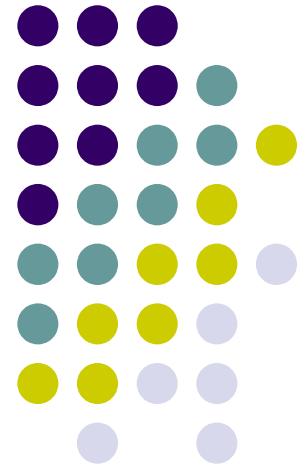


Chapter 10

Rotation of a Rigid Object about a Fixed Axis





Outline for W10,D2

Finishing center of mass (Ch. 9)

Rotation of a rigid solid (Ch. 10)

θ , ω , and α

Homework

Ch. 10 P. 1,4-6,19-21,25,28-30,34,35,37,53,54,55,64,67,69

Due Fri

Notes:

Lab this week is “2D Collisions”

See “NEW STUFF” for Ch. 10.

Rigid Object

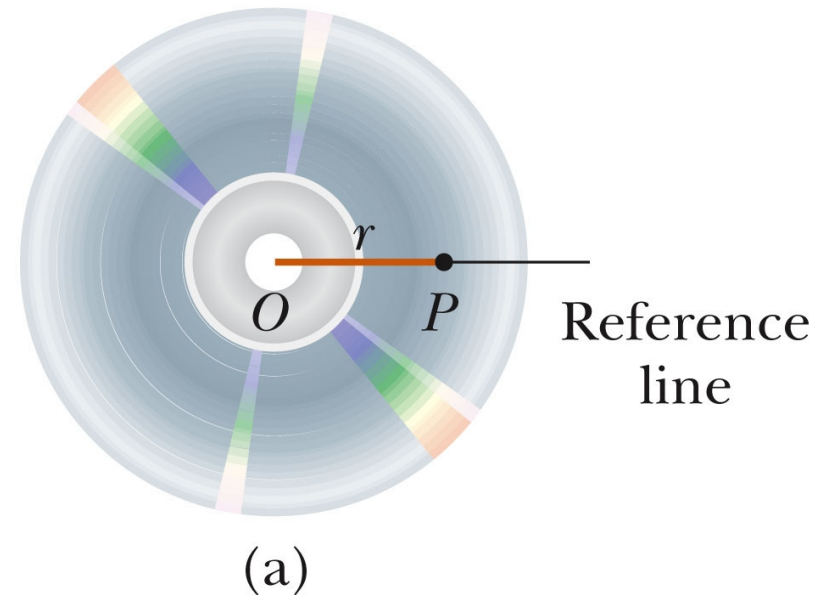


- A rigid object is one that is nondeformable
 - The relative locations of all particles making up the object remain constant
 - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible
- This simplification allows analysis of the motion of an extended object



Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin
 - A small element of the disc can be modeled as a particle at P





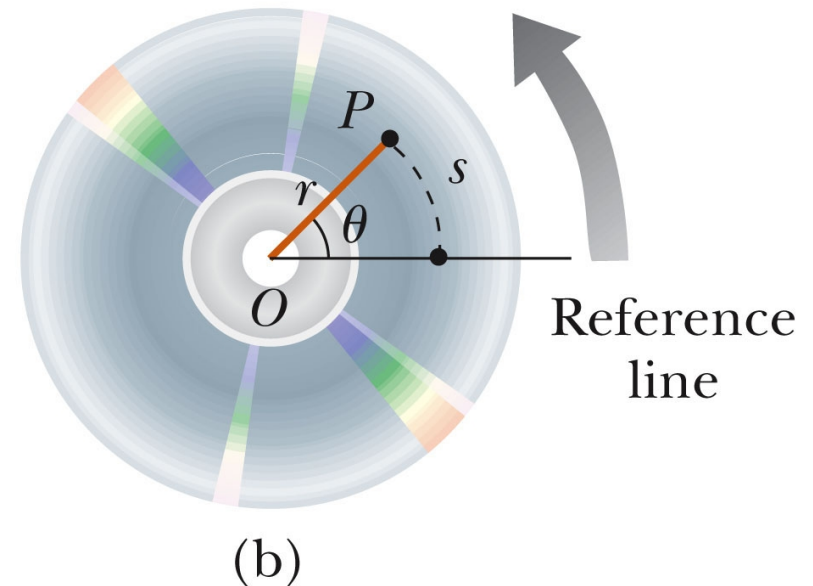
Angular Position, 2

- Point P will rotate about the origin in a circle of radius r
- **Every** particle on the disc undergoes circular motion about the origin, O
- P is located at (r, θ) where r is the distance from the origin to P and θ is the measured counterclockwise from the reference line

Angular Position, 3



- As the particle moves, the only coordinate that changes is θ
- As the particle moves through θ , it moves through an arc length s .
- The arc length and r are related:
 - $s = \theta r$
 - where θ is in radians



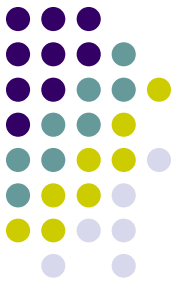


Radian

- This can also be expressed as

$$\theta = \frac{s}{r}$$

- θ is a pure number, but commonly is given the artificial unit, radian
- One radian is the angle subtended by an arc length equal to the radius of the arc
- Whenever using rotational equations, you must use angles expressed in radians



Conversions

- Comparing degrees and radians

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

- Converting from degrees to radians

$$\theta \text{ (rad)} = \frac{\pi}{180^\circ} \theta \text{ (degrees)}$$



Angular Position, final

- We can associate the angle θ with the entire rigid object as well as with an individual particle
 - Remember every particle on the object rotates through the same angle
- The angular position of the rigid object is the angle θ between the reference line on the object and the fixed reference line in space
 - The fixed reference line in space is often the x-axis

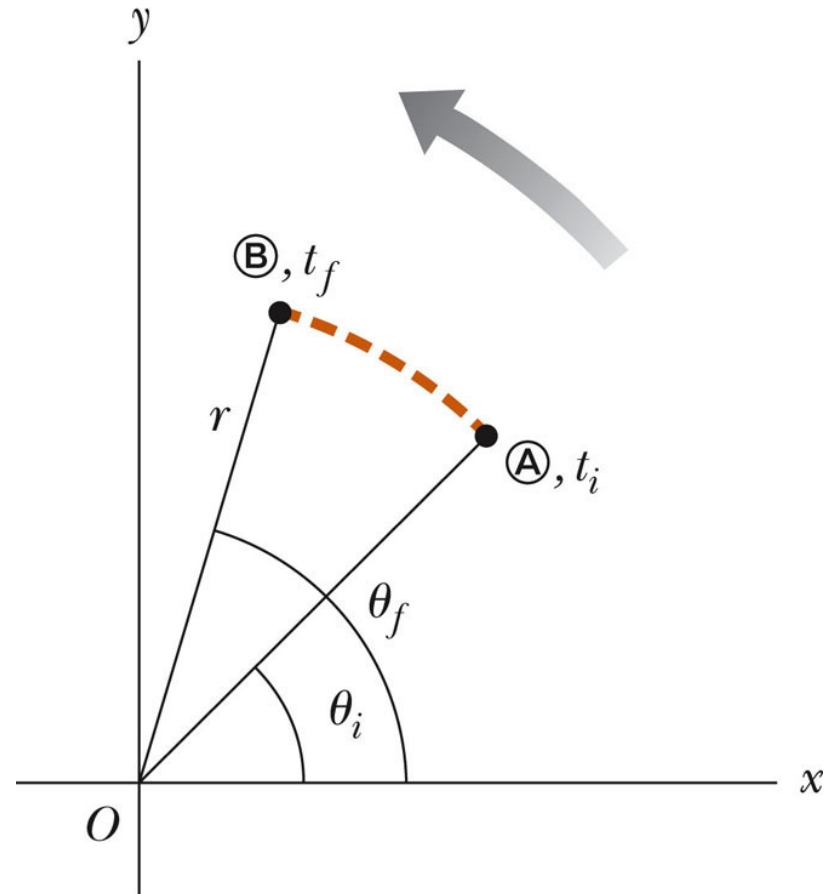


Angular Displacement

- The *angular displacement* is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- This is the angle that the reference line of length r sweeps out





Average Angular Speed

- The *average* angular speed, ω_{avg} , of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Speed



- The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$



Angular Speed, final

- Units of angular speed are radians/sec
 - rad/s or s^{-1} since radians have no dimensions
- Angular speed will be positive if θ is increasing (counterclockwise)
- Angular speed will be negative if θ is decreasing (clockwise)



Average Angular Acceleration

- The average angular acceleration, α , of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

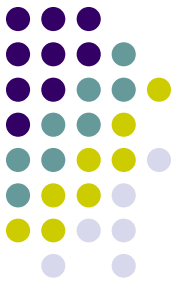
$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration



- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$



Angular Acceleration, final

- Units of angular acceleration are rad/s^2 or s^{-2} since radians have no dimensions
- Angular acceleration will be positive if an object rotating counterclockwise is speeding up
- Angular acceleration will also be positive if an object rotating clockwise is slowing down

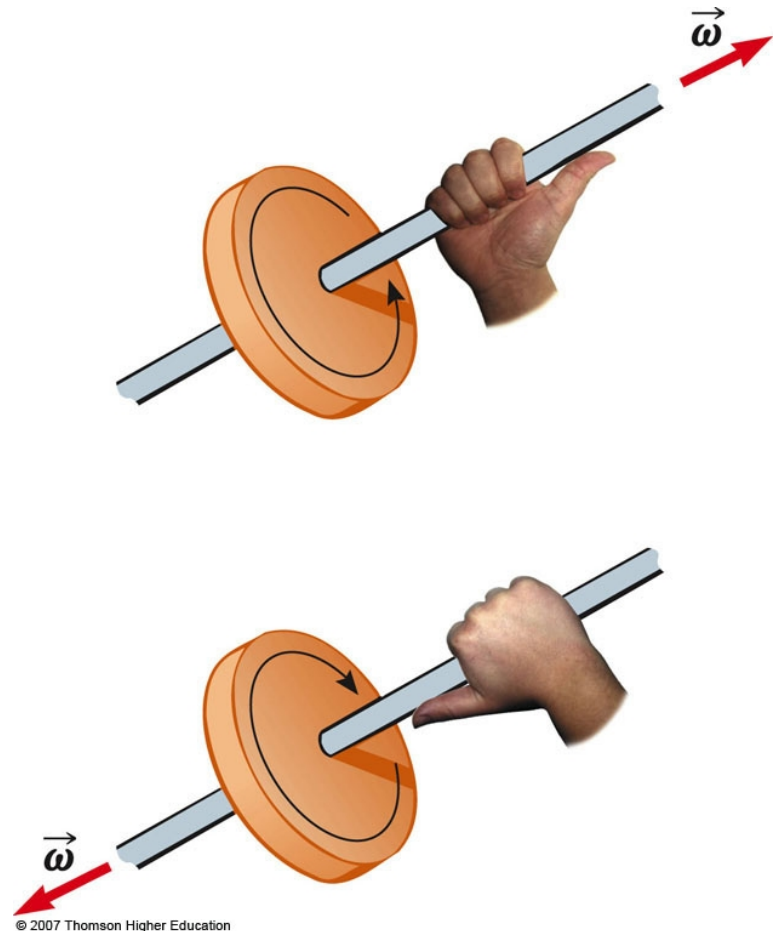
Angular Motion, General Notes



- When a rigid object rotates about a fixed axis in a given time interval, every portion on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration
 - So θ , ω , α all characterize the motion of the entire rigid object as well as the individual particles in the object

Directions, details

- Strictly speaking, the speed and acceleration (ω , α) are the magnitudes of the velocity and acceleration vectors
- The directions are actually given by the right-hand rule





Hints for Problem-Solving

- Similar to the techniques used in linear motion problems
 - With constant angular acceleration, the techniques are much like those with constant linear acceleration
- There are some differences to keep in mind
 - For rotational motion, define a rotational axis
 - The choice is arbitrary
 - Once you make the choice, it must be maintained
 - In some problems, the physical situation may suggest a natural axis
 - The object keeps returning to its original orientation, so you can find the number of revolutions made by the body



Rotational Kinematics

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
 - These are similar to the kinematic equations for linear motion
 - The rotational equations have the same mathematical form as the linear equations
- The new model is a **rigid object under constant angular acceleration**
 - Analogous to the particle under constant acceleration model

Rotational Kinematic Equations



$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha (\theta_f - \theta_i)$$

$$\theta_f = \theta_i + \frac{1}{2} (\omega_i + \omega_f) t$$

all with constant α

Comparison Between Rotational and Linear Equations

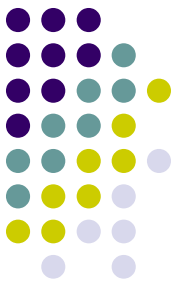


TABLE 10.1

Kinematic Equations for Rotational and Translational Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis	Translational Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Relationship Between Angular and Linear Quantities



- Displacements

$$s = \theta r$$

- Speeds

$$v = \omega r$$

- Accelerations

$$a = \alpha r$$

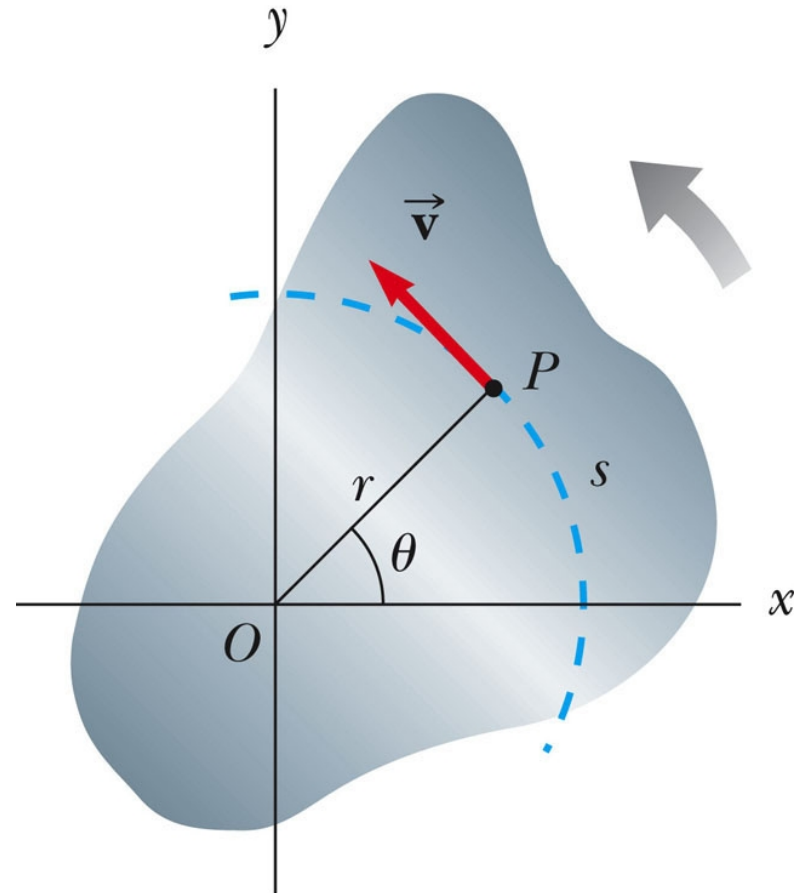
- Every point on the rotating object has the same angular motion
- Every point on the rotating object does **not** have the same linear motion

Speed Comparison



- The linear velocity is always tangent to the circular path
 - Called the tangential velocity
- The magnitude is defined by the tangential speed

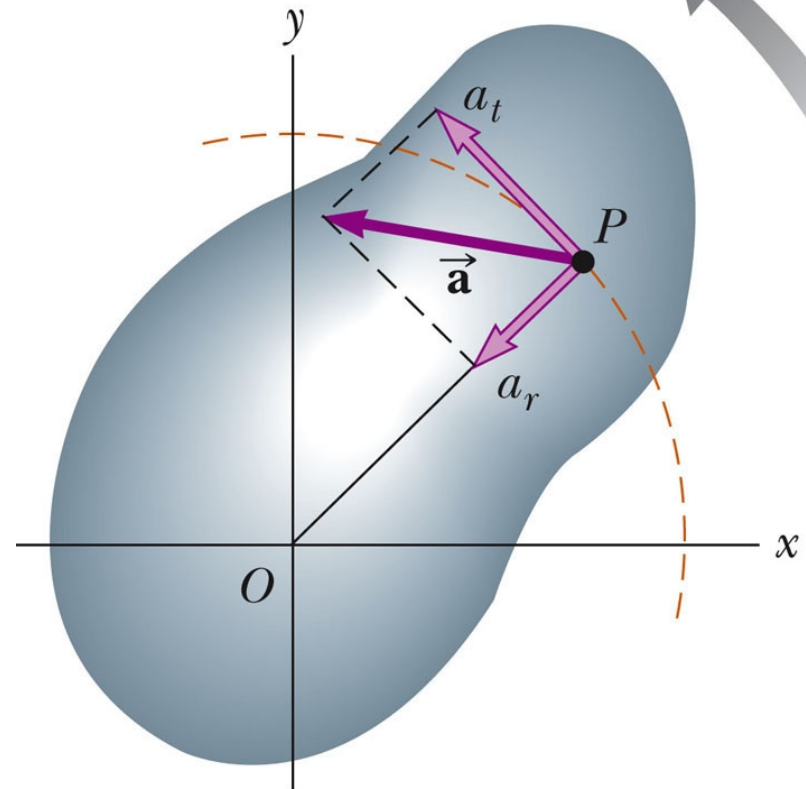
$$v = \frac{ds}{dt} = r \frac{d\theta}{dt} = r\omega$$



Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity

$$a_t = \frac{dv}{dt} = r \frac{d\omega}{dt} = r\alpha$$



Speed and Acceleration Note



- All points on the rigid object will have the same angular speed, but not the same tangential speed
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration
- The tangential quantities depend on r , and r is not the same for all points on the object



Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
 - Therefore, each point on a rotating rigid object will experience a centripetal acceleration

$$a_c = \frac{v^2}{r} = r\omega^2$$



Resultant Acceleration

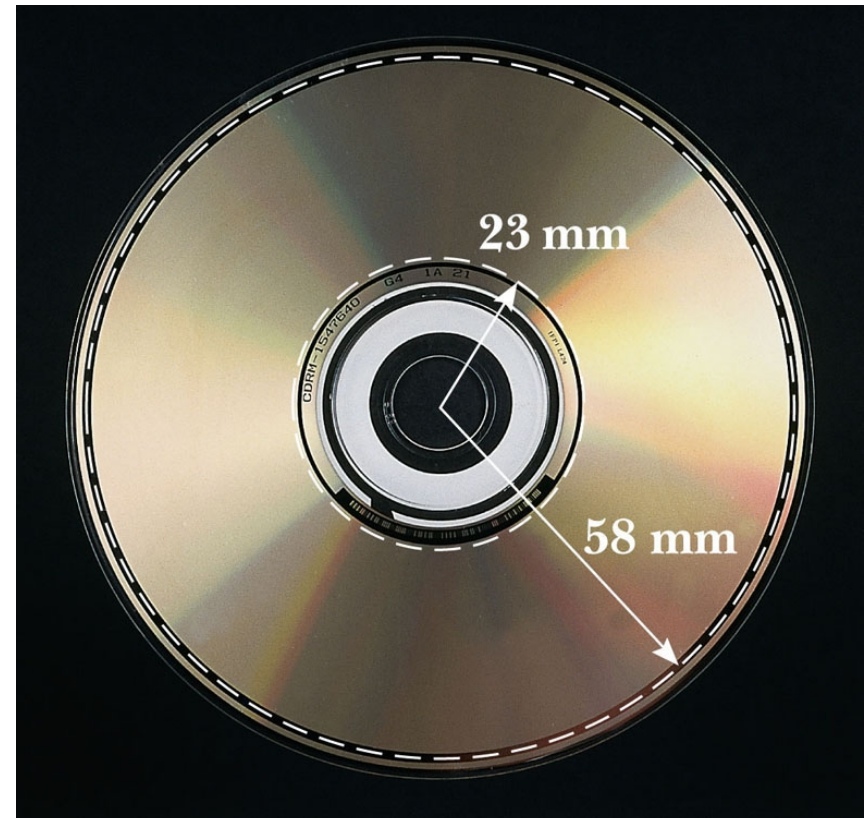
- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2 \alpha^2 + r^2 \omega^4} = r \sqrt{\alpha^2 + \omega^4}$$



Rotational Motion Example

- For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant ($v_t = \omega r$)
- At the inner sections, the angular speed is faster than at the outer sections





Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, ω , has rotational kinetic energy even though it may not have any translational kinetic energy
- Each particle has a kinetic energy of
 - $K_i = \frac{1}{2} m_i v_i^2$
- Since the tangential velocity depends on the distance, r , from the axis of rotation, we can substitute $v_i = \omega_i r$



Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

- Where I is called the moment of inertia

Rotational Kinetic Energy, final



- There is an analogy between the kinetic energies associated with linear motion ($K = \frac{1}{2} m v^2$) and the kinetic energy associated with rotational motion ($K_R = \frac{1}{2} I \omega^2$)
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object
- The units of rotational kinetic energy are Joules (J)



Moment of Inertia

- The definition of moment of inertia is

$$I = \sum_i r_i^2 m_i$$

- The dimensions of moment of inertia are ML^2 and its SI units are $kg \cdot m^2$
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass Δm_i



Moment of Inertia, cont

- We can rewrite the expression for I in terms of Δm

$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- With the small volume segment assumption,

$$I = \int \rho r^2 dV$$

- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known



Notes on Various Densities

- Volumetric Mass Density \rightarrow mass per unit volume: $\rho = m / V$
- Surface Mass Density \rightarrow mass per unit thickness of a sheet of uniform thickness, t :
 $\sigma = \rho t$
- Linear Mass Density \rightarrow mass per unit length of a rod of uniform cross-sectional area: $\lambda = m / L = \rho A$

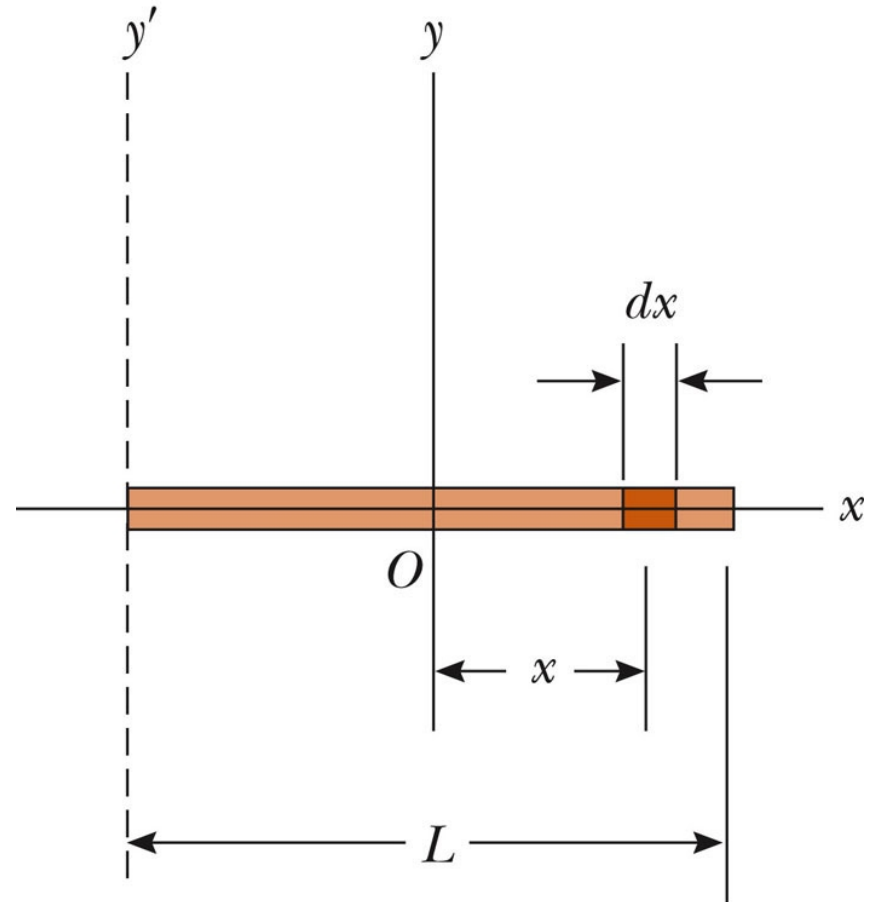
Moment of Inertia of a Uniform Rigid Rod



- The shaded area has a mass
 - $dm = \lambda dx$
- Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

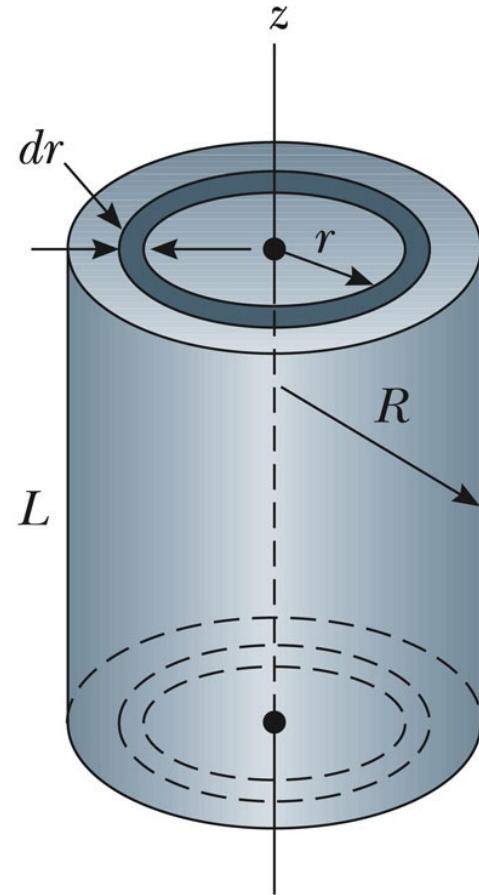
$$I = \frac{1}{12} ML^2$$



Moment of Inertia of a Uniform Solid Cylinder



- Divide the cylinder into concentric shells with radius r , thickness dr and length L
- $dm = \rho dV = 2\pi\rho Lr dr$
- Then for I
$$I_z = \int r^2 dm = \int r^2 (2\pi\rho Lr dr)$$
$$I_z = \frac{1}{2}MR^2$$



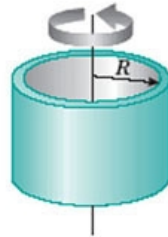
Moments of Inertia of Various Rigid Objects



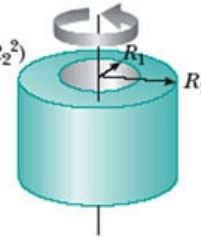
TABLE 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

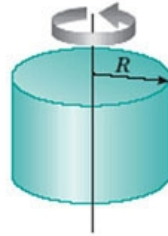
Hoop or thin
cylindrical shell
 $I_{\text{CM}} = MR^2$



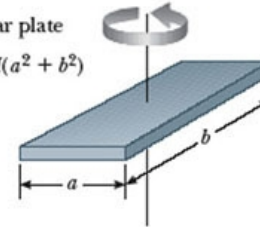
Hollow cylinder
 $I_{\text{CM}} = \frac{1}{2} M(R_1^2 + R_2^2)$



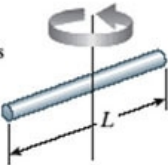
Solid cylinder
or disk
 $I_{\text{CM}} = \frac{1}{2} MR^2$



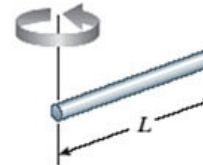
Rectangular plate
 $I_{\text{CM}} = \frac{1}{12} M(a^2 + b^2)$



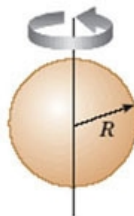
Long, thin rod
with rotation axis
through center
 $I_{\text{CM}} = \frac{1}{12} ML^2$



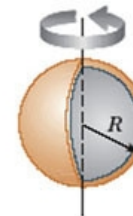
Long, thin
rod with
rotation axis
through end
 $I = \frac{1}{3} ML^2$



Solid sphere
 $I_{\text{CM}} = \frac{2}{5} MR^2$



Thin spherical
shell
 $I_{\text{CM}} = \frac{2}{3} MR^2$





Parallel-Axis Theorem

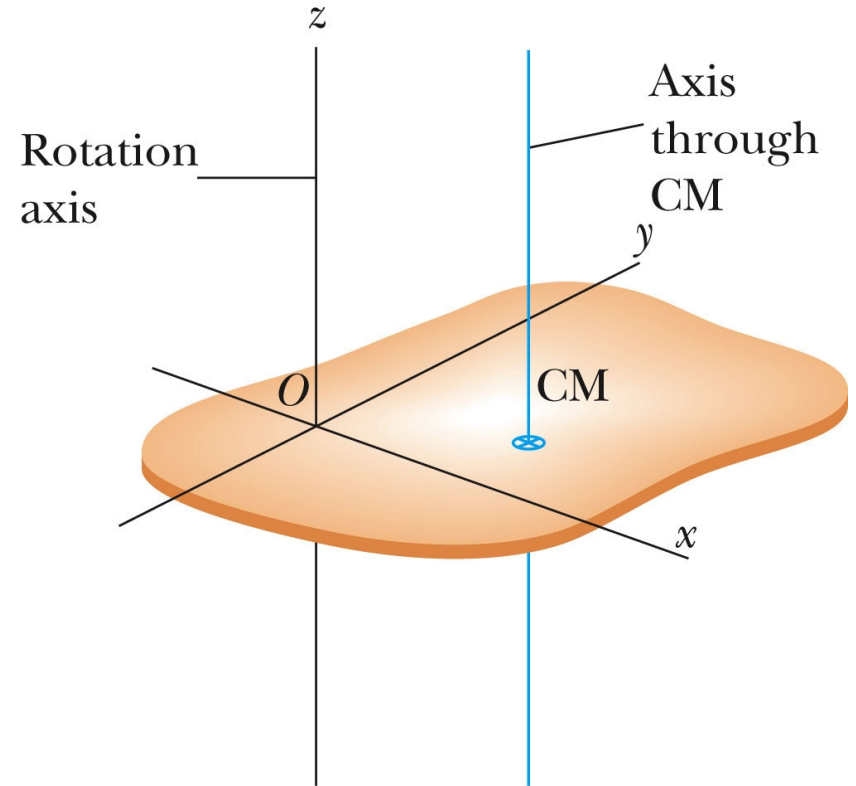
- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations
- The theorem states $I = I_{\text{CM}} + MD^2$
 - I is about any axis parallel to the axis through the center of mass of the object
 - I_{CM} is about the axis through the center of mass
 - D is the distance from the center of mass axis to the arbitrary axis

Parallel-Axis Theorem

Example



- The axis of rotation goes through O
- The axis through the center of mass is shown
- The moment of inertia about the axis through O would be $I_O = I_{CM} + MD^2$



(b)

Moment of Inertia for a Rod Rotating Around One End



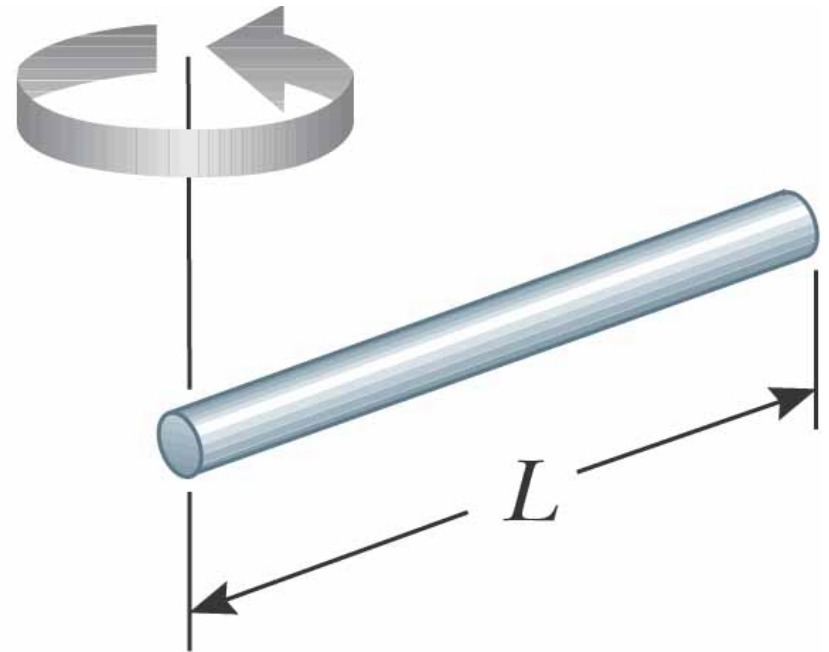
- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12} ML^2$$

- D is $\frac{1}{2} L$
- Therefore,

$$I = I_{CM} + MD^2$$

$$I = \frac{1}{12} ML^2 + M \left(\frac{L}{2} \right)^2 = \frac{1}{3} ML^2$$



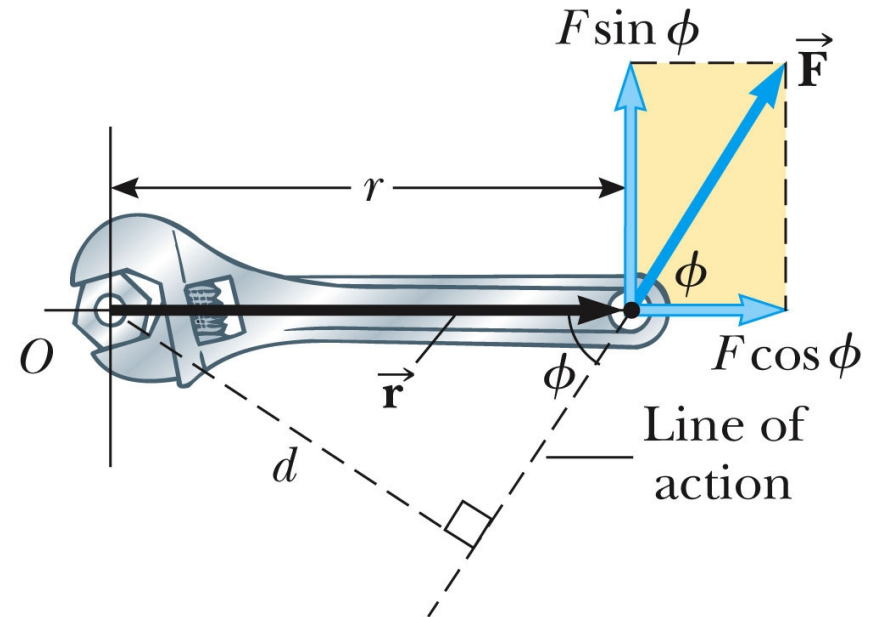


Torque

- Torque, τ , is the tendency of a force to rotate an object about some axis
 - Torque is a vector, but we will deal with its magnitude here
 - $\tau = r F \sin \phi = F d$
 - F is the force
 - ϕ is the angle the force makes with the horizontal
 - d is the *moment arm* (or lever arm) of the force

Torque, cont

- The moment arm, d , is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force
 - $d = r \sin \phi$





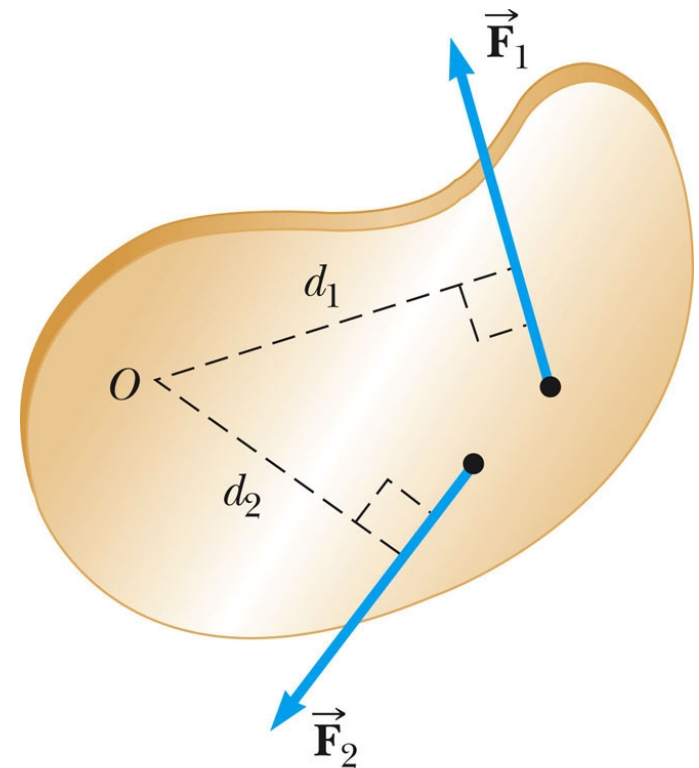
Torque, final

- The horizontal component of the force ($F \cos \phi$) has no tendency to produce a rotation
- Torque will have direction
 - If the turning tendency of the force is counterclockwise, the torque will be positive
 - If the turning tendency is clockwise, the torque will be negative



Net Torque

- The force \vec{F}_1 will tend to cause a counterclockwise rotation about O
- The force \vec{F}_2 will tend to cause a clockwise rotation about O
- $\Sigma \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$





Torque vs. Force

- Forces can cause a change in translational motion
 - Described by Newton's Second Law
- Forces can cause a change in rotational motion
 - The effectiveness of this change depends on the force and the moment arm
 - The change in rotational motion depends on the torque



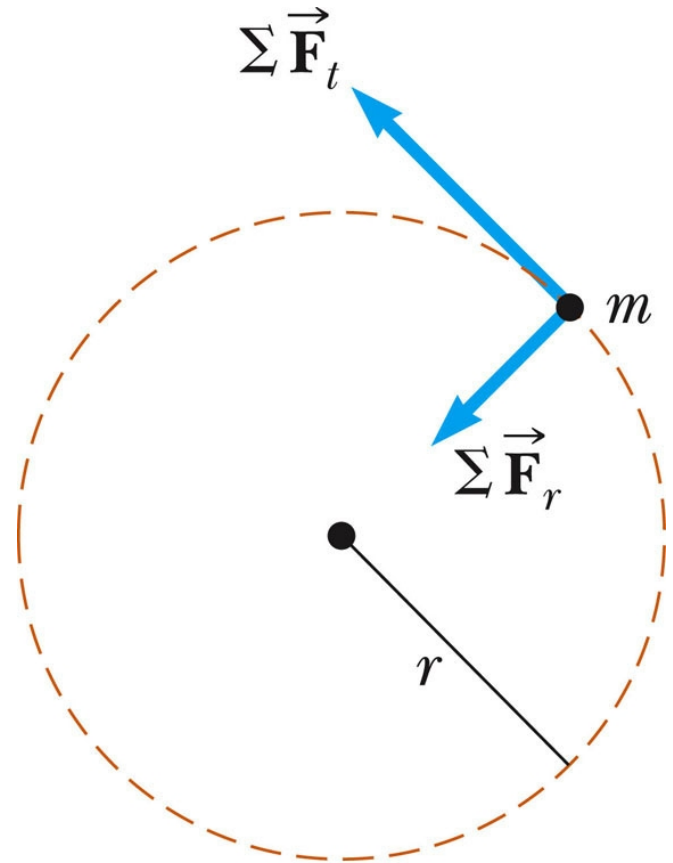
Torque Units

- The SI units of torque are N·m
 - Although torque is a force multiplied by a distance, it is very different from work and energy
 - The units for torque are reported in N·m and not changed to Joules

Torque and Angular Acceleration



- Consider a particle of mass m rotating in a circle of radius r under the influence of tangential force \mathbf{F}_t
- The tangential force provides a tangential acceleration:
 - $F_t = ma_t$
- The radial force, \mathbf{F}_r causes the particle to move in a circular path



Torque and Angular Acceleration, Particle cont.

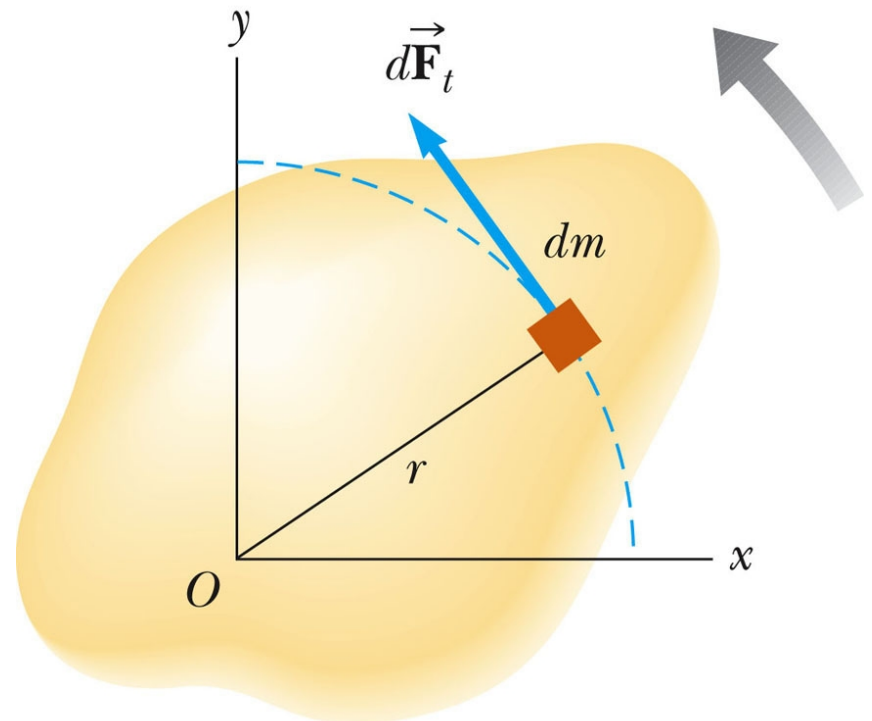


- The magnitude of the torque produced by $\sum \vec{F}_t$ around the center of the circle is
 - $\sum \tau = \sum F_t r = (ma_t) r$
- The tangential acceleration is related to the angular acceleration
 - $\sum \tau = (ma_t) r = (mr\alpha) r = (mr^2) \alpha$
- Since mr^2 is the moment of inertia of the particle,
 - $\sum \tau = I\alpha$
 - The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia

Torque and Angular Acceleration, Extended



- Consider the object consists of an infinite number of mass elements dm of infinitesimal size
- Each mass element rotates in a circle about the origin, O
- Each mass element has a tangential acceleration



Torque and Angular Acceleration, Extended cont.



- From Newton's Second Law
 - $dF_t = (dm) a_t$
- The torque associated with the force and using the angular acceleration gives
 - $d\tau = r dF_t = a_t r dm = \alpha r^2 dm$
- Finding the net torque
 - $\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$
 - This becomes $\sum \tau = I\alpha$

Torque and Angular Acceleration, Extended final

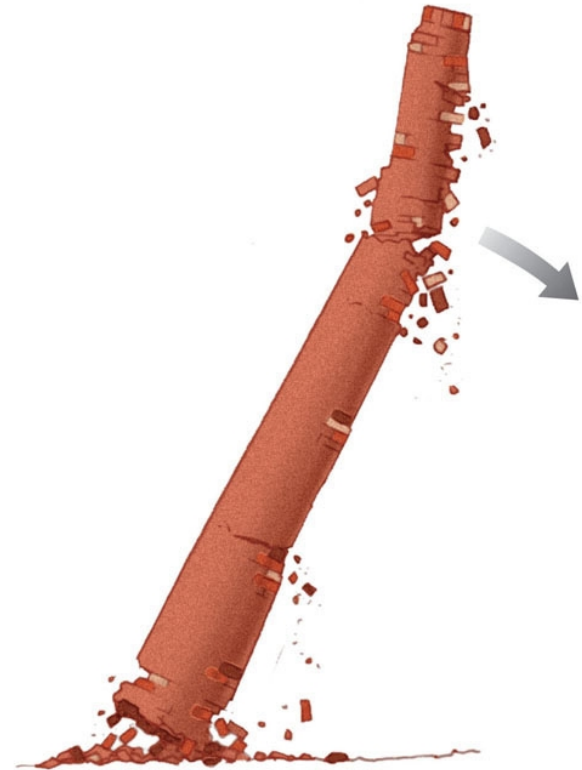


- This is the same relationship that applied to a particle
- This is the mathematic representation of the analysis model of a rigid body under a net torque
- The result also applies when the forces have radial components
 - The line of action of the radial component must pass through the axis of rotation
 - These components will produce zero torque about the axis



Falling Smokestack Example

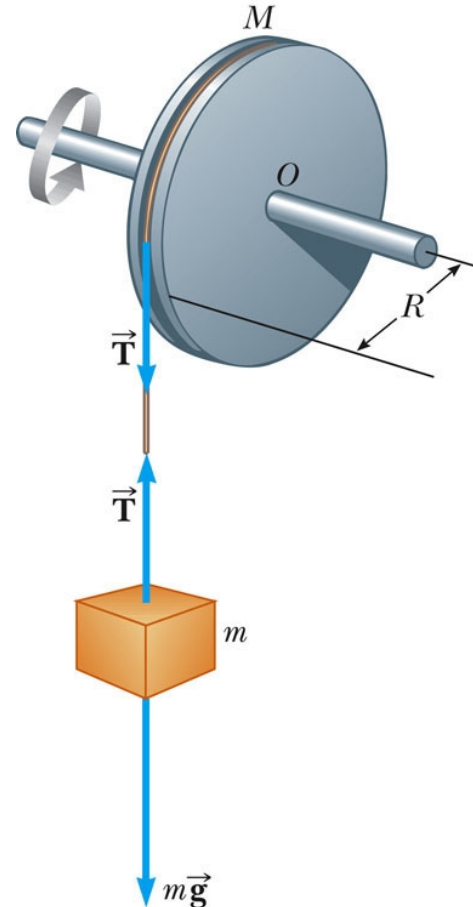
- When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground
- Each higher portion of the smokestack has a larger tangential acceleration than the points below it
- The shear force due to the tangential acceleration is greater than the smokestack can withstand
- The smokestack breaks



Torque and Angular Acceleration, Wheel Example



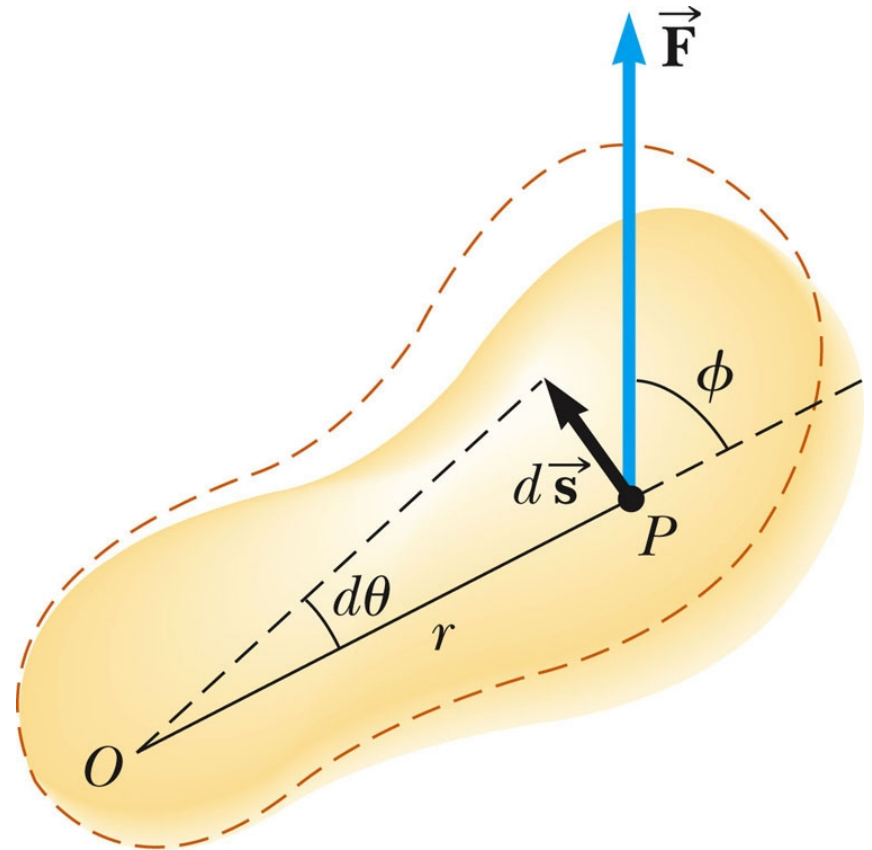
- Analyze:
- The wheel is rotating and so we apply $\Sigma \tau = I\alpha$
 - The tension supplies the tangential force
- The mass is moving in a straight line, so apply Newton's Second Law
 - $\Sigma F_y = ma_y = mg - T$

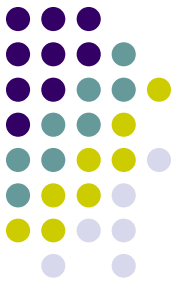




Work in Rotational Motion

- Find the work done by \vec{F} on the object as it rotates through an infinitesimal distance $ds = r d\theta$
$$dW = \vec{F} \cdot d\vec{s}$$
$$= (F \sin \phi) r d\theta$$
- The radial component of the force does no work because it is perpendicular to the displacement





Power in Rotational Motion

- The rate at which work is being done in a time interval dt is

$$\text{Power} = \wp = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $\wp = Fv$ in a linear system

Work-Kinetic Energy Theorem in Rotational Motion



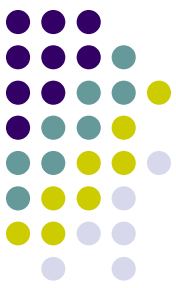
- The work-kinetic energy theorem for rotational motion states that *the net work done by external forces in rotating a symmetrical rigid object about a fixed axis equals the change in the object's rotational kinetic energy*

$$\sum W = \int_{\dot{\theta}_i}^{\dot{\theta}_f} I \dot{\theta} \, d\dot{\theta} = \frac{1}{2} I \dot{\theta}_f^2 - \frac{1}{2} I \dot{\theta}_i^2$$

Work-Kinetic Energy Theorem, General



- The rotational form can be combined with the linear form which indicates *the net work done by external forces on an object is the change in its **total** kinetic energy, which is the sum of the translational and rotational kinetic energies*



Summary of Useful Equations

TABLE 10.3

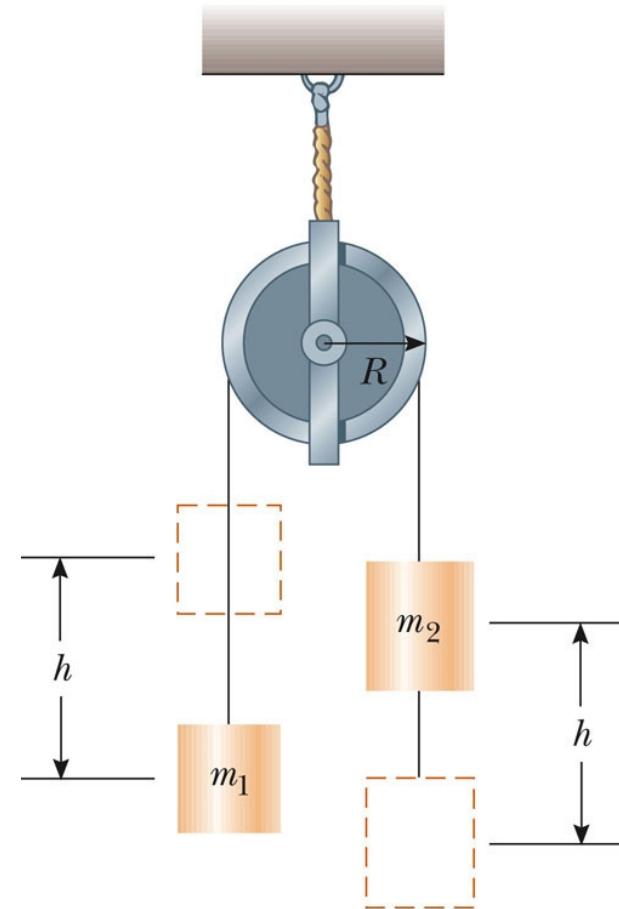
Useful Equations in Rotational and Translational Motion

Rotational Motion About a Fixed Axis	Translational Motion
Angular speed $\omega = d\theta/dt$	Translational speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Translational acceleration $a = dv/dt$
Net torque $\Sigma \tau = I\alpha$	Net force $\Sigma F = ma$
If $\alpha = \text{constant}$ $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a = \text{constant}$ $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau d\theta$	Work $W = \int_{x_i}^{x_f} F_x dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma \tau = dL/dt$	Net force $\Sigma F = dp/dt$

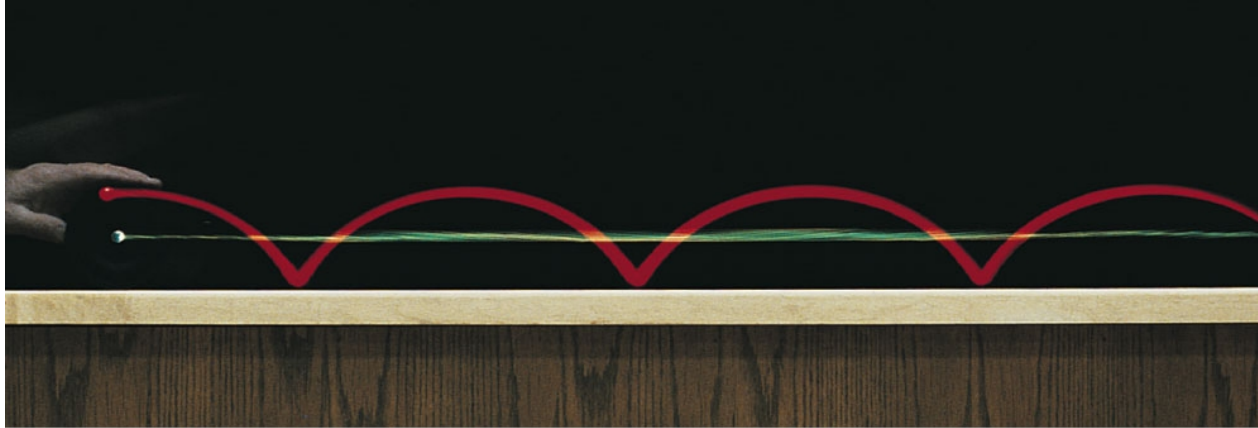
Energy in an Atwood Machine, Example



- The blocks undergo changes in translational kinetic energy and gravitational potential energy
- The pulley undergoes a change in rotational kinetic energy
- Use the active figure to change the masses and the pulley characteristics



Rolling Object



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- The red curve shows the path moved by a point on the rim of the object
 - This path is called a ***cycloid***
- The green line shows the path of the center of mass of the object



Pure Rolling Motion

- In pure rolling motion, an object rolls without slipping
- In such a case, there is a simple relationship between its rotational and translational motions

Rolling Object, Center of Mass

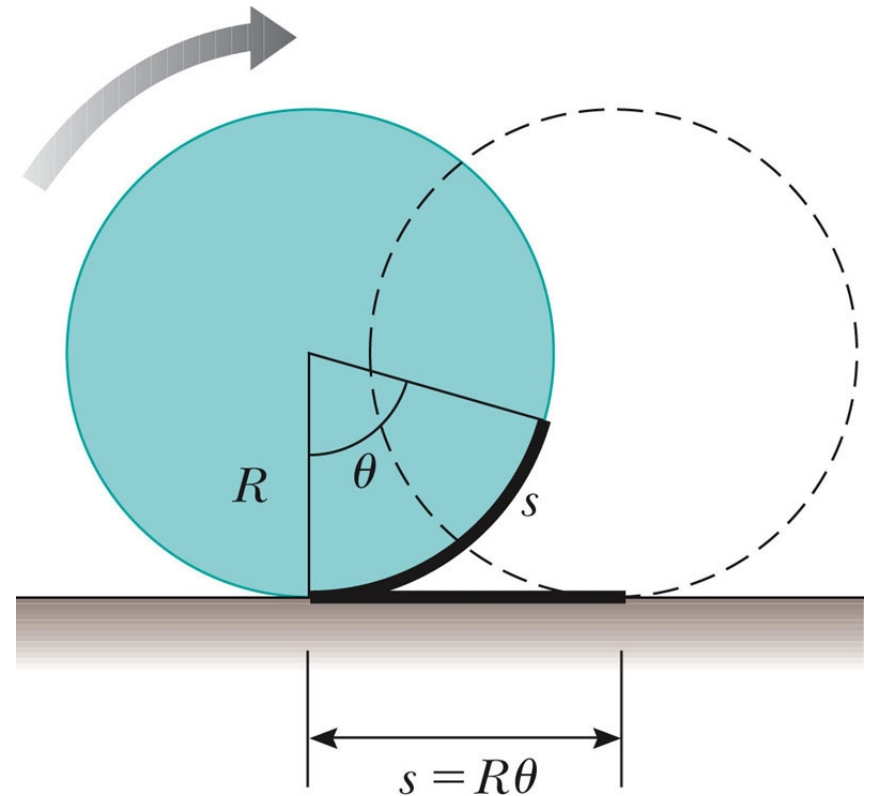


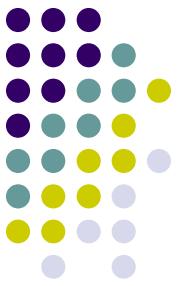
- The velocity of the center of mass is

$$v_{\text{CM}} = \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega$$

- The acceleration of the center of mass is

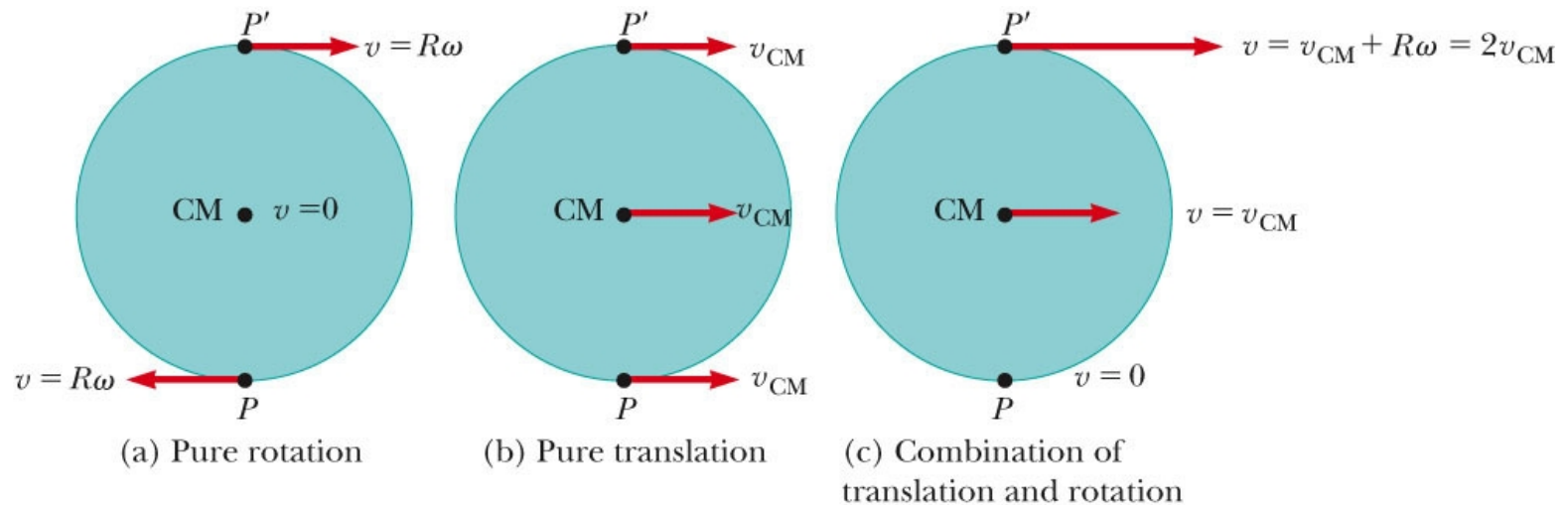
$$a_{\text{CM}} = \frac{dv_{\text{CM}}}{dt} = R \frac{d\omega}{dt} = R\alpha$$





Rolling Motion Cont.

- Rolling motion can be modeled as a combination of pure translational motion and pure rotational motion
- The contact point between the surface and the cylinder has a translational speed of zero (c)



Total Kinetic Energy of a Rolling Object

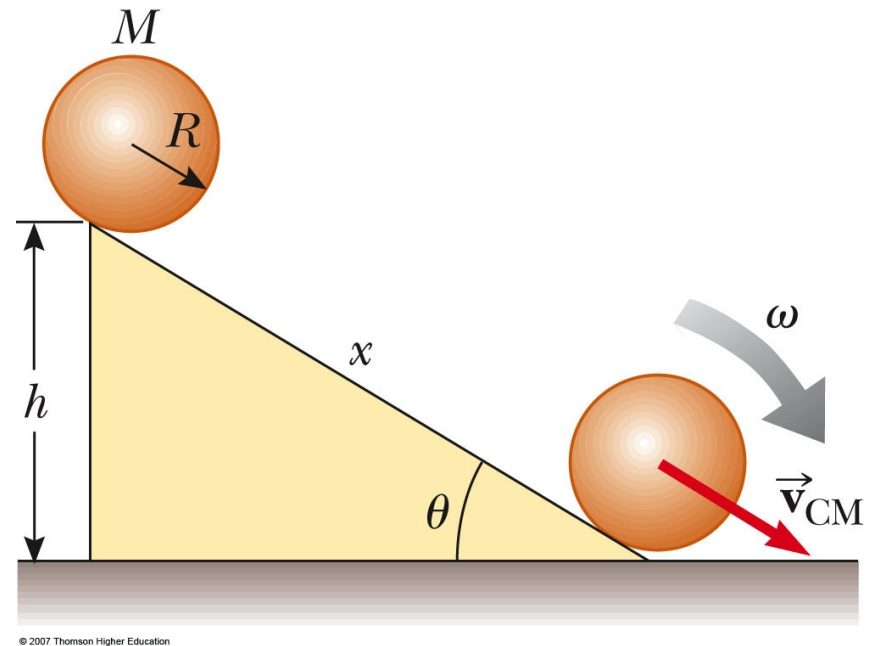


- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass
 - $K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$
 - The $\frac{1}{2} I_{CM} \omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass
 - The $\frac{1}{2} M v^2$ represents the translational kinetic energy of the cylinder about its center of mass

Total Kinetic Energy, Example



- Accelerated rolling motion is possible only if friction is present between the sphere and the incline
 - The friction produces the net torque required for rotation
 - No loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant
 - Use the active figure to vary the objects and compare their speeds at the bottom



Total Kinetic Energy, Example cont



- Apply Conservation of Mechanical Energy

- Let $U = 0$ at the bottom of the plane

- $K_f + U_f = K_i + U_i$

- $K_f = \frac{1}{2} (I_{CM} / R^2) v_{CM}^2 + \frac{1}{2} M v_{CM}^2 = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$

- $U_i = Mgh$

- $U_f = K_i = 0$

- Solving for v

$$v = \left[\frac{2gh}{1 + \left(\frac{I_{CM}}{MR^2} \right)} \right]^{\frac{1}{2}}$$

Sphere Rolling Down an Incline, Example



- **Conceptualize**

- A sphere is rolling down an incline

- **Categorize**

- Model the sphere and the Earth as an isolated system
 - No nonconservative forces are acting

- **Analyze**

- Use Conservation of Mechanical Energy to find v
 - See previous result

Sphere Rolling Down an Incline, Example cont



- **Analyze, cont**
 - Solve for the acceleration of the center of mass
- **Finalize**
 - Both the speed and the acceleration of the center of mass are independent of the mass and the radius of the sphere
- **Generalization**
 - *All homogeneous solid spheres experience the same speed and acceleration on a given incline*
 - Similar results could be obtained for other shapes