Exercises

(1) Find the Möbius transformation that maps the points

$$z_1 = \infty, \quad z_2 = i, \quad z_3 = 0$$

onto the points

$$w_1 = 0, \quad w_2 = i, \quad w_3 = \infty.$$

Solution.

We need to find the values a, b, c and d in

$$T(z) = \frac{az+b}{cz+d}, \text{ with } ad - bc \neq 0.$$

Since $T(\infty) = 0$, then

$$\frac{a}{c} = 0 \implies a = 0 \quad (c \neq 0).$$

Now, since $T(0) = \infty$, then

$$\frac{-d}{c} = 0 \implies d = 0 \quad (c \neq 0)$$

Finally, since $T(i) = i$, we have

$$\frac{ai+b}{ci+d} = i.$$

Using previous values, we obtain

$$\frac{b}{ci} = i \implies b = i^2 c = -c.$$

Hence, the general Möbius transformation is

$$T(z) = \frac{-c}{cz}$$

In particular, for $c = 1$ we have

$$T(z) = -\frac{1}{z}.$$

(2) Find the Möbius transformation that maps the points

$$z_1 = -1, \quad z_2 = \infty, \quad z_3 = i$$

onto the points

$$w_1 = \infty, \quad w_2 = i, \quad w_3 = 1.$$

Solution.

Since $T(\infty) = i$, then

$$\frac{a}{c} = i \implies a = ic \implies ai = -c \quad (c \neq 0).$$

Now, since $T(-1) = \infty$, then

$$\frac{-d}{c} = -1 \implies d = c \quad (c \neq 0).$$

Finally, since $T(i) = 1$, we have

$$\frac{ai + b}{ci + d} = 1.$$

Using previous values, we obtain

$$\frac{-c + b}{ci + c} = 1 \implies b - c = c(i + 1) \implies b = c(i + 2).$$

Hence, the general Möbius transformation is

$$T(z) = \frac{icz + c(i + 2)}{cz + c}.$$

In particular, for $c = 1$ we have

$$T(z) = \frac{iz + (i + 2)}{z + 1}.$$