Chapter 4

Motion in Two Dimensions

With modifications by Pinkney

“Kinematics in Two Dimensions” covers:

the vector nature of position, velocity and acceleration in greater detail

projectile motion – a special case of 2D motion

uniform circular motion - another special case of 2D motion

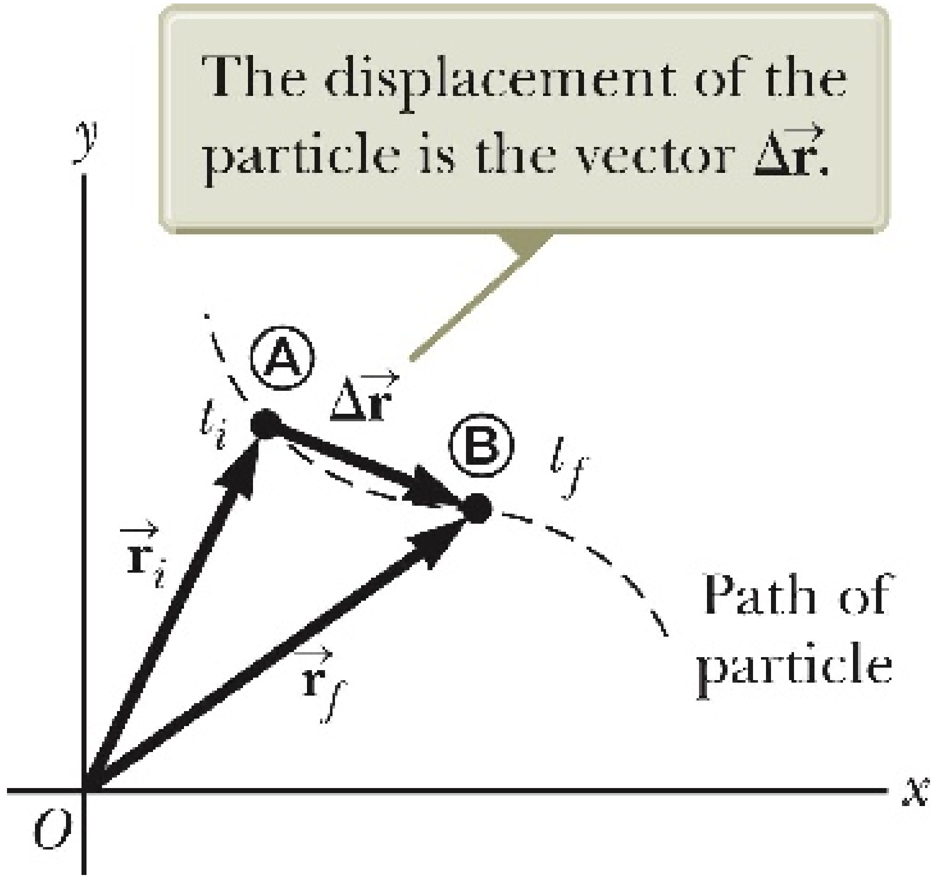
relative motion

Position and Displacement

The position of an object is described by its position vector, Δ**r.**

The **displacement** of the object is defined as the ***change in its position.***

$$\Delta \vec{\mathbf{r}} \equiv \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i$$



General 2-D Motion Ideas

- Use full vector notation, otherwise much like 1D motion.
- Positive and negative signs are no longer sufficient to determine the direction.

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

Average Velocity

$\mathbf{v}_{avg}$ is the ratio of the displacement to the time interval for the displacement.

$$\vec{\mathbf{v}}_{avg} \equiv \frac{\Delta \vec{\mathbf{r}}}{\Delta t}$$

The direction of $\mathbf{v}_{avg}$ is the same as the displacement vector.

The average velocity between points is *independent of the path* taken, just as is the displacement.

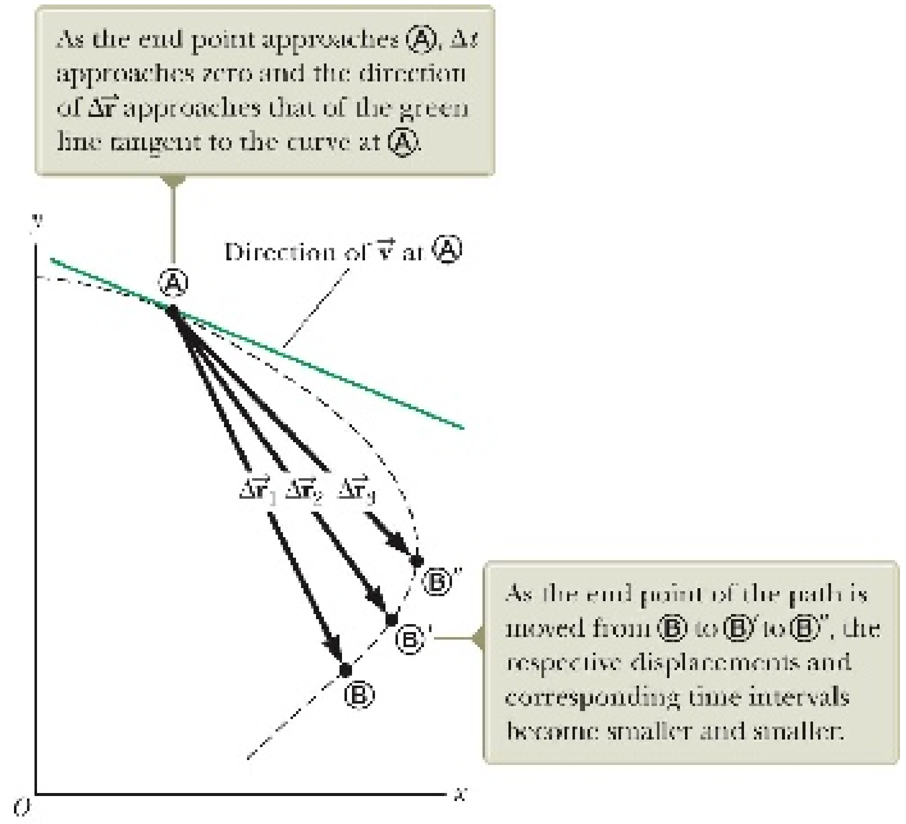
Try the "ice cream truck" problem.

Instantaneous Velocity

The instantaneous velocity is the limit of the average velocity as Δt approaches zero.

$$\vec{\mathbf{v}} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt}$$

- As the time interval becomes smaller, the direction of the displacement approaches that of the line tangent to the curve.



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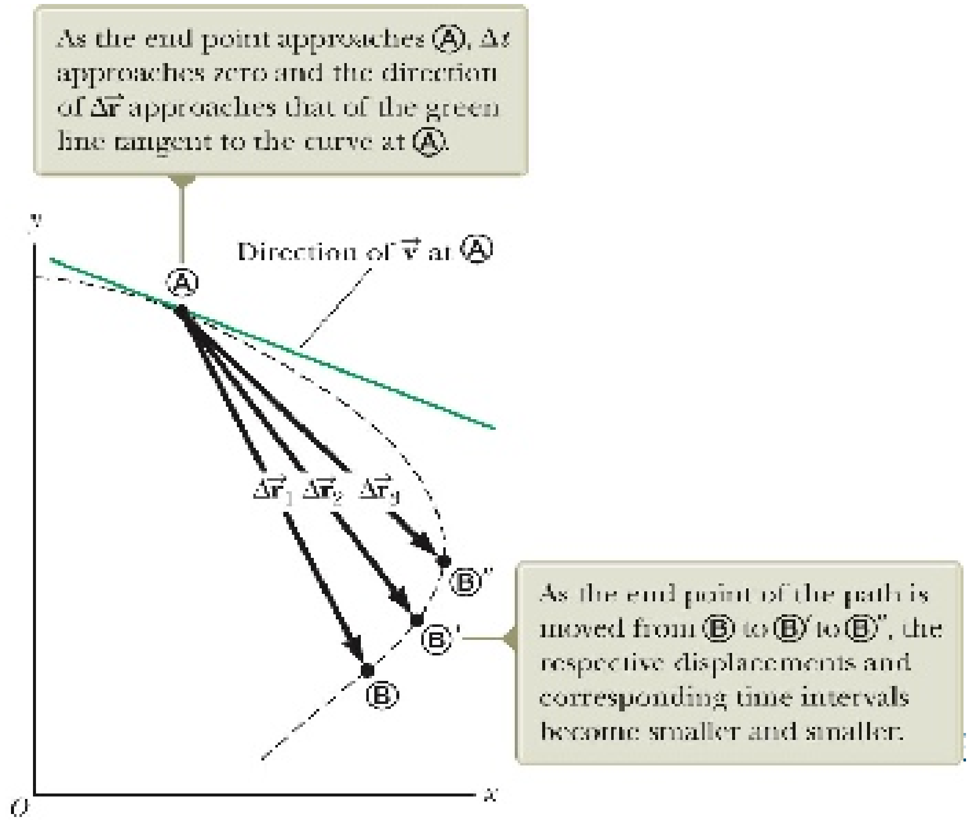
Instantaneous Velocity, cont

The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion.

The magnitude of the instantaneous velocity vector is the speed.

- The speed is a scalar quantity.



Try example problem in which a functional form is given for *r(t)*.

Average Acceleration

The average acceleration of a particle is defined as the change in the instantaneous velocity divided by the time interval.

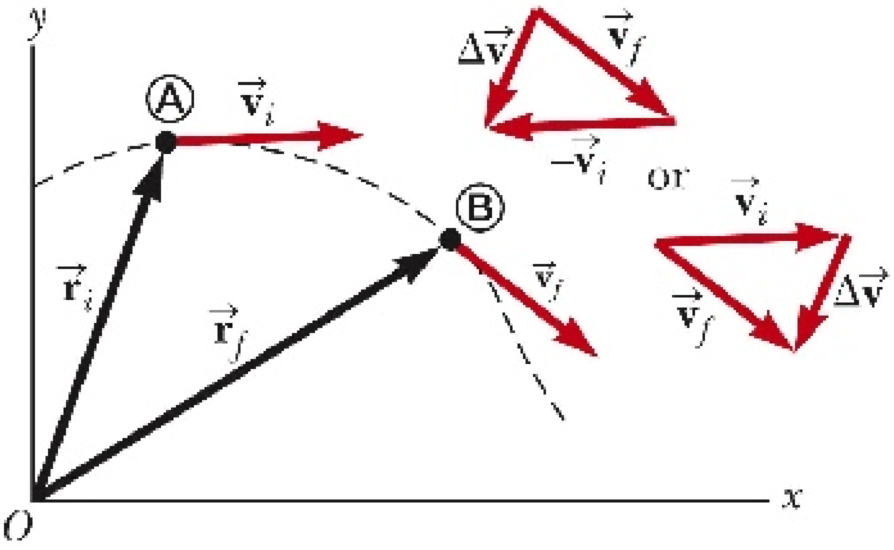
$$\vec{a}_{avg} \equiv \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

Average Acceleration, cont

As a particle moves, the direction of the change in velocity is found by vector subtraction.

$$\Delta \vec{\mathbf{v}} = \vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i$$

The average acceleration is a vector quantity directed along $\Delta \vec{\mathbf{v}}$.



Instantaneous Acceleration

The instantaneous acceleration is the limiting value of the ratio $\Delta \vec{\mathbf{v}} / \Delta t$ as Δt approaches zero.

$$\vec{\mathbf{a}} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt}$$

- The instantaneous ***a*** equals the derivative of the velocity vector with respect to time.

Producing An Acceleration

Various changes in a particle's motion may produce an acceleration.

- The magnitude of the velocity vector may change.
- The direction of the velocity vector may change.
  - Even if the magnitude remains constant!
- Both may change simultaneously

(See Transparency.)

Kinematic Equations for Uniform Acceleration in 2D

similar to those of one-dimensional kinematics.

Motion in two dimensions can be modelled as two *independent* motions, one parallel to the x-axis and one parallel to the y-axis.

- Any influence in the y direction does not affect the motion in the x direction.

Kinematic Equations, 2

Position vector for a particle moving in the xy plane.

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

The velocity vector can be found from the position vector.

$$\vec{\boldsymbol{v}} = \frac{d\,\vec{\boldsymbol{r}}}{dt} = v_x\,\hat{\boldsymbol{i}} + v_y\,\hat{\boldsymbol{j}}$$

- Since acceleration is constant, we can also find an expression for the velocity as a function of time:

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t$$

Kinematic Equations, 3

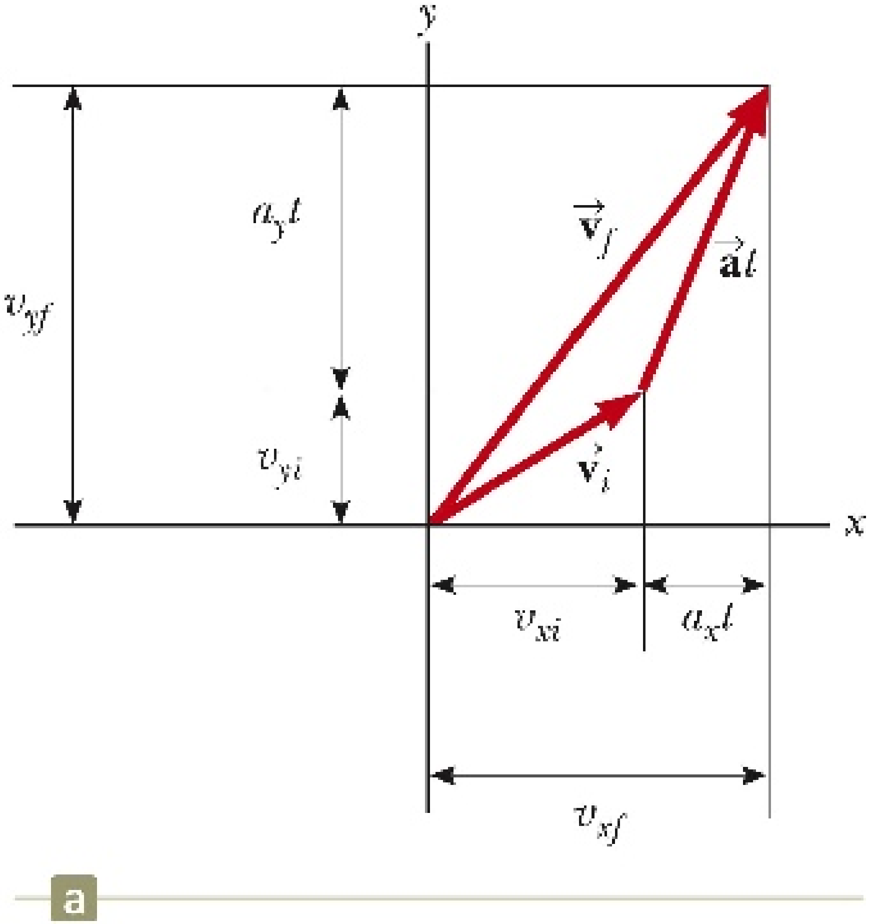
The position vector can also be expressed as a function of time:

- $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_i + \vec{\boldsymbol{v}}_i t + \frac{1}{2} \vec{\boldsymbol{a}} t^2$
- This indicates that the position vector is the sum of three other vectors:
  - The initial position vector
  - The displacement resulting from the initial velocity
  - The displacement resulting from the acceleration

Kinematic Equations, Graphical Representation of Final Velocity

The velocity vector can be represented by its components.

$\vec{\mathbf{v}}_f$ is generally not along the direction of either $\vec{\mathbf{v}}_i$ or $\vec{\mathbf{a}}$

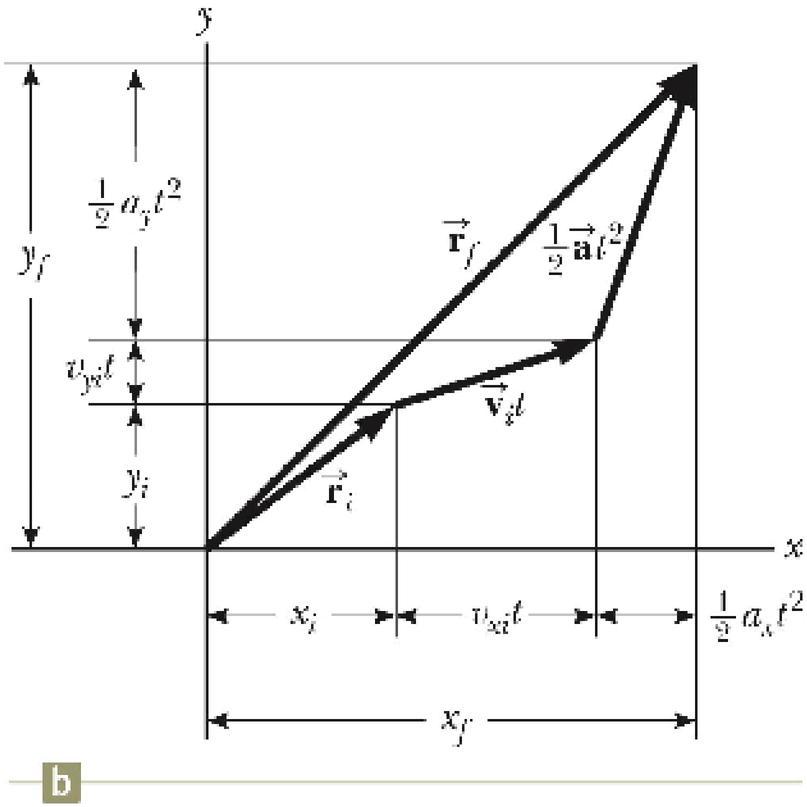


Kinematic Equations, Graphical Representation of Final Position

The vector representation of the position vector

$\vec{\mathbf{r}}_f$ is generally not along the same direction as $\vec{\mathbf{r}}_i$, $\vec{\mathbf{v}}_i$ or $\vec{\mathbf{a}}$

$\vec{\mathbf{v}}_f$ and $\vec{\mathbf{r}}_f$ are generally not in the same direction



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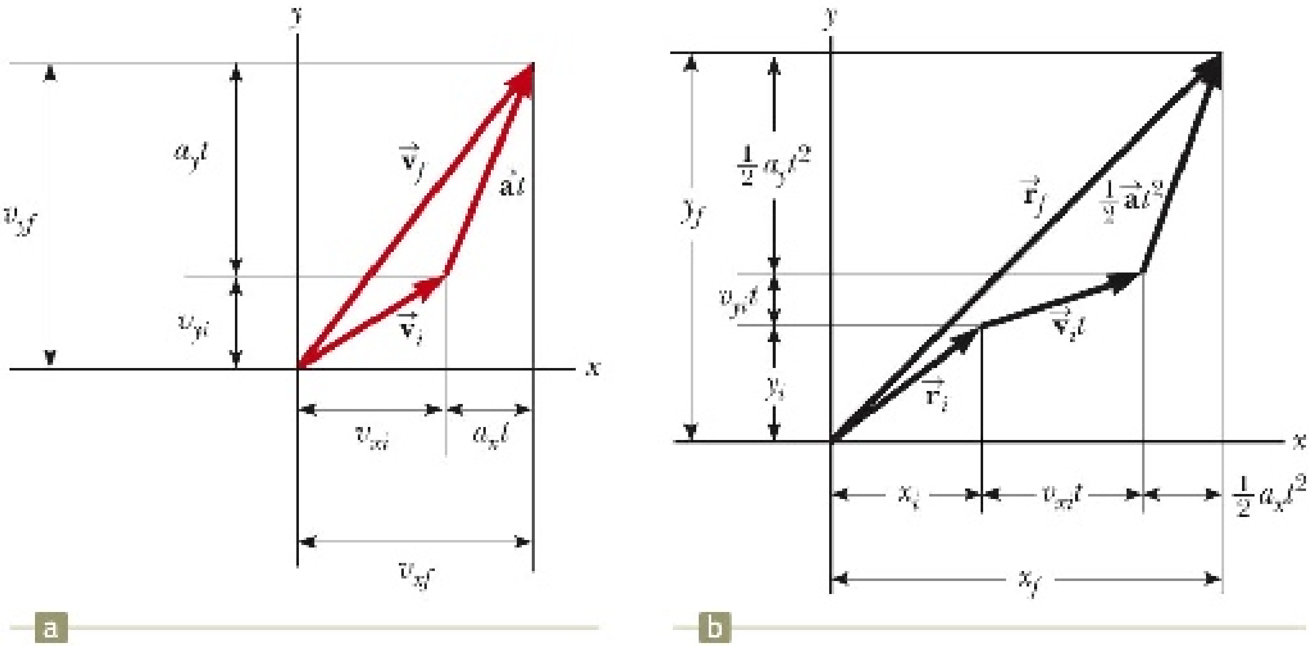
Graphical Representation Summary

Various starting positions and initial velocities can be chosen.

Here is the general case where a has a non-zero x and y component.



Projectile Motion

An object may move in both the *x* and *y* directions simultaneously.

The form of two-dimensional motion we will deal with is called **projectile motion.**

Assumptions of Projectile Motion

The free-fall acceleration is constant over the range of motion.

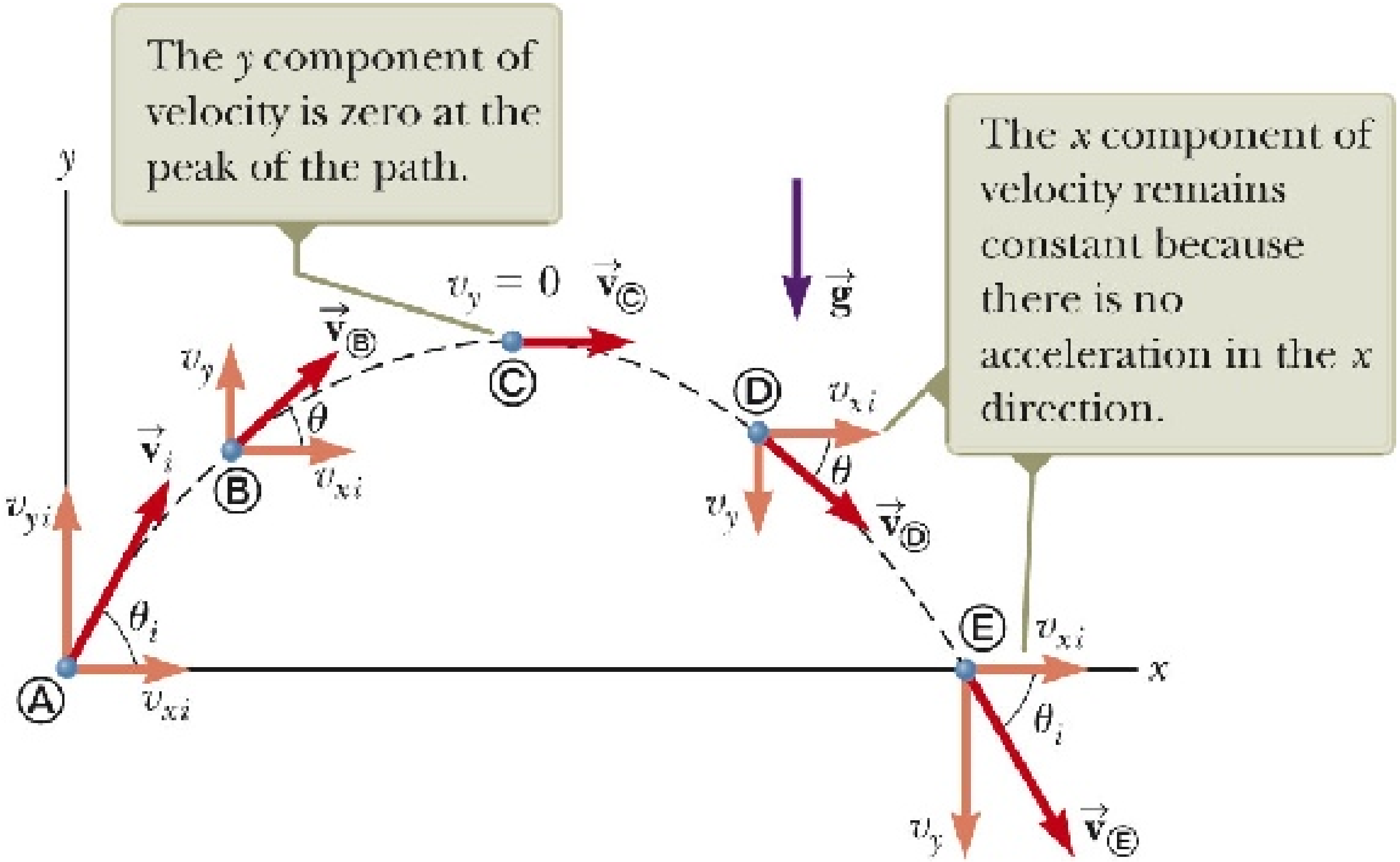
- It is directed downward.
- This is the same as assuming a flat Earth over the range of the motion.
- It is reasonable as long as the range is small compared to the radius of the Earth.

The effect of air friction is negligible.

With these assumptions, an object in projectile motion will follow a parabolic path.

- This path is called the ***trajectory.***

Projectile Motion Diagram – a *trajectory*



Acceleration at the Highest Point

The vertical velocity is zero at the top.

The acceleration is not zero anywhere along the trajectory.

- If the projectile experienced zero acceleration at the highest point, its velocity at the point would not change.
  - The projectile would move with a constant horizontal velocity from that point on.

Analyzing Projectile Motion

Consider the motion as the superposition of the motions in the *x*- and *y*-directions.

The actual position at any time is given by:

$$\vec{\rho}_f = \vec{\rho}_i + \vec{\varpi}_i t + \frac{1}{2}\vec{\gamma} t^2$$

The initial velocity can be expressed in terms of its components.

- $v_{xi} = v_i \cos\theta$ and $v_{yi} = v_i \sin\theta$

The *x*-direction has constant velocity.

- $a_x = 0$

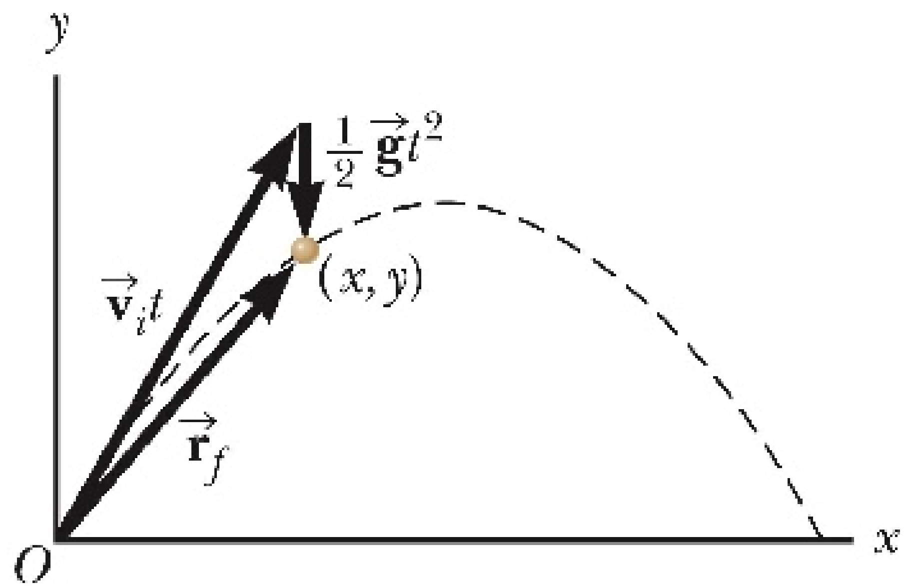
The y-direction is free fall.

- $a_y = -g$

Projectile Motion Vectors

$$\vec{\rho}_f = \vec{\rho}_i + \vec{\varpi}_i t + \frac{1}{2}\vec{\gamma}\, t^2$$

The final position is the vector sum of the initial position, the position resulting from the initial velocity and the position resulting from the acceleration.

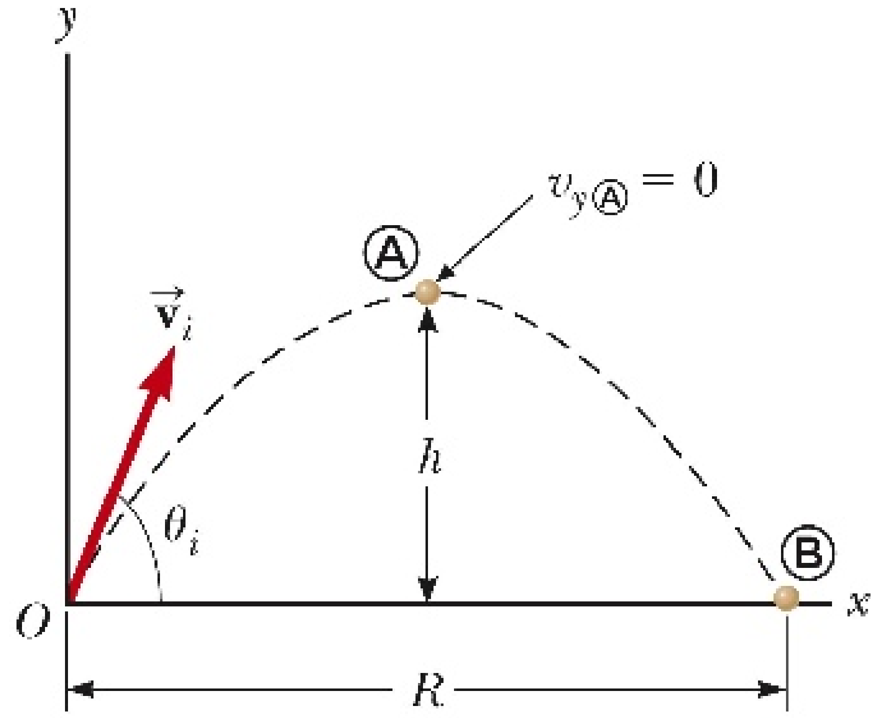


Range and Maximum Height of a Projectile

When analyzing projectile motion, two characteristics are of special interes.t

The range, *R*, is the horizontal distance of the projectile.

The maximum height the projectile reaches is *h.*



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Height of a Projectile, equation

The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

This equation is valid only for symmetric motion.

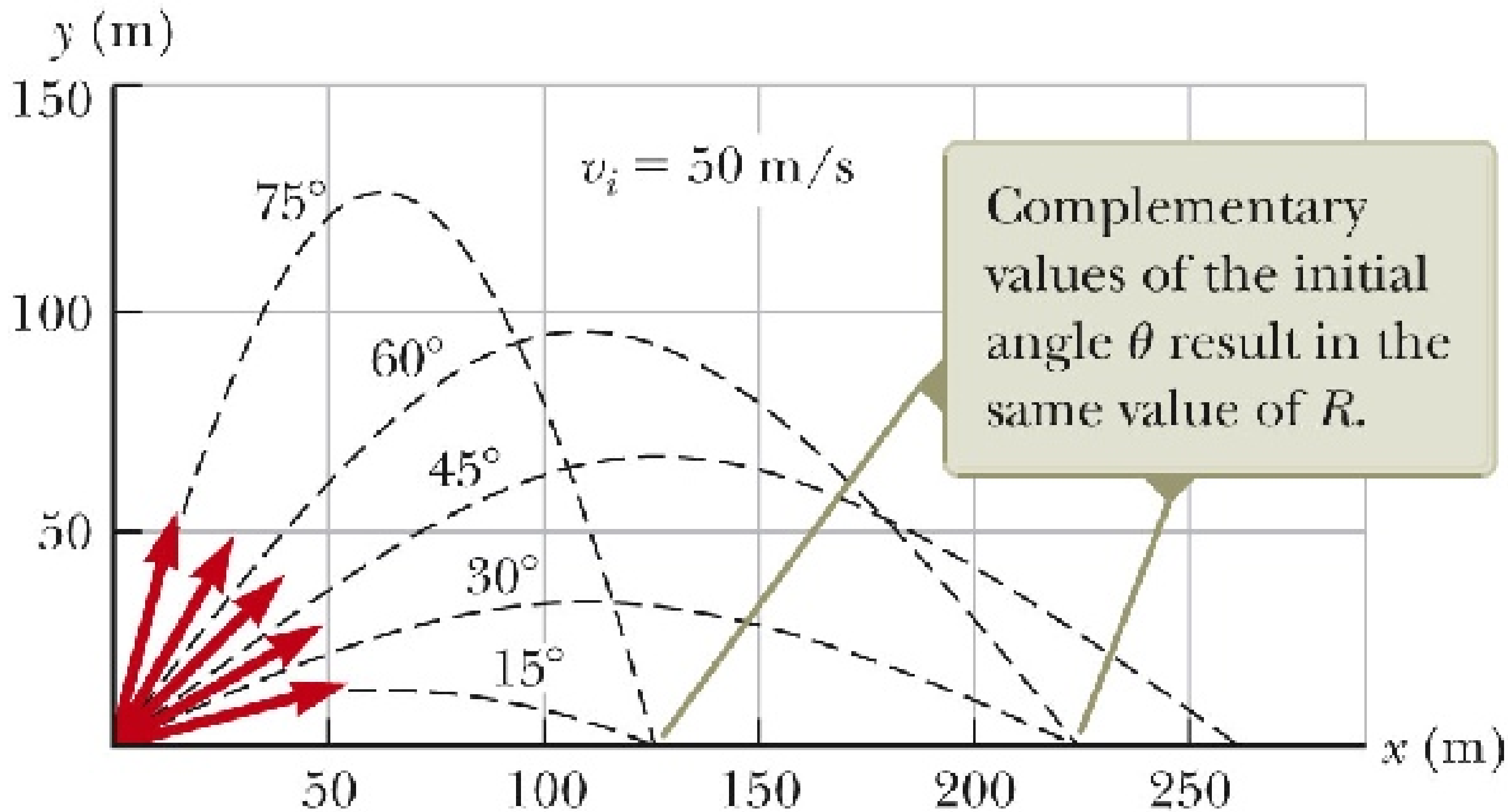
Range of a Projectile, equation

The range of a projectile can be expressed in terms of the initial velocity vector:

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

This is valid only for symmetric trajectory.

More About the Range of a Projectile



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Range of a Projectile, final

The maximum range occurs at $\theta_i = 45^o$.

Complementary angles will produce the same range.

- The maximum height will be different for the two angles.
- The times of the flight will be different for the two angles.

Projectile Motion – Problem Solving Hints

Conceptualize

- Establish the mental representation of the projectile moving along its trajectory.

Categorize

- Confirm air resistance is neglected.
- Select a coordinate system with x in the horizontal and y in the vertical direction.

Analyze

- If the initial velocity is given, resolve it into x and y components.
- Treat the horizontal and vertical motions independently.

Projectile Motion – Problem Solving Hints, cont.

Analysis, cont.

- Analyze the horizontal motion with the particle-under-constant-velocity model.
- Analyze the vertical motion with the particle-under-constant-acceleration model.
- Remember that both directions share the same time.

Finalize

- Check to see if your answers are consistent with the mental and pictorial representations.
- Check to see if your results are realistic.

Non-Symmetric Projectile Motion

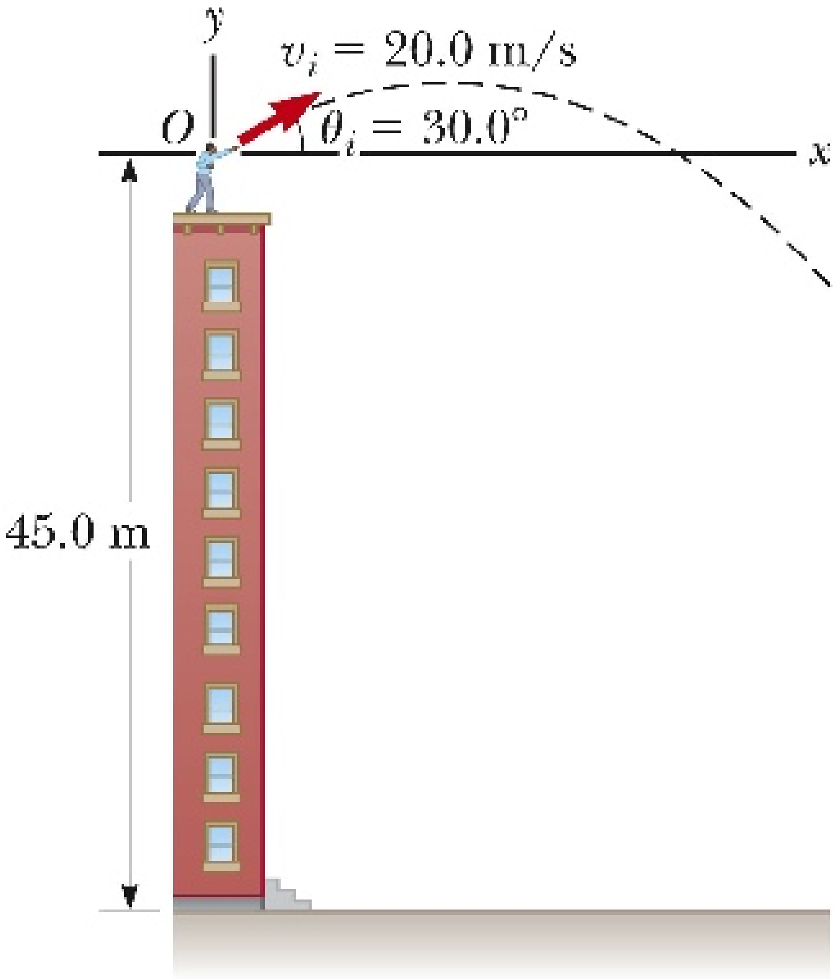
Follow the general rules for projectile motion.

Break the *y*-direction into parts.

- up and down *or*
- symmetrical back to initial height and then the rest of the height

Apply the problem solving process to determine and solve the necessary equations.

May be non-symmetric in other ways



Uniform Circular Motion

Uniform circular motion occurs when an object moves in a circular path with a constant speed.

The associated **analysis model** is a *particle in uniform circular motion.*

An acceleration exists since the *direction* of the motion is changing .

- This change in velocity is related to an acceleration.

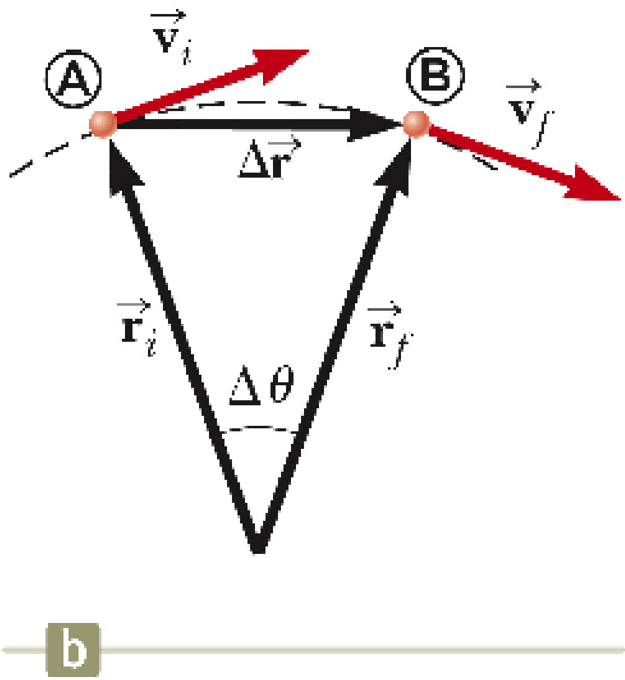
The constant-magnitude velocity vector is always tangent to the path of the object.

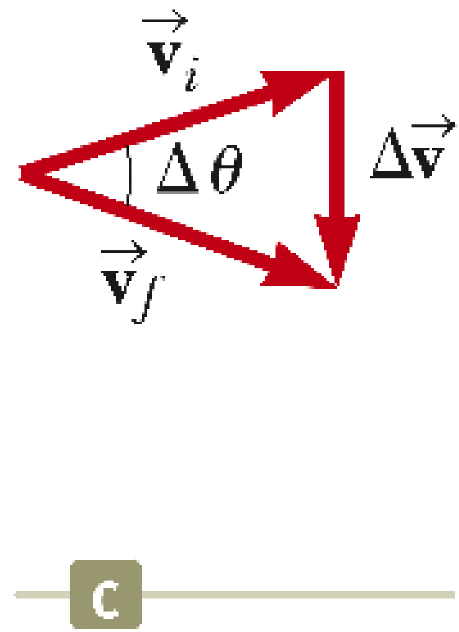
Changing Velocity in Uniform Circular Motion

The change in the velocity vector is due to the change in direction.

The direction of the change in velocity is toward the center of the circle.

The vector diagram shows $\vec{\varpi}_f = \vec{\varpi}_i + \Delta\vec{\varpi}$





Centripetal Acceleration

The acceleration is always perpendicular to the path of the motion.

The acceleration always points toward the center of the circle of motion.

This acceleration is called the ***centripetal acceleration.***

Centripetal Acceleration, cont

The magnitude of the centripetal acceleration vector is given by

$$a_C = \frac{v^2}{r}$$

The direction of the centripetal acceleration vector is always changing, to stay directed toward the center of the circle of motion.

Period

The ***period***, *T*, is the time required for one complete revolution.

The speed of the particle would be the circumference of the circle of motion divided by the period.

Therefore, the period is defined as

$$T \equiv \frac{2\pi r}{v}$$

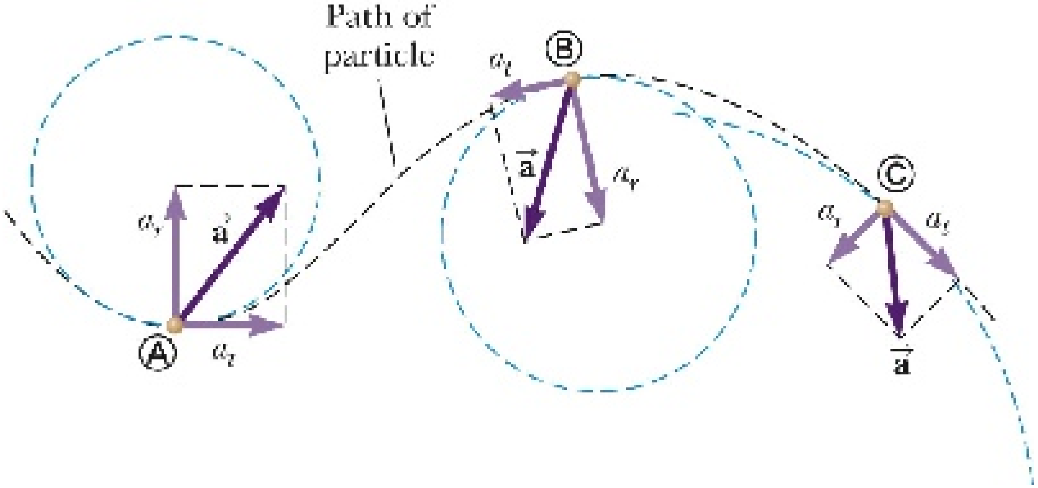
Tangential Acceleration

The magnitude of the velocity could also be changing.

In this case, there would be a ***tangential acceleration.***

The motion would be under the influence of both tangential and centripetal accelerations.

- Note the changing acceleration vectors



Total Acceleration

The tangential acceleration causes the change in the speed of the particle.

The radial acceleration comes from a change in the direction of the velocity vector.

Total Acceleration, equations

The tangential acceleration:

$$a_t = \left|\frac{dv}{dt}\right|$$

The radial acceleration:

$$a_r = -a_C = -\frac{v^2}{r}$$

The total acceleration:

- Magnitude

  $$a = \sqrt{a_r^2 + a_t^2}$$

- Direction
  - Same as velocity vector if v is increasing, opposite if v is decreasing

Relative Velocity

Two observers moving relative to each other generally do not agree on the outcome of an experiment.

However, the observations seen by each are related to one another.

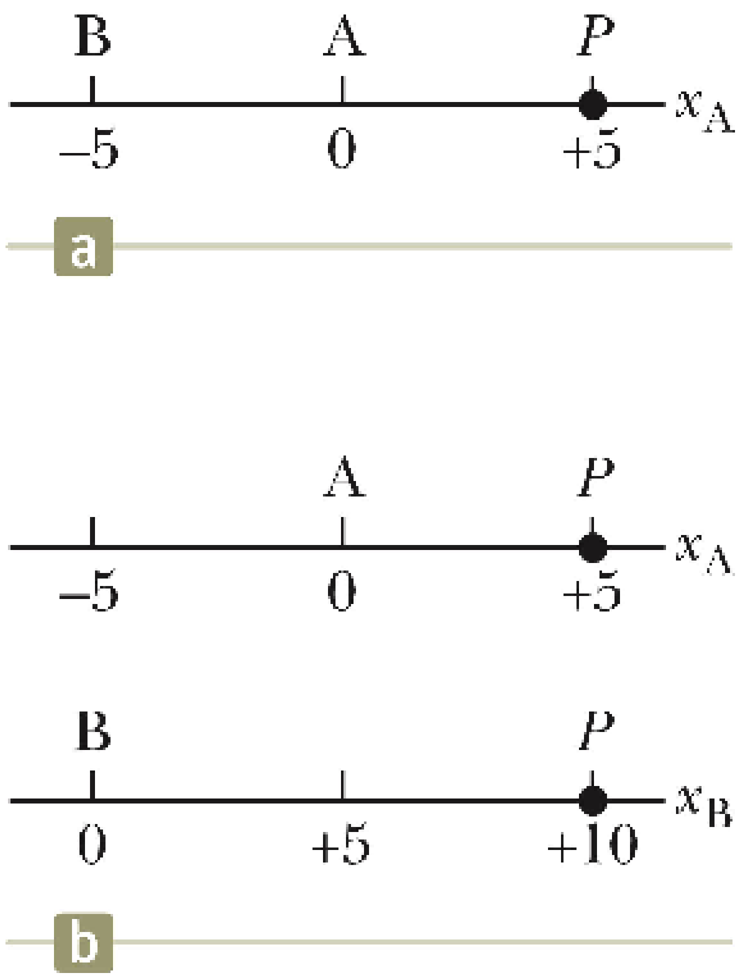
A frame of reference can described by a Cartesian coordinate system for which an observer is at rest with respect to the origin.

Different Measurements, example

Observer A measures point P at +5 m from the origin

Observer B measures point P at +10 m from the origin

The difference is due to the different frames of reference being used.



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Different Measurements, another example

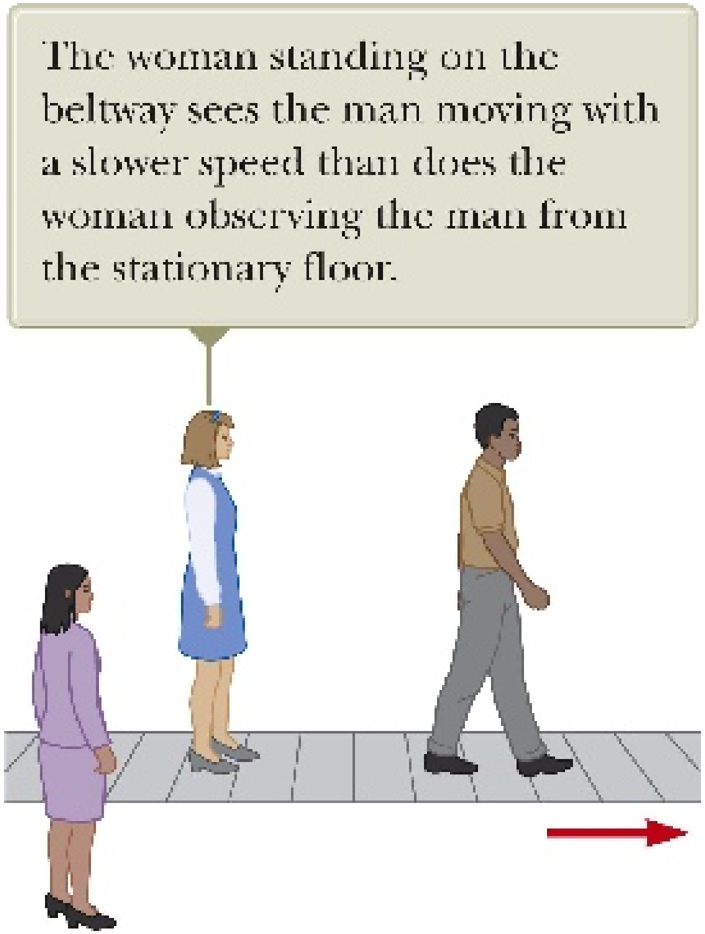
The man is walking on the moving beltway.

The woman on the beltway sees the man walking at his normal walking speed.

The stationary woman sees the man walking at a much higher speed.

- The combination of the speed of the beltway and the walking.

The difference is due to the relative velocity of their frames of reference.



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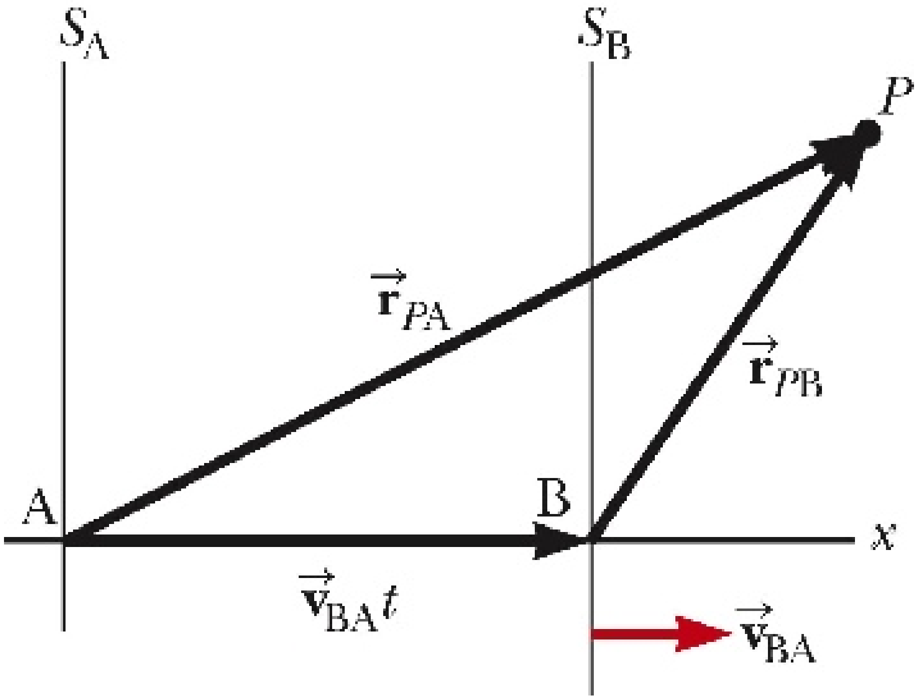
Relative Velocity, generalized

Reference frame S_A is stationary

Reference frame S_B is moving to the right relative to S_A at $\vec{\mathbf{v}}_{AB}$

- This also means that S_A moves at $-\vec{\mathbf{v}}_{BA}$ relative to S_B

Define time $t = 0$ as that time when the origins coincide



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Notation

The first subscript represents what is being observed.

The second subscript represents who is doing the observing.

Example

- The velocity of B (and attached to frame S_B) as measured by observer A

$$\vec{\mathbf{v}}_{BA}$$

Relative Velocity, equations

The positions as seen from the two reference frames are related through the velocity

- $\vec{\rho}_{PA} = \vec{\rho}_{PB} + \vec{\varpi}_{BA} t$

The derivative of the position equation will give the velocity equation

- $\vec{\upsilon}_{PA} = \vec{\upsilon}_{PB} + \vec{\varpi}_{BA}$
  - $\vec{\mathbf{u}}_{PA}$ is the velocity of the particle P measured by observer A
  - $\vec{\mathbf{u}}_{PB}$ is the velocity of the particle P measured by observer B

These are called the **Galilean transformation equations.**

Acceleration in Different Frames of Reference

The derivative of the velocity equation will give the acceleration equation.

The acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving at a *constant velocity* relative to the first frame.

Acceleration, cont.

Calculating the acceleration gives

$$\frac{d\vec{\upsilon}_{PA}}{dt} = \frac{d\vec{\upsilon}_{PB}}{dt} + \frac{d\vec{\varpi}_{BA}}{dt}$$

Since

$\vec{\varpi}_{BA}$ is contant, $d\vec{\varpi}_{BA} / dt = 0$

Therefore,

$$\vec{\alpha}_{PA} = \vec{\alpha}_{PB}$$