Exercises

(1) Find a function harmonic in the upper half of the z-plane, $\mathbf{Im}\ z > 0$, which takes the prescribe values on the x axis given by $G(x) = 1$ for $x > 0$, and $G(x) = 0$ for $x < 0$.

Solution:

We need to find $\Phi(x, y)$ such that

$$\nabla^2\Phi = 0, \quad -\infty < x < \infty, \ y > 0.$$
$$\Phi = 1, \ x > 0; \quad \Phi = 0, \ x < 0.$$

This is a Dirichlet problem for the upper half plane.

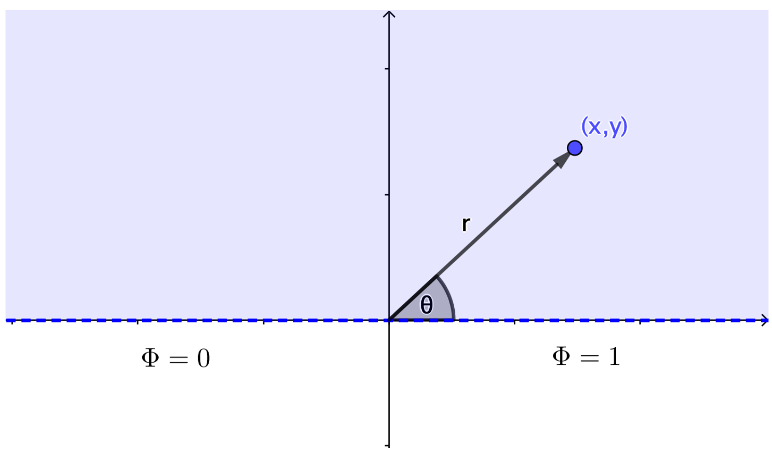

Figure 1: Upper half plane.

The function $A\theta + B$, where A and B are real constants, is harmonic since it is the imaginary part of $A\log(z) + Bi$.

To determine A and B note that the boundary conditions are $\Phi = 1$ for $x > 0$, that is, $\theta = 0$ and $\Phi = 0$ for $x < 0$, that is $\theta = \pi$. Thus

$$1 = A(0) + B, \qquad 0 = A(\pi) + B$$

from which $A = -1/\pi$, $B = 1$. Then the required solution is

$$\Phi = A\theta + B = 1 - \frac{\theta}{\pi} = 1 - \frac{1}{\pi}\mathbf{Arg}(z) = 1 - \frac{1}{\pi}\arctan\left(\frac{y}{x}\right).$$

(2) Find a function harmonic inside the unit circle $|z| = 1$ and taking the prescribed values given by $F(\theta) = 1$ for $0 < \theta < \pi$, and $F(\theta) = 0$ for $\pi < \theta < 2\pi$ on its circumference.

Solution: This is a Dirichlet problem for the unit circle in which we need to find a function satisfying Laplace's equation inside $|z| = 1$ and taking the values 1 on the upper arc of the circle and 0 on the lower arc of the circle.

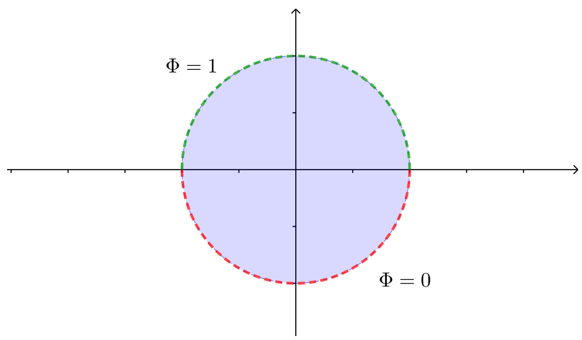

Figure 2: Unit circle.

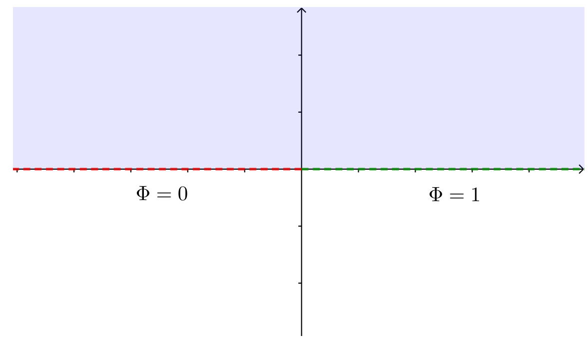

Figure 3: Image under $w = i(1-z)/(1+z)$.

We map the interior of the circle $|z| = 1$ on to the upper half of the w plane by using the mapping

$$w = i\frac{1-z}{1+z}.$$

Under this transformation, the upper and lower arcs are mapped on to the positive and negative real axis on the w-plane respectively. This means that the boundary conditions $\Phi = 1$ on the upper arc of the circle and $\Phi = 0$ on the lower arc of the circle become respectively $\Phi = 1$ for $u > 0$ and $\Phi = 0$ for $u < 0$.

Thus we have reduced the problem to finding a function Φ harmonic in the upper half w-plane and taking the values 0 for $u < 0$ and 1 for $u > 0$. But this problem has already been solved in the previous exercise, and the solution (replacing x by u and y by v) is given by

$$\Phi = 1 - \frac{1}{\pi}\arctan\left(\frac{v}{u}\right). \tag{1}$$

Now from $w = i(1-z)/(z+1)$, we find

$$u = \frac{2y}{(1+x)^2 + y^2}, \quad v = \frac{1-(x^2+y^2)}{(1+x)^2+y^2}.$$

Then substituting these in (1), we find the required solution

$$\Phi = 1 - \frac{1}{\pi}\arctan\left(\frac{1-(x^2+y^2)}{2y}\right)$$

or in polar coordinates (r, θ)

$$\Phi = 1 - \frac{1}{\pi}\arctan\left(\frac{1-r^2}{2r\sin\theta}\right).$$