Exercises

(1) Evaluate the following integrals, justifying your procedures:

(a) $\displaystyle\int_C \frac{2dz}{z^2-1}$, where C is the circle with radius $1/2$, centre 1, positively oriented;

(b) $\displaystyle\int_0^i ze^{z^2}dz$.

Solution (a): Notice that

$$\frac{2}{z^2-1} = \frac{1}{z-1} - \frac{1}{z+1}.$$

Thus

$$\int_C \frac{2}{z^2-1}dz = \int_C \frac{1}{z-1}dz - \int_C \frac{1}{z+1}dz.$$

Now, using the Cauchy Integral Formula (CIF)

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz,$$

with $z_0 = 1$ and $f(z) = 1$, we have

$$\int_C \frac{1}{z-1}dz = 2\pi i.$$

On the other hand, since $1/(z+1)$ is analytic on and inside C, then

$$\int_C \frac{1}{z+1}dz = 0$$

by Cauchy-Goursat Theorem. Therefore,

$$\int_C \frac{2}{z^2-1}dz = 2\pi i.$$

Solution (b): Since the integrand $f(z) = ze^{z^2}$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is

$$F(z) = \frac{e^{z^2}}{2}.$$

Thus

$$\int_C ze^{z^2}dz = \left.\frac{e^{z^2}}{2}\right|_0^i = \frac{1}{2e} - \frac{1}{2}.$$

(2) Let D be the annulus $6 < |z| < 8$, and let C be any simple closed contour inside D. Show that:

$$\int_C \frac{dz}{z^2+1} = 0$$

Solution:

The function $f(z) = \frac{1}{z^2+1}$ is analytic on $\mathbb{C} \setminus \{-i, i\}$. In particular, it is analytic on C.

Since

$$\begin{aligned} \frac{1}{z^2+1} &= \frac{1/2i}{z+i} - \frac{1/2i}{z-i} \\ &= \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i} \end{aligned}$$

we have

$$\int_C \frac{dz}{z^2+1} = \frac{1}{2i}\int_C \frac{dz}{z-i} - \frac{1}{2i}\int_C \frac{dz}{z+i}.$$

Now, since $C \subset D$, we have exactly three cases:

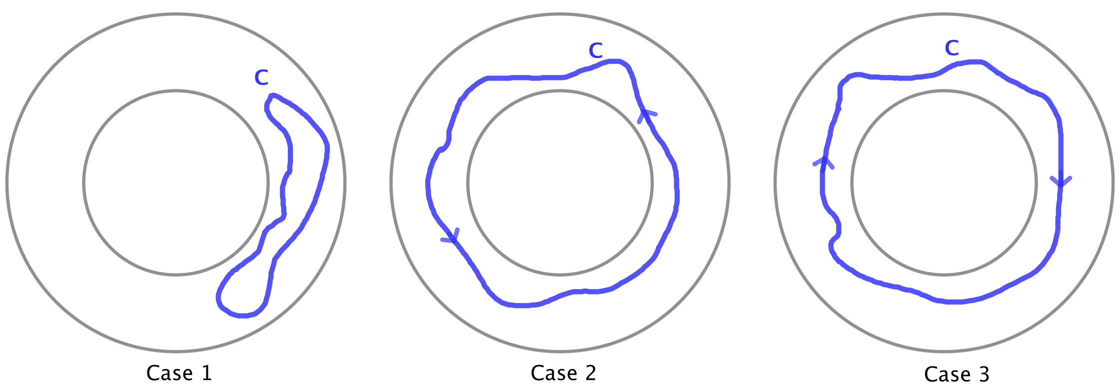

Figure 1: Cases.

Observe that

- Case 1: $-i, i \notin \text{int}(C)$
- Case 2: $-i, i \in \text{int}(C)$, and C is positively oriented.
- Case 3: $-i, i \in \text{int}(C)$, and C is negatively oriented.

Thus

- In case 1, using Cauchy-Goursat Theorem, we have

$$\int_C \frac{dz}{z^2+1} = \frac{1}{2i}\int_C \frac{dz}{z-i} - \frac{1}{2i}\int_C \frac{dz}{z+i} = 0+0=0.$$

- In case 2, using CIF, we obtain

$$\begin{aligned}\int_C \frac{dz}{z^2+1} &= \frac{1}{2i}\int_C \frac{dz}{z-i} - \frac{1}{2i}\int_C \frac{dz}{z+i}\\ &= \frac{1}{2i}(2\pi i) - \frac{1}{2i}(2\pi i)\\ &= \pi - \pi = 0.\end{aligned}$$

- Finally, in case 3, we use CIF again to obtain

$$\begin{aligned}\int_C \frac{dz}{z^2+1} &= \frac{1}{2i}\int_C \frac{dz}{z-i} - \frac{1}{2i}\int_C \frac{dz}{z+i}\\ &= -\frac{1}{2i}\int_{-C} \frac{dz}{z-i} + \frac{1}{2i}\int_{-C} \frac{dz}{z+i}\\ &= -\frac{1}{2i}(2\pi i) + \frac{1}{2i}(2\pi i)\\ &= -\pi + \pi = 0.\end{aligned}$$

(3) Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \le \frac{\pi R\,(2R^2+1)}{(R^2-1)\,(R^2-4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

Solution: Note that if $|z| = R$ $(R > 2)$, then

$$|2z^2-1| \le 2|z|^2+1 = 2R^2+1$$

and

$$|z^4+5z^2+4| = |z^2+1|\cdot|z^2+4| \ge \left||z|^2-1\right|\cdot\left||z|^2-4\right| = \left(R^2-1\right)\left(R^2-4\right).$$

Thus

$$\left|\frac{2z^2-1}{z^4+5z^2+4}\right| = \frac{|2z^2-1|}{|z^4+5z^2+4|} \le \frac{2R^2+1}{(R^2-1)\,(R^2-4)}$$

when $|z| = R$ $(R > 2)$. Since the length of C_R is πR, then

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \le \frac{\pi R\,(2R^2+1)}{(R^2-1)\,(R^2-4)} = \frac{\dfrac{\pi}{R}\left(2+\dfrac{1}{R^2}\right)}{\left(1-\dfrac{1}{R^2}\right)\left(1-\dfrac{4}{R^2}\right)}$$

Hence we can conclude that the value of the integral tends to zero as R tends to infinity.