Exercises

(1) Evaluate the following integrals:

a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

b) $\int_0^{\pi/6} e^{i2t} dt$

Solution:

$$\begin{aligned} \int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left(\frac{1}{t^2} - 2\right) dt - 2i \int_1^2 \frac{1}{t} dt \\ &= -\frac{1}{2} - 2\ln(2) = -\frac{1}{2} - \ln(4) \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/6} e^{i2t} dt &= \left[\frac{e^{i2t}}{2i}\right]_0^{\pi/6} \\ &= \frac{1}{2i}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i}{4} \end{aligned}$$

(2) Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0, & \text{when } m \neq n, \\ 2\pi, & \text{when } m = n. \end{cases}$$

Solution: First, notice that

$$I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta.$$

If $m = n$, I becomes

$$I = \int_0^{2\pi} d\theta = 2\pi.$$

If $m \neq n$, then

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

(3) Evaluate $\int_C f(z)dz$ for $f(z) = (z+2)/z$ and C is

a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Solution: For part a) we have

$$\begin{aligned}
\int_C \frac{z+2}{z} dz &= \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^\pi \left(1 + \frac{2}{2e^{i\theta}}\right) 2e^{i\theta} d\theta \\
&= 2i \int_0^\pi \left(e^{i\theta} + 1\right) d\theta \\
&= \left[\frac{e^{i\theta}}{i} + \theta\right]_0^\pi = 2i(i + \pi + i) \\
&= -4 + 2\pi i
\end{aligned}$$

For part b) we have

$$\begin{aligned}
\int_C \frac{z+2}{z} dz &= 2i \int_\pi^{2\pi} \left(e^{i\theta} + 1\right) d\theta \\
&= \left[\frac{e^{i\theta}}{i} + \theta\right]_0^\pi = 2i(-i + 2\pi - i - \pi) \\
&= 4 + 2\pi i
\end{aligned}$$

For part c) we just add the previous results to obtain $4\pi i$.

(4) Evaluate

$$\int_{-1}^{1} z^i dz$$

Analyse the function before doing any computations. Is it single valued? Multiple valued?

Solution: This function is multiple valued. So we need to choose a branch. We cannot choose the principal branch (Why?). Instead we use

$$f(z) = \exp(i \log z), \qquad \left(r > 0, \frac{-\pi}{2} < \mathbf{arg}\ z < \frac{3\pi}{2}\right).$$

In this case, the function is well defined and continuous. There must be an antiderivative. Thus

$$\begin{aligned}
\int_{-1}^{1} z^i dz &= \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} = \frac{1}{i+1}\left[(1)^{i+1} - (-1)^{i+1}\right] \\
&= \frac{1}{i+1}\left[e^{(i+1)\log 1} - e^{(i+1)\log(-1)}\right] \\
&= \frac{1}{i+1}\left[e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)}\right] \\
&= \frac{1}{i+1}\left(1 - e^{-\pi}e^{i\pi}\right) = \frac{1+e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} \\
&= \frac{1+e^{-\pi}}{2}(1-i).
\end{aligned}$$