Chapter 10

Rotation of a Rigid Object about a Fixed Axis

Outline for W10,D3

Finish center of mass (Ch. 9)

Rotation of a rigid solid (Ch. 10)

θ, ω, and α

Relation between linear (s,v,a) and angular quantities

Torque

Homework

Ch. 10 P. 1,4-6,19-21,25,28-30,34,35,37,53,54,55,64,67,69

Do for Mon/Wed

Notes:

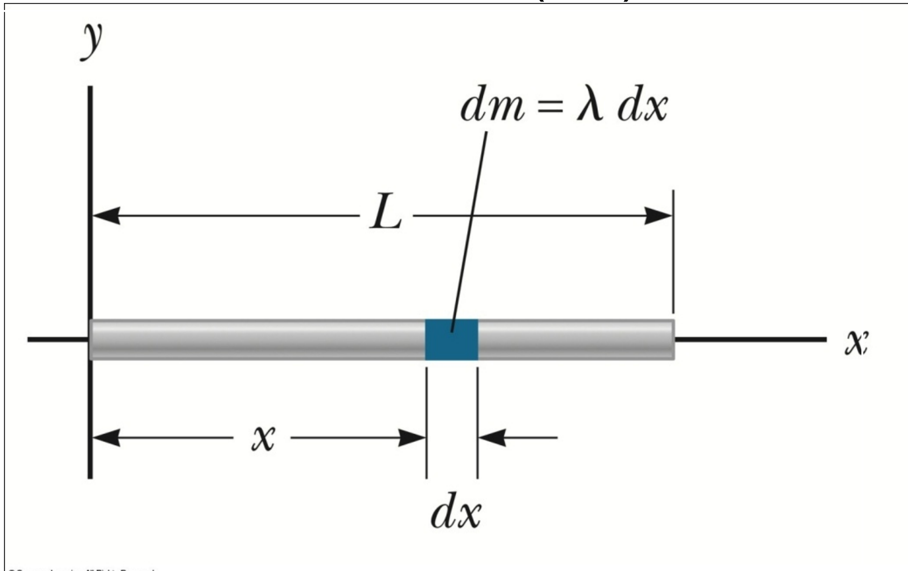
Lab this week is “2D Collisions”

See “NEW STUFF” for Ch. 10.

Center of Mass, Rod

Ex) Find the COM of a non-uniform rod of length 1.0 m if its linear mass distribution is $\lambda(x)=3x+1$ kg/m, where x=0 at the origin.

- As before, rod is aligned with the x-axis, with one end on (0,0), and $y_{COM} = z_{COM} = 0$.



Do integral using $\lambda = 3x+1$

$$x_{COM} = \frac{1}{M}\int_0^L x\lambda\, dx$$

$$x_{COM} = \frac{1}{M}\int_0^{1.0} x(3x+1)\, dx$$

$$x_{COM} = \frac{1}{M}\left(x^3 + \frac{x^2}{2}\right)\Big|_0^{1.0}$$

So x_{com} = 1.5/M, but what is M? M is the total mass.

M= $\int_0^{1.0} \lambda\, dx = \int_0^{1.0} (3x+1)\, dx = \left(\frac{3x^2}{2} + x\right)\Big|_0^{1.0}$ = 5/2.

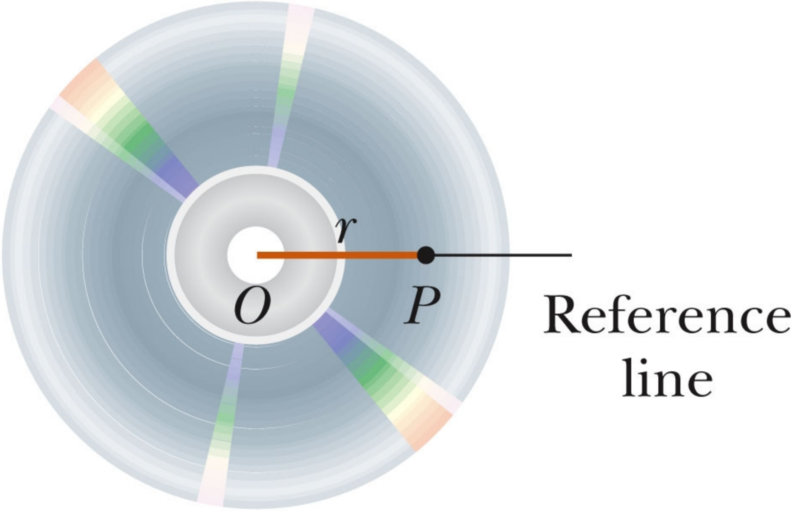
So x_{com}=(3/2)/(5/2) = 3/5 =0.6 m

Rigid Object

- A rigid object is one that is nondeformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible
- This simplification allows analysis of the motion of an extended object

Angular Position

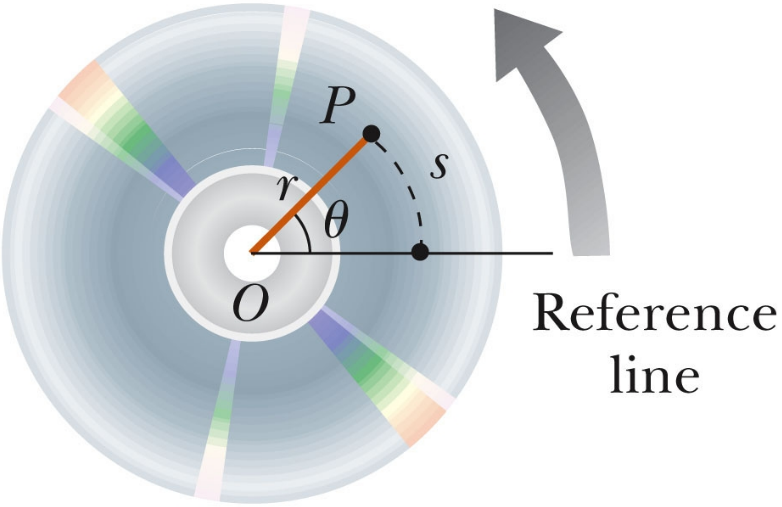
- Axis of rotation runs through the center of the disc, $\perp$ the disk.
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin



(a)

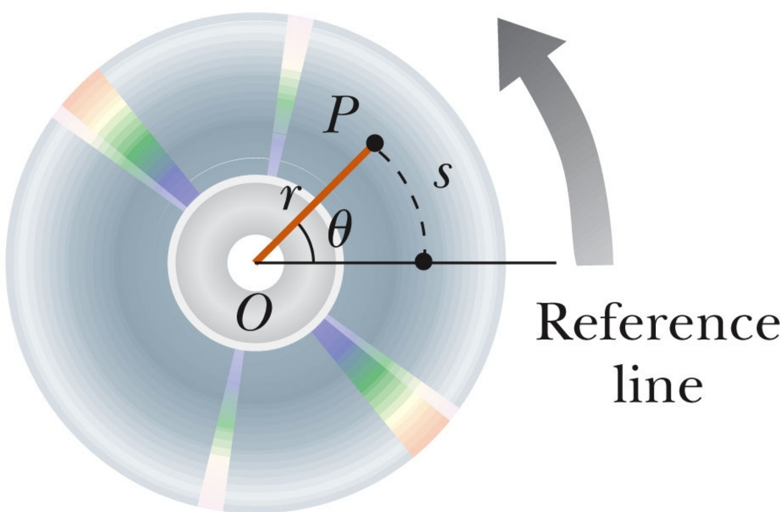
Angular Position, 2

- Point *P* will rotate about the origin in a circle of radius *r*
- ***Every*** point on the disc undergoes circular motion about the center.
- Specify the position of point *P* in polar coordinates (r, θ) where θ is the measured counterclockwise from the reference line.



Angular Position, 3

- As the particle moves through θ, it moves though an arc length s.
- The arc length and r are related:
  - $s = \theta r$
  - where θ is in radians



(b)

The Radian

- This can also be expressed as

$$\theta = \frac{s}{r}$$

- θ is dimensionless, but is expressed in units of *radians* (rad).
- Ex) How many radians are subtended by an arc length of 6 inches if the radius of the arc is 3 in?
- Ex) How many radians are subtended by an arclength of 3 in if the radius is 3 in?
  - Try to estimate how many degrees this is!

Conversions

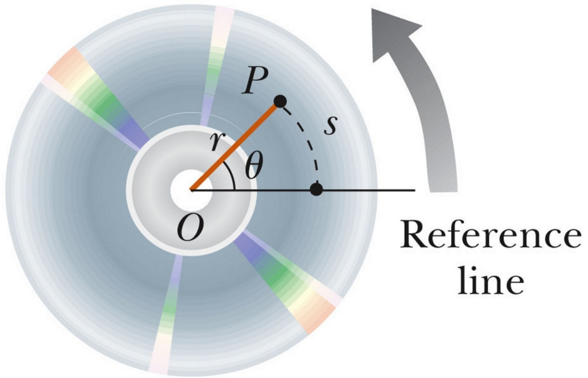
- Comparing degrees and radians

$$1\,rad = \frac{360^\circ}{2\pi} \simeq 57.3^\circ$$

- Converting from degrees to radians

$$\theta(rad) = \frac{\pi}{180^\circ}\,\theta(degrees)$$

Angular Position, final



- So the *angular position* of a point P on an object is the angle θ, measured in radians or degrees.
- θ is the angle between a radial line running from the spin axis to P, and a reference line (usually the x-axis) also running through the spin axis.

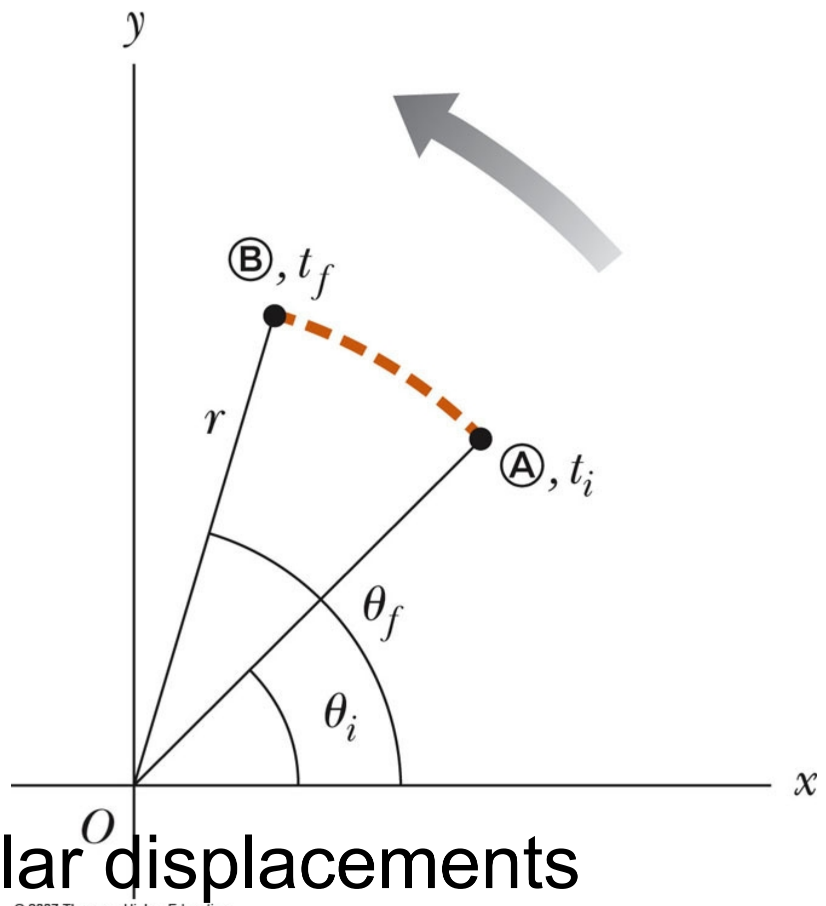
DEMO: My CD has two points along the same radial line. How do their angular positions compare?

Angular Displacement

- The *angular displacement* is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- This is the angle that the radial line of length *r* sweeps out.



DEMO: How do the angular displacements of the two dots on the CD compare?

Average Angular Speed

- The *average* angular speed, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Speed

- The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular Speed, final

- Units of angular speed are radians/sec
  - rad/s or s^{-1} since radians have no dimensions
- Angular speed will be positive if θ is increasing (counterclockwise)
- Angular speed will be negative if θ is decreasing (clockwise)

Average Angular Acceleration

- The average angular acceleration, α,

  of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

Angular Acceleration, final

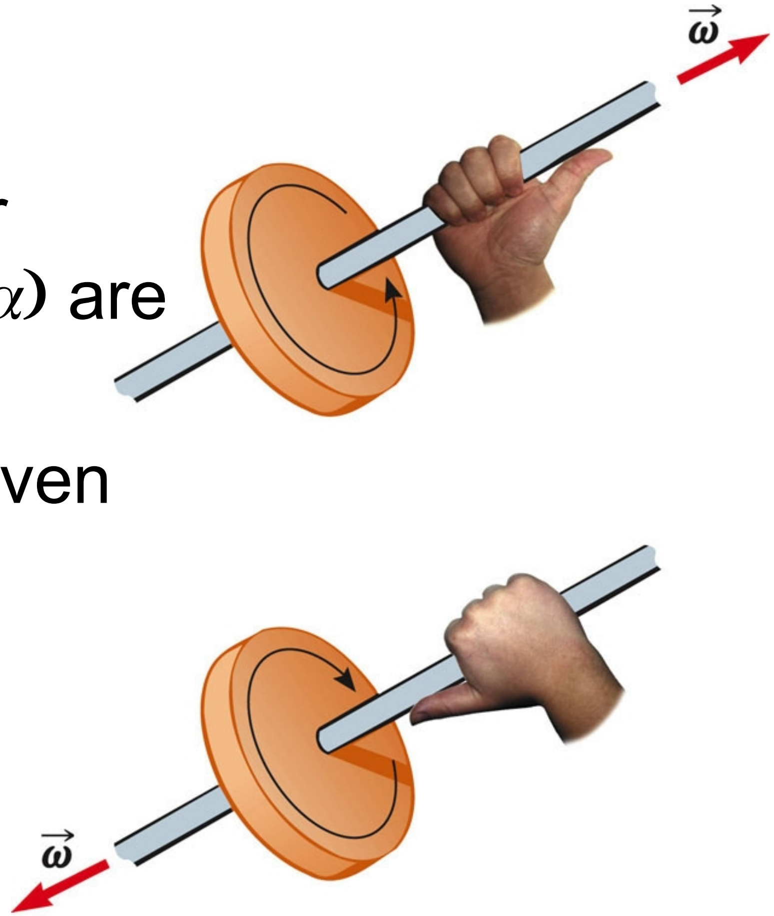
- Units of angular acceleration are rad/s² or s^{-2} since radians have no dimensions
- Angular acceleration will be positive if an object rotating counterclockwise is speeding up
- Angular acceleration will also be positive if an object rotating clockwise is slowing down

Angular Motion, mini-quiz

- T or F. The $\Delta\theta$, ω, and α are the same for every point on a rigid solid.
- T or F. The θ, $\Delta\theta$, ω, and α are the same for every point on a rigid solid.
- What is the ω_{avg} (in rad/sec) of a wheel that rotates 1 revolution in 2 seconds?
- If a CD spins up from 0 to 50 rad/s in 5 seconds, what is the α_{avg}?

Directions, details

- Strictly speaking, the angular speed and acceleration (ω, α) are the magnitudes of vectors
- The directions are actually given by the right-hand rule.



Rotational Kinematics

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations
- The new model is a **rigid object under constant angular acceleration**
  - Analogous to the particle under constant acceleration model

Rotational Kinematic Equations

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

$$\theta_f = \theta_i + \frac{1}{2}\left(\omega_i + \omega_f\right)t$$

all with consant α

Comparison Between Rotational and Linear Equations

TABLE 10.1

Kinematic Equations for Rotational and Translational Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis	Translational Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Relationship Between Angular and Linear Quantities

- Path lengths

$$s = \theta r$$

- Speeds

$$v = \omega r$$

- Accelerations

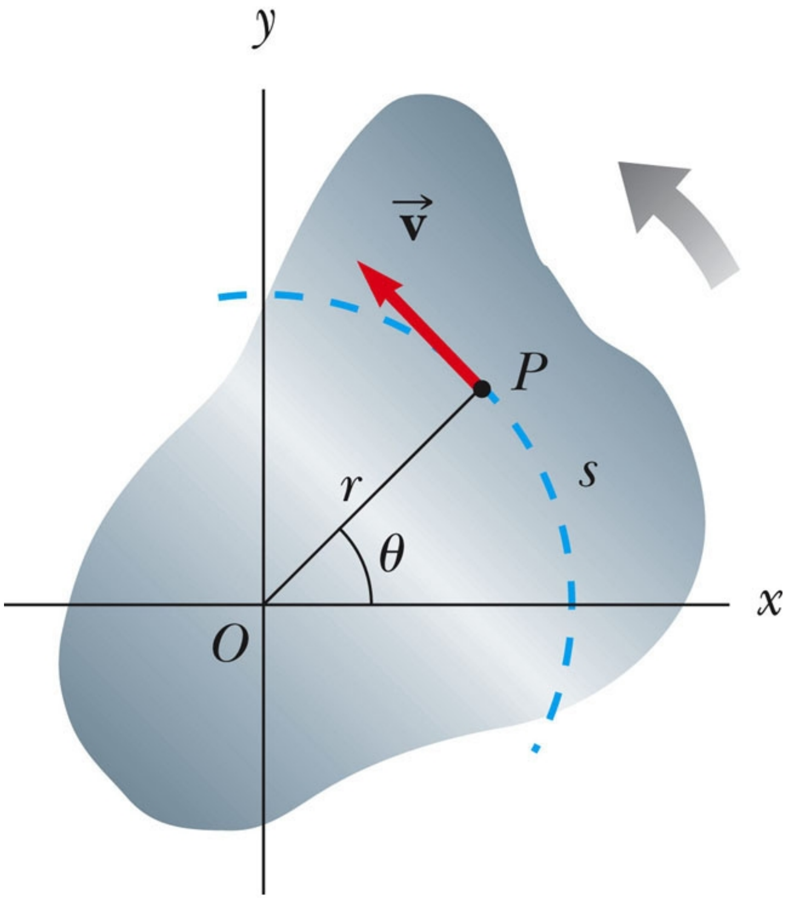
$$a = \alpha r$$

- Every point on the rotating object has the same angular motion
- Every point on the rotating object does **not** have the same linear motion

Speed Comparison

- The linear velocity is always tangent to the circular path
  - Called the tangential velocity
- The magnitude is defined by the tangential speed

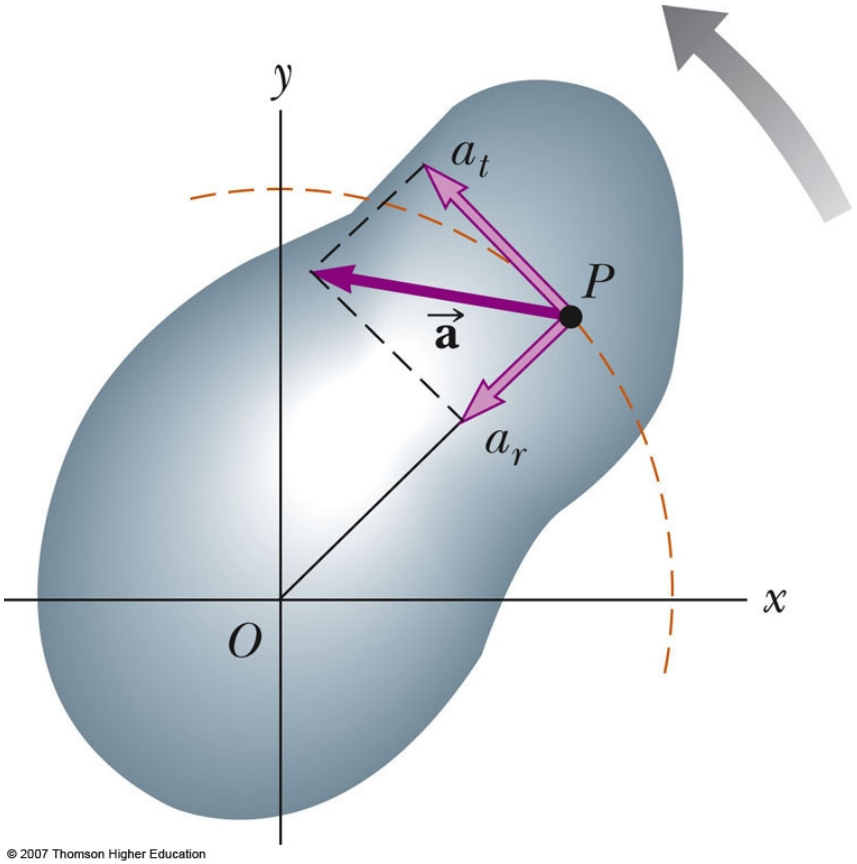
$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$



Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
  - Therefore, each point on a rotating rigid object will experience a centripetal acceleration

$$a_C = \frac{v^2}{r} = r\omega^2$$

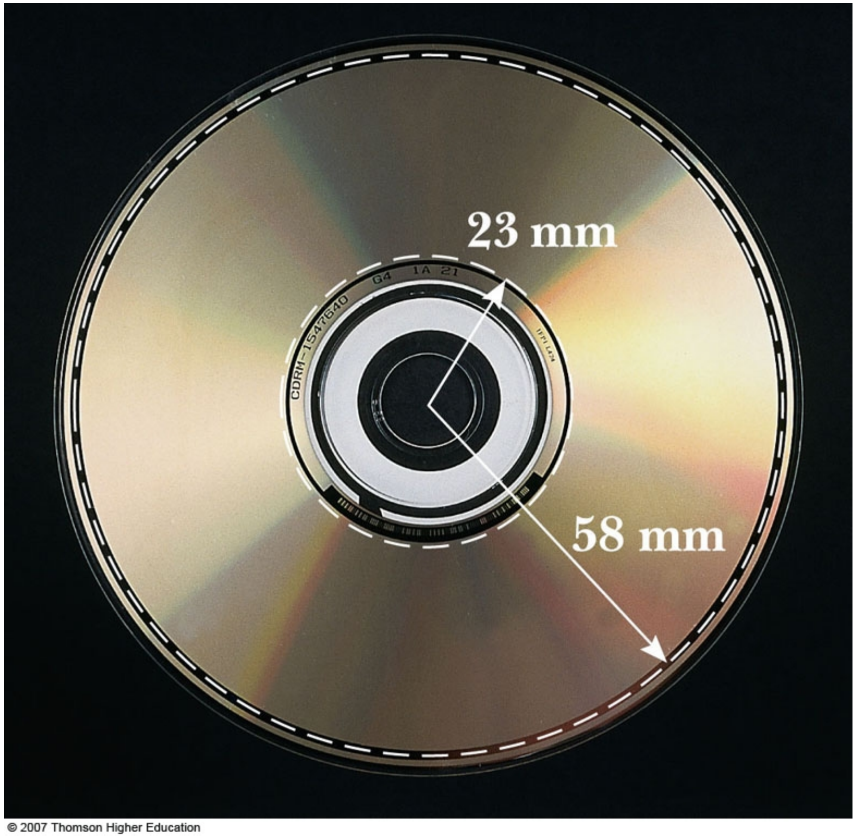
Resultant Acceleration

- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Rotational Motion Example

- For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant ($v_t = \omega r$)
- At the inner sections, the angular speed is faster than at the outer sections



Ex) Find v_t at r=23mm if it spins at 500 RPM. v_t=52.4*.023=1.21m/s

Ex) Find v_t at r=58mm if it spins at 200 RPM. v_t=20.9*.058=1.21m/s

Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, ω, has rotational kinetic energy even though it may not have any translational kinetic energy
- Each particle has a kinetic energy of
  - $K_i = \frac{1}{2} m_i v_i^2$
- Since the tangential velocity depends on the distance, r, from the axis of rotation, we can substitute $v_i = \omega_i r$

Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I \omega^2$$

- Where *I* is called the moment of inertia

Rotational Kinetic Energy, final

- There is an analogy between the kinetic energies associated with linear motion ($K = ½\ mv^2$) and the kinetic energy associated with rotational motion ($K_R = ½\ I\omega^2$)
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object
- The units of rotational kinetic energy are Joules (J)

Moment of Inertia

- The definition of moment of inertia is
$$I = \sum_i r_i^2 m_i$$
- The dimensions of moment of inertia are ML^2 and its SI units are $kg\cdot m^2$
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass Δm_i

Moment of Inertia, cont

- We can rewrite the expression for *I* in terms of Δm

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- With the small volume segment assumption,

$$I = \int \rho r^2 dV$$

- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known

Notes on Various Densities

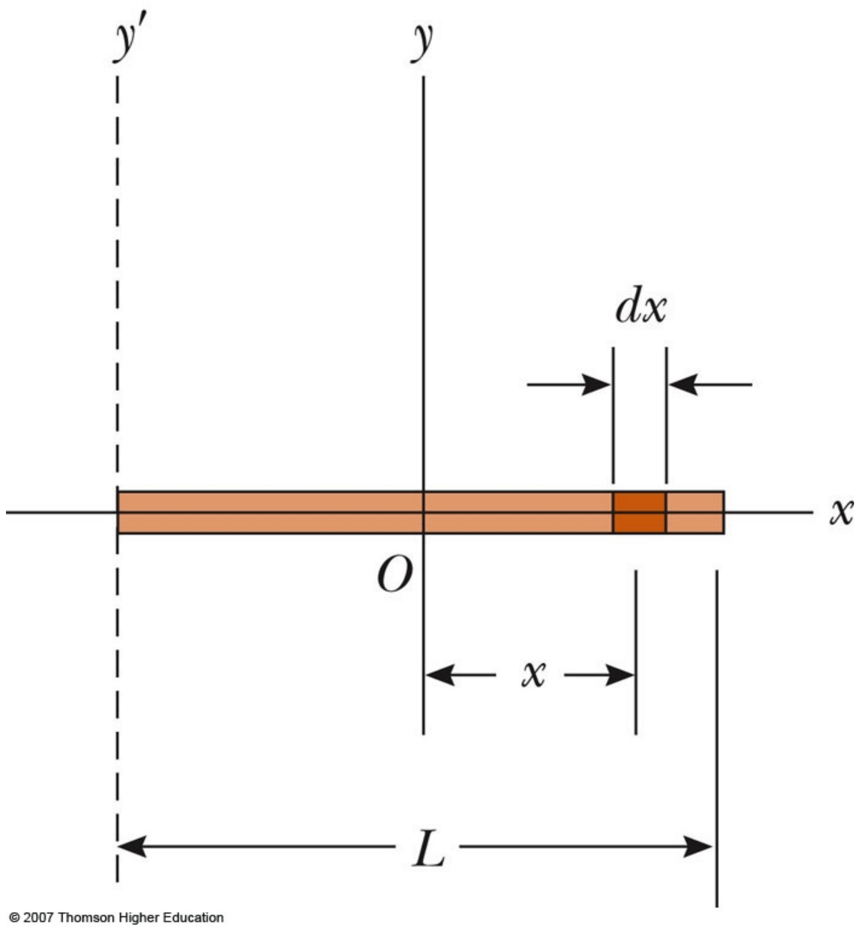
- Volumetric Mass Density → mass per unit volume: $\rho = m / V$
- Surface Mass Density → mass per unit thickness of a sheet of uniform thickness, t : $\sigma = \rho t$
- Linear Mass Density → mass per unit length of a rod of uniform cross-sectional area: $\lambda = m / L = \rho A$

Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  - $dm = \lambda\, dx$
- Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

$$I = \frac{1}{12} ML^2$$

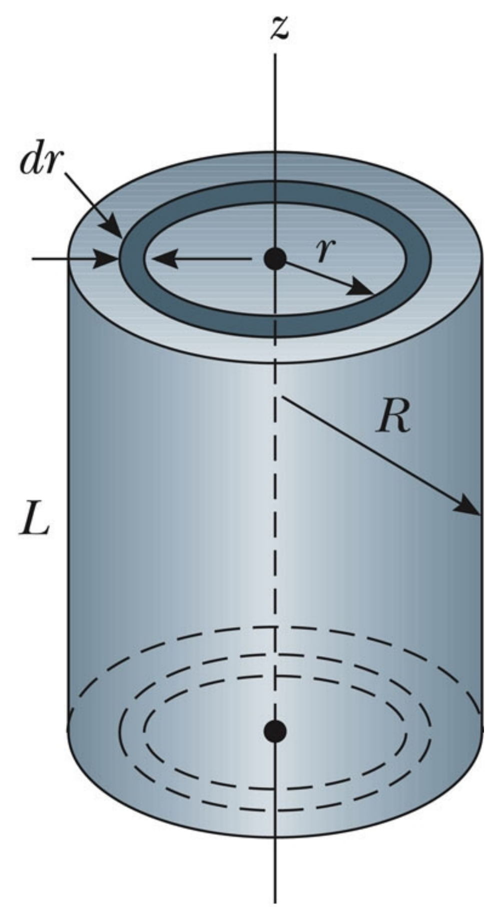


Moment of Inertia of a Uniform Solid Cylinder

- Divide the cylinder into concentric shells with radius *r*, thickness *dr* and length *L*
- $dm = \rho\, dV = 2\pi\rho L r\, dr$
- Then for *I*

$$I_z = \int r^2 dm = \int r^2 (2\pi\rho L r \; dr)$$

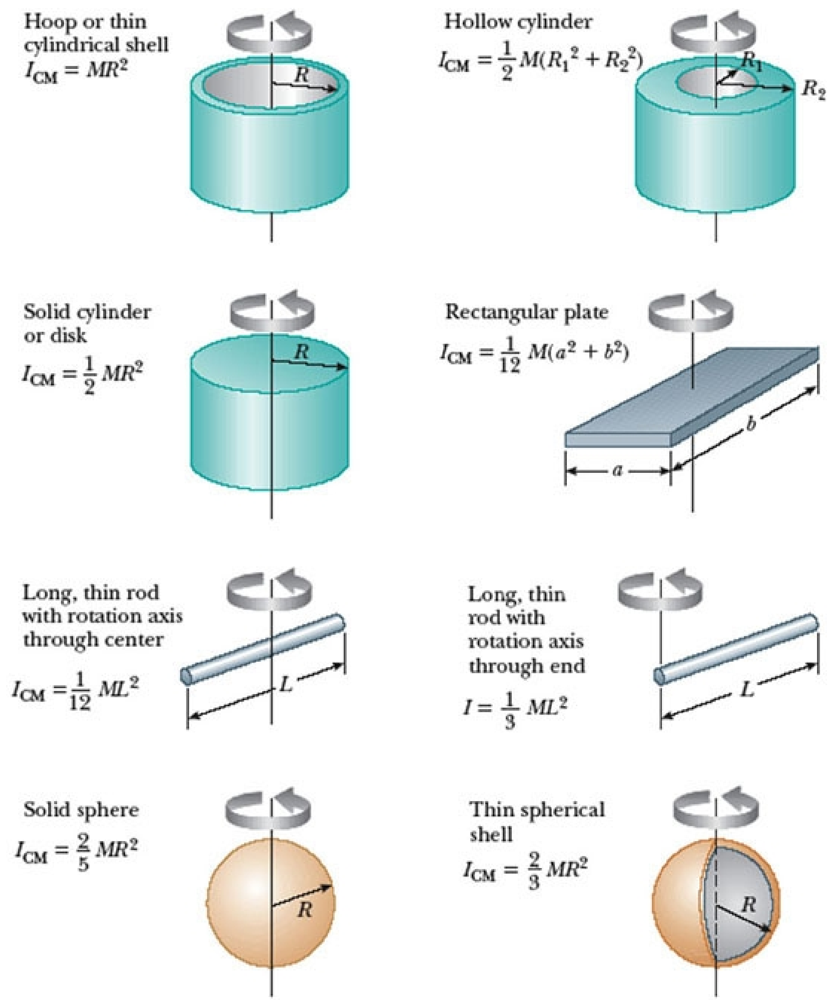
$$I_z = \frac{1}{2} M R^2$$



Moments of Inertia of Various Rigid Objects

TABLE 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

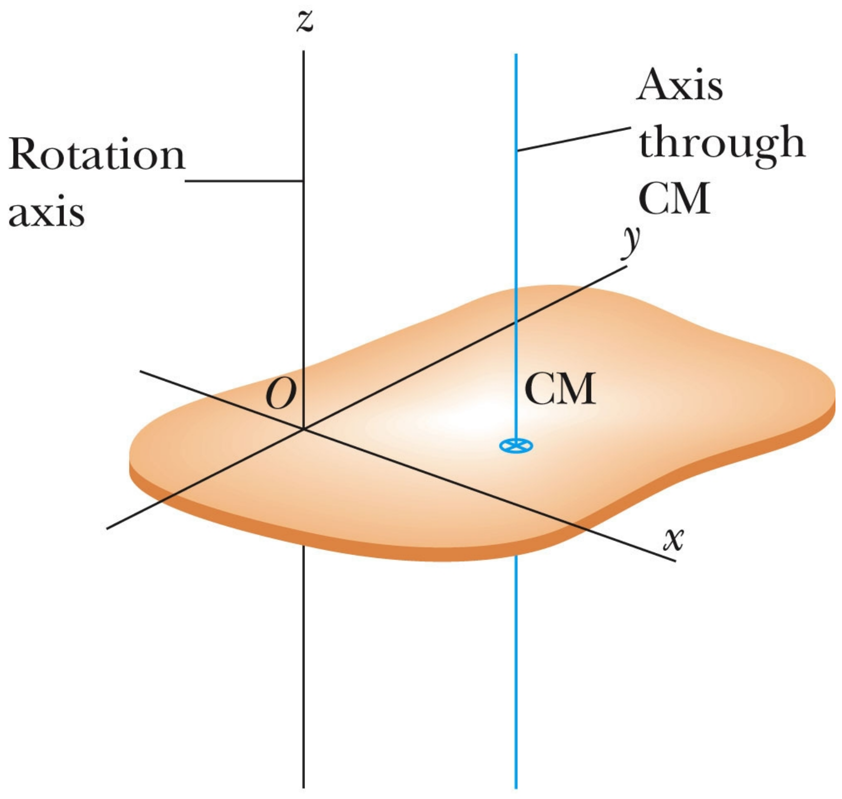


Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations
- The theorem states $I = I_{CM} + MD^2$
  - I is about any axis parallel to the axis through the center of mass of the object
  - I_{CM} is about the axis through the center of mass
  - D is the distance from the center of mass axis to the arbitrary axis

Parallel-Axis Theorem Example

- The axis of rotation goes through *O*
- The axis through the center of mass is shown
- The moment of inertia about the axis through *O* would be $I_O = I_{CM} + MD^2$



(b)

Moment of Inertia for a Rod Rotating Around One End

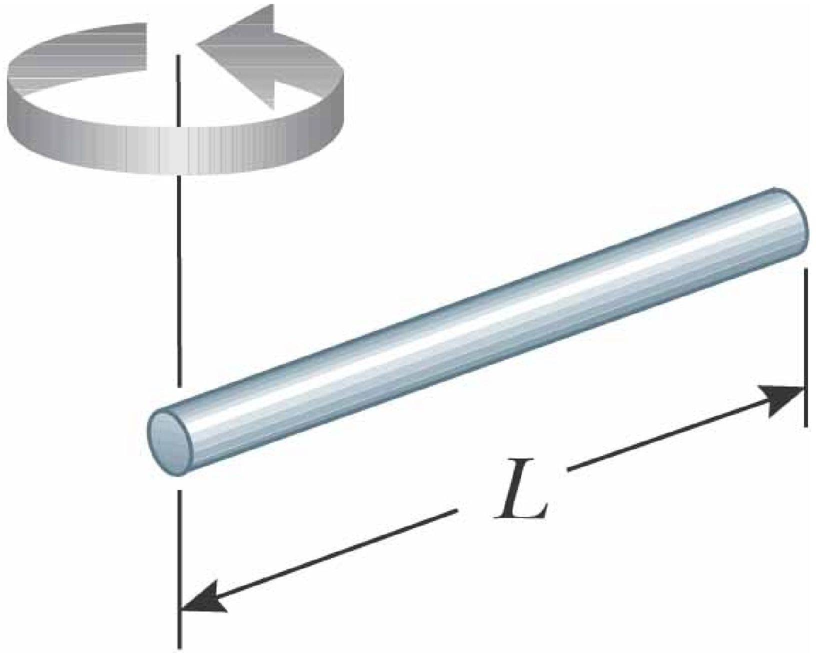
- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12} ML^2$$

- D is ½ L
- Therefore,

$$I = I_{\mathrm{CM}} + MD^2$$

$$I = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$$

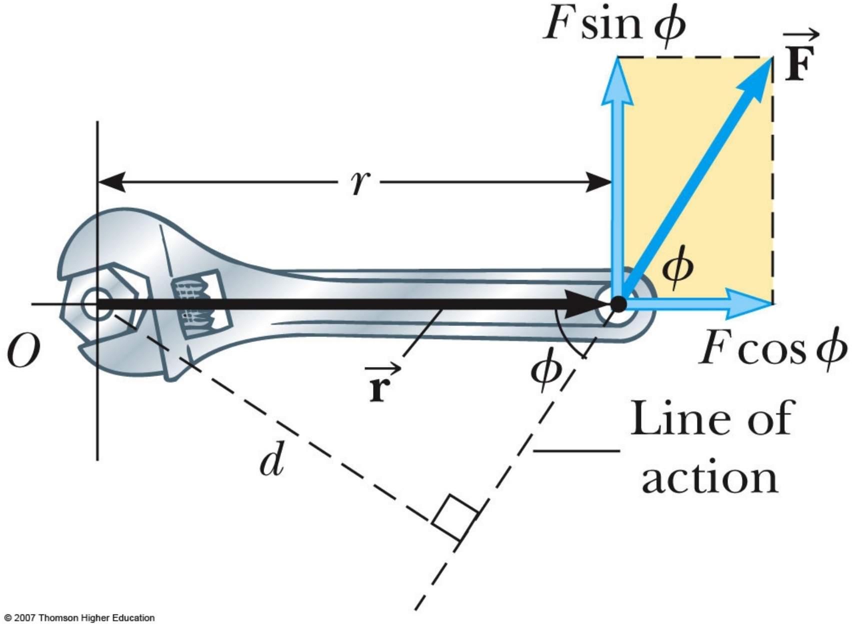


Torque

- Torque, τ, is the tendency of a force to rotate an object about some axis
  - Torque is a vector, but we will deal with its magnitude here
  - $\tau = r F \sin \phi = F d$
    - F is the force
    - ϕ is the angle the force makes with the horizontal
    - d is the *moment arm* (or lever arm) of the force

Torque, cont

- The moment arm, *d*, is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force
  - $d = r \sin \Phi$

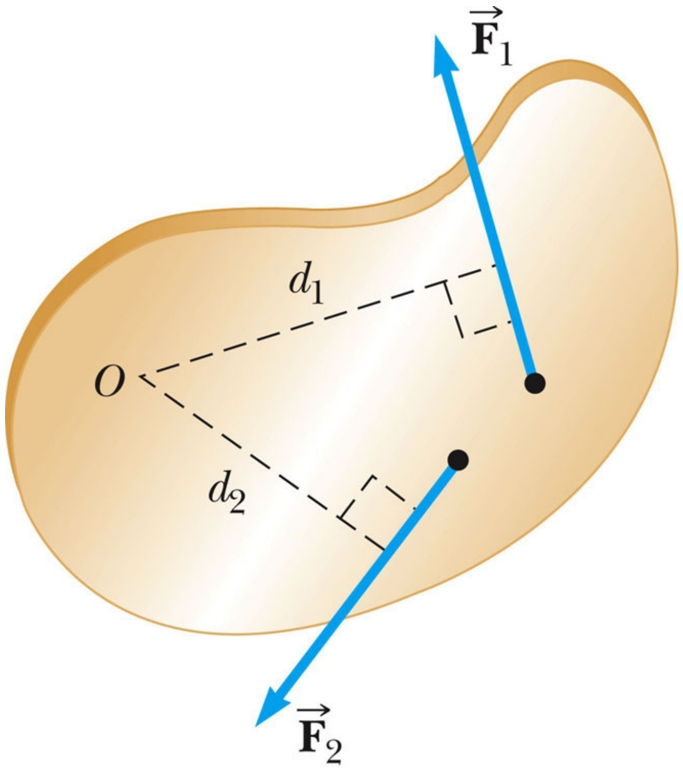


Torque, final

- The horizontal component of the force (F cos ϕ) has no tendency to produce a rotation
- Torque will have direction
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative

Net Torque

- The force $\mathbf{F}_1$ will tend to cause a counterclockwise rotation about *O*
- The force $\mathbf{F}_2$ will tend to cause a clockwise rotation about *O*
- $\Sigma\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$



Torque vs. Force

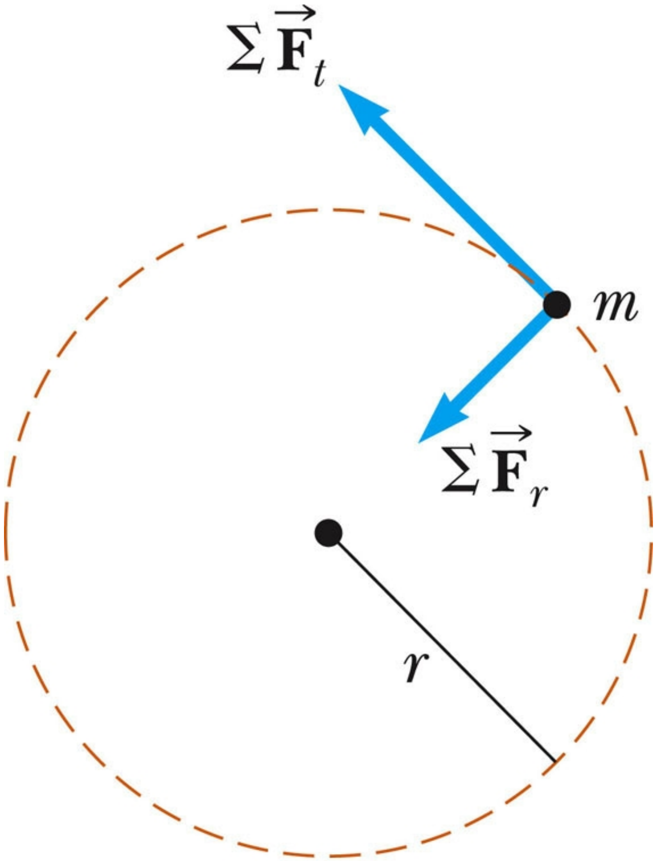
- Forces can cause a change in translational motion
  - Described by Newton's Second Law
- Forces can cause a change in rotational motion
  - The effectiveness of this change depends on the force and the moment arm
  - The change in rotational motion depends on the torque

Torque Units

- The SI units of torque are N·m
  - Although torque is a force multiplied by a distance, it is very different from work and energy
  - The units for torque are reported in N·m and not changed to Joules

Torque and Angular Acceleration

- Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of tangential force $\mathbf{F}_t$
- The tangential force provides a tangential acceleration:
  - $F_t = ma_t$
- The radial force, $\mathbf{F}_r$ causes the particle to move in a circular path

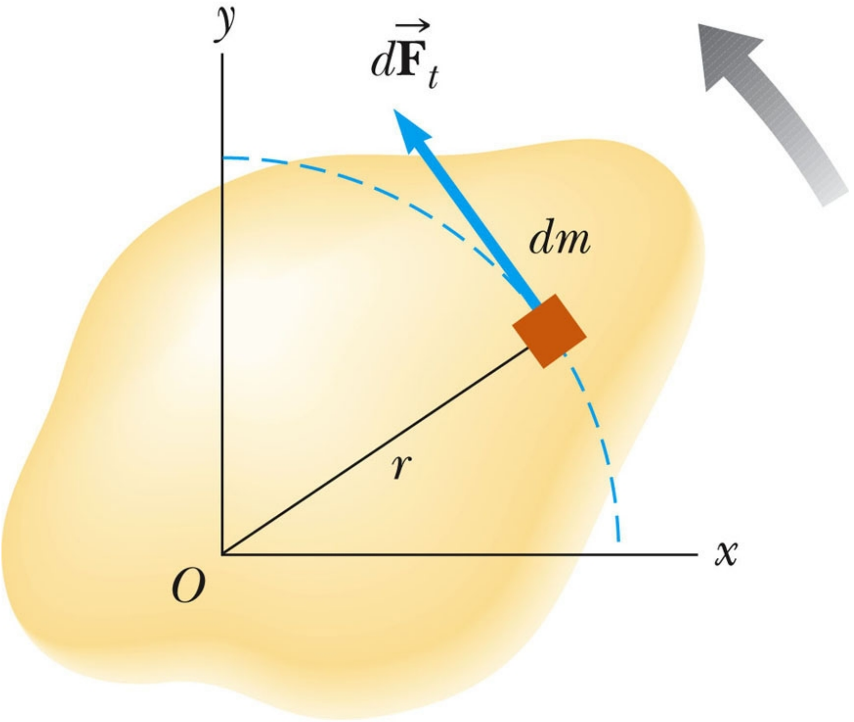


Torque and Angular Acceleration, Particle cont.

- The magnitude of the torque produced by $\sum \Phi_t$ around the center of the circle is
  - $\Sigma\tau = \Sigma F_t\, r = (m a_t)\, r$
- The tangential acceleration is related to the angular acceleration
  - $\Sigma\tau = (m a_t)\, r = (m r \alpha)\, r = (m r^2)\, \alpha$
- Since mr^2 is the moment of inertia of the particle,
  - $\Sigma\tau = I\alpha$
  - The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia

Torque and Angular Acceleration, Extended

- Consider the object consists of an infinite number of mass elements *dm* of infinitesimal size
- Each mass element rotates in a circle about the origin, *O*
- Each mass element has a tangential acceleration



Torque and Angular Acceleration, Extended cont.

- From Newton's Second Law
  - $dF_t = (dm)\, a_t$
- The torque associated with the force and using the angular acceleration gives
  - $d\tau = r\, dF_t = a_t r\, dm = \alpha r^2\, dm$
- Finding the net torque
  - $\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$
  - This becomes $\Sigma\tau = I\alpha$

Torque and Angular Acceleration, Extended final

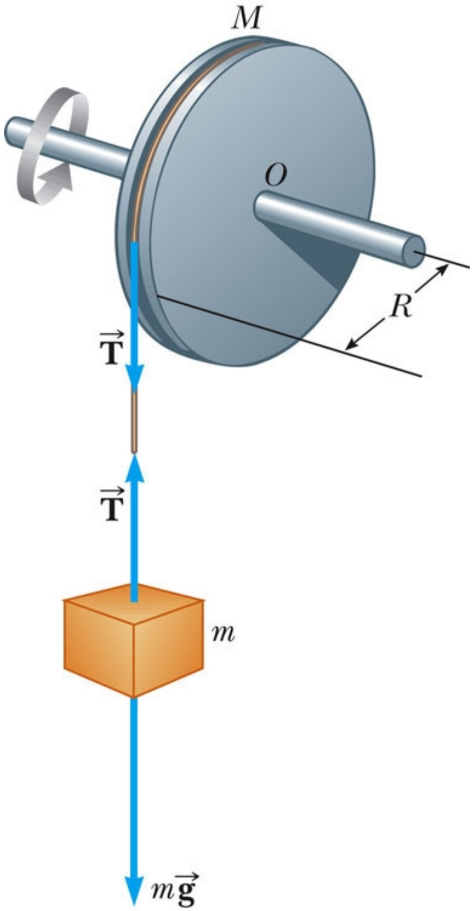
- This is the same relationship that applied to a particle
- This is the mathematic representation of the analysis model of a rigid body under a net torque
- The result also applies when the forces have radial components
  - The line of action of the radial component must pass through the axis of rotation
  - These components will produce zero torque about the axis

Falling Smokestack Example

- When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground
- Each higher portion of the smokestack has a larger tangential acceleration than the points below it
- The shear force due to the tangential acceleration is greater than the smokestack can withstand
- The smokestack breaks

Torque and Angular Acceleration, Wheel Example

- Analyze:
- The wheel is rotating and so we apply $\Sigma\tau = I\alpha$
  - The tension supplies the tangential force
- The mass is moving in a straight line, so apply Newton's Second Law
  - $\Sigma F_y = ma_y = mg - T$

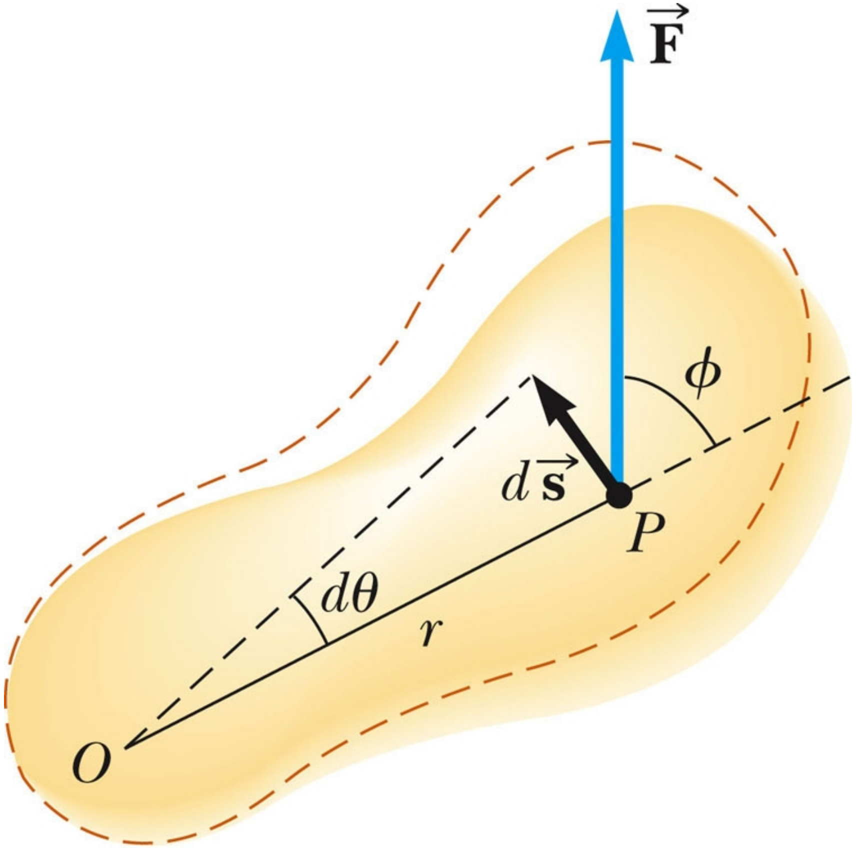


Work in Rotational Motion

- Find the work done by $\mathbf{F}$ on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$

$$dW = \mathbf{F} \cdot d\mathbf{s}$$
$$= (F \sin\varphi) r\, d\theta$$

- The radial component of the force does no work because it is perpendicular to the displacement



Power in Rotational Motion

- The rate at which work is being done in a time interval *dt* is

$$\text{Power} = \wp = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $\wp = Fv$ in a linear system

Work-Kinetic Energy Theorem in Rotational Motion

- The work-kinetic energy theorem for rotational motion states that *the net work done by external forces in rotating a symmetrical rigid object about a fixed axis equals the change in the object's rotational kinetic energy*

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

Work-Kinetic Energy Theorem, General

- The rotational form can be combined with the linear form which indicates *the net work done by external forces on an object is the change in its* ***total*** *kinetic energy, which is the sum of the translational and rotational kinetic energies*

Summary of Useful Equations

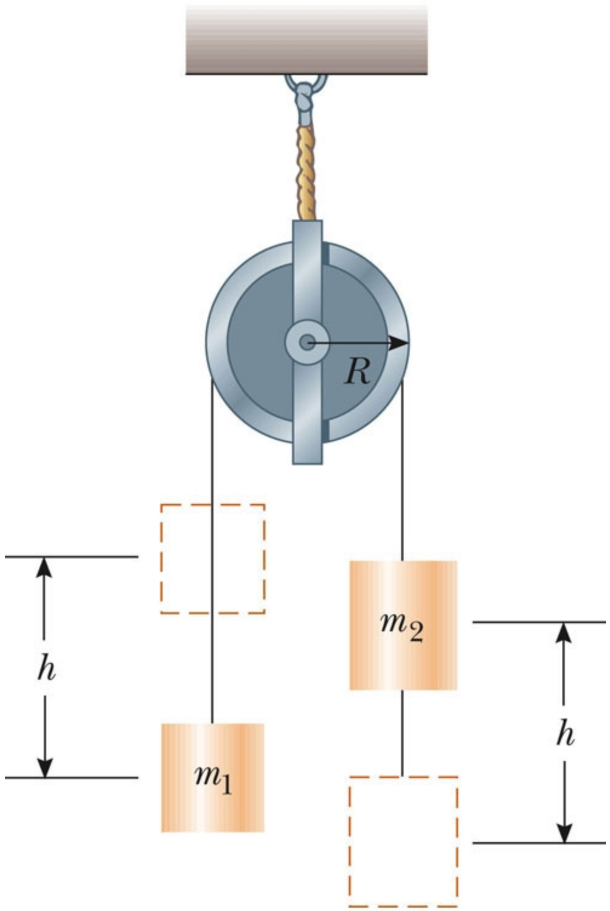
TABLE 10.3

Useful Equations in Rotational and Translational Motion

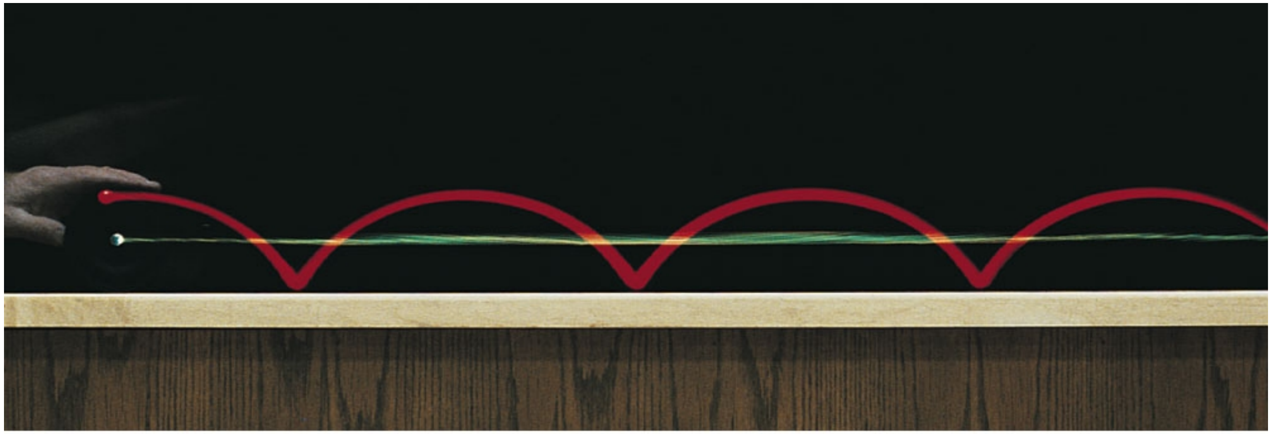
Rotational Motion About a Fixed Axis	Translational Motion
Angular speed $\omega = d\theta/dt$	Translational speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Translational acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Energy in an Atwood Machine, Example

- The blocks undergo changes in translational kinetic energy and gravitational potential energy
- The pulley undergoes a change in rotational kinetic energy
- Use the active figure to change the masses and the pulley characteristics



Rolling Object



- The red curve shows the path moved by a point on the rim of the object
  - This path is called a ***cycloid***
- The green line shows the path of the center of mass of the object

Pure Rolling Motion

- In pure rolling motion, an object rolls without slipping
- In such a case, there is a simple relationship between its rotational and translational motions

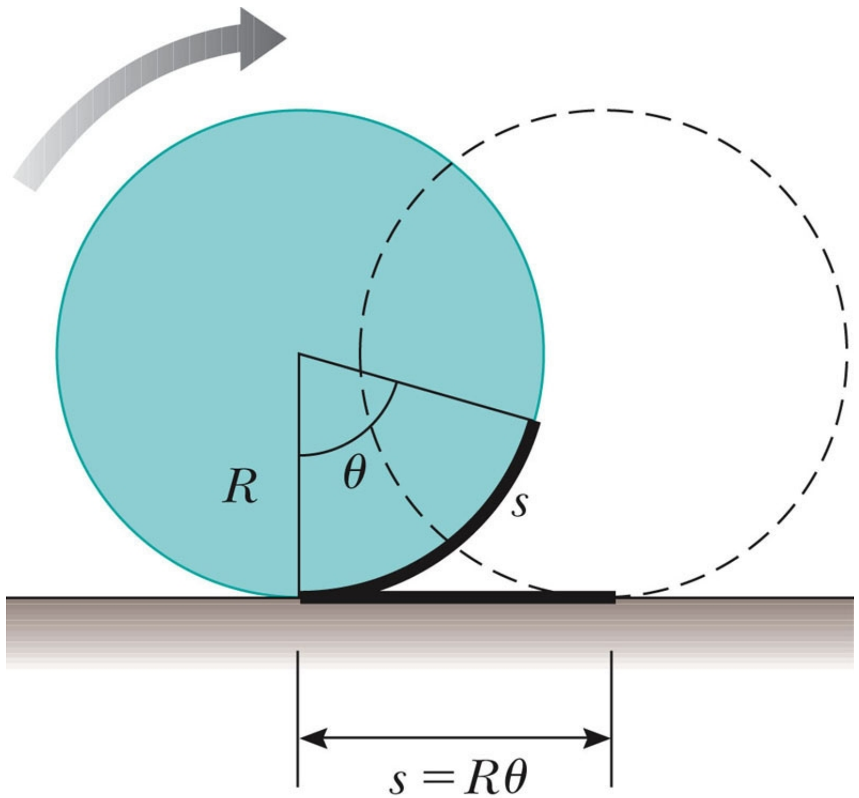
Rolling Object, Center of Mass

- The velocity of the center of mass is

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

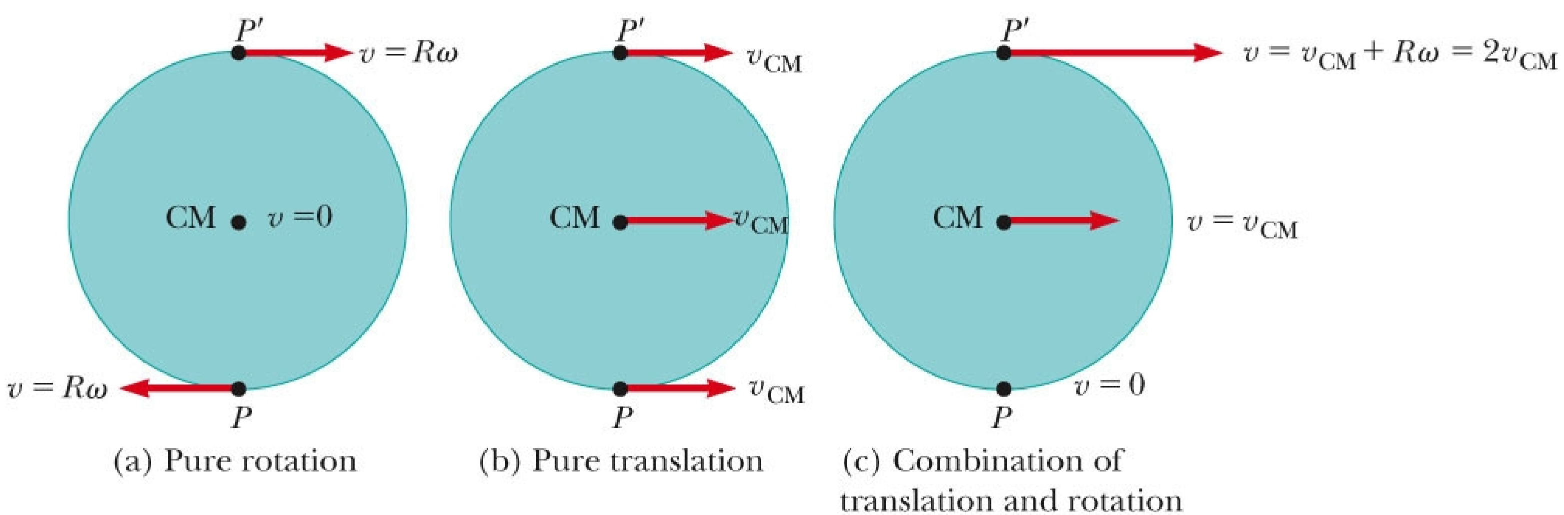
- The acceleration of the center of mass is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



Rolling Motion Cont.

- Rolling motion can be modeled as a combination of pure translational motion and pure rotational motion
- The contact point between the surface and the cylinder has a translational speed of zero (c)

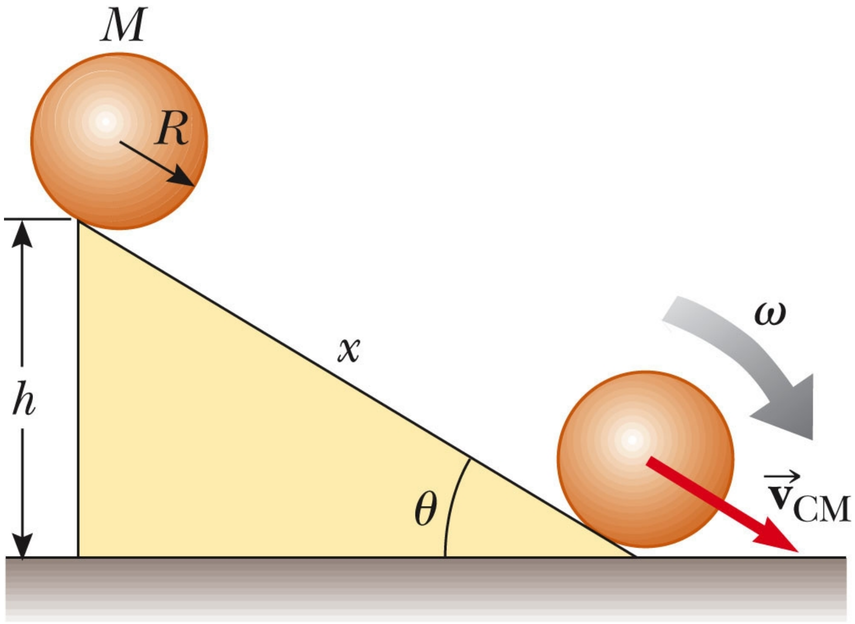


Total Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass
  - $K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$
    - The $\frac{1}{2} I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass
    - The $\frac{1}{2} Mv^2$ represents the translational kinetic energy of the cylinder about its center of mass

Total Kinetic Energy, Example

- Accelerated rolling motion is possible only if friction is present between the sphere and the incline
  - The friction produces the net torque required for rotation
  - No loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant
  - Use the active figure to vary the objects and compare their speeds at the bottom



Total Kinetic Energy, Example cont

- Apply Conservation of Mechanical Energy
  - Let $U = 0$ at the bottom of the plane
  - $K_f + U_f = K_i + U_i$
  - $K_f = \frac{1}{2}(I_{CM} / R^2)\, v_{CM}^2 + \frac{1}{2} M v_{CM}^2$ $\quad = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$
  - $U_i = Mgh$
  - $U_f = K_i = 0$
- Solving for v

$$v = \left[\frac{2gh}{1 + \left(I_{CM} / MR^2\right)}\right]^{1/2}$$

Sphere Rolling Down an Incline, Example

- **Conceptualize**
  - A sphere is rolling down an incline
- **Categorize**
  - Model the sphere and the Earth as an isolated system
  - No nonconservative forces are acting
- **Analyze**
  - Use Conservation of Mechanical Energy to find v
    - See previous result

Sphere Rolling Down an Incline, Example cont

- **Analyze**, cont
  - Solve for the acceleration of the center of mass
- **Finalize**
  - Both the speed and the acceleration of the center of mass are independent of the mass and the radius of the sphere
- Generalization
  - *All homogeneous solid spheres experience the same speed and acceleration on a given incline*
    - Similar results could be obtained for other shapes