Exercise

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(1) Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate when

(a) $u(x,y) = 2x(1-y)$
(b) $u(x,y) = 2x - x^3 + 3xy^2$
(c) $u(x,y) = \sinh x \sin y$
(c) $u(x,y) = \dfrac{x}{x^2+y^2}$

Solutions:

(a) When $u(x,y) = 2x(1-y)$, we have that

$$u_x = 2 - 2y, \qquad u_y = -2x$$

and

$$u_{xx} = 0, \qquad u_{yy} = 0.$$

Thus

$$u_{xx} + u_{yy} = 0.$$

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2-2y$. Now, using Cauchy-Riemann equations

$$u_x = v_y \implies v_y = 2 - 2y \implies v(x,y) = 2y - y^2 + g(x).$$

Then

$$u_y = -v_x \implies -2x = -g'(x) \implies g'(x) = 2x \implies g(x) = x^2 + c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x,y) = 2y - y^2 + (x^2 + c) = x^2 - y^2 + 2y + c \quad (c \in \mathbb{R}).$$

(b) When $u(x,y) = 2x - x^3 + 3xy^2$, we have that

$$u_x = 2 - 3x^2 + 3y^2, \qquad u_y = 6xy$$

and

$$u_{xx} = -6x, \qquad u_{yy} = 6x.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \implies v_y = 2 - 3x^2 + 3y^2 \implies v(x, y) = 2y - 3x^2y + y^3 + g(x).$$

Then

$$u_y = -v_x \implies 6xy = 6xy - g'(x) \implies g'(x) = 0 \implies g(x) = c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x, y) = 2y - 3x^2y + y^3 + c \quad (c \in \mathbb{R}).$$

(c) When $u(x, y) = \sinh x \sin y$, we have that

$$u_x = \cosh x \sin y, \qquad u_y = \sinh x \cos y$$

and

$$u_{xx} = \sinh x \sin y, \qquad u_{yy} = -\sinh x \sin y.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = \cosh x \sin y$. Now

$$u_x = v_y \implies v_y = \cosh x \sin y \implies v(x, y) = -\cosh x \cos y + g(x).$$

Then

$$u_y = -v_x \implies \sinh x \cos y = \sinh x \cos y - g'(x) \implies g'(x) = 0 \implies g(x) = c \quad (c \in \mathbb{R}).$$

Consequently

$$v(x, y) = -\cosh x \cos y + c \quad (c \in \mathbb{R}).$$

(d) Finally for $u(x, y) = \dfrac{x}{x^2 + y^2}$, we have that

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad u_y = \frac{-2xy}{(x^2 + y^2)^2}$$

and

$$u_{xx} = 2x\frac{x^2 - 3y^2}{(x^2 + y^2)^3}, \qquad u_{yy} = -2x\frac{x^2 - 3y^2}{(x^2 + y^2)^3}.$$

Thus $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate $v(x, y)$, we start with $u_x(x, y) = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Now

$$u_x = v_y \implies v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} \implies v(x, y) = -\frac{y}{x^2 + y^2} + g(x).$$

Then

$$\begin{aligned} u_y = -v_x \implies \frac{-2xy}{(x^2 + y^2)^2} &= -\frac{2xy}{(x^2 + y^2)^2} - g'(x) \\ \implies g'(x) = 0 \implies g(x) &= c \quad (c \in \mathbb{R}). \end{aligned}$$

Consequently

$$v(x, y) = -\frac{y}{x^2 + y^2} + c \quad (c \in \mathbb{R}).$$