Exercise

(1) Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9}$$

Solution: We want the Maclaurin series for

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 - (-z^4/9)}$$

Replace just z by $(-z^4)/9$ in

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

as well as its condition of validity, to get

$$\frac{1}{1 + (-z^4/9)} = \sum_{n=0}^{\infty} \left(\frac{-z^4}{9} \right)^n = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n}\, z^n}_{\text{Final result}} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n}}_{\text{Extra simplification}}, \quad |z| < \sqrt{3}.$$

Then if we multiply through this last equation by $z/9 = z/3^2$, we have the desired expansion:

$$f(z) = \frac{z}{3^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1}, \quad |z| < \sqrt{3}.$$

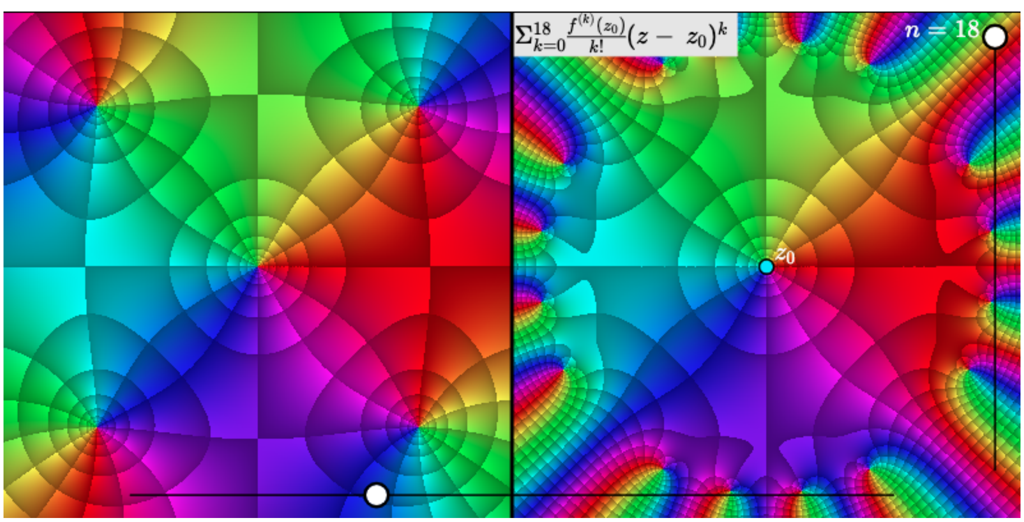

Figure 1: Maclaurin series for $f(z)$. Link: Taylor Series

(2) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Solution: Suppose that $0 < |z| < 4$. Then $0 < |z/4| < 1$, and we can use the know expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

That is, when $0 < |z| < 4$

$$\begin{aligned}
\frac{1}{4z - z^2} &= \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}} \\
&= \frac{1}{4z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}} \\
&= \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}
\end{aligned}$$

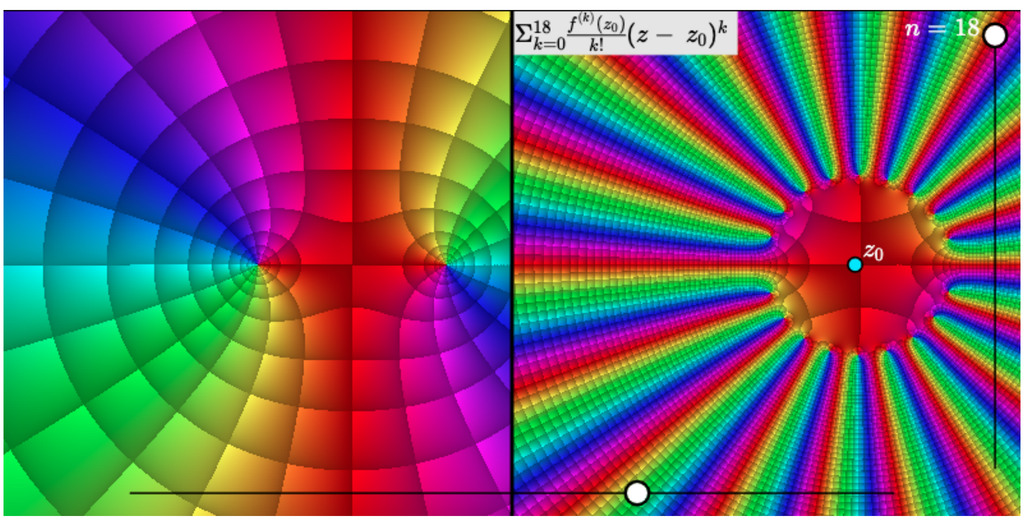

Figure 2: Series representation for $1/(4z - z^2)$. Link: Taylor Series

(3) Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Hint 1: For one domain you should get

$$\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z}.$$

For the other domain, you should get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

Hint 2: Observe that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.

Solution:

The function $f(z)$ has isolated singularities at $z = 0$ and $z = \pm i$, as indicated in the figure below.

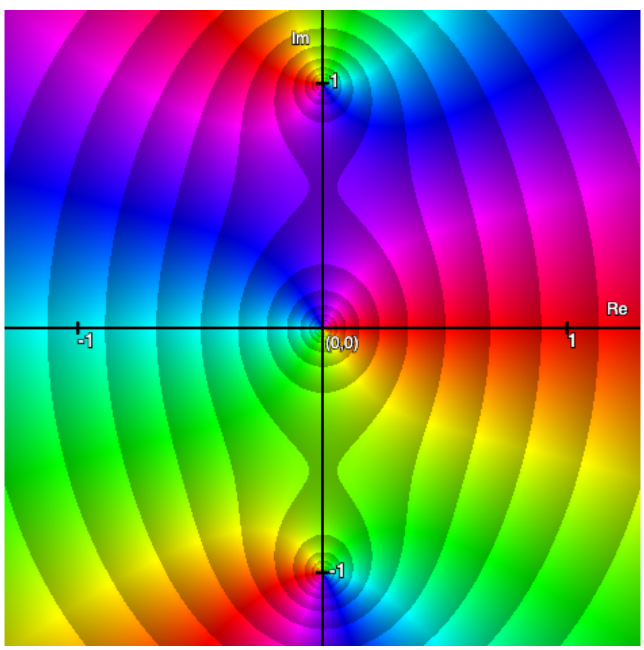

Figure 3: Domain coloring plot of $f(z)$. Link: Domain coloring

Hence there is a Laurent series representation for the domain $0 < |z| < 1$ and also one for the domain $1 < |z| < \infty$, which is exterior to the circle $|z| = 1$.

To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

For the domain $0 < |z| < 1$, we have

$$\begin{aligned}
f(z) &= \frac{1}{z}\frac{1}{1+z^2} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-z^2\right)^n \\
&= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{2n-1} \\
&= \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1} + \frac{1}{z}.
\end{aligned}$$

On the other hand, when $1 < |z| < \infty$,

$$\begin{aligned}
f(z) &= \frac{1}{z^3}\frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}
\end{aligned}$$

In this last part we use the fact that $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$.