Exercises

(1) In each case, find the Laurent series of the function at its isolated singular point. Determine whether that point is a pole (determine its order), a removable singular point or an essential singularity. Finally, determine the corresponding residue.

$$(a)\ z\exp\left(\frac{1}{z}\right); \quad (b)\ \frac{z^2}{1+z}; \quad (c)\ \frac{\cos z}{z}; \quad (d)\ \frac{1-\cosh z}{z^3}; \quad (e)\ \frac{1}{(2-z)^3}.$$

Suggestion 1: Use the know series

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}, \qquad \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}, \qquad \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}, \qquad (|z|<\infty).$$

Suggestion 2: For part (b) notice that $z^2=(z+1)^2-2z-1=(z+1)^2-2(z+1)+1$

(2) Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

taken counterclockwise around the circle (a) $|z-2|=2$; (b) $|z|=4$.

Ans. (a) πi; *(b)* $6\pi i$.

(3) (Extra challenge) Use residues to evaluate the improper integral:

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

Ans. $\pi/4$.