Exercises

(1) Let $f(z) = (\overline{z})^3 + 3\overline{z}$.

a) Find all points $z \in \mathbb{C}$ at which f is differentiable. Make sure you justify your answer.

b) Show that f is nowhere analytic in C.

c) Explain why there is no contradiction between your answers to (a) and (b).

Solution:

Part a) (*Method one*) For $z = x + iy$, we have that

$$f(z) = (\overline{z})^3 + 3\overline{z} = (x - iy)^3 + 3(x - iy)$$

Expanding the cubic and simplifying, we get

$$\begin{aligned} f(z) &= x^3 - 3ix^2y - 3xy^2 + iy^3 + 3x - i3y \\ &= \underbrace{x^2 - 3xy^2 + 3x}_{u(x,y)} + i\underbrace{(y^3 - 3x^2y - 3y)}_{v(x,y)} \end{aligned}$$

Then

$$u(x, y) = x^2 - 3xy^2 + 3x \quad \text{and} \quad v(x, y) = y^3 - 3x^2y - 3y$$

and

$$\begin{aligned} u_x &= 3x^2 - 3y^2 + 3, \quad & v_x &= -6xy \\ u_y &= -6xy, & v_y &= 3y^2 - 3x^2 - 3. \end{aligned}$$

The Cauchy-Riemann equations hold if and only if

$$\begin{cases} 3x^2 - 3y^2 + 3 = 3y^2 - 3x^2 - 3, \\ -6xy = 6xy. \end{cases}$$

That is

$$\begin{cases} y^2 - x^2 = 1, \\ xy = 0. \end{cases}$$

The Cauchy-Riemann equations hold only at the points $x = 0, y = 1$ and $x = 0, y = -1$, that is, $z_1 = i$ and $z_2 = -i$, see Figure 1.

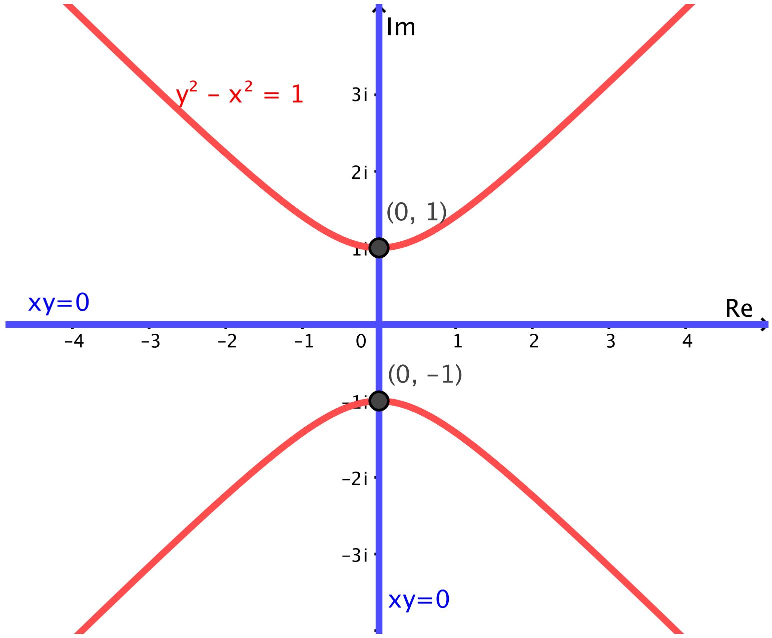

Figure 1: Plot of system.

Notice that the functions u_x, u_y, v_x and v_y are all continuous on $\mathbb{R}^2$ (since they are polynomials of x, y). Thus f satisfies the sufficient conditions for differentiability at $(0, 1)$ and $(0, -1)$. Hence f is differentiable only at those points.

Since Cauchy-Riemann equations are necessary for differentiability, f is not differentiable anywhere else.

Part a) (*Method two*) Now let's use the operator

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then we have

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z) &= \frac{1}{2}\left(\frac{\partial}{\partial x} f(z) + i\frac{\partial}{\partial y} f(z)\right) \\
&= \frac{1}{2}\left[\frac{\partial}{\partial x}\left((\overline{z})^3 + 3\overline{z}\right) + i\frac{\partial}{\partial y}\left((\overline{z})^3 + 3\overline{z}\right)\right] \\
&= \frac{1}{2}\left[\frac{\partial}{\partial x}(\overline{z})^3 + 3\frac{\partial}{\partial x}\overline{z} + i\frac{\partial}{\partial y}(\overline{z})^3 + 3i\frac{\partial}{\partial y}\overline{z}\right] \\
&= \frac{1}{2}\left[3\frac{\partial}{\partial x}(\overline{z})^2 \cdot \frac{\partial}{\partial x}\overline{z} + 3\frac{\partial}{\partial x}\overline{z} + 3i\frac{\partial}{\partial y}(\overline{z})^2 \cdot \frac{\partial}{\partial y}\overline{z} + 3i\frac{\partial}{\partial y}\overline{z}\right]
\end{aligned}$$

Since

$$\frac{\partial}{\partial x}\overline{z} = 1 \quad \text{and} \quad \frac{\partial}{\partial y}\overline{z} = -i, \quad \text{(Why?)}$$

we obtain

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z) &= \frac{1}{2}\left[3(\overline{z})^2 + 3 + 3i(\overline{z})^2(-i) + 3i(-i)\right] \\
&= \frac{1}{2}\left[6(\overline{z})^2 + 6\right] = 3(\overline{z})^2 + 3
\end{aligned}$$

If the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z) = 0.$$

By solving the equation

$$\begin{aligned}
3(\overline{z})^2 + 3 &= 0 \\
(\overline{z})^2 &= -1 = i^2 \\
\overline{z} &= (i^2)^{1/2}
\end{aligned}$$

We can conclude that f is differentiable only at two points: $z = i$ and $z = -i$, as we already found using the previous method.

Part b)

From part a), we saw that$f$ is differentiable only at $(0, 1)$ and $(0, -1)$. This means that f is not differentiable on any neighbourhood of any point $z \in \mathbb{C}$.

Part c)

We know that if $f(z)$ is analytic at z_0, then $f(z)$ is differentiable at each point in some neighbourhood of z_0. However, $f(z)$ differentiable at z_0 does not imply that $f(z)$ is analytic at z_0.

In other words, analytic implies differentiability, not vice versa.

(2) Use the Cauchy-Riemann equations to show that the function

$$f(z) = \exp \overline{z}$$

is not analytic anywhere.

Solution: (*Method one*) For $z = x + iy$, we have that

$$f(z) = \exp \overline{z} = \exp(x - iy) = e^x (\cos y - i \sin y) \,.$$

Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = -e^x \sin y$$

and

$$\begin{array}{ll} u_x = e^x \cos y, & v_x = -e^x \sin y \\ u_y = -e^x \sin y, & v_y = -e^x \cos y. \end{array}$$

The Cauchy-Riemann equations hold if and only if

$$\begin{cases} 2e^x \cos y = 0 \\ 2e^x \sin y = 0 \end{cases}$$

that is, if and only if $\sin y = 0 = \cos y$. However, this is not possible, since $\sin y = 0$ for $y = \pi n$ with $n \in \mathbb{Z}$ and $\cos y = 0$ for $y = \pi n - \pi/2$ with $n \in \mathbb{Z}$.

Therefore, there are no points $z \in \mathbb{C}$ for which $f(z) = \exp(\overline{z})$ is differentiable, and so no points $z \in \mathbb{C}$ at which f is analytic.

(*Method two*). Try using the operator

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Are there any solutions for the equation $\exp \overline{z} = 0$?

(3) Calculate $\dfrac{d}{dz}(1 - 2i)^z$. Explain any restrictions you need to make for your answer to be valid.

Solution: We know that

$$(1 - 2i)^z = \exp\left(z \log(1 - 2i)\right).$$

Here we need to specify a single value of $\log(1-2i)$. We can choose the principal branch, that is, Log(1-2i). Thus

$$\begin{aligned} \frac{d}{dz}(1 - 2i)^z &= \text{Log}(1 - 2i) \exp\left(z\text{Log(1-2i)}\right) \\ &= \text{Log}(1 - 2i)(1 - 2i)^z. \end{aligned}$$

(4) (*Bonus*) Differentiate $f(z) = \sqrt{e^z + 1}$, giving the appropriate region on which $f(z)$ is analytic.

Solution: First choose a branch of the square root. For example, the function $w \mapsto \sqrt{w}$ is analytic on

$$\mathbb{C} \setminus \{x + iy \,|\, x \leq 0, y = 0\}.$$

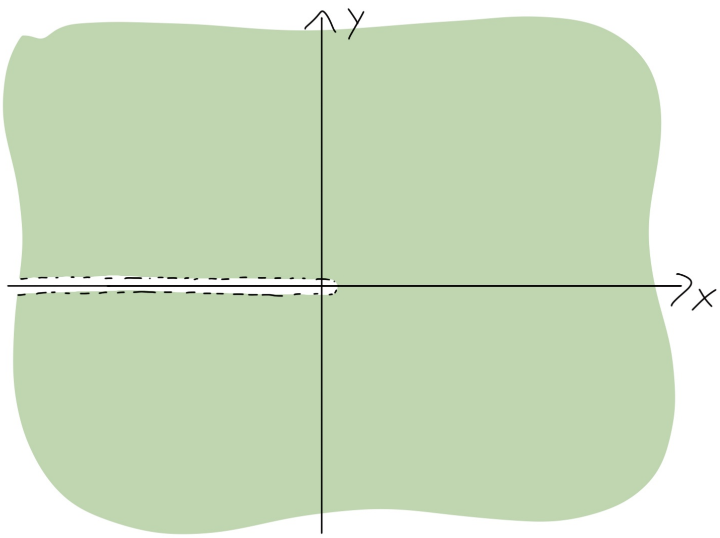

Figure 2: Region of analyticity of $\sqrt{w}$ (Sketch).

Thus we must choose a region A such that if $z \in A$, then $e^z + 1$ is not both real and ≤ 0. Now recall that $e^z = e^x(\cos y + i \sin y)$. Then e^z is real if and only if $y = \mathbf{Im}(z) = n\pi$ for some integer. In this case we write

$$e^z = e^x \cos y. \tag{1}$$

Notice also that (1) is positive when $y = 2n\pi$, and negative when $y = (2n+1)\pi$. Here $|e^z| = e^x$, where $x = \mathbf{Re}(z)$ and $e^x \geq 1$ if and only if $x \geq 0$. Thus, we have that

$$e^z + 1 \;\text{ is real and }\; e^z + 1 = e^x \cos y + 1 \leq 0$$

if and only if $x \geq 0$ and $y = (2n+1)\pi$, $n \in \mathbb{Z}$.

Therefore if we define the set

$$A = \mathbb{C} \setminus \{x + iy \,|\, x \geq 0, y = (2n+1)\pi, n \in \mathbb{Z}\},$$

then $e^z + 1$ is not both real and ≤ 0 if and only if $z \in A$, see Figure 3.

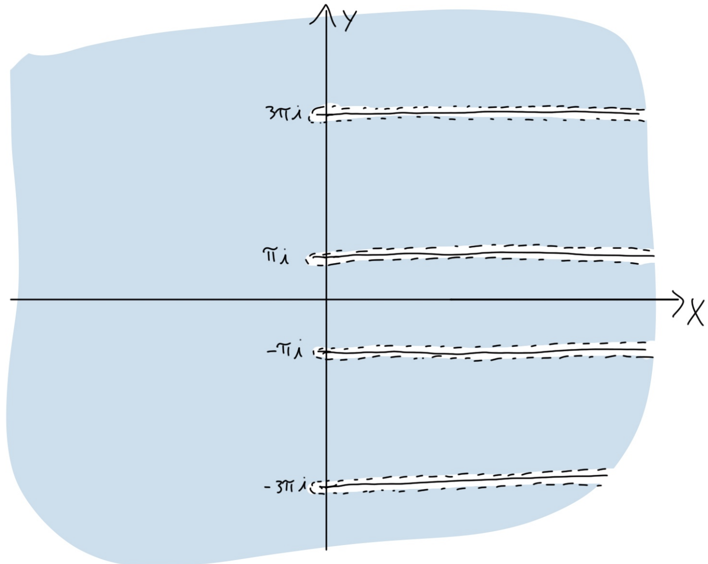

Figure 3: Region of analyticity of $\sqrt{e^z+1}$ (Sketch).

Since $e^z + 1$ is entire (analytic everywhere), it is certainly analytic on A. Hence $\sqrt{e^z+1}$ is analytic on A with derivative at z given by

$$f'(z) = \frac{1}{2}\frac{e^z}{\sqrt{e^z+1}}$$