Exercises

(1) Prove that

$$\tan^{-1} z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right).$$

Solution:

Let $w = \tan^{-1} z$. Using the equations

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}, \quad \cos w = \frac{e^{iw} + e^{-iw}}{2},$$

we have the following

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{\frac{e^{iw} - e^{-iw}}{2i}}{\frac{e^{iw} + e^{-iw}}{2}} = \frac{1}{i} \cdot \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i \cdot \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}.$$

Then

$$ze^{iw} + ze^{-iw} = -ie^{iw} + ie^{-iw}.$$

Multiplying by e^{iw}, we obtain

$$ze^{2iw} + z = -ie^{2iw} + i.$$

That is

$$\begin{aligned}
(z+i)e^{2iw} &= i - z \\
e^{2iw} &= \frac{i-z}{i+z} \\
2iw &= \log\left(\frac{i-z}{i+z}\right) \\
w &= \frac{1}{2i} \log\left(\frac{i-z}{i+z}\right) \\
w &= \frac{i}{2} \log\left(\frac{i+z}{i-z}\right).
\end{aligned}$$

The last part is obtained using the equation

$$\frac{1}{2i} = -\frac{i}{2}$$

and the properties of the logarithm

$$-\log\left(\frac{i-z}{i+z}\right) = \log\left[\left(\frac{i-z}{i+z}\right)^{-1}\right] = \log\left(\frac{i+z}{i-z}\right).$$