Exercises

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(1) Are the following functions conformal? To answer this, analyse their domains and draw some sketches to map specific regions.

- $f(z) = e^z$
- $f(z) = z^2$
- $f(z) = z + \frac{1}{z}$

Solution

The function $f(z) = e^z$ is conformal throughout the entire z plane since the function is entire and $(e^z)' = e^z \neq 0$ for each z. For details about this mapping see Section 14 from Churchill's book.

We already know that the function $f(z) = z^2$ maps a quarter plane to a half plane, and therefore doubles the angle between the coordinate axes at the origin (see Figures 1 and 2).

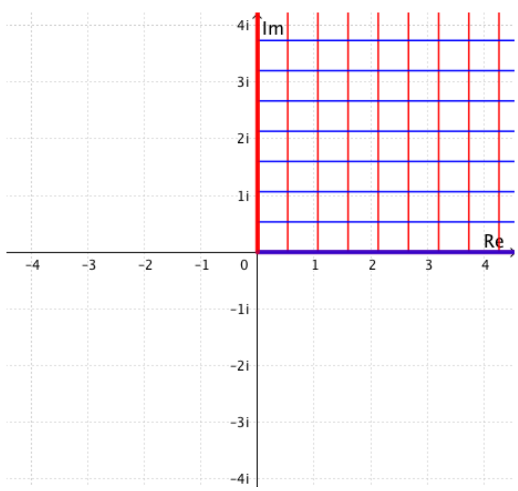

Figure 1: A quarter plane.

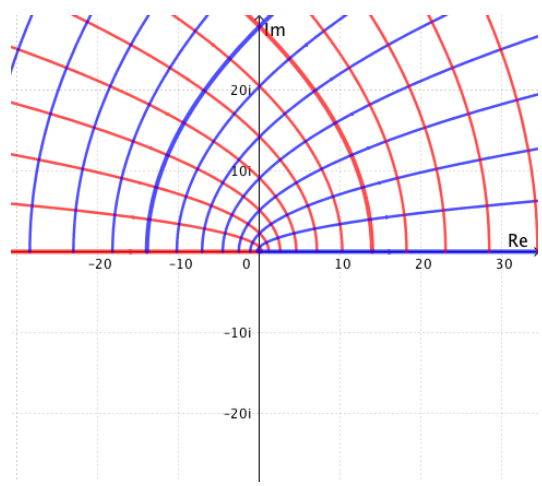

Figure 2: Image under the map $f(z) = z^2$.

This means that $f(z) = z^2$ is not conformal at $z_0 = 0$. The explanation is, of course, that $z_0 = 0$ is a critical point of f, that is $f'(0) = 0$. Amazingly, the map preserves angles everywhere else.

Although $f(z) = z^2$ is not conformal at $z_0 = 0$, we can find a region that will be mapped conformally. For example, consider the right half-plane $\{\mathbf{Re}(z) > 0\}$. This region is mapped conformally by $w = z^2$ onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$, as illustrated in Figure 3.

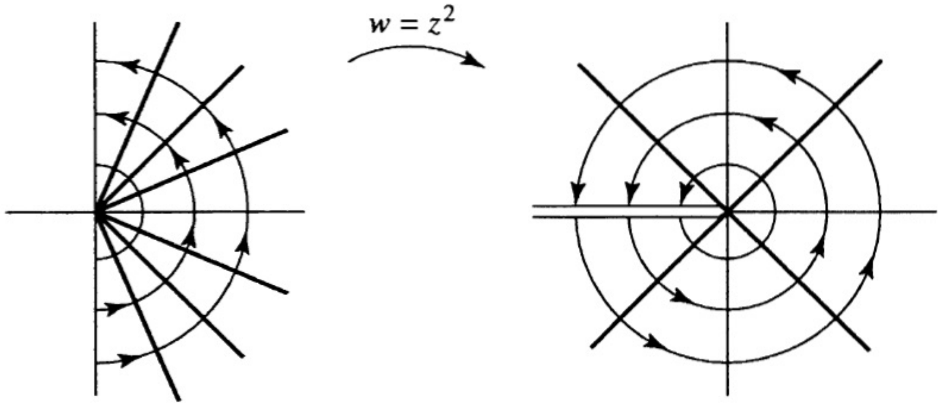

Figure 3: Mapping the region $\{\mathbf{Re}(z) > 0\}$.

Consider now the **Joukowsky map**

$$w = z + \frac{1}{z} \tag{1}$$

Since

$$\frac{d}{dz} w = 1 - \frac{1}{z^2} = 0 \quad \text{if and only if} \quad z = \pm 1,$$

the **Joukowsky map** is conformal except at the critical points $z = \pm 1$ as well as the singularity $z = 0$, where it is not defined.

If $z = e^{i\theta}$ lies on the unit circle, then

$$w = e^{i\theta} + e^{-i\theta} = 2\cos\theta,$$

lies on the real axis, with $-2 \leq w \leq 2$. Thus, the **Joukowsky map** squashes the unit circle down to the real line segment $[-2, 2]$. The images of points outside the unit circle fill the rest of the w plane, as do the images of the (nonzero) points inside the unit circle. Indeed, if we solve (1) for z, we have

$$z = \frac{1}{2}\left(w \pm \sqrt{w^2 - 4}\right).$$

We see that every w except ± 2 comes from two different points z; for w not on the critical line segment $[-2, 2]$, one point (with the minus sign) lies inside and one (with the plus sign) lies outside the unit circle, whereas if $-2 < w < 2$, both points lie on the unit circle and a common vertical line.

Therefore, the **Joukowski map**

$$f(z) = z + \frac{1}{z}$$

defines a one-to-one conformal mapping from the exterior of the unit circle $\{|z| > 1\}$ onto the exterior of the line segment $\mathbb{C} \setminus [-2, 2]$.

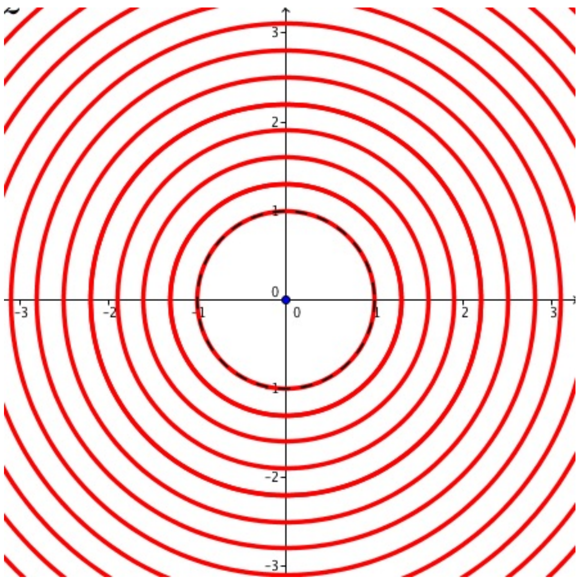

Figure 4: Concentric circles $|z| = r \geq 1$.

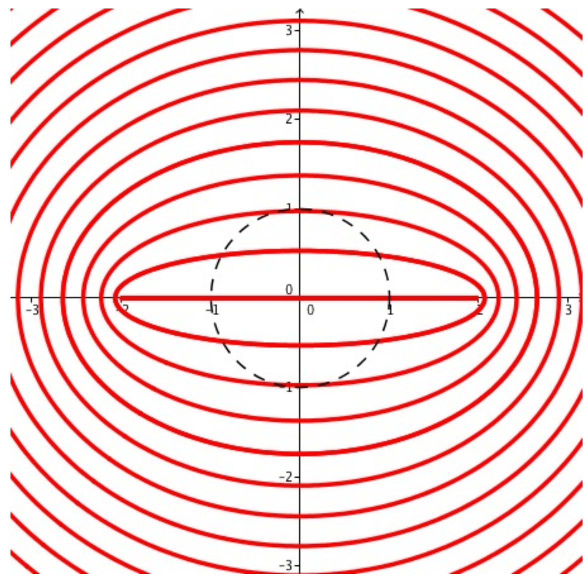

Figure 5: Image under the **Joukowski map**.