Exercises

Show that

(1) $\displaystyle\lim_{z\to\infty}\frac{3z^4+2z^2-z+1}{z^4+1}=3.$

(2) $\displaystyle\lim_{z\to\infty}\frac{z^{12}-42z}{z^8+17z}=\infty.$

(3) $\displaystyle\lim_{z\to -i}\frac{a}{(z+i)^2}=\infty$, if $a\neq 0$.

Solution:

(1) Notice that

$$\lim_{z\to\infty}\frac{3z^4+2z^2-z+1}{z^4+1}=3 \quad\Longleftrightarrow\quad \lim_{z\to 0}\frac{3\left(1/z\right)^4+2\left(1/z\right)^2-(1/z)+1}{\left(1/z\right)^4+1}=3.$$

Simplifying we obtain

$$\lim_{z\to 0}\frac{z^4-z^3+2z^2+3}{z^4+1}=3.$$

This limit holds. In fact the new expression

$$\frac{z^4-z^3+2z^2+3}{z^4+1}$$

is continuous at $z=0$.

Alternatively, we have

$$\begin{aligned}
\lim_{z\to\infty}\frac{3z^4+2z^2-z+1}{z^4+1} &= \lim_{z\to\infty}\frac{z^4\left(3+2\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}\right)}{z^4\left(1+\frac{1}{z^4}\right)}\\
&= \lim_{z\to\infty}\frac{3+2\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}}{1+\frac{1}{z^4}}\\
&= \frac{\lim\limits_{z\to\infty}(3)+2\lim\limits_{z\to\infty}\dfrac{1}{z^2}-\lim\limits_{z\to\infty}\dfrac{1}{z^3}+\lim\limits_{z\to\infty}\dfrac{1}{z^4}}{\lim\limits_{z\to\infty}(1)+\lim\limits_{z\to\infty}\dfrac{1}{z^4}}\\
&= 3.
\end{aligned}$$

In the last part, we use the fact that

$$\lim_{z\to\infty}\frac{1}{z^n}=0.$$

(2) Notice that

$$\lim_{z\to\infty} \frac{z^{12}-42z}{z^8+17z} = \infty \quad \Longleftrightarrow \quad \lim_{z\to 0} \frac{1}{\dfrac{(1/z)^{12}-42(1/z)}{(1/z)^8+17(1/z)}} = 0.$$

Simplifying we obtain

$$\lim_{z\to 0} \frac{17z^{11}+z^4}{1-42z^{11}} = 0.$$

This limit holds. In fact the new expression

$$\frac{17z^{11}+z^4}{1-42z^{11}}$$

is continuous at $z=0$.

Alternatively, we have

$$\begin{aligned}
\lim_{z\to\infty} \frac{z^{12}-42z}{z^8+17z} &= \lim_{z\to\infty} \frac{z^8\left(z^4-42\frac{1}{z^7}\right)}{z^8\left(1+17\frac{1}{z^7}\right)} \\
&= \lim_{z\to\infty} \frac{z^4-42\frac{1}{z^7}}{1+17\frac{1}{z^7}} \\
&= \frac{\lim\limits_{z\to\infty} z^4 - 42 \lim\limits_{z\to\infty} \dfrac{1}{z^7}}{\lim\limits_{z\to\infty}(1) + 17 \lim\limits_{z\to\infty} \dfrac{1}{z^7}} \\
&= \infty.
\end{aligned}$$

In the last part, we use again the fact that

$$\lim_{z\to\infty} \frac{1}{z^n} = 0.$$

(3) Notice that

$$\lim_{z\to -i} \frac{a}{(z+i)^2} = \infty \iff \lim_{z\to -i} \frac{1}{\frac{a}{(z+i)^2}} = 0$$

Simplifying

$$\lim_{z\to -i} \frac{(z+i)^2}{a} = 0$$

This limit holds because

$$\lim_{z\to -i} \frac{(z+i)^2}{a} = \frac{(-i+i)^2}{a} = 0, \quad (a \neq 0).$$