, we calculate the directional derivative of $|K|$ and $\mathcal{I}(\mathbf{x}_K)$ separately.

The derivative of the area $|K|$ is:

Theorem 5.23. *Let K and K' be arbitrary convex bodies. Then we have the following.*

$$
\left.\frac{d}{d\lambda}\right|_{\lambda=0} |(1-\lambda)K + \lambda K'| = \langle p_{K'} - p_K, \beta_K \rangle_{S^1}
$$

Proof. For any convex body K we have $|K| = V(K, K)$ where V is the mixed volume of two planar convex bodies. So by applying Lemma 5.11 to $|K| = V(K, K)$ and using that $V(K, K') = V(K', K)$, we have the following.

$$
\left.\frac{d}{d\lambda}\right|_{\lambda=0} |(1-\lambda)K + \lambda K'| = 2V(K', K) - 2V(K, K)
$$

By applying Theorem A.18 we get the result. $\square$

We move on to the derivative of $\mathcal{I}(\mathbf{x}_K)$. We have the following general calculation the curve area functional $\mathcal{I}(\mathbf{x})$ of any curve $\mathbf{x}$.

Theorem 5.24. *Let $\mathbf{x}_1, \mathbf{x}_2 : [a,b] \to \mathbb{R}^2$ be two rectifiable curves. Then the following holds.*

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{I}((1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2) = \left[\int_a^b (\mathbf{x}_2(t) - \mathbf{x}_1(t)) \times d\mathbf{x}_1(t)\right] + \mathcal{I}(\mathbf{x}_1(b), \mathbf{x}_2(b)) - \mathcal{I}(\mathbf{x}_1(a), \mathbf{x}_1(a))$$

Proof. Consider the bilinear form $\mathcal{J}(\mathbf{x}_1, \mathbf{x}_2) = \int_a^b \mathbf{x}_1(t) \times d\mathbf{x}_2(t)$ on absolutely continuous $\mathbf{x}_1, \mathbf{x}_2 : [a,b] \to \mathbb{R}^2$. Apply Lemma 5.11 to $2\mathcal{I}(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbf{x})$ to get the following.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} 2\mathcal{I}((1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2) = \mathcal{J}(\mathbf{x}_1, \mathbf{x}_2) + \mathcal{J}(\mathbf{x}_2, \mathbf{x}_1) - 2\mathcal{J}(\mathbf{x}_1, \mathbf{x}_1)$$

Using the integration by parts, we have the following.

$$\int_a^b \mathbf{x}_1(t) \times d\mathbf{x}_2(t) = \mathbf{x}_1(b) \times \mathbf{x}_2(b) - \mathbf{x}_1(a) \times \mathbf{x}_2(a) + \int_a^b \mathbf{x}_2(t) \times d\mathbf{x}_1(t)$$

Plug this back in to get the result. $\square$

To calculate the derivative of $\mathcal{I}(\mathbf{x}_K)$ with respect to cap K, we introduce the following definitions.

Definition 5.13. For any cap $K \in \mathcal{K}_\omega$, define the function $i_K : J_\omega \to \mathbb{R}$ as $i_K(t) = h_K^+(t) - 1$ and $i_K(t + \pi/2) = g_K^+(t) - 1$ for every $t \in [0, \omega]$. Define ι_K as the measure on J_ω derived from the density function i_K. That is, $\iota_K(dt) = i_K(t)dt$.

Definition 5.13 is motivated by the following lemma.

Lemma 5.25. *Let $I \subseteq [0, \omega]$ be an arbitrary Borel subset. Let $K_1, K_2 \in \mathcal{K}_\omega$ be arbitrary. Then the following holds.*

$$\int_{t \in I} \mathbf{x}_{K_1}(t) \times d\mathbf{x}_{K_2}(t) = \langle p_{K_1} - 1, \iota_{K_2} \rangle_{I \cup (I + \pi/2)}$$

Proof. By Theorem 5.7, the derivative of $\mathbf{x}_{K_2}(t)$ with respect to t exists almost everywhere and is the following.

$$\mathbf{x}'_{K_2}(t) = -(g_{K_2}^+(t) - 1)u_t + (h_{K_2}^+(t) - 1)v_t$$

Meanwhile, we have the following.

$$\mathbf{x}_{K_1}(t) = (p_{K_1}(t) - 1)u_t + (p_{K_1}(t + \pi/2) - 1)v_t$$

So the cross-product $\mathbf{x}_{K_1}(t) \times \mathbf{x}'_{K_2}(t)$ is equal to the following almost everywhere.

$$(h_{K_2}^+(t) - 1)(p_{K_1}(t) - 1) + (g_{K_2}^+(t) - 1)(p_{K_1}(t + \pi/2) - 1)$$

Now the left-hand side is equal to

$$\int_{t \in J} (h_{K_2}^+(t) - 1)(p_{K_1}(t) - 1) + (g_{K_2}^+(t) - 1)(p_{K_1}(t + \pi/2) - 1)\, dt$$

and by Definition 5.13 this integral is equal to $\langle p_{K_1} - 1, \iota_{K_2} \rangle_{I \cup (I + \pi/2)}$. $\square$

Theorem 5.26. *Let K_1 and K_2 be two caps in $\mathcal{K}_\omega$. Then we have the following.*

$$D\mathcal{A}_1(K_1; K_2) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{A}_1((1-\lambda)K_1 + \lambda K_2) = \langle p_{K_2} - p_{K_1}, \beta_{K_1} - \iota_{K_1} \rangle_{J_\omega}$$

Proof. We have $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K)$. Apply Theorem 5.23 to the term $|K|$ to have the following.

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} |(1-\lambda)K_1 + \lambda K_2| = \langle p_{K_2} - p_{K_1}, \beta_{K_1} \rangle_{S^1}$$

Note that β_{K_1} and β_{K_2} are supported on the set $J_\omega \cup \{\pi+\omega, 3\pi/2\}$, and both p_{K_2} and p_{K_1} have function value equal to 1 on the set $\{\pi+\omega, 3\pi/2\}$. So we have $\langle p_{K_2} - p_{K_1}, \beta_{K_1} \rangle_{S^1} = \langle p_{K_2} - p_{K_1}, \beta_{K_1} \rangle_{J_\omega}$.

Apply Theorem 5.24 to the term $\mathcal{I}(\mathbf{x}_K)$, and use that the points O, $\mathbf{x}_K(0)$, and $A_K^-(0)$ (respectively, the points O, $\mathbf{x}_K(\omega)$, and $C_K^+(\omega)$) are colinear to get the following.

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathcal{I}((1-\lambda)\mathbf{x}_{K_1} + \lambda \mathbf{x}_{K_2}) &= \int_0^\omega (\mathbf{x}_{K_2}(t) - \mathbf{x}_{K_1}(t)) \times d\mathbf{x}_{K_1}(t) \\ &= \langle p_{K_2} - p_{K_1}, \iota_{K_1} \rangle\end{aligned}$$

The second equality comes from applying Lemma 5.25 twice. Subtract the derivates of $|K|$ and $\mathcal{I}(\mathbf{x}_K)$ above to conclude the proof. ☐

We explain the intuitive meaning of Theorem 5.26 by comparing it to the local optimization argument of Theorem 2 in [11]. Assume for the sake of explanation that S is a monotone sofa of rotation angle $\pi/2$ with cap K, such that the niche $\mathcal{N}(K)$ is exactly the region bounded by the curve $\mathbf{x}_K(t)$. Assume that our cap K has vertices $A_K(t) = A_K^\pm(t)$ and $C_K(t) = C_K^\pm(t)$ continuously differentiable with respect to t. Assume also that and $\mathbf{x}_K(t)$ is continuously differentiable.

Take an arbitrary angle t_0 and fix small positive δ and ϵ. In [11], Romik pertubed the sofa S to obtain a new sofa S' as the following. The monotone sofa $S = H \cap \bigcap_{0 \leq t \leq \pi/2} L_K(t)$ is the intersection of rotating hallways $L_K(t)$. For every $t \in [0, \pi/2]$, Romik pertubed each hallway $L_K(t)$ to a new hallways $L'(t)$ as the following.

- For every $t \in [t_0, t_0+\delta]$, let $L'(t) = L_K(t) + \epsilon u_t$.
- For every other t, let $L'(t) = L_K(t)$.

That is, we move $L_S(t)$ in the direction of ϵu_t for only $t \in [t_0, t_0+\delta]$. Now define $S' = H \cap \bigcap_{0 \leq u \leq \pi/2} L'(u)$ so that S' is a sofa which is slight perturbation of S. If S attains the maximum area, it should be that S' has area equal to or less than S.

We now compare the area of S and S'. As we perturb S to S', some area is gained near $A_K(t_0)$ as the walls $a_K(t)$ are pushed in the direction of ϵu_t for $t \in [t_0, t_0+\delta]$. The gain near $A_K(t_0)$ is approximately $\epsilon\delta \left\| A_K'(t_0) \right\|$ as the shape of the gain is approximately a rectangle of sides $A_K(t_0+\delta) - A_K(t_0) \simeq \delta \left\| A_K'(t_0) \right\| v_0$ and ϵu_{t_0}. Likewise, some area is lost near $\mathbf{x}_K(t_0)$ as we perturb S to S' as the corners $\mathbf{x}_K(t)$ are pushed in the direction of ϵu_t for $t \in [t_0, t_0+\delta]$. The loss near $\mathbf{x}_K(t_0)$ is $\epsilon\delta \mathbf{x}_K'(t_0) \cdot v_{t_0}$ as the shape of the loss is approximately a parallelogram of sides $\delta \mathbf{x}_K'(t_0)$ and ϵu_{t_0}. So the total gain of area from S to S' is approximately $\epsilon\delta \left(\left\| A_K'(t_0) \right\| - \mathbf{x}'(t_0) \cdot v_{t_0} \right)$. In [11], Romik solved for the critical condition $\left\| A_K'(t_0) \right\| = \mathbf{x}_K'(t_0) \cdot v_{t_0}$ (and another condition $\left\| C_K'(t_0) \right\| = -\mathbf{x}_K'(t_0) \cdot u_{t_0}$ obtained by perturbing each $L_K(t)$ in the orthogonal direction of ϵv_{t_0}) to derive an ordinary differential equation of $\mathbf{x}_K$ (ODE3 of Theorem 2, [11]).

We now observe that this total gain of $\epsilon\delta \left\| A_K'(t) \right\| - \epsilon\delta \mathbf{x}'(t) \cdot v_t$ for $t = t_0$ is captured in Theorem 5.26. The perturbation of hallways from $L_K(t)$ to $L'(t)$ can be described in terms of their support functions as $p_{K'} = p_K + \epsilon 1_{[t,t+\delta]}$. Correspondingly, the value $\langle p_{K'} - p_K, \beta_{K_1} \rangle_{[0,\pi]} = \epsilon \beta_{K_1}((t, t+\delta])$ is approximately $\epsilon\delta \left\| A_K'(t) \right\|$ which is equal to the gain near $A_K(t)$. The value $\langle p_{K'} - p_K, \iota_{K_1} \rangle_{[0,\pi]} = \epsilon \iota_{K_1}((t, t+\delta])$, by Theorem 5.7 and Definition 5.13, is approximately $\epsilon\delta \mathbf{x}_K'(t) \cdot v_t$ which is equal to the loss near $\mathbf{x}_K(t)$.

To summarize, the values $(p_{K'} - p_K)(t)$ and $(p_{K'} - p_K)(t + \pi/2)$ measures the movement of $\mathbf{x}_K(t)$ along the direction u_t and v_t respectively. Then the measure β_K near t and $t + \pi/2$ respectively measures the differential side lengths of the boundary of K near $A_K(t)$ and $C_K(t)$ respectively. Likewise, $i_K(t)$ and $i_K(t + \pi/2)$ measures the component of $\mathbf{x}'(t)$ in direction of v_t and $-u_t$ respectively.

5.6 Maximizer of A1

We now solve for the maximizer $K = K_{\omega,1}$ of our concave quadratic upper bound $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$. We do this *a priori* by solving for the cap K that satisfies $\beta_K = \iota_K$ on the set $J_\omega \setminus \{\omega, \pi/2\}$. Once we find such K, then it happens that the directional derivative $D\mathcal{A}_1(K; K') = 0$ for every other cap K' by Theorem 5.26 because $p_{K'}(\omega) = p_K(\omega) = p_{K'}(\pi/2) = p_K(\pi/2) = 1$. Then by Theorem 5.12 the cap K attains the maximum value of $\mathcal{A}_1$.

As we have seen in the previous subsection, the equation $\beta_K = \iota_K$ on the set $J_\omega \setminus \{\omega, \pi/2\}$ can be compared to the local optimality equation (ODE3) in Theorem 2 of [11]. However, unlike [11] which solved the equation for the inner corner $\mathbf{x}_K$, we will solve for the arm lengths $g_K^\pm(t)$ and $h_K^\pm(t)$ to find K. This will lead to a much simpler set of differential equations to solve. From now on, let $K \in \mathcal{K}_\omega$ be an arbitrary cap with rotation angle $\omega \in (0, \pi/2]$.

The following theorem shows that under sufficient conditions, we don't need to differentiate $g_K^+(t)$ and $g_K^+(t)$ (resp. $h_K^+(t)$ and $h_K^-(t)$) and also can calculate the derivatives of g_K and h_K in terms of β_K.

Theorem 5.27. *Assume that there is a open interval U in $(0, \pi/2)$ and a continuous function $f : U \cup (U + \pi/2) \to \mathbb{R}$ such that the measure β_K has density function f on $U \cup (U+\pi/2)$. That is, we have $\beta_K(X) = \int_X f(x)\,dx$ for every Borel subset $X \subseteq U \cup (U + \pi/2)$. Then we have $g_K^+(t) = g_K^-(t)$ and $h_K^+(t) = h_K^-(t)$ for every $t \in U$ so the function $g_K(t)$ and $h_K(t)$ are well-defined on $t \in U$. Moreover, $g_K'(t) = -f(t) + h_K(t)$ and $h_K'(t) = f(t + \pi/2) - g_K(t)$ for every $t \in U$.*

Proof. We have $g_K(t) = g_K^\pm(t)$ and $h_K(t) = h_K^\pm(t)$ for all $t \in U$ by Proposition 5.8. To calculate the derivatives, first apply Theorem A.19 to Proposition 5.6 to get the following equations for all $t \in U$.

$$g_K(t) = \int_t^{t+\pi/2} \cos(u - t)\,\beta_K(du)$$

$$h_K(t) = \int_t^{t+\pi/2} \sin(u - t)\,\beta_K(du)$$

Differentiate them at $t \in U$ using Leibniz integral rule to complete the proof.

$$g_K'(t) = -f(t) + \int_t^{t+\pi/2} \sin(u-t)\,\beta_K(du) = -f(t) + h_K(t)$$

$$h_K'(t) = f(t + \pi/2) - \int_t^{t+\pi/2} \cos(u-t)\,\beta_K(du) = f(t+\pi/2) - g_K(t)$$

□

This theorem is a converse of Theorem 5.27 that calculates β_K from continuously differentiable g_K and h_K.

Theorem 5.28. *Assume there is a open interval U in $(0, \pi/2)$ so that $g_K(t)$ and $h_K(t)$ are well-defined and continuously differentiable on U. Define the continous function f on $U \cup (U + \pi/2)$ as $f(t) = h_K(t) - g_K'(t)$ and $f(t+\pi/2) = g_K(t) + h_K'(t)$ for all $t \in U$. Then the measure β_K has density function f on $U \cup (U + \pi/2)$.*

Proof. By Theorem 5.7, $\mathbf{y}_K(t)$ has continuous differentiation $-g_K(t)u_t + h_K(t)v_t$ on $t \in U$. So $A_K^\pm(t) = \mathbf{y}_K(t) - g_K(t)u_t$ and $C_K^\pm(t) = \mathbf{y}_K(t) - h_K(t)v_t$ are continuously differentiable on $t \in U$ too. Then by Corollary A.20 the measure β_K has a continuous density function f_0 on $U \cup (U+\pi/2)$, where $f_0(t) = A_K'(t) \cdot v_t$ and $f_0(t + \pi/2) = -C_K'(t) \cdot v_t$. Now by Theorem 5.27 we should have $f_0 = f$, completing the proof. □

Now we solve the equation $\beta_K = \iota_K$ on any open set $(t_1, t_2) \cup (t_1 + \pi/2, t_2 + \pi/2)$ in terms of β_K.

Theorem 5.29. *Let $0 \leq t_1 < t_2 \leq \omega$ be two arbitrary angles. Let $U = (t_1, t_2)$ and $X = U \cup (U + \pi/2)$. Then the followings are equivalent.*

1. *We have $\beta_K = \iota_K$ on the set X*
2. *We have $g_K(t) = a + t$ and $h_K(t) = b - t$ for some constants $a, b \in \mathbb{R}$ on $t \in U$.*

Proof. Assume (1) that $\beta_K = \iota_K$ on X. The measure of $\beta_K = \iota_K$ is zero for every point of X by Definition 5.13. So we have $g_K(t) = g_K^{\pm}(t)$ and $h_K(t) = h_K^{\pm}(t)$ for every $t \in U$ by Proposition 5.8. Also, $g_K(t)$, $h_K(t)$ are continuous with respect to $t \in U$ by Corollary 5.9. Now since $\beta_K = \iota_K$ has the continuous density function i_K on X, we can apply Theorem 5.27 to K. By applying so, we have

$$g_K'(t) = -i_K(t) + h_K(t) = 1 \tag{9}$$
$$h_K'(t) = i_K(t + \pi/2) - g_K(t) = -1 \tag{10}$$

on $t \in U$ and this immediately proves (2).

Now assume (2). By Theorem 5.28 the measure β_K should have density function $f(t) = b - t - 1$ and $f(t + \pi/2) = a + t + 1$ over all $t \in U$. This matches with the density function of ι_K in Definition 5.13, completing the proof of (1). ☐

We now solve for the equation $\beta_K = \iota_K$ on the set $J_\omega \setminus \{\omega, \pi/2\}$ by solving for β_K. Note that our derivation aims to not solve the equation completely, but to derive enough properties of such K to motivate a single definite solution K that we will define shortly. First, we should have $\beta_K(\{0\}) = \beta_K(\{\omega + \pi/2\}) = 0$ because $0, \omega + \pi/2 \in J_\omega \setminus \{\omega, \pi/2\}$ and ι_K have measure zero on every singleton. So $g_K(0)$ and $h_K(\omega)$ exist, and $g_K(0) = h_K(\omega) = 1$ as the width of K measured in the direction of u_ω and v_0 are one. So by Theorem 5.29 with $t_1 = 0$ and $t_2 = \omega$ we should have $g_K(t) = t + 1$ and $h_K(t) = \omega - t + 1$ on the set $t \in (0, \omega)$. Now by Definition 5.13, the measure β_K has density $\beta_K(dt) = (\omega - t)dt$ and $\beta_K(dt + \pi/2) = tdt$ on $t \in (0, \omega)$. It remains to find the values of β_K on the points ω and $\pi/2$. The measure β_K has to satisfy the equations in Theorem 5.19. Since we have calculations

$$\int_{t\in[0,\omega)} (\omega - t)\cos t \, dt = 1 - \cos\omega$$
$$\int_{t\in(\pi/2,\omega+\pi/2]} (t - \pi/2)\cos(\omega + \pi/2 - t) \; dt = 1 - \cos\omega$$

we should have $\beta_K(\{\omega\}) = \beta_K(\{\pi/2\}) = 1$ if $\omega < \pi/2$. Motivated by the calculations made here, we define the following candidate $K = K_{\omega,1}$

Definition 5.14. Define the cap $K = K_{\omega,1} \in \mathcal{K}_\omega$ with rotation angle $\omega \in (0, \pi/2]$ as the unique cap with following boundary measure β_K.

1. $\beta_K(dt) = (\omega - t)dt$ on $t \in [0, \omega)$ and $\beta_K(dt + \pi/v) = tdt$ on $t \in (0, \omega]$.
2. If $\omega < \pi/2$, $\beta_K(\{\omega\}) = \beta_K(\{\pi/2\}) = 1$.
3. If $\omega = \pi/2$, $\beta_K(\{\pi/2\}) = 2$ and $v_K^+(\pi/2) = (-1, 1)$ and $v_K^-(\pi/2) = (1, 1)$.

Let us justify the unique existence of such $K_{\omega,1}$. Observe that the two equations in Theorem 5.20 are true by Section 5.6. So if $\omega < \pi/2$, the unique existence of $K_{\omega,1}$ comes from the statement of Theorem 5.20 immediately. If $\omega = \pi/2$, then the additional constraints $v_K^+(\pi/2) = (-1, 1)$ and $v_K^-(\pi/2) = (1, 1)$ fixes the single K among all possible horizontal translates. So the uniqueness is also guaranteed.

We now check back that the solution $K = K_{\omega,1}$ indeed satisfies the equation $\beta_K = \iota_K$ on $J_\omega \setminus \{\omega, \pi/2\}$.

Theorem 5.30. *The cap $K_{\omega,1}$ maximizes $\mathcal{A}_1 : \mathcal{K}_\omega \to \mathbb{R}$.*

Proof. Let $K = K_{\omega,1}$. First we verify that K satisfies the equation $\beta_K = \iota_K$ on $J_\omega \setminus \{\omega, \pi/2\}$. Because $\beta_K(\{0\}) = \beta_K(\{\omega + \pi/2\}) = 0$, the values $g_K(0)$ and $h_K(\omega)$ exist. We have $g_K(0) = 1$ and $h_K(\omega) = 1$ as the width of K measured in the direction of u_ω and v_0 are one. Then by Theorem 5.27 on the interval $(0, \omega)$ we have

$$g_K'(t) = -(\omega - t) + h_K(t)$$
$$h_K'(t) = t - g_K(t)$$

on $t \in (0, \omega)$. This imply $g_K''(t) = 1 + t - g_K(t)$ on $t \in (0, \omega)$. So we have $g_K(t) = 1 + t + C_1 \sin t + C_2 \cos t$ for some constants C_1, C_2 on $t \in (0, \omega)$. Because $g_K(t) \to g_K(0) = 1$ as $t \to 0$, we should have $C_2 = 0$. Because $h_K(t) \to h_K(\omega) = 1$ as $t \to \omega$, we have $g_K'(t) \to 1$ as $t \to \omega$. This then imply $1 + C_1 \sin \omega = 1$ and $C_1 = 0$. Now $g_K(t) = 1 + t$ and correspondingly $h_K(t) = 1 + \omega - t$ on $t \in (0, \omega)$. As ι_K is defined from the values of g_K and h_K (Definition 5.13), we can verify $\beta_K = \iota_K$ on $J_\omega \setminus \{\omega, \pi/2\}$. □

Theorem 5.31. *The maximum value $\mathcal{A}_1(K_{\omega,1})$ of $\mathcal{A}_1$ is $1 + \omega^2/2$.*

Proof. Let $K = K_{\omega,1}$. We will exploit the mirror symmetry of $K_{\omega,1}$ along the line l connecting O to o_ω. The line divides $K_{\omega,1}$ into two pieces which are mirror images to each other. Call the piece on the right side of l as K_h. Observe that the boundary of K_h consists of the curve from $A_K^-(0)$ to o_ω, and two segments from O to $A_K^-(0)$ and o_ω respectively. So $p_K(t) = p_{K_h}(t)$ for $t \in [0, \omega]$ and β_K and β_{K_h} agree on $[0, \omega)$. What is good about K_h is that $\beta_{K_h}(\{\omega\}) = 1$ no matter if either $\omega < \pi/2$ or $\omega = \pi/2$, unlike $\beta_K(\{\omega\})$ that may change depending on ω. We also have $|K| = 2|K_h|$.

Now we compute the value of $p_K(t) = p_{K_h}(t)$ for $t \in [0, \omega]$. For the second equality, we are using Corollary A.22 with $t_1 = t$ and $t_2 = \omega$.

$$
\begin{aligned}
p_{K_h}(t) - o_\omega \cdot u_t &= (A_{K_h}^-(t) - o_\omega) \cdot u_t = \\
&= \sin(\omega - t) + \int_{u \in [t,\omega]} (\omega - u) \sin(u - t)\, du \\
&= \omega - t
\end{aligned}
$$

So $p_K(t) = p_{K_h}(t) = \omega - t + o_\omega \cdot u_t$.

Now use the symmetry of K along the line from O to o_ω to calculate half the area of K.

$$
\begin{aligned}
|K_h| = \frac{1}{2} \int_{t \in [0,\omega]} p_{K_h}(t)\, \beta(dt) &= \frac{1}{2} + \frac{1}{2} \int_{t \in [0,\omega]} (\omega - t + o_\omega \cdot u_t)\, (\omega - t)\, dt \\
&= \frac{1}{2} + \frac{1}{2} \left(\omega^3/3 + o_\omega \cdot \int_0^\omega u_t(\omega - t)\, dt \right)
\end{aligned}
$$

Define $R := o_\omega \cdot \int_0^\omega u_t(\omega - t)\, dt$, and by multiplying by 2 we have the following.

$$
|K| = 1 + \omega^3/3 + R
$$

Next, we compute the curve area functional $\mathcal{I}(\mathbf{x}_K)$. We have

$$
\mathbf{x}_K(t) = (p_K(t) - 1)u_t + (p_K(t + \pi/2) - 1)v_t
$$

and

$$
\mathbf{x}_K' = -(g_K^+(t) - 1) \cdot u_t + (h_K^+(t) - 1) \cdot v_t = -t \cdot u_t + (\omega - t) \cdot v_t
$$

so

$$
\begin{aligned}
\mathcal{I}(\mathbf{x}_K) &= \frac{1}{2} \int_0^\omega (p_K(t) - 1)(\omega - t) + (p_K(t + \pi/2) - 1)t\, dt \\
&= \frac{1}{2} \int_0^\omega (\omega - t + o_\omega \cdot u_t - 1)(\omega - t)\, dt + \frac{1}{2} \int_0^\omega (t + o_\omega \cdot v_t - 1)t\, dt \\
&= \omega^3/3 - \omega^2/2 + R
\end{aligned}
$$

Finally, we compute $\mathcal{A}_1(K) = |K| - \mathcal{I}(\mathbf{x}_K) = 1 + \omega^2/2$. □

Now Theorem 5.5, Theorem 5.30, Theorem 5.31 imply Theorem 4.3, and thus the main Theorem 1.4.

A Convex bodies

Fix an arbitrary planar convex body K, which by Definition 2.12 is a nonempty, compact and convex subset of $\mathbb{R}^2$. This appendix defines and proves numerous properties of K. For the ease of understanding it is recommended to access the parts this appendix . However, an interested reader can read this appendix from beginning to end to verify the correctness of all proofs and theorems.

We will allow the case where K have empty interior in most theorems, and we will make the condition explicit in a theorem if K is required to have nonempty interior.

A.1 Vertex and support function

Here, we prove numerous properties on the support function p_K (Definition 2.8) and the vertices $v_K^{\pm}(t)$ (Definition 2.14) of a convex body K.

A.1.1 Continuity and linearity

Definition A.1. For any function f that maps an arbitrary convex body K to a value $f(K)$ in a vector space V, say that f is *linear* with respect to K if the followings hold.

1. For any $a \geq 0$ and a convex body K, we have $f(aK) = af(K)$.
2. For any $a, b \geq 0$ and convex bodies K_1, K_2, we have $f(K_1 + K_2) = f(K_1) + f(K_2)$.

Note that the sum used in Definition A.1 is the Minkowski sum of convex bodies. That is, $aK = \{ap : p \in K\}$ and $K_1 + K_2 = \{p_1 + p_2 : p_1 \in K_1, p_2 : K_2\}$.

Theorem A.1. *For any convex body K, its support function p_K is Lipschitz.*

Proof. If K is a single point $z \in \mathbb{R}^2$, then $p_K = p_z$ is a sinusoidal function with amplitude $|z|$ where $|z|$ denotes the distance of z from the origin. For a general convex body K, take $R \geq 0$ so that K is contained in a closed ball of radius R centered at zero. Then

$$p_K(t) = \max_{z \in K} z \cdot u_t = \sup_{z \in K} p_z(t)$$

and note that each function $p_z : S^1 \to \mathbb{R}$ is R-Lipschitz. So the supremum p_K of p_z over all z is also R-Lipschitz. $\square$

Proposition A.2. *The support function p_K is linear with respect to K.*

Proof. First condition of Definition A.1 on p_K follows from a direct argument.

$$p_{aK}(t) = \max_{z \in aK} z \cdot u_t = \max_{z' \in K} (az') \cdot u_t = ap_K(t)$$

For arbirary convex bodies K_1, K_2 and a fixed $t \in S^1$,

$$\begin{aligned} p_{K_1+K_2}(t) &= \max_{z \in K_1+K_2} z \cdot u_t = \max_{z_1 \in K_1, z_2 \in K_2} (z_1 + z_2) \cdot u_t \\ &= \max_{z_1 \in K_1} z_1 \cdot u_t + \max_{z_2 \in K_2} z_2 \cdot u_t = p_{K_1}(t) + p_{K_2}(t) \end{aligned}$$

so the second condition of Definition A.1 is true. $\square$

We will soon show that the vertex $v_K^+(t)$ (resp. $v_K^-(t)$) is right-continuous (resp. left-continuous) respect to t and is linear with respect to K. To do so, we compute the limit of vertices via Theorem A.4.

Definition A.2. For every $t_1, t_2 \in S^1$ such that $t_2 \neq t_1, t_1 + \pi$, define $v_K(t_1, t_2)$ as the intersection $l_K(t_1) \cap l_K(t_2)$.

Lemma A.3. *For any fixed $t_1, t_2 \in S^1$ such that $t_2 \neq t_1, t_1 + \pi$, the point $v_K(t_1, t_2)$ is linear with respect to K.*

Proof. The point $p = v_K(t_1, t_2)$ is the unique point such that $p \cdot u_{t_1} = p_K(t_1)$ and $p \cdot u_{t_2} = p_K(t_2)$ holds. By solving the linear equations, observe that the coordinates of p are linear combinations of $p_K(t_1)$ and $p_K(t_2)$. By Proposition A.2 the result follows. $\square$

Theorem A.4. *Let K be a convex body and t be an arbitrary angle. We have the following right limits all converging to $v_K^+(t)$. In particular, the vertex $v_K^+(t)$ is a right-continuous function on $t \in S^1$.*

$$\lim_{t' \to t^+} v_K^+(t) = \lim_{t' \to t^+} v_K^-(u) = \lim_{t' \to t^+} v_K(t, t') = v_K^+(t)$$

Similarly, we have the following left limits.

$$\lim_{t' \to t^-} v_K^+(t') = \lim_{t' \to t^-} v_K^-(t') = \lim_{t' \to t^-} v_K(t', t) = v_K^-(t)$$

Proof. We only compute the right limits. Left limits can be shown using a symmetric argument.

Let $\epsilon > 0$ be arbitrary. Let $p = v_K^+(t) + \epsilon v_t$. Then by the definition of $v_K^+(t)$ the point p is not in K. As $\mathbb{R}^2 \setminus K$ is open, any sufficiently small open neighborhood of p is disjoint from K, so we can take some positive $\epsilon' < \epsilon$ such that the closed line segment connecting p and $q = p - \epsilon' u_t$ is disjoint from K as well. Define the closed right-angled triangle T with vertices $v_K^+(t)$, p, and q. Take the line l that passes through both q and $v_K^+(t)$. Call the two closed half-planes divided by the line l as H_T and H', where H_T is the half-plane containing T and H' is the other one. By definition of H', the half-plane H' contains $v_K^+(t)$ and q on its boundary and does not contain the point q. And consequently H' has normal angle $t' \in (t, t + \pi/2)$ (Definition 2.7) because $p = v_K^+(t) + \epsilon v_t$ and $q = p - \epsilon' u_t$.

We show that $K \cap H_T \subseteq T$. Assume by contradiction that there is $r \in K \cap H_T$ not in T. As $r \in K$, r should be in the tangential half-plane $H_K(t)$. So r is in the cone $H_T \cap H_K(t)$ sharing the vertex $v_K^+(t)$ and two edge $l_K(t)$, l with T. Since $r \notin T$, the line segment connecting r and the vertex $v_K^+(t)$ of T should cross the edge of T connecting p and q at some point s. As $r, v_K^+(t) \in K$ we also have $s \in K$ by convexity. But the line segment connecting p and q is disjoint from K by the definition of q, so we get contradiction. Thus we have $K \cap H_T \subseteq T$.

Now take arbitrary $t_0 \in (t, t')$. We show that the edge $e_K(t_0)$ should lie inside T. It suffices to show that any point z in K that attains the maximum value of $z \cdot u_{t_0}$ is in T. Define the fan $F := H_K(t) \cap H'$, so that F is bounded by lines $l_K(t)$ and l with the vertex $v_K^+(t)$. If $z \in F$, it should be that $z = v_K^+(t) \in T$, because $v_K^+(t) \in K$ and $v_K^+(t) \cdot u_{t_0} > z \cdot u_{t_0}$ for every point z in F other than $z = v_K^+(t)$. If $z \in K \setminus F$ on the other hand, we have $K \setminus F = K \setminus H' \subseteq K \cap H_T \subseteq T$ so $z \in T$. This completes the proof of $e_K(t_0) \subseteq T$.

Observe that the triangle T contains $v_K^+(t)$ and has diameter $< 2\epsilon$ because the two perpendicular sides of T containing p have length $\leq \epsilon$. So the endpoints $v_K^+(u)$ and $v_K^-(u)$ of the edge $e_K(t_0) \subseteq T$ are distance at most 2ϵ away from $v_K^+(t)$. This completes the epsilon-delta argument for $\lim_{t' \to t^+} v_K^+(t') = \lim_{t' \to t^+} v_K^-(t') = v_K^+(t)$.

From $e_K(t_0) \subseteq T$ and that the vertex p of T maximizes the value of $z \cdot u_{t_0}$ over all $z \in T$, we get that p is either on $l_K(t_0)$ or outside the half-plane $H_K(t_0)$. On the other hand we have $v_K^+(t) \in H_K(t_0)$. So the line $l_K(t_0)$ passes through the segment connecting p and $v_K^+(t)$, and the intersection $v_K(t, t_0) = l_K(t) \cap l_K(t_0)$ is inside T. This with that the diameter of T is less than 2ϵ proves $\lim_{t' \to t^+} v_K(t, t') = v_K^+(t)$. $\square$

The vertex $v_K^+(t)$ is right-continuous by Theorem A.4.

Corollary A.5. *The vertex $v_K^+(t)$ is right-continuous with respect to $t \in S^1$. Likewise, the vertex $v_K^-(t)$ is left-continuous with respect to $t \in S^1$.*

From Lemma A.3 and Theorem A.4, we have the linearlity of vertices $v_K^\pm(t)$.

Corollary A.6. *For a fixed $t \in S^1$, the vertices $v_K^\pm(t)$ are linear with respect to K.*

A.1.2 Parametrization of tangent line

We calculate $v_K(t,t')$ as the following.

Lemma A.7. *Let $t,t' \in S^1$ be arbitrary such that $t' \neq t, t+\pi$. The following calculation holds.*

$$v_K(t,t') = p_K(t)u_t + \left(\frac{p_K(t') - p_K(t)\cos(t'-t)}{\sin(t'-t)}\right) v_t$$

Proof. Because the point $p = v_K(t,t') = l_K(t) \cap l_K(t')$ is on the line $l_K(t)$, we have $p = p_K(t)u_t + \beta v_t$ for some constant $\beta \in \mathbb{R}$. We use $p \cdot u_{t'} = p_K(t')$ to derive the unique value β. Write $t'-t$ as θ.

$$\begin{aligned} p_K(t') &= p_K(t)(u_t \cdot u_{t'}) + \beta(v_t \cdot u_{t'}) \\ &= p_K(t)\cos\theta + \beta\sin\theta \end{aligned}$$

This gives $\beta = p_C(t')\csc\theta - p_C(t)\cot\theta$ as claimed and completes the calculation. The value α is continuous on $(-\pi,\pi) \setminus \{0\}$ by the formula. ☐

Using Lemma A.7, we can show that $v_K(t,t')$ parametrizes the half-lines in $l_K(t)$ emanating from $v_K^{\pm}(t)$.

Theorem A.8. *Take any $t \in S^1$ and assume that the width $p_K(t+\pi)+p_K(t)$ of K measured in the direction of u_t is strictly positive (e.g. when K has nonempty interior). Define $\mathbf{p} : [t,t+\pi) \to \mathbb{R}^2$ as $\mathbf{p}(t) = v_K^+(t)$ and $\mathbf{p}(t') = v_K(t,t')$ for all $t' \in (t,t+\pi)$. Then the followings hold.*

1. *$\mathbf{p}$ is absolutely continuous on any closed and bounded interval.*
2. *$\mathbf{p}(t') = v_K^+(t') + \alpha(t')v_t$ where $\alpha(t) = 0$ and the function $\alpha : [t,t+\pi) \to \mathbb{R}$ is monotonically increasing.*
3. *$\mathbf{p}$ is a parametrization of the half-line emanating from $v_K^+(t)$ in the direction of v_t.*

Proof. (1) The function $\mathbf{p}$ is continuous at t because of Theorem A.4. The function $\mathbf{p}$ is absolutely continuous on any closed subinterval of $(t,t+\pi)$ by Lemma A.7 (note that p_K is Lipschitz). So the derivative $\mathbf{p}'(u)$ of $\mathbf{p}$ on $u \in (t,t+\pi)$ exists almost everywhere and $\mathbf{p}(u_2) - \mathbf{p}(u_1) = \int_{u_1}^{u_2} \mathbf{p}'(u)\,du$ for every $t < u_1 < u_2 < t+\pi$. Take the limit $u_1 \to t^+$ to obtain $\mathbf{p}(u_2) - \mathbf{p}(t) = \int_t^{u_2} \mathbf{p}'(u)\,du$. So $\mathbf{p}$ is absolutely continuous on any closed subinterval of $[t,t+\pi)$ including the endpoint t.

(1) comes from the geometric fact that for every $t < t_1 < t_2 < t+\pi$, the point $v_K(t,t_1)$ lies in the segment connecting $v_K^+(t)$ and $v_K(t,t_2)$.

By taking the limit $u \to t+\pi^-$ in Lemma A.7, we have $\alpha(u) \to \infty$ (note that we use the fact that the width $p_K(t+\pi)+p_K(t)$ is positive). Now (3) follows from (1), (2), and $\alpha(u) \to \infty$. ☐

An argument very similar to the proof above proves the following variant Theorem A.8 of Theorem A.8. We omit the details.

Theorem A.9. *Take any $t \in S^1$ and assume that the width $p_K(t+\pi)+p_K(t)$ of K measured in the direction of u_t is strictly positive (e.g. when K has nonempty interior). Define $\mathbf{q} : (t-\pi,t] \to \mathbb{R}^2$ as $\mathbf{q}(t') = v_K(t',t)$ for all $t' \in (t-\pi,t)$ and $\mathbf{q}(t) = v_K^-(t)$. Then the followings hold.*

1. *$\mathbf{q}$ is absolutely continuous on any closed and bounded interval.*
2. *$\mathbf{q}(t') = v_K^-(t') - \beta(t')v_t$ where $\beta(t) = 0$ and the function $\beta : (t-\pi,t] \to \mathbb{R}$ is monotonically decreasing.*
3. *$\mathbf{q}$ is a parametrization of the half-line terminating with the endpoint $v_K^-(t)$ in the direction of v_t.*

Theorem A.10. *Let K be any convex body. Let $t_0 \in \mathbb{R}$ be any angle. On the interval $t \in [t_0,t_0+\pi]$, the value $v_K^+(t) \cdot v_t$ is monotonically decreasing. On the interval $t \in [t_0-\pi,t_0]$, the value $v_K^+(t) \cdot v_t$ is monotonically increasing.*

Proof. Take two arbitrary values $t_1 < t_2$ in the interval $[t_0, t_0+\pi]$. By Theorem A.8, the points $v_K^+(t_1), l_K(t_1) \cap l_K(t_2), v_K^-(t_2), v_K^+(t_2)$ goes further in the direction of $-v_t$ in the increasing order. This shows that the value $v_K^+(t) \cdot v_t$ is monotonically decreasing on the interval $t \in [t_0, t_0 + \pi]$. A symmetric argument proves the other claim. $\square$

Theorem A.11. *On any bounded interval $t \in I$ of $\mathbb{R}$, $v_K^+(t)$ is of bounded variation.*

Proof. The x and y coordinates of $v_K^+(t)$ either monotonically increases or decreases on each of the intervals $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$ by Theorem A.10. So the coordinates are of bounded variation on each interval. $\square$

A.2 Lebesgue-Stieltjes measure

For any right-continuous, real-valued function F of bounded variation on domain $\mathbb{R}$ or $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, we will define its *Lebesgue-Stieltjes measure* dF that measures the difference of F. The standard statement on dF that can be found in most literature is the following.

Theorem A.12. *For any right-continuous and monotonically increasing function $F : \mathbb{R} \to \mathbb{R}$, there exists a unique measure dF on $\mathbb{R}$ such that $dF((c, d]) = F(d) - F(c)$ for any half-open interval $(c, d]$ of $\mathbb{R}$.*

We can restrict the

Theorem A.13. *For any right-continuous function $F : [a, b] \to \mathbb{R}$ of bounded variation, there exists a unique finite signed measure dF such that $dF(a) = 0$ and for any half-open interval $(c, d] \subseteq [a, b]$, $dF((c, d]) = F(d) - F(c)$.*

Proof. Define the extension $F_0 : \mathbb{R} \to \mathbb{R}$ of F as $F_0(x) = F(a)$ for all $x < a$, $F_0(x) = F(b)$ for all $x > b$, and observe that F_0 is also right-continuous and bounded variation. Restrict the measure dF_0 on $\mathbb{R}$ to $[a, b]$ to construct dF. For the uniqueness of such dF, note that for any two candidates μ_1, μ_2 of dF, the difference measure $\lambda = \mu_1 - \mu_2$ is zero for all half-open intervals $(c, d]$. Use Haan decomposition to represent λ with positive and negative part. The positive and negative part all match on half-open intervals. So λ is zero by Caratheodory extension. $\square$

Define $q : [0, 2\pi] \to S^1$ as the quotient map identifying 0 and 2π.

Theorem A.14. *For any right-continuous function $F : S^1 \to \mathbb{R}$ of bounded variation, there exists a unique finite signed measure dF on S^1 such that for any half-open interval $(t_1, t_2]$ of S^1 with $t_1 < t_2 \leq t_1 + 2\pi$, we have $dF((t_1, t_2]) = F(t_2) - F(t_1)$. Moreover, such dF is the pushforward of $d(F \circ q)$ under q.*

Proof. We first show the existence of such dF. Let μ be the pushforward of $d(F \circ q)$ under the map q. Then for any $\square$

Proposition A.15. *The Lebesgue-Stieltjes measure dF is linear with respect to F.*

Proof. Use the uniqueness of Theorem A.14 to show that $\square$

Proposition A.16. *Let F, G be real-valued functions on either S^1 or a closed interval, so that F is continuous and G is right-continuous. Then $d(FG) = GdF + FdG$.*

Proof. Theorem 21.67(iv) of $\square$

A.3 Surface area measure

Definition A.3. For any convex body K, denote the *surface area measure* of K as σ_K.

A full definition of σ_K is given in p214 of [12], which is also denoted as $S_1(K, -)$ or S_K in p464 of [12]. For any convex body K, the surface area measure σ_K is a measure on S^1 that essentially measures the side lengths of K. For example, if K is a convex polygon, then σ_K is a discrete measure such that the measure $\sigma_K(\{t\})$ at point t is the length of the edge $e_K(t)$. On the other hand, assume the case where for every

$t \in S^1$ the tangent line $l_K(t)$ meets K at a single point $v(t)$, so that ∂K is parametrized smoothly by $v_K(t)$ for $t \in S^1$. Then it turns out that $\sigma_K(dt) = R(t)dt$ where $R(t) = \|v'(t)\|$ is the radius of curvature of ∂K at $v(t)$.

We now collect theorems on σ_K.

Theorem A.17. *(Equation (8.23), p464 of [12]) The surface area measure σ_K is convex-linear with respect to K.*

Note that for any measurable function f on a space X and a measure σ on X, we denote the integral of f with respect to σ as $\langle f, \sigma \rangle_X = \int_{x \in X} f(x)\, \sigma(dx)$.

Theorem A.18. *(Theorem 5.1.7 in p280 of [12]) The mixed volume $V(K_1, K_2)$ of any two planar convex bodies K_1 and K_2 can be represented as the following.*

$$V(K_1, K_2) = \langle p_{K_1}, \sigma_{K_2} \rangle_{S^1} / 2$$

Consequently, the area $|K|$ of any planar convex body K can be represented as the following.

$$|K| = V(K, K) = \langle p_K, \sigma_K \rangle_{S^1} / 2$$

We prove the following important *vertex equality* of K using σ_K.

Theorem A.19. *For every interval $(t_1, t_2]$ in S^1 of length $\leq 2\pi$, we have the following equality.*

$$v_K^+(t_2) - v_K^+(t_1) = \int_{t \in (t_1, t_2]} v_t \, \sigma_K(dt)$$

Proof. First we observe that the equality holds when K is a polygon. In this case, for every t the value $\sigma_K(\{t\})$ is nonzero if and only if it measures the length of a proper edge $e_K(t)$. So the right-hand side measures the sum of all vectors from vertex $v_K^-(t)$ to $v_K^+(t)$ along the proper edges $e_K(t)$ with angles $t \in (t_1, t_2]$. The sum in the right-hand side is then the vector from $v_K^+(t_1)$ to $v_K^+(t_2)$, justifying the equality for polygon K.

Now we prove the equality for general K. As in the proof of Theorem 8.3.3, p466 of [12], we can take a series $K_1, K_2, \dots$ of polygons converging to K in the Hausdorff distance such that $e_{K_n}(t_i) = e_K(t_i)$ for all $n = 1, 2, \dots$ and $i = 1, 2$. In particular, we have $v_{K_n}^\pm(t_i) = v_K^\pm(t_i)$ and $\sigma_{K_n}(\{t_i\}) = \sigma_K(\{t_i\})$ for all $n = 1, 2, \dots$ and $i = 1, 2$. By Theorem 4.2.1, p212 of [12], the measures σ_{K_n} converge to σ_K weakly as $n \to \infty$.

For any measure σ on S^1 and a Borel subset A of S^1, define the *restriction* $\sigma|_A$ of σ to A as the measure defined as $\sigma|_A(X) = \sigma(A \cap X)$. Define U as the open set $S^1 \setminus \{t_1, t_2\}$ of S, and V as the open interval (t_1, t_2) of S^1. Define μ_n and μ as the restriction of σ_{K_n} and σ_K to U, then μ_n converges to μ weakly as $n \to \infty$ because $\sigma_{K_n}(\{t_i\}) = \sigma_K(\{t_i\})$ for $i = 1, 2$. Define λ_n and λ as the restriction of σ_{K_n} and σ_K to V. We want to prove that $\lambda_n \to \lambda$ weakly as $n \to \infty$. Take any continuity set X of λ so that $\lambda(\partial X) = 0$ and thus $\mu(\partial X \cap V) = 0$. Because $\partial(X \cap V) \subseteq (\partial X \cap V) \cup \partial V$, and both $\mu(\partial X \cap V)$ and $\mu(\partial V)$ are zero, the set $X \cap V$ is a continuity set of μ. So $\mu_n(X \cap V) \to \mu(X \cap V)$ and thus $\lambda_n(X) \to \lambda(X)$ as $n \to \infty$. This completes the proof that $\lambda_n \to \lambda$ weakly as $n \to \infty$.

We finally take the limit $n \to \infty$ to the equality

$$v_{K_n}^+(t_2) - v_{K_n}^+(t_1) = \int_{t \in (t_1, t_2]} v_t \, \sigma_{K_n}(dt)$$

for polygons K_n. The left-hand side is equal to $v_K^+(t_2) - v_K^+(t_1)$ by the way how we took K_n. The right-hand side is equal to

$$(v_{K_n}^+(t_2) - v_{K_n}^-(t_2)) + \int_{t \in S^1} v_t \, \lambda_n(dt)$$

and by $v_{K_n}^\pm(t_i) = v_K^\pm(t_i)$ and the weak convergence $\lambda_n \to \lambda$, the expression converges to

$$(v_K^+(t_2) - v_K^-(t_2)) + \int_{t \in S^1} v_t \, \lambda(dt) = \int_{t \in (t_1, t_2]} v_t \, \sigma_K(dt)$$

thus completing the proof for general K. □

Theorem A.19 has the following concise representation in differentials via the Lebesgue-Stieltjes measure.

Corollary A.20. *We have $dv_K^+(t) = v_t \sigma_K(dt)$.*

That is, if we write the coordinates of $v_K^+(t)$ as $(x(t), y(t))$, then the Lebesgue-Stieltjes measure dx and dy of $x(t)$ and $y(t)$ are $-\sin t \cdot \sigma_K(dt)$ and $\cos t \cdot \sigma_K(dt)$ respectively. Note that dx and dy are well-defined because $v_K^+(t)$ is of bounded variation (Theorem A.11) and right-continuous (Corollary A.5). The proof of Corollary A.20 is immediate from checking that Theorem A.19 matches with the definition of $dv_K^+(t)$.

Surface area measure at a single point t measures the length of the edge $e_K(t)$.

Theorem A.21. *$\sigma_K(\{t\})$ is equal to the length of the edge $e_K(t)$. Moreover, $v_K^+(t) = v_K^-(t) + \sigma_K(\{t\})v_t$.*

Proof. Let $t_2 = t$ and send $t_1 \to t^-$ in Theorem A.19. Then by Theorem A.4 we get the equation $v_K^+(t) = v_K^-(t) + \sigma_K(\{t\})v_t$. $\square$

We also have the following closed and open interval versions of Theorem A.19. They are corollaries of Theorem A.19 and Theorem A.21.

Corollary A.22. *For every interval $[t_1, t_2]$ in S^1 of length $< 2\pi$, we have the following equality.*

$$v_K^+(t_2) - v_K^-(t_1) = \int_{t \in [t_1, t_2]} v_t \, \sigma_K(dt)$$

Corollary A.23. *For every interval (t_1, t_2) in S^1 of length $\leq 2\pi$, we have the following equality.*

$$v_K^-(t_2) - v_K^+(t_1) = \int_{t \in (t_1, t_2)} v_t \, \sigma_K(dt)$$

Apply $t_1 = t, t_2 = t + \pi$ to Theorem A.19 and take the dot product with $-u_t$ to get the following.

Corollary A.24. *For any angle $t \in S^1$, the width $p_K(t) + p_K(t + \omega)$ of K measured in the direction of u_t is equal to the following.*

$$\int_{u \in (t, t+\pi)} \sin(u - t) \, \sigma_K(dt)$$

A.4 Parametrization of boundary

If K has nonempty interior, it occurs naturally that the boundary ∂K is a Jordan curve bounding K in its interior. So for any different $p, q \in \partial K$, we can think of a Jordan arc $\mathbf{b}$ connecting p and q along the boundary ∂K. However, to show that the curve area functional $\mathcal{I}(\mathbf{b})$ of $\mathbf{b}$ is well-defined and relates to the surface area measure σ_K (see Theorem A.28), we need to contruct an explicit rectifiable parametrization of $\mathbf{b}$ and this requires some work.

For every $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi]$, we will define $\mathbf{b}_K^{t_0,t_1}$ as essentially the arc-length parametrization of the curve connecting $v_K^+(t_0)$ to $v_K^+(t_1)$ along the boundary ∂K counterclockwise. The full Definition A.10 of $\mathbf{b}_K^{t_0,t_1}$ is technical will be given much later. Instead, we start by stating all the properties that agrees naturally with our intuition that we will prove rigorously. We will also soon show that $\mathbf{b}_K^{t_0,t_1}$ is one of: a Jordan arc, a Jordan curve, or a single point (Corollary A.31). Note that in the theorems below we allow K to have empty interior.

Theorem A.25. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi]$. Then $\mathbf{b}_K^{t_0,t_1}$ is an arc-length parametrization of the $\left\{v_K^+(t_0)\right\} \bigcup \cup_{t \in (t_0, t_1]} e_K(t)$ from point $v_K^+(t_1)$ to $v_K^+(t_2)$.*

Theorem A.26. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi]$. Then the curve $\mathbf{b}_K^{t_0,t_1}$ have length $\sigma_K((t_0, t_1])$.*

Theorem A.27. *Assume arbitrary t_0, t_1, t_2 such that $t_0 \leq t_1 \leq t_2 \leq t_0 + 2\pi$. Then $\mathbf{b}_K^{t_0,t_2}$ is the concatenation of $\mathbf{b}_K^{t_0,t_1}$ and $\mathbf{b}_K^{t_1,t_2}$.*

Theorem A.28. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi]$. Then the curve area functional of $\mathbf{b}_K^{t_0,t_1}$ can be represented in two different ways:*

$$\mathcal{I}\left(\mathbf{b}_K^{t_0,t_1}\right) = \frac{1}{2}\int_{(t_0,t_1]} p_K(t)\,\sigma_K(dt) = \frac{1}{2}\int_{(t_0,t_1]} v_K^+(t) \times dv_K^+(t)$$

The following is a corollary of Theorem A.18 and Theorem A.28.

Theorem A.29. *For every $t \in \mathbb{R}$, we have $|K| = \mathcal{I}\left(\mathbf{b}_K^{t,t+2\pi}\right)$.*

We recall the difference between a Jordan arc and curve (p170 of [1]).

Definition A.4. A *Jordan curve* is a curve parametrized by continuous $\mathbf{p} : [a, b] \to \mathbb{R}^2$ such that $a < b$, $\mathbf{p}(a) = \mathbf{p}(b)$ and $\mathbf{p}$ being injective on $[a, b)$.

Definition A.5. A *Jordan arc* is a curve parametrized by continuous and injective $\mathbf{p} : [a, b] \to \mathbb{R}^2$ such that $a < b$.

In order for ∂K to be a Jordan curve, K has to have nonempty interior. For the notion of the orientation of a Jordan curve, we refer to p170 of [1]. The following theorem shows that $\mathbf{b}_K^{t,t+2\pi}$ is the only parametrization of ∂K as a positively-oriented Jordan curve.

Theorem A.30. *Assume that K have nonempty interior. For every $t \in \mathbb{R}$, the curve $\mathbf{b}_K^{t,t+2\pi}$ is a positively oriented arc-length parametrization of the boundary ∂K as a Jordan curve that starts and ends with the point $v_K^+(t)$.*

Now that $\mathbf{b}_K^{t_0,t_1}$ is injective is a corollary of Theorem A.27 and Theorem A.30.

Corollary A.31. *Assume that K have nonempty interior. Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi]$. Then $\mathbf{b}_K^{t_0,t_1}$ is one of: a Jordan arc, a Jordan curve, or a single point.*

A.4.1 Definition of parametrization

Fix an arbitrary convex body K and the starting angle $t_0 \in \mathbb{R}$.

Definition A.6. Denote the *perimeter* of K as $B_K = \sigma_K\left(S^1\right)$.

Our goal to construct an arc-length parametrization $\mathbf{b}_K^{t_0} : [0, B_K] \to \mathbb{R}^2$ of the boundary ∂K starting with the point $v_K^+(t_0)$.

Take an arbitrary point p on the boundary ∂K. Let $s \in [0, B_K]$ be the arc length from $v_K^+(t_0)$ to p along ∂K, so that we want $p = \mathbf{b}_K^{t_0}(s)$ in the end. As p is in ∂K, it is inside the tangent line $l_K(t)$ for some angle $t \in (t_0, t_0 + 2\pi]$. Now the arc length $s \in [0, B_K]$ and the tangent line angle $t \in (t_0, t_0 + 2\pi]$ are the two different variables attached to $p \in \partial K$.

Unfortunately, the relation between s and t cannot be simply described as a function from one to another. A single value of s may correspond to multiple values of t (when p is a sharp corner of angle $< \pi$), and likewise a single value of t may correspond to multiple values of s (when p is on the edge $e_K(t)$ which is a proper line segment). We need the language of generalized inverse (e.g. [3]) to describe the relationship between s and t.

The map $g_K^{t_0}$ is defined so that it sends t to the largest possible corresponding s.

Definition A.7. Define $g_K^{t_0} : [t_0, t_0 + 2\pi] \to [0, B_K]$ as $g_K^{t_0}(t) = \sigma_K((t_0, t])$.

The map $f_K^{t_0}$ is defined so that sends s to the smallest possible corresponding t.

Definition A.8. Define $f_K^{t_0} : [0, B_K] \to [t_0, t_0 + 2\pi]$ as $f_K^{t_0}(s) = \min\{t \geq t_0 : \sigma_K((t_0, t]) \geq s\}$.

It is rudimentary to check that $f_K^{t_0}$ is well-defined. We remark that $f_K^{t_0}$ is the minimum inverse $g_K^{t_0\wedge}$ of $g_K^{t_0}$ in [3]. Note the following.

Proposition A.32. *The functions $f_K^{t_0}$ and $g_K^{t_0}$ are monotonically increasing.*

Proof. That $g_K^{t_0}(t)$ is increasing is immediate from Definition A.7. For any $t_1 < t_2$, observe

$$\{t_1 \geq t_0 : \sigma_K((t_0, t]) \geq s\} \subseteq \{t_2 \geq t_0 : \sigma_K((t_0, t]) \geq s\}$$

so by Definition A.8 we have $f_K^{t_0}(t_1) \leq f_K^{t_0}(t_2)$. ☐

The following can be checked using Definition A.8.

Proposition A.33. *We have $f_K^{t_0}(0) = t_0$ and $f_K^{t_0}(s) > t_0$ for all $s > 0$.*

Proof. That $f_K^{t_0}(0) = t_0$ is immediate from Definition A.8. If $s > 0$, then any $t \geq t_0$ satisfying $\sigma_K((t_0, t]) \geq s$ has to satisfy $t > t_0$, so we have $f_K^{t_0}(s) > t_0$. ☐

We will often write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g in proofs because our K and t_0 are fixed. With the converstions between s and t prepared (f maps s to t, and g maps t to s), the path $\mathbf{b}_K^{t_0}$ can be defined by integrating the unit vector u_t for each s.

Definition A.9. Define $\mathbf{b}_K^{t_0} : [0, B_K] \to \mathbb{R}^2$ as the absolutely continuous (and thus rectifiable) function with the initial condition $\mathbf{b}_K^{t_0}(0) = v_K^+(t_0)$ and the derivative $\left(\mathbf{b}_K^{t_0}\right)'(s) = v_{f_K^{t_0}(s)}$ almost everywhere. That is:

$$\mathbf{b}_K^{t_0}(s) := v_K^+(t_0) + \int_{s' \in (0,s]} v_{f_K^{t_0}(s')}\, ds'$$

Note that in Definition A.9, the function $f_K^{t_0}$ is monotone so the integral is well-defined.

Proposition A.34. *The function $\mathbf{b}_K^{t_0} : [0, B_K] \to \mathbb{R}^2$ is an arc-length parametrization.*

Proof. Length of an absolutely continuous curve $\mathbf{x} : [a, b] \to \mathbb{R}^2$ is the integral of $||\mathbf{x}'(s)||$ from $s = a$ to $s = b$ [6]. For $\mathbf{x} = \mathbf{b}_K^{t_0}$, we have $||\mathbf{x}'(s)|| = 1$ for almost every s by Definition A.9, thus completing the proof. ☐

We define $\mathbf{b}_K^{t_0,t_1}$ as an initial segment of $\mathbf{b}_K^{t_0}$.

Definition A.10. For any $t_0, t_1 \in \mathbb{R}$ such that $t_1 \in [t_0, t_0 + 2\pi]$, define $\mathbf{b}_K^{t_0,t_1}$ as the curve $\mathbf{b}_K^{t_0}(s)$ restricted on the interval $s \in [0, g_K^{t_0}(t_1)]$.

A.4.2 Theorems on parametrization

We now show that $\mathbf{b}_K^{t_0}$ does parametrize our boundary ∂K as intended. We prepare three technical lemmas that handle conversions between s and t.

Lemma A.35. *The followings hold.*

1. *For any $t_1 \in (t_0, t_0 + 2\pi]$, we have $\left(f_K^{t_0}\right)^{-1}([t_0, t_1]) = [0, \sigma_K((t_0, t_1])] = [0, g_K^{t_0}(t_1)]$.*
2. *Moreover, the set $\left(f_K^{t_0}\right)^{-1}(\{t_1\})$ is either $[g_K^{t_0}(t_1-), g_K^{t_0}(t_1)]$ or $(g_K^{t_0}(t_1-), g_K^{t_0}(t_1)]$.*

Proof. Write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g. The first statement comes from manipulating the definitions as the following.

$$\begin{aligned} f^{-1}([t_0, t_1]) &= \{s \in [0, B_K] : \min\{t \geq t_0 : \sigma_K((t_0, t]) \geq s\} \in [t_0, t_1]\} \\ &= \{s \in [0, B_K] : \sigma((t_0, t_1]) \geq s\} \\ &= [0, \sigma_K((t_0, t_1])] = [0, g(t_1)] \end{aligned}$$

Now send $t \to t_1^-$ in the equality $f^{-1}([t_0, t]) = [0, g(t)]$ to obtain that $f^{-1}([t_0, t_1)) = \bigcup_{t < t_1} [0, g(t)]$ is either $[0, g(t_1-))$ or $[0, g(t_1-)]$. Then use $f^{-1}(\{t_1\}) = f^{-1}([t_0, t_1]) \setminus f^{-1}([t_0, t_1))$ to get the second statement. ☐

Lemma A.36. *The measure σ_K on $(t_0, t_0 + 2\pi]$ is the pushforward of the Lebesgue measure on $(0, B_K]$ with respect to the map $f_K^{t_0} : (0, B_K] \to (t_0, t_0 + 2\pi]$ restricted to $(0, B_K]$.*

Proof. Write $f_K^{t_0}$ as f. Observe that f restricted to $(0, B_K]$ has range in $(t_0, t_0 + 2\pi]$ by Proposition A.33. The first statement of Lemma A.35 then shows that the measure σ_K on $(t_0, t_0 + 2\pi]$ and the pushforward of the Lebesgue measure on $(0, B_K]$ with respect to $f : (0, B_K] \to (t_0, t_0 + 2\pi]$ agree on every closed interval $(t_0, t]$ for all $t \in (t_0, t_0 + 2\pi]$. ☐

Lemma A.37. $\mathbf{b}_K^{t_0}(g_K^{t_0}(t)) = v_K^+(t)$ *for all* $t \in [t_0, t_0 + 2\pi]$ *and* $\mathbf{b}_K^{t_0}(g_K^{t_0}(t-)) = v_K^-(t)$ *for all* $t \in (t_0, t_0 + 2\pi]$. *Moreover, for all $t \in (t_0, t_0 + 2\pi]$ the function $\mathbf{b}_K^{t_0}$ restricted to the interval $[g_K^{t_0}(t_1-), g_K^{t_0}(t_1)]$ is the arc-length parametrization of the edge $e_K(t)$ from vertex $v_K^-(t)$ to $v_K^+(t)$.*

Proof. Write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g. By Lemma A.36 and Theorem A.19, we have the following calculation.

$$\begin{aligned}
\mathbf{b}_K^{t_0}(g(t)) &= v_K^+(t_0) + \int_{s' \in (0, g(t)]} v_{f(s')}\, ds' \\
&= v_K^+(t_0) + \int_{s' \in f^{-1}([t_0, t])} v_{f(s')}\, ds' \\
&= v_K^+(t_0) + \int_{t \in (t_0, t]} v_t\, \sigma(dt) = v_K^+(t)
\end{aligned}$$

For the proof of $\mathbf{b}_K^{t_0}(g_K^{t_0}(t-)) = v_K^-(t)$, send $t' \to t^-$ in the equation $\mathbf{b}_K^{t_0}(g_K^{t_0}(t')) = v_K^+(t')$ and use Theorem A.4. By the second statement of Lemma A.35, the value $f(s')$ on the interval $s' \in (g(t-), g(t)]$ is always equal to t. So the derivative of $\mathbf{b}_K^{t_0}(s')$ restricted to the interval $[g(t-), g(t)]$ is almost everywhere equal to v_t, and $\mathbf{b}_K^{t_0}$ is the arc-length parametrization of the edge $e_K(t)$ from vertex $v_K^-(t)$ to $v_K^+(t)$ on the interval $[g(t-), g(t)]$. ☐

We now prove the claimed theorems on $\mathbf{b}_K^{t_0,t_1}$. That $\mathbf{b}_K^{t_0,t_1}$ is injective will be proved later.

Proof. (of Theorem A.25) Write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g. By Lemma A.35, the interval $[0, g(t_1)]$ is equal to the inverse image $f^{-1}([t_0, t_1])$, and so is the disjoint union of the singleton $f^{-1}(\{t_0\}) = \{0\}$ and the intervals $f^{-1}(\{t\})$ whose closure is $[g(t-), g(t)]$ for all $t \in (t_0, t_1]$. Under the map $\mathbf{b}_K^{t_0}$, the singleton $\{0\}$ maps to $\{v_K^+(t_0)\}$ and the set $[g(t-), g(t)]$ maps to $e_K(t)$ for all $t \in (t_0, t_1]$ by Lemma A.37. This proves that the image of the interval $[0, g(t_1)]$ under the map $\mathbf{b}_K^{t_0}$ is the set $\{v_K^+(t_0)\} \cup \bigcup_{t \in (t_0, t_1]} e_K(t)$. ☐

Proof. (of Theorem A.26) This comes from Proposition A.34 and that the domain $[0, g_K^{t_0}(t_1)]$ of $\mathbf{b}_K^{t_0,t_1}$ has length $\sigma_K((t_0, t_1])$. ☐

Proof. (of Theorem A.27) Write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g. The curve $\mathbf{b}_K^{t_0,t_1}$ is an initial part of the curve $\mathbf{b}_K^{t_0,t_2}$. So it remains to show that $\mathbf{b}_K^{t_0}$ restricted to the interval $[g(t_1), g(t_2)]$ is the same as $\mathbf{b}_K^{t_1}$ restricted to $[0, g_K^{t_1}(t_2)]$, with the domain shifted to right by $g(t_1)$. Observe $g(t_1) + g_K^{t_1}(t_2) = g(t_2)$ by Definition A.7 and additivity of σ_K. The initial point of the two curves is equal to $v_K^+(t_1)$ by Lemma A.37. We show that the derivatives of $\mathbf{b}_K^{t_0}(t + g(t_1))$ and $\mathbf{b}_K^{t_1}(t)$ match for all $t \in [0, g(t_2) - g(t_1)]$. By Definition A.9, we only need to check $f(t + g(t_1)) = f_K^{t_1}(t)$. This immediately follows from Definition A.8. ☐

Proof. (of Theorem A.28) Write $f_K^{t_0}$ and $g_K^{t_0}$ as simply f and g. Take any $s \in (0, g(t_1)]$ and let $t = f(s)$. Observe that by Proposition A.33, we have $t \in (t_0, t_1]$ and s is in $f^{-1}(\{t\})$ which is either $(g(t_1-), g(t_1)]$ or $[g(t_1-), g(t_1)]$ by Lemma A.35. Then as $\mathbf{b}_K^{t_0}(s) \in e_K(t)$ by Lemma A.37, we have $\mathbf{b}_K^{t_0}(s) \times v_t = p_K(t)$. So we have the following.

$$
\begin{aligned}
\mathcal{I}\left(\mathbf{b}_K^{t_0,t_1}\right) &= \frac{1}{2}\int_{s\in(0,g(t_1)]} \mathbf{b}_K^{t_0}(s)\times v_{f(s)}\,ds \\
&= \frac{1}{2}\int_{s\in f^{-1}((t_0,t_1])} p_K(f(s))\,ds \\
&= \frac{1}{2}\int_{t\in(t_0,t_1]} p_K(t)\,\sigma(dt)
\end{aligned}
$$

The first equality above uses Definition A.9. The second equality above uses Lemma A.35 and $\mathbf{b}_K^{t_0}(s)\times v_t = p_K(t)$. The last equality above uses Lemma A.36. This proves the first equality stated in the theorem. To show the second stated equality, check $v_K(t)\times dv_K^+(t) = v_K^+(t)\times v_t\sigma_K(dt) = p_K(t)\sigma(dt)$ by Corollary A.20. $\square$

A.4.3 Injectivity of parametrization

Proof of Theorem A.30 requires a bit of preparation. The boundary ∂K is the union of all the edges.

Theorem A.38. *Let K be any convex body. Then the topological boundary ∂K of K as a subset of $\mathbb{R}^2$ is the union of all edges $\cup_{t\in S^1} e_K(t)$.*

Proof. Let $E = \cup_{t\in S^1} e_K(t)$. $E\subseteq \partial K$ comes from $E\subseteq K$ and that any point in E is on some tangent line of K so its neighborhood contains a point outside K. Now take any point $p\in\partial K$. As K is closed we have $p\in K$. So $p\cdot u_t \le p_K(t)$ for any $t\in S^1$. Assume that the equality does not hold for any $t\in S^1$. Then by compactness of S^1 and continuity of p_K there is some $\epsilon>0$ such that $\epsilon\le p_K(t)-p\cdot u_t$ for any t. This implies that the ball of radius ϵ centered at p is contained in K. This contradicts $p\in\partial K$. So it should be that there is some $t\in S^1$ such that $p\cdot u_t = p_K(t)$. That is, $p\in e_K(t)$. $\square$

We use the following lemma to determine the orientation of a Jordan curve.

Lemma A.39. *Let p and q be two different points of $\mathbb{R}^2$. Define the closed half-planes H_0 and H_1 as the closed half-planes separated by the line l connecting p and q, so that for any point x in the interior of H_0 (resp. H_1) the points x,p,q are in clockwise (resp. counterclockwise) order. If a Jordan curve J consists of the join of two arcs Γ_0 and Γ_1, where Γ_0 connects p to q inside H_0, and Γ_1 connects q to p inside H_1, then J is positively oriented.*

Proof. (sketch) We first show that it is safe to assume the case where J only intersects l at two points p and q. Observe that H_i has a deformation retract to some subset $S_i\subseteq H_i$ with $S_i\cap l=\{p,q\}$ (push the three segments of $l\setminus\{p,q\}$ towards the interior of H_i). Using the retracts, we may continuously deform the arcs Γ_0 and Γ_1 inside S_0 and S_1 respectively without chainging the orientation of J. Now take any point r inside the segment connecting p and q. Continuously move a point x inside J in the orientation of J starting with $x=p$. As x moves along Γ_0 from p to q the argument of x with respect to r increases by π. And as x moves along Γ_1 the argument of x with respect to r again increases by π. So the total increase in the argument of $x\in J$ is 2π and J is positively oriented. $\square$

We define the following segment of ∂K as well.

Definition A.11. For any $t_0,t_1\in\mathbb{R}$ such that $t_1\in[t_0,t_0+2\pi]$, define $\mathbf{b}_K^{t_0,t_1-}$ as the curve $\mathbf{b}_K^{t_0}(s)$ restricted on the interval $s\in[0,g_K^{t_0}(t_1-)]$.

The following is a corollary of Lemma A.37.

Corollary A.40. *For any $t_0,t_1\in\mathbb{R}$ such that $t_1\in[t_0,t_0+2\pi]$, $\mathbf{b}_K^{t_0,t_1}$ is the concatenation of $\mathbf{b}_K^{t_0,t_1-}$ and the arc-length parametrization of $e_K(t_1)$ from $v_K^-(t_1)$ to $v_K^+(t_1)$.*

By Definition A.9 we have $\left(\mathbf{b}_K^{t_0}\right)'(s) = u_{f_K^{t_0}(s)}$ for almost every s, and by Proposition A.33 and Lemma A.35 we have $t_0 < f_K^{t_0}(s) < t_1$ for every $0<s<g_K^{t_0}(t_1-)$. Thus we have the following:

Corollary A.41. *Let $t_0, t_1 \in \mathbb{R}$ be arbitrary such that $t_1 \in [t_0, t_0 + 2\pi]$. Then for almost every s, the derivative $\left(\mathbf{b}_K^{t_0,t_1-}\right)'(s)$ is equal to u_t for some $t \in (t_0, t_1)$.*

Now we are ready to prove Theorem A.30.

Proof. (of Theorem A.30)

That $\mathbf{b}_K^{t,t+2\pi}$ is an arc-length parametrization of ∂K comes from Theorem A.25 and Theorem A.38.

We now show that $\mathbf{b}_K^{t,t+2\pi}$ is a Jordan curve. By Theorem A.27 the curve $\mathbf{b}_K^{t,t+2\pi}$ is the concatenation of two curves $\mathbf{b}_K^{t,t+\pi}$ and $\mathbf{b}_K^{t+\pi,t+2\pi}$ connecting $p = v_K^+(t)$ and $q = v_K^+(t+\pi)$ and vice versa. As K has nonempty interior, the width of K measured in the direction of u_t is strictly positive, and the point p is strictly further than the point q in the direction of u_t.

We first show that the curve $\mathbf{b}_K^{t,t+\pi}$ is a Jordan arc from p to q. The curve $\mathbf{b}_K^{t,t+\pi}$ is the join of the curve $\mathbf{b}_K^{t,t+\pi-}$ and $e_K(t+\pi)$ by Corollary A.40. Also, by Corollary A.41, the derivative of $\mathbf{b}_K^{t,t+\pi-}(s) \cdot u_t$ with respect to s is strictly positive for almost every s, so the curve $\mathbf{b}_K$ is moving strictly in the direction of $-u_t$. This with the fact that $e_K(t+\pi)$ is parallel to v_t shows that the curve $\mathbf{x}_{K,t,t+\pi}$ is injective and thus a Jordan arc. A similar argument shows that $\mathbf{b}_K^{t+\pi,t+2\pi}$ is also a Jordan arc.

Define the closed half-planes H_0 and H_1 as the half-planes divided by the line l connecting p and q, so that for any point x in the interior of H_0 (resp. H_1) the points x, p, q are in clockwise (resp. counterclockwise) order. Observe that $\mathbf{b}_K^{t,t+\pi}$ (resp. $\mathbf{b}_K^{t+\pi,t+2\pi}$) is in H_0 (resp. H_1) by Theorem A.25. Let $\mathbf{b}$ be either of the curves $\mathbf{b}_K^{t,t+\pi}$ or $\mathbf{b}_K^{t+\pi,t+2\pi}$. Call the line segment connecting p and q as pq. Then $\mathbf{b}$ is either pq (in case $\mathbf{b}$ passes through a point r strictly on pq) or a curve connecting p and q through the interior of H_0 (or H_1). In any case, the curves $\mathbf{b}_K^{t,t+\pi}$ and $\mathbf{b}_K^{t+\pi,t+2\pi}$ only overlap at the endpoints p and q because K has nonempty interior, showing that $\mathbf{b}_K^{t,t+2\pi}$ is a Jordan curve. That $\mathbf{b}_K^{t,t+2\pi}$ is positively oriented is a consequence of Lemma A.39. □

A.4.4 Closed interval

We define a closed-interval variant $\mathbf{b}_K^{t_0-,t_1}$ of $\mathbf{b}_K^{t_0,t_1}$. With following theorems, $\mathbf{b}_K^{t_0-,t_1}$ is essentially the arc-length parametrization of the curve connecting $v_K^-(t_0)$ to $v_K^+(t_1)$ along the boundary ∂K counterclockwise.

Definition A.12. For every $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0+2\pi)$ define $\mathbf{b}_K^{t_0-,t_1}$ as the concatenation of the arc-length parametrization of the edge $e_K(t_0)$ from $v_K^-(t_0)$ to $v_K^+(t_1)$ and the curve $\mathbf{b}_K^{t_0,t_1}$.

This follows from Theorem A.25.

Corollary A.42. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0+2\pi)$. Then $\mathbf{b}_K^{t_0-,t_1}$ is an arc-length parametrization of the set $\cup_{t\in[t_0,t_1]} e_K(t)$ from point $v_K^-(t_0)$ to $v_K^+(t_1)$.*

This follows from Theorem A.26.

Corollary A.43. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi)$. Then the curve $\mathbf{b}_K^{t_0-,t_1}$ have length $\sigma_K([t_0, t_1])$.*

This follows from Theorem A.27.

Corollary A.44. *Assume arbitrary t_0, t_1, t_2 such that $t_0 \le t_1 \le t_2 < t_0 + 2\pi$. Then $\mathbf{b}_K^{t_0-,t_2}$ is the concatenation of $\mathbf{b}_K^{t_0-,t_1}$ and $\mathbf{b}_K^{t_1,t_2}$.*

This follows from Theorem A.28 and Theorem A.21.

Corollary A.45. *Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi)$. Then we have:*

$$\mathcal{I}\left(\mathbf{b}_K^{t_0-,t_1}\right) = \frac{1}{2}\int_{[t_0,t_1]} p_K(t)\,\sigma_K(dt)$$

Theorem A.46. *Assume that K have nonempty interior. Assume arbitrary $t_0 \in \mathbb{R}$ and $t_1 \in [t_0, t_0 + 2\pi)$. Then $\mathbf{b}_K^{t_0-,t_1}$ is one of: a Jordan arc, a Jordan curve, or a single point.*

Proof. Take an arbitrary t_{-1} so that $t_{-1} < t_0 \le t_1 < t_{-1} + 2\pi$. Then $\mathbf{b}_K^{t_{-1},t_1}$ is the concatenation of $\mathbf{b}_K^{t_{-1},t_0-}$, $e_K(t_0)$, and $\mathbf{b}_K^{t_0,t_1}$ by Theorem A.27 and Corollary A.40. Then by Definition A.12 the concatenation of $e_K(t_0)$, and $\mathbf{b}_K^{t_0,t_1}$ is $\mathbf{b}_K^{t_0-,t_1}$. Now by Corollary A.31 and that $\mathbf{b}_K^{t_0-,t_1}$ is a part of $\mathbf{b}_K^{t_{-1},t_1}$, we prove the theorem. □

A.5 Normal angles

This section focuses on the set of *normal angles* of a convex body K defined as the following.

Definition A.13. Define the set of *normal angles* $\mathbf{n}(K)$ as the support of the surface area measure σ_K on S^1.

If K is a convex polygon $\mathbf{n}(K)$ is the collection of all angles t such that each u_t is a normal vector of a proper edge of K. The notion $\mathbf{n}(K)$ generalizes this to arbitrary convex body K. For example, take the semicircle $K = \{(x, y) : x^2 + y^2 \le 1, y \ge 0\}$. Then the normal angles of K is the set $[0, \pi] \cup 3\pi/2$.

We now collect theorems on $\mathbf{n}(K)$.

Lemma A.47. *Let (t_1, t_2) be any open interval of S^1 of length $< \pi$. Then for every $t \in S^1 \setminus (t_1, t_2)$, we have $v_K(t_1, t_2) \in H_K(t)$.*

Proof. Let $p = v_K(t_1, t_2)$. We can either assume $t_1 - \pi < t < t_1$ or $t_2 < t < t_2 + \pi$. In the first case, the points $v_K(t_1, t)$ and p are on the line $l_K(t_1)$ and p is further than $v_K(t_1, t)$ in the direction of v_{t_1}. Since $v_K(t_1, t) \in H_K(t)$ we now should have $p \in H_K(t)$. In the second case, the points $v_K(t, t_2)$ and p are on the line $l_K(t_2)$ and $v_K(t, t_2)$ is further than p in the direction of v_{t_2}. Since $v_K(t, t_2) \in H_K(t)$ we now have $p \in H_K(t)$. □

Theorem A.48. *Let K be a convex body, and let (t_1, t_2) be any open interval of S^1 of length $< \pi$. The followings are equivalent.*

1. *(t_1, t_2) is disjoint from $\mathbf{n}(K)$*
2. *There is a single point p so that we have $v_K^+(t) = v_K^-(t) = p$ for all $t \in (t_1, t_2)$.*
3. *Every tangent line $l_K(t)$ passes through a common point p for $t \in [t_1, t_2]$.*
4. *$v_K(t_1, t_2) \in K$*

Proof. $(1 \Rightarrow 2)$ Let $p = v_K^+(t_1)$. Then $v_K^-(t_2) = p$ as well by Corollary A.23. We also have $p = v_K^{\pm}(t)$ for every $t \in (t_1, t_2)$ because σ is zero on the intervals $(t_1, t]$, (t_1, t) and Theorem A.19 and Corollary A.23 holds on those intervals.

$(2 \Rightarrow 1)$ By Theorem A.4 we also have $v_K^+(t_1) = v_K^-(t_2) = p$. By Corollary A.23 we have the integral $\int_{t \in (t_1,t_2)} v_t \, \sigma_K(dt) = v_K^-(t_2) - v_K^+(t_1)$ equal to 0. Now σ_K has to be zero on (t_1, t_2), or otherwise the integral taken the dot product with $-u_{t_1}$ should be nonzero as well.

$(2 \Rightarrow 3)$ comes from that every edge $e_K(t) = l_K(t) \cap K$ is the segment connecting $v_K^-(t)$ to $v_K^+(t)$.

$(3 \Rightarrow 4)$ The point p that every tangent line $l_K(t)$ of $t \in [t_1, t_2]$ pass through should be $l_K(t_1) \cap l_K(t_2) = v_K(t_1, t_2)$. So we have $p \in H_K(t)$ for all $t \in [t_1, t_2]$. We also have $p \in H_K(t)$ for all $t \in S^1 \setminus (t_1, t_2)$ by Lemma A.47. Now $p \in \bigcap_{t \in S^1} H_K(t) = K$.

$(4 \Rightarrow 2)$ Let $p := v_K(t_1, t_2)$ and define the cone $F = H_K(t_1) \cap H_K(t_2)$ pointed at p. Take any $t \in (t_1, t_2)$. Then the value of $z \cdot u_t$ over all $z \in F$ has a unique maximum at $z = p$. Because $p \in K \subseteq F$, the value of $z \cdot u_t$ over all $z \in K$ also has a unique maximum at $z = p$. This means that $e_K(t) = \{p\}$, completing the proof. □

Theorem A.49. *Let Π be any closed subset of S^1 such that $S^1 \setminus \Pi$ is a disjoint union of open intervals of length $< \pi$. Then for any convex body K, the followings are equivalent.*

1. *$K = \bigcap_{t \in \Pi} H_K(t)$*
2. *$\mathbf{n}(K)$ is contained in Π.*

Proof. (1 ⇒ 2) Let (t_1, t_2) be any connected component of $S^1 \setminus \Pi$. Then the interval has length $< \pi$ by assumption. Now take any $t \in (t_1, t_2)$. The vertex $v_K(t_1, t_2)$ is in $\bigcap_{t\in\Pi} H_K(t) = K$ by Lemma A.47. So by Theorem A.48, (t_1, t_2) is disjoint from $\mathbf{n}(K)$. Since (t_1, t_2) was an arbitrary connected component of $S^1 \setminus \Pi$, we are done.

(2 ⇒ 1) It suffices to show that $\bigcap_{u\in\Pi} H_K(u) \subseteq H_K(t)$ for all $t \in S^1$. Once this is shown, we can take intersection over all $t \in S^1$ to conclude $K \subseteq \bigcap_{u\in\Pi} H_K(u) \subseteq K$.

If $t \in \Pi$, then we obviously have $\bigcap_{u\in\Pi} H_K(u) \subseteq H_K(t)$ so the proof is done. Now take any $t \in S^1 \setminus \Pi$ and let (t_1, t_2) be the connected component of $S^1 \setminus \Pi$ containing t. By condition 4 of Theorem A.48 the half-plane $H_K(t)$ contains the intersection $F := H_K(t_1) \cap H_K(t_2)$. Observe $t_1, t_2 \in \Pi$. So $\bigcap_{u\in\Pi} H_K(u) \subseteq F \subseteq H_K(t)$ for all $t \in S^1 \setminus \Pi$, completing the proof. □

The following theorem is known as the Gauss-Minkowski theorem ([8] or Theorem 8.3.1, p465 of [12]). It gives a bijection between any convex body K and its boundary measure σ_K.

Theorem A.50. *(Gauss-Minkowski) For any finite Borel measure σ on S^1 with $\int_{S^1} v_t \, \sigma(dt) = 0$ there is a unique convex body K with $\sigma_K = \sigma$ up to translations of K.*

By Definition A.13, we immediately get the following restriction of Theorem A.50.

Corollary A.51. *Let Π be any closed subset of S^1. For any finite Borel measure σ on Π such that $\int_\Pi v_t \, \sigma(dt) = 0$, there is a convex body K with normal angles $\mathbf{n}(K)$ in Π such that $\sigma_K|_\Pi = \sigma$, which is unique up to translations of K.*

A.6 Mamikon's theorem

We prove a version Mamikon's theorem [9] that works for the boundary segment $\mathbf{b}_K^{t_0,t_1}$ of any convex body K that may not be differentiable.

Theorem A.52. *(Mamikon) Let K be an arbitrary convex body. Let $t_0, t_1 \in \mathbb{R}$ be any angles such that $t_0 < t_1 \le t_0 + 2\pi$. Note that $\mathbf{b}_K^{t_0,t_1}$ is the counterclockwise curve along ∂K from $p := v_K^+(t_0)$ to $q := v_K^+(t_1)$. Let $\mathbf{y} : [t_0, t_1] \to \mathbb{R}^2$ be any curve that is continous and rectifiable, such that for all $t \in [t_0, t_1]$ the point $\mathbf{y}(t)$ is always on the tangent line $l_K(t)$. Consequently, there is a measurable function $f : [t_0, t_1] \to \mathbb{R}$ such that $\mathbf{y}(t) = v_K^+(t) + f(t)v_t$ for all $t \in [t_0, t_1]$. Then the following holds.*

$$\mathcal{I}(\mathbf{y}) + \mathcal{I}(\mathbf{y}(t_1), q) - \mathcal{I}\left(\mathbf{b}_K^{t_0,t_1}\right) - \mathcal{I}(\mathbf{y}(t_0), p) = \frac{1}{2}\int_{t_0}^{t_1} f(t)^2 \, dt$$

Proof. For this proof only, let $\mathbf{x} := v_K^+$ be the alias of $v_K^+ : [t_0, t_1] \to \mathbb{R}^2$. First, we prove a differential version of the theorem by calculating differentials on the interval $(t_0, t_1]$. Note that $\mathbf{y}$ is continuous by definition and $\mathbf{x}$ is right-continuous by Theorem A.4, so that $\mathbf{y} \times \mathbf{x}$ is also right-continuous on $(t_0, t_1]$. So the Lebesgue-Stieltjes measure $d(\mathbf{y} \times \mathbf{x})$ makes sense as a Lebesgue-Stieltjes measure on $(t_0, t_1]$. Now the following is a chain of equality of measures on $(t_0, t_1]$.

$$\begin{aligned}
&\mathbf{y}(t) \times d\mathbf{y}(t) - \mathbf{x}(t) \times d\mathbf{x}(t) + d\left(\mathbf{y}(t) \times \mathbf{x}(t)\right) \\
&= \mathbf{y}(t) \times d\mathbf{y}(t) - \mathbf{x}(t) \times d\mathbf{x}(t) + \left(d\mathbf{y}(t) \times \mathbf{x}(t) + \mathbf{y}(t) \times d\mathbf{x}(t)\right) \\
&= \left(\mathbf{y}(t) - \mathbf{x}(t)\right) \times d\left(\mathbf{y}(t) + \mathbf{x}(t)\right) \\
&= \left(\mathbf{y}(t) - \mathbf{x}(t)\right) \times d\left(\mathbf{y}(t) - \mathbf{x}(t)\right) \\
&= f(t)u_t \times d\left(f(t)u_t\right) = f(t)u_t \times \left(u_t df(t) + f(t)v_t dt\right) = f(t)^2 dt
\end{aligned}$$

The first equality uses the product rule of differentials. The second equality is an rearrangement using linearity (note that $d\mathbf{y}(t) \times \mathbf{x}(t) = -\mathbf{x}(t) \times d\mathbf{y}(t)$ by antisymmetry of $\times$). As we have $d\mathbf{x}(t) = \sigma(dt)v_t$ by Corollary A.20 and $\mathbf{y}(t) - \mathbf{x}(t) = f(t)v_t$, they are parallel and we get $(\mathbf{y}(t) - \mathbf{x}(t)) \times d\mathbf{x}(t) = 0$ which is used in the third equality. The last chain of equalities are basic calculations.

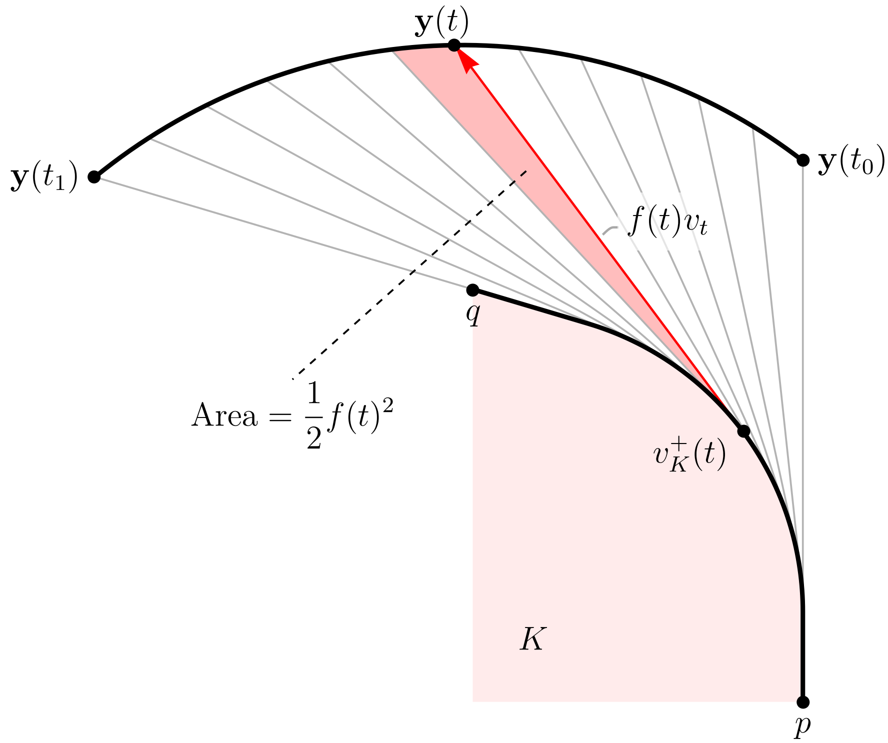

Figure 7: An illustration of Mamikon's theorem (Theorem A.52).

If we integrate the differential formula above on the whole interval $(t_0, t_1]$, the terms $\mathbf{y}(t) \times d\mathbf{y}(t)$ and $\mathbf{y}(t) \times d\mathbf{y}(t)$ becomes $2\mathcal{I}(\mathbf{y})$ and $2\mathcal{I}(\mathbf{x})$ respctively by Definition 5.2 and . The Lebesgue-Stieltjes measure $d(\mathbf{y}(t) \times \mathbf{x}(t))$ integrates to the difference $2\mathcal{I}\left(\mathbf{y}(t_1), v_K^+(t_1)\right) - 2\mathcal{I}\left(\mathbf{y}(t_0), v_K^+(t_0)\right)$. So the integral matches twice the left-hand side of the claimed equality. We conclude the proof by dividing by two. $\square$

We have the following variant on the curve $\mathbf{b}_K^{t_0-,t_1}$ as well.

Theorem A.53. *Let K be an arbitrary convex body. Let $t_0, t_1 \in \mathbb{R}$ be any angles such that $t_0 < t_1 < t_0 + 2\pi$. Note that $\mathbf{b}_K^{t_0-,t_1}$ is a curve along ∂K from $p := v_K^-(t_0)$ to $q := v_K^+(t_1)$. Let $\mathbf{y} : [t_0, t_1] \to \mathbb{R}^2$ and $f : [t_0, t_1] \to \mathbb{R}$ be as in Theorem A.52. Then the following holds.*

$$\mathcal{I}(\mathbf{y}) + \mathcal{I}\left(\mathbf{y}(t_1), q\right) - \mathcal{I}(\mathbf{b}_K^{t_0-,t_1}) - \mathcal{I}\left(\mathbf{y}(t_0), p\right) = \frac{1}{2}\int_{t_0}^{t_1} f(t)^2\, dt$$

Proof. Apply Theorem A.52 to $\mathbf{b}_K^{t_0,t_1}$, and use that $\mathbf{b}_K^{t_0-,t_1}$ is the join of $e_K(t_0)$ and $\mathbf{b}_K^{t_0,t_1}$. $\square$

A.6.1 Mamikon's theorem on tangent lines

Here, we describe a variant of Mamikon's theorem (Theorem A.55) where $\mathbf{y}(t)$ parametrizes a segment of a tangent line of K. To do so, we need to prepare some notation.

Definition A.14. Let $t, t' \in S^1$ be arbitrary such that $t' \neq t, t + \pi$. Define $\tau_K(t, t')$ as the unique value α such that $v_K(t, t') = v_K^+(t) + \alpha v_t$.

Note that $v_K(t, t')$ was defined as the intersection of $l_K(t)$ and $l_K(t')$ (Definition A.2). So fixing the line $l_K(t')$ at angle t', the value $\tau_K(t, t')$ measures the distance from $v_K^+(t)$ to $v_K(t, t')$ along the line $l_K(t)$ at angle t. Such a value α exists because the points $v_K(t, t')$ and $v_K^+(t)$ are on the line $l_K(t)$. Linearity of $\tau_K(t, t')$ comes from Lemma A.3 and Corollary A.6.

Corollary A.54. *Let $t, t' \in S^1$ be arbitrary such that $t' \neq t, t+\pi$. Then $\tau_K(t,t')$ is linear with respect to K.*

Theorem A.55. *Let K be an arbitrary convex body. Let $t_0, t_1 \in \mathbb{R}$ be the angles such that $t_0 < t_1 < t_0 + \pi$. Note that $\mathbf{b}_K^{t_0,t_1}$ is the counterclockwise curve along ∂K from $p := v_K^+(t_0)$ to $q := v_K^+(t_1)$. Let $r = l_K(t_0) \cap l_K(t_1)$. The following holds.*

$$\mathcal{I}(r,q) - \mathcal{I}\left(\mathbf{b}_K^{t_0,t_1}\right) - \mathcal{I}\left(r,p\right) = \frac{1}{2}\int_{t_0}^{t_1} \tau_K(t_0,t_1)^2\, dt$$

Proof. Define $\mathbf{y} : [t_0,t_1] \to \mathbb{R}^2$ as $\mathbf{y}(t) = l_K(t) \cap l_K(t_1)$ for every $t < t_1$ and $\mathbf{y}(t_1) = v_K^-(t_1)$. Then $\mathbf{y}$ is absolutely continuous by Theorem A.8 and parametrizes the line segment from r to $v_K^-(t_1)$. So $f(t)$ is integrable as well. Apply Theorem A.52 to the curves $\mathbf{b}_K^{t_0,t_1}$ and $\mathbf{y}$ to get the following.

$$\mathcal{I}(\mathbf{y}) + \mathcal{I}\left(v_K^-(t_1), q\right) - \mathcal{I}\left(\mathbf{b}_K^{t_0,t_1}\right) - \mathcal{I}\left(r,p\right) = \frac{1}{2}\int_{t_0}^{t_1} \tau_K(t_0,t_1)^2\, dt$$

Now use $\mathcal{I}(\mathbf{y}) = \mathcal{I}(r, v_K^-(t_1))$ to conclude the proof. ∎

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