CME 108 / MATH 114
Problem Set 7
due: Friday, March 2, 2018

Recommended Reading:

1. *Composite Simpson rule:* Consider a quadratic interpolant $p(x)$ to the function $y = f(x)$ passing through the three points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}), where the function values are $y_i = f(x_i)$ and similarly for the other points. Define the grid spacings $h_i = x_i - x_{i-1}$ and $h_{i+1} = x_{i+1} - x_i$; do not assume that $h_i = h_{i+1}$. Last week you wrote the interpolant in terms of the Lagrange polynomials; start from that expression.

   (a) (2 pt) Analytically calculate
   $$\int_{x_{i-1}}^{x_{i+1}} p(x)dx. \tag{1}$$
   Write this in terms of the function values and grid spacings only. This expression is a generalization of the Simpson rule for unevenly spaced data points. *Hint: Avoid lots of algebra by changing the integration variable to* $z = x - x_{i-1}$.

   (b) (1 pt) Check that your formula simplifies to the familiar one for uniform grid spacing: $h_i = h_{i+1}$.

   (c) (5 pt) Now consider approximating
   $$I = \int_a^b f(x)dx \tag{2}$$
   using a composite Simpson rule for $N + 1$ (possibly unevenly spaced) points ($x_0 = a, x_1, \ldots, x_{N-1}, x_N = b$) based on the basic formula you derived in (a). Assume N is even. Implement your method in MATLAB and use the function to approximate
   $$\int_0^1 \sin(4\pi x^2)dx \tag{3}$$
   with $N = 16$. Do this for both evenly spaced points and unevenly spaced points, `x=sqrt(linspace(0,1,N+1))`. Which result is more accurate and why?

2. *Quadrature rule for singular integrands:* (6 pt) Numerically approximate

$$I = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{g(x)dx}{\sqrt{1-x^2}} \tag{4}$$

for smooth and nonsingular $g(x)$ using a quadrature rule of the form

$$I \approx w_{-1}g_{-1} + w_0 g_0 + w_1 g_1, \tag{5}$$

where $g_{-1} = g(-1)$, $g_0 = g(0)$, $g_1 = g(1)$, and the w_i are weights that are obtained by requiring that the method exactly integrate all functions of the form

$$f(x) = \frac{a + bx + cx^2}{\sqrt{1-x^2}}. \tag{6}$$

Determine the weights and then use your method to approximate

$$\int_{4}^{9} \frac{\sin(x)dx}{\sqrt{25x-(x+6)^2}}. \tag{7}$$

You might find the following useful:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi, \quad \int_{-1}^{1} \frac{xdx}{\sqrt{1-x^2}} = 0, \quad \int_{-1}^{1} \frac{x^2dx}{\sqrt{1-x^2}} = \frac{\pi}{2}. \tag{8}$$

3. *Hiking:* How long does it take to climb a mountain? Let

$$z(x,y) = -0.1e^{y-(x-1)^2}\sin(3\pi y/2) \tag{9}$$

be the elevation as a function of the two horizontal coordinates x and y. You start at $(0,0)$ and climb to $(1,1)$ along the path given by the vectors `X`, `Y` (load them by putting `hiking_trail.mat` in your path and typing `load hiking_trail` or using the ASCII file `hiking_trail.txt`). Assume that your velocity v depends on the slope m as $v = e^{-m}$ (steeper=slower).

   (a) (2 pt) Write an expression for the hiking time in terms of an integral over the path. *Hint: What is the distance between (x,y,z) and $(x+dx, y+dy, z+dz)$ in the limit that dx, dy, dz approach zero?*

   (b) (7 pt) Devise a numerical method to approximate this integral. Describe this method in your written report. Implement the method and calculate the time it takes to hike this trail. *Hint: Approximate derivatives with finite differences.*