Square Wheels Derivations

Square Wheel

To solve the problem the system is parameterised using the variable t. In the road frame this gives,

$$\begin{cases} x_r = x(t) \\ y_r = y(t) < 0. \end{cases} \tag{1}$$

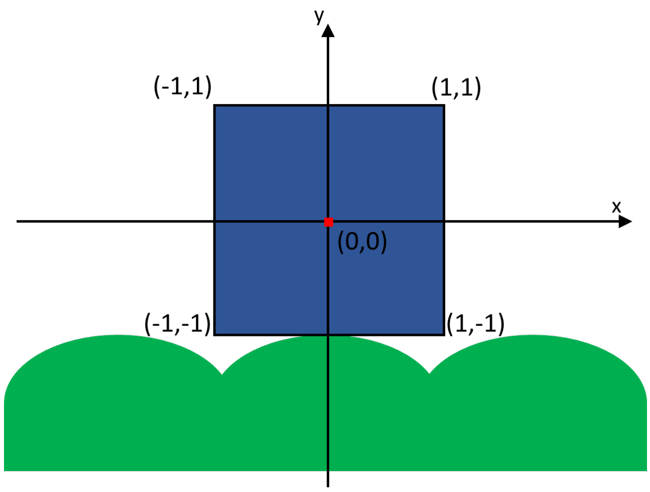

Figure 1: Square wheel parameterised to the give the coordinates of the square in the frame of the road.

The axle must stay confined to a horizontal line for smooth motion and so the x-axis is chosen to be this line. Therefore, the y-position of the road must always be negative.

In the frame of the wheel, placing the axle at the origin of the wheel's local coordinate system, the position on the rim can be parameterised as,

$$\begin{cases} r_w = r(t) \\ \theta_w = \theta(t). \end{cases} \tag{2}$$

By allowing the parameter t to be time, the location of the road-wheel contact point at a time t can be determined. Using the vertical alignment property, the distance between the axle and the contact point in the road space is $y(t)$. While in the wheel space, using the axle as the origin, the distance from the axle to the contact point is $r(t)$. Therefore,

$$y(t) = -r(t) \tag{3}$$

with the negation required as the road position lays below the axis $(y(t) < 0)$.

By employing the stationary rim property, it is observed that in the frame of the axle, all points on the rim are constantly in motion while the rim is stationary with respect to the road at the contact point. Therefore, the rim speed relative to the axle is equivalent to the axle speed relative to the road.

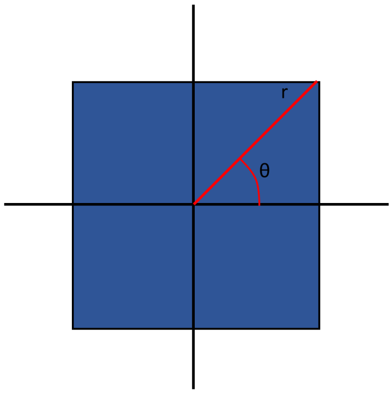

Figure 2: Square wheel parameterised to the give the coordinates of the square in the frame of the wheel.

Using this relation, the speed of the axle relative to the road is equivalent to the horizontal speed of the contact point on the road. Equating the road frame to the axle frame,

$$\frac{dx}{dt} = r\frac{d\theta}{dt}. \tag{4}$$

Eq. 3 and Eq. 4 form a set of equations which can be denoted the road-wheel equations which are equivalent to Gregory's transformation. These equations hold for any shape of wheel and road surface.

For a square of length 2 centred on the origin in the Cartesian frame, the shape of the road can be deduced. The line segment from (1,-1) to (1,1) is isolated, with the solution for the entire square obtained from repeating the obtained shape fourfold.

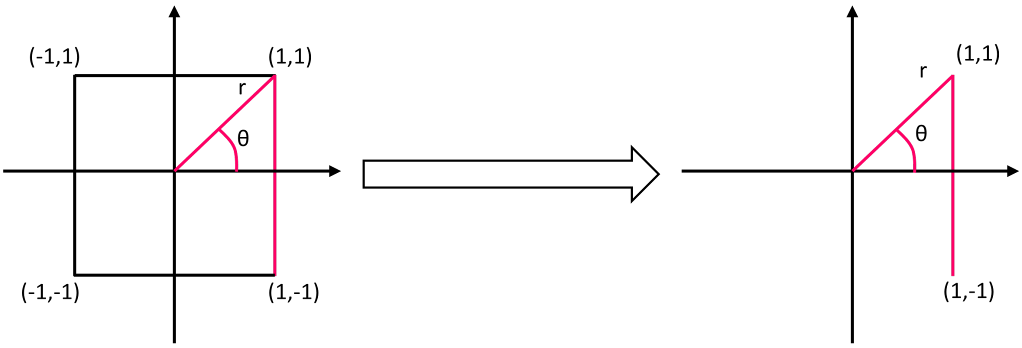

Figure 3: A square of arbitrary unit length 2, with a line segment from (1,-1) to (1,1) is isolated. Parametric coordinate equations are applied to isolated segment to derive a quarter of the equation for the road.

This line segment can be parameterised as,

$$\begin{cases} x_w(t) = 1 \\ y_w(t) = t, \text{ where } -1 \leq t \leq 1, \end{cases} \tag{5}$$

and converting to polar coordinates gives,

$$\begin{cases} r = \sqrt{x_w^2 + y_w^2} = \sqrt{1+t^2} \\ \theta = \arctan\left(\frac{y_w}{x_w}\right) = \arctan(t). \end{cases} \tag{6}$$

Substituting into Eq. 3, the expression for the y position is found as,

$$y(t) = -\sqrt{1+t^2}. \tag{7}$$

Using this result in Eq. 4 and calculation of the required derivative,

$$\frac{d\theta}{dt} = \frac{1}{1+t^2}, \tag{8}$$

the x solution is given as the following,

$$x = \int \frac{1}{\sqrt{1+t^2}} dt = \sinh^{-1}(t) + C. \tag{9}$$

By letting $x = 0$ at $t = 0$ the constant $C = 0$.

Solving for t and substituting into Eq. 7, the shape of the road surface for a line segment is given as,

$$y = -\cosh(x), \tag{10}$$

which is an inverted catenary.

Elliptical wheel with axle at centre

Placing the axle at the centre of the wheel, the polar form of the ellipse is given as,

$$r = \frac{b}{\sqrt{1-\varepsilon^2\cos^2(\theta)}}, \text{ with } \varepsilon = \sqrt{1-\frac{b^2}{a^2}} \tag{11}$$

where a and b are the semi-major and semi-minor axis of the ellipse respectfully and ε the eccentricity. Replacing the parameter t in Eq. 4 with θ an elliptical integral with no closed form solution is obtained,

$$x = \int \frac{b}{\sqrt{1-\varepsilon^2\cos^2(\theta)}} d\theta. \tag{12}$$

Elliptical wheel with axle at focus

The polar equation of an ellipse with a focus at the origin and the other focus on the positive y-axis is given by,

$$r = \frac{k\varepsilon}{(1-\varepsilon\sin(\theta))}, \tag{13}$$

where k is the distance from the origin to the corresponding directrix. By replacing the parameter t with θ as before, the x-position is found to be,

$$x = k\varepsilon \int \frac{1}{1-\varepsilon\sin(\theta)} d\theta. \tag{14}$$

Therefore,

$$x = \frac{-2k\varepsilon \arctan\left(\frac{\varepsilon-\tan\left(\frac{\theta}{2}\right)}{\sqrt{1-\varepsilon^2}}\right)}{\sqrt{1-\varepsilon^2}} + C. \tag{15}$$

As the secondary focus lays on the positive y-axis, for $\theta = \frac{3\pi}{2}$, $x = 0$. Substituting into Eq. 15 gives,

$$C = \frac{2k\varepsilon}{\sqrt{1-\varepsilon^2}} \arctan\left(\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}\right), \tag{16}$$

and so, using that $\arctan$ is an odd function,

$$x = \frac{2k\varepsilon}{\alpha}\left[\arctan\left(\frac{\tan\left(\frac{\theta}{2}\right)-\varepsilon}{\alpha}\right) + \arctan\left(\frac{1+\varepsilon}{\alpha}\right)\right], \tag{17}$$

where

$$\alpha = \sqrt{1-\varepsilon^2}. \tag{18}$$

Taking the tangent of Eq. 17 and rearranging for $\sin(\theta)$ and substituting into Eq. 3 gives the equation of the road to be,

$$y = \left(\frac{k\varepsilon}{\alpha^2}\right)\left(1 - \varepsilon\cos\left(\frac{\alpha}{k\varepsilon}x\right)\right). \tag{19}$$

Setting $k = 1$ and $\varepsilon = \frac{1}{\sqrt{2}}$, yields a road function of

$$y = -\sqrt{2} + \cos x. \tag{20}$$

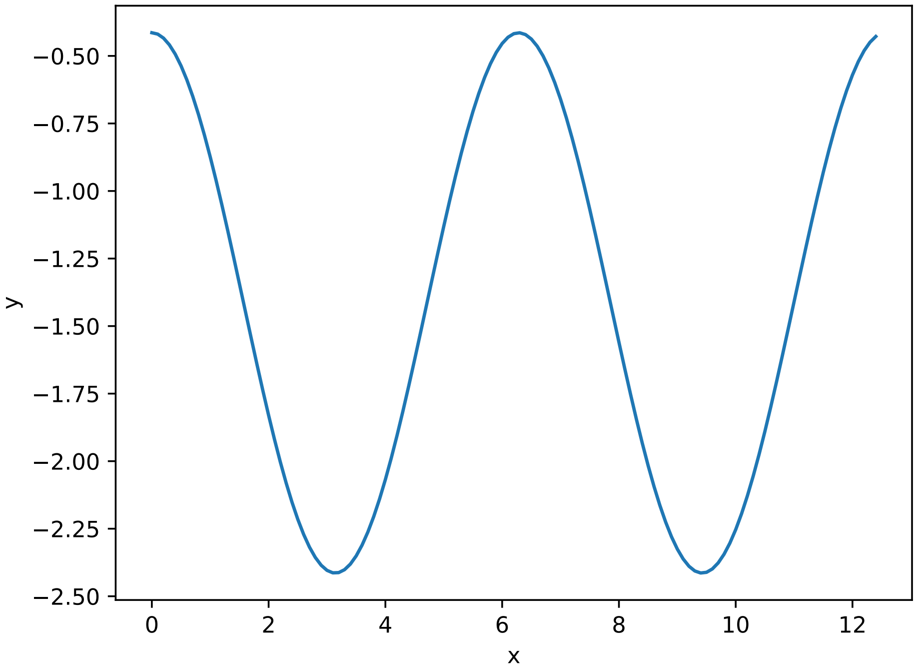

Figure 4: Plot of shape of road for an ellipse with the axle at one focus with $k = 1$ and $\varepsilon = \frac{1}{\sqrt{2}}$