## Exercises

**(1)** In each case, find the Laurent series of the function at its isolated singular point. Determine whether that point is a pole (determine its order), a removable singular point or an essential singularity. Finally, determine the corresponding residue.

$$(a)\ z\exp\left(\frac{1}{z}\right); \quad (b)\ \frac{z^2}{1+z}; \quad (c)\ \frac{\cos z}{z}; \quad (d)\ \frac{1-\cosh z}{z^3}; \quad (e)\ \frac{1}{(2-z)^3}.$$

*Suggestion 1: Use the know series*

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}, \qquad \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}, \qquad \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}, \qquad (|z|<\infty).$$

*Suggestion 2: For part (b) notice that* $z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1$

**Solution:**

*(a)* From the expansion

$$\exp(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

we see that

$$f(z) = z\exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} + \cdots$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z=0$ is then

$$\frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} + \cdots$$

and $z=0$ is an essential singularity. Finally,

$$b_1 = \operatorname*{\mathbf{Res}}_{z=0} f(z) = \frac{1}{2!}.$$

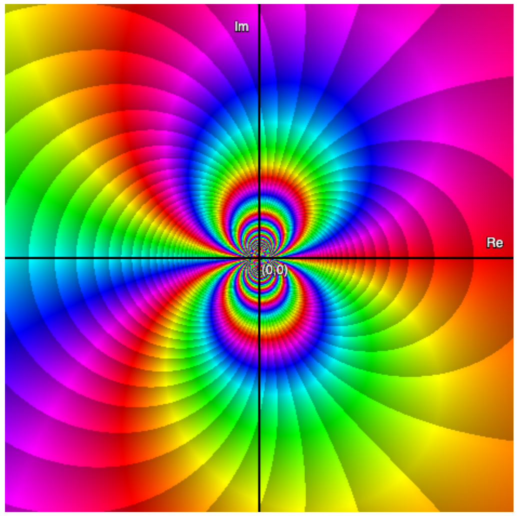


Figure 1: Domain coloring for $z\exp(1/z)$. Link: Domain Coloring

*(b)* The isolated singular point of

$$f(z) = \frac{z^2}{1+z}$$

is at $z = -1$. Using suggestion 2 we have that

$$\frac{z^2}{1+z} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}$$

In this case the principal part is $\dfrac{1}{z+1}$, and the point $z = -1$ is a simple pole.

Finally,

$$b_1 = \operatorname*{\mathbf{Res}}_{z=-1} f(z) = 1.$$

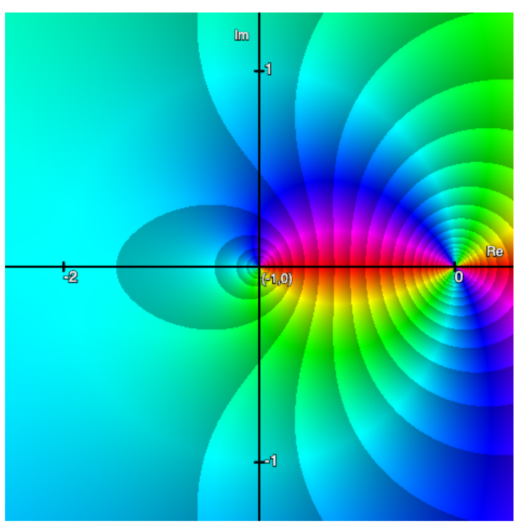


Figure 2: Domain coloring for $z^2/(1+z)$. Link: Domain Coloring

*(c)* The isolated singular point of

$$f(z) = \frac{\cos z}{z}$$

is $z = 0$. Using the known series

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

we have

$$\frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$$

Thus the principal part is $\frac{1}{z}$. This means that $z = 0$ is a simple pole. Finally

$$b_1 = \operatorname*{\mathbf{Res}}_{z=0} f(z) = 1.$$

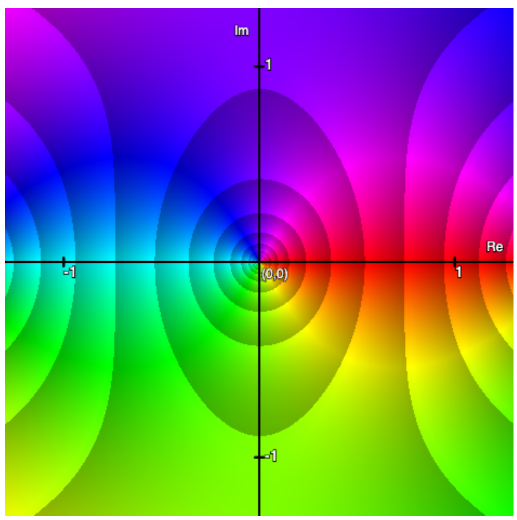


Figure 3: Domain coloring for $\cos z/z$. Link: Domain Coloring

*(d)* The singular point in this case is $z = 0$. Using the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

we have

$$\begin{aligned} \frac{1 - \cosh z}{z^3} &= \frac{1}{z^3}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)\right] \\ &= -\frac{1}{2!} \cdot \frac{1}{z} - \frac{z}{4!} - \frac{z^3}{6!} - \cdots \end{aligned}$$

Thus the principal part is $-\dfrac{1}{2} \cdot \dfrac{1}{z}$. This means that $z = 0$ is a simple pole. Finally

$$b_1 = \operatorname*{\mathbf{Res}}_{z=0} f(z) = -\frac{1}{2!}.$$

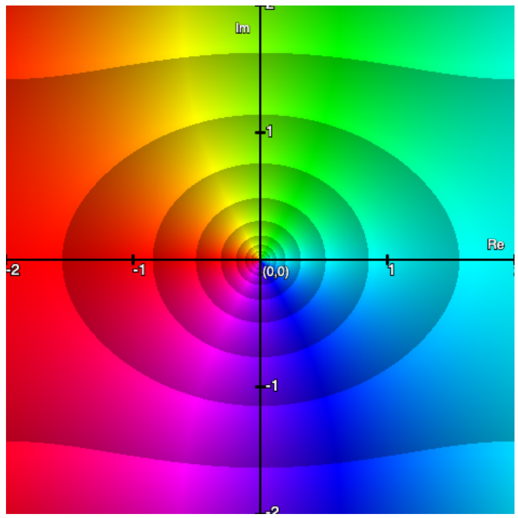


Figure 4: Domain coloring for $(1 - \cosh z)/z^3$. Link: Domain Coloring

*(e)* The function $f(z) = \dfrac{1}{(2-z)^3}$ has a singular point at $z = 2$. Notice also that

$$\frac{1}{(2-z)^3} = \frac{-1}{(z-2)^3}.$$

In this case the principal part of $f$ is the function itself. The singular point is a pole of order 3 and

$$b_1 = \operatorname*{\mathbf{Res}}_{z=2} f(z) = 0.$$

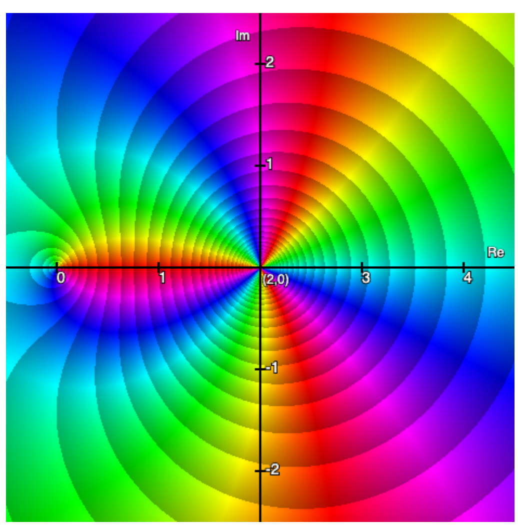


Figure 5: Domain coloring for $1/(2-z)^3$. Link: Domain Coloring

**(2)** Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

taken counterclockwise around the circle (a) $|z-2|=2$; (b) $|z|=4$.

*Ans. (a)* $\pi i$; *(b)* $6\pi i$.

**Solution - Part (a):**

Observe that the point $z_0 = 1$, which is the only singularity inside $C$, is a simple pole of the integrand.

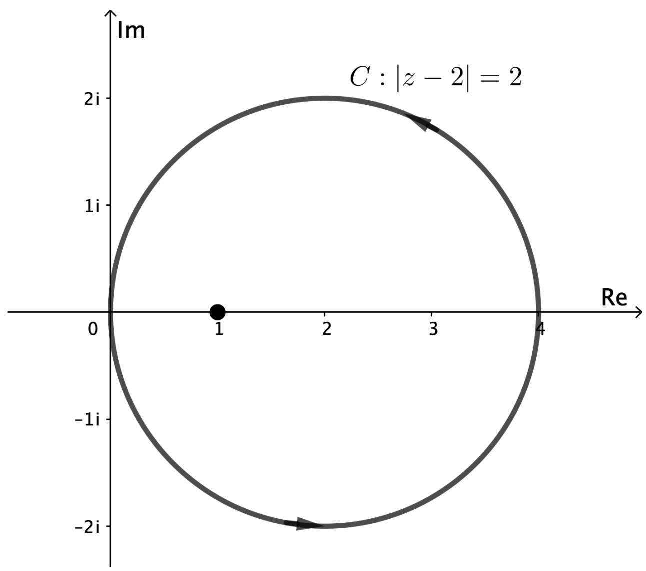


Figure 6: Circle $|z-2|=2$.

Notice that

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z-1} \quad \text{with} \quad \phi(z) = \frac{3z^3+2}{z^2+9}.$$

Since $\phi(z)$ is analytic at $z_0 = 1$ and $\phi(z_0) \neq 0$, then

$$\operatorname*{\mathbf{Res}}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{z^2+9}\right|_{z=1} = \frac{3(1)^3+2}{(1)^2+9} = \frac{5}{10} = \frac{1}{2}.$$

Hence

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \cdot \frac{1}{2} = \pi i.$$

**Solution - Part (b):**

In this case the singularities $z_0 = 1, z_1 = 3i, z_2 = -3i$ of the integrand are inside $C$.

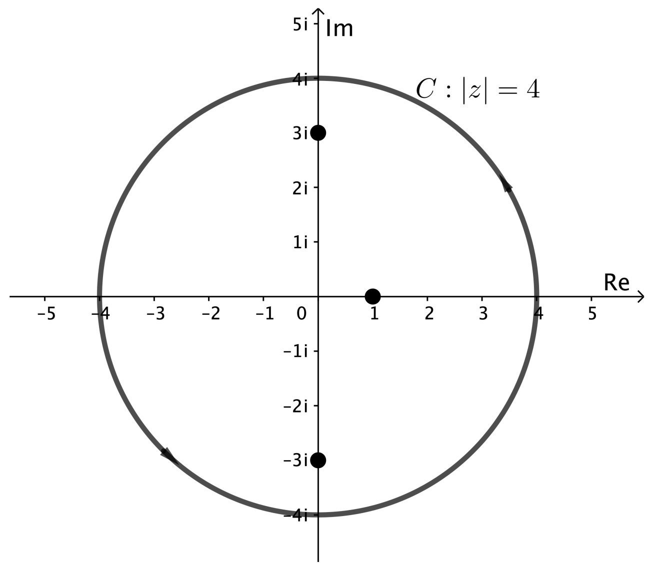


Figure 7: Circle $|z| = 4$.

From part (a)

$$\underset{z=1}{\mathbf{Re}} \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{1}{2}.$$

Now, notice that

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

Thus for $z_1 = 3i$ we have

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z-3i} \quad \text{with} \quad \phi(z) = \frac{3z^3+2}{(z-1)(z+3i)}.$$

Since $\phi(z)$ is analytic at $z_1 = 3i$ and $\phi(z_1) \neq 0$, then

$$\underset{z=3i}{\mathbf{Res}} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{(z-1)(z+3i)}\right|_{z=3i} = \frac{3(3i)^3+2}{((3i)-1)((3i)+3i)} = \frac{15+49i}{12}.$$

On the other hand, for $z_2 = -3i$ we have

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z+3i} \quad \text{with} \quad \phi(z) = \frac{3z^3+2}{(z-1)(z-3i)}.$$

Since $\phi(z)$ is analytic at $z_2 = -3i$ and $\phi(z_2) \neq 0$, then

$$\operatorname*{\mathbf{Res}}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{(z-1)(z-3i)}\right|_{z=-3i} = \frac{3(-3i)^3+2}{((-3i)-1)((-3i)-3i)} = \frac{15-49i}{12}.$$

Therefore, using Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{\mathbf{Res}}_{z=z_k} f(z),$$

we find that

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \left(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12}\right) = 2\pi i(3) = 6\pi i.$$

**(3)** Use residues to evaluate the improper integral:

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

*Ans.* $\pi/4$.

**Solution:** First notice that the function $1/(x^2+1)^2$ is even. Then

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}.$$

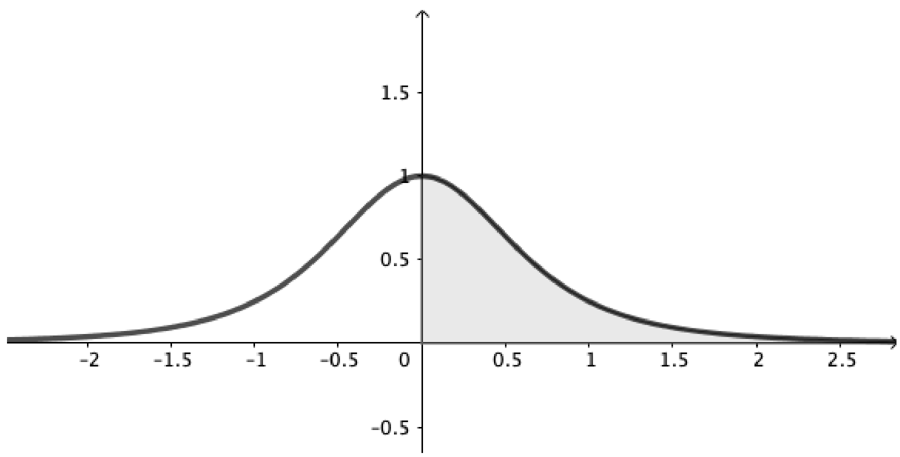


Figure 8: Improper integral.

Now we need to calculate the integral

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}.$$

To do this we will calculate the integral of the complex function

$$f(z) = \frac{1}{(z^2+1)^2}$$

around the simple closed contour consisting of:

(i) the segment of the real axis from $z = -R$ to $z = R$, and

(ii) the top half of the circle $|z| = R$, described counterclockwise and denoted by $C_R$

with $R > 1$, see Figure 9.

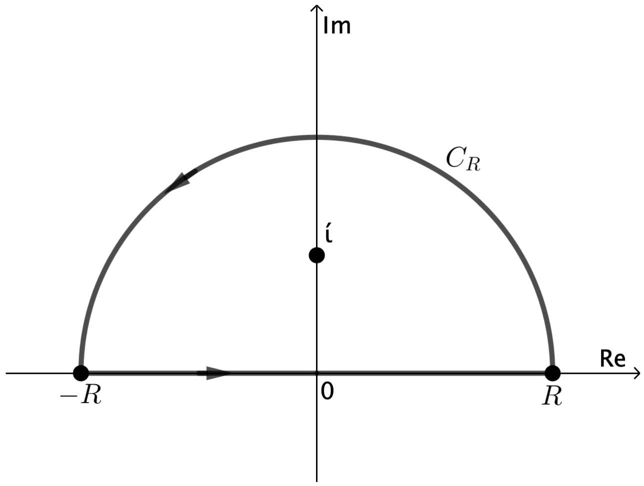


Figure 9: Simple closed contour.

Since the singularity $z_0 = i$ lies in the interior of $C_R\,(R > 1)$, we have that

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i B,$$

where

$$B = \mathop{\mathbf{Res}}_{z=i} \frac{1}{(z^2+1)^2}.$$

Since

$$\frac{1}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{(z+i)^2},$$

we can find that $B = \phi^{(1)}(i) = \dfrac{1}{4i}$ (Why?). Thus

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.$$

Observe that if $z \in C_R$,

$$|z^2+1| \geq ||z|^2 - 1| > R^2 - 1.$$

Thus

$$\left| \int_{C_R} \frac{dz}{(z^2+1)^2} \right| \leq \frac{\pi R}{(R^2+1)^2} = \frac{\dfrac{\pi}{R^3}}{\left(1 - \dfrac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Then

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}.$$

Therefore

$$\int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} = \frac{\pi}{4}.$$