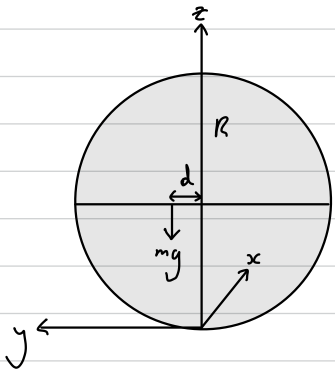
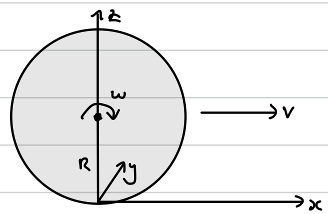
Flat Green Bowling Derivation





Assume the bowl is spherical and initially projected along x-axis
d is the offset distance of the COM from the geometrical centre

Weighted ball precesses like a gyroscope or spinning top but axis of rotation not anchored to a fixed point.
Precessional motion combined with linear motion, generating a curved trajectory

Frictional force on rolling bowl

$$F_r = -\mu F$$
$$= -\mu mg \quad \Rightarrow \quad a = -\mu g$$

Ignore any initial sliding component

For an unbiased ball : $v^2 = v_0^2 + 2as$

$$\Rightarrow s = \frac{v_0^2}{2\mu g}$$

When rolling $v = R\omega$

For the bowl to move slowly along a curved path, the axis of rotation must pass through the COM. If it doesn't, the COM will rise and fall generating a wobble in motion.

Assume axis of rotation remains horizontal throughout and ignore tilt of bowl

A frictional force is required for the bowl to move in a curved path

$$F_y = \frac{mv^2}{r}$$

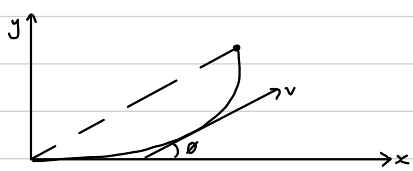
Torque : $\bar{\tau} = \bar{r} \times \bar{F}$

x-component : $\tau_x = F_y R - \underbrace{mgd}$ acting about the geometric centre acts to change the direction but not magnitude of the angular momentum $\left(\bar{\tau} = \frac{d\bar{L}}{dt}\right)$

offset component of mass

$L = I_{CM}\,\omega$ where I_{CM} is the moment of inertia about the rotation axis passing through the COM

At any point on trajectory, a tangent to the path makes an angle ϕ with the x-axis



The change in angular momentum in x-direction as a result of small rotation $d\phi$ in time dt:

$$dL_x = -L\,d\phi$$

Thus the x-component of torque is: (+ve x into page)

$$\frac{mv^2R}{r} - mgd = L\frac{d\phi}{dt} = -I_{cm}\,\omega\,\omega_p$$

where $\omega_p = \frac{d\phi}{dt}$ is the angular velocity of precession

When the ball is rolling $v = r\omega_p = R\omega$

$$\Rightarrow \omega_p = \frac{d\phi}{dt} = \frac{mgdR}{I_0 v}$$

Using Steiner's parallel axis theorem:

$I_0 = I_{cm} + mR^2$ is the moment of inertia about a horizontal axis through the edge of the ball

For a solid sphere $I_{cm} = \frac{2}{5}mR^2$

The radius of curvature r:

$$r = \frac{v}{\omega_p} = \frac{v^2 I_0}{mgdR}$$

so $$\frac{dr}{dt} = \frac{I_0}{mgdR}\frac{d}{dt}v^2 \qquad \left(\frac{dv^2}{dv}\frac{dv}{dt}\right)$$

$$= \frac{I_0}{mgdR}\,2v\cdot(-\mu g) \qquad \frac{dv}{dt} = -\mu g$$

$$= -2\mu \cancel{g} \frac{I_0 v}{m \cancel{g} d R}$$

$$= -2\mu \frac{I_0}{m d R} r \frac{d\phi}{dt}$$

$$\frac{dr}{dt} = -\alpha r \frac{d\phi}{dt} \quad \text{where} \quad \alpha = \frac{2 I_0 \mu}{m d R}$$

Integrating:

$$\int \frac{1}{r} dr = -\alpha \int d\phi$$

$$\ln r = -\alpha \phi + c$$

$$r = e^{-\alpha\phi + c}$$

$$r = C e^{-\alpha\phi}$$

when $\phi = 0$, $r = r_0 = \frac{v_0^2 I_0}{m g d R} = \frac{I_0}{m d R} \frac{v_0^2}{g} = \frac{\alpha v_0^2}{2\mu g}$

$$\Rightarrow \quad r = r_0 e^{-\alpha\phi}$$

where $r_0 = \frac{\alpha v_0^2}{2\mu g}$ is the initial radius of curvature

Path length of bowl:

$$s = \int_0^{\phi} r \, d\phi = r_0 \int_0^{\phi} e^{-\alpha\phi} d\phi$$

$$= \frac{(1 - e^{-\alpha\phi}) r_0}{\alpha} \qquad (*)$$

From $\frac{d\phi}{dt} = \frac{mgdR}{I_0 v}$

$(v = v_0 - \mu g t)$

$$\int d\phi = \frac{mgdR}{I_0}\int \frac{1}{v}\,dt$$

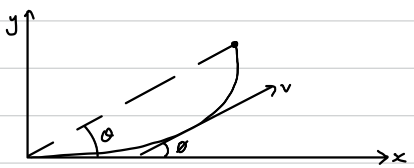
$$\phi = \frac{2}{\alpha}\ln\frac{v_0}{v}$$

$\therefore\ \phi \to \infty$ as $v \to 0$

From (*), the total path length is given as

$S = \frac{r_0}{\alpha} = \frac{v_0^2}{2\mu g}$ which is independent of α $\therefore$ same as unbiased ball

Trajectory



θ is the angle of projection to the end point
Must throw at angle θ to where you want the ball to land

From the diagram: $\frac{dx}{dt} = v\cos\phi$ and $\frac{dy}{dt} = v\sin\phi$

Now using complex numbers

$$ve^{i\phi} = v\cos\phi + iv\sin\phi$$

$$= \frac{dx}{dt} + i\frac{dy}{dt}$$

$$\Rightarrow \quad x + iy = \int_0^t v e^{i\phi}\,dt = \int_0^{\phi} r e^{i\phi}\,d\phi$$

as $v\,dt = ds = r\,d\phi$

Equating real and imaginary components:

$$x + iy = \int_0^{\phi} r_0 e^{-\alpha\phi} e^{i\phi}\,d\phi$$

$$= r_0 \int_0^{\phi} e^{(-\alpha+i)\phi}\,d\phi$$

$$= r_0 \left[\frac{1}{-\alpha+i} e^{(-\alpha+i)\phi} \right]_0^{\phi}$$

$$= r_0 \left[\frac{1}{-\alpha+i} e^{(-\alpha+i)\phi} - \frac{1}{-\alpha+i} \right]$$

$$= r_0 \frac{-\alpha-i}{-\alpha-i} \left(\frac{1}{-\alpha+i} e^{(-\alpha+i)\phi} - \frac{1}{-\alpha+i} \right)$$

$$= r_0 \frac{-\alpha-i}{\alpha^2+1} \left(e^{(-\alpha+i)\phi} - 1 \right)$$

$$= \frac{r_0}{\alpha^2+1} \left[-\alpha e^{(-\alpha+i)\phi} + \alpha - i e^{(-\alpha+i)\phi} + i \right]$$

$$= \frac{r_0}{\alpha^2+1} \left[-\alpha\beta e^{i\phi} + \alpha - i\beta e^{i\phi} + i \right] \quad \text{where } \beta = e^{-\alpha\phi}$$

$$= \frac{r_0}{\alpha^2+1} \left[-\alpha\beta\cos\phi - i\alpha\beta\sin\phi + \alpha - i\beta\cos\phi + \beta\sin\phi + i \right]$$

$$= \frac{r_0}{\alpha^2+1} \left[(\alpha - \alpha\beta\cos\phi + \beta\sin\phi) + i(1 - \beta\cos\phi - \alpha\beta\sin\phi) \right]$$

$$\Rightarrow \quad x = \frac{r_0}{\alpha^2+1}\left(\alpha - \alpha\beta\cos\phi + \beta\sin\phi\right)$$

$$y = \frac{r_0}{\alpha^2+1}\left(1 - \beta\cos\phi - \alpha\beta\sin\phi\right)$$

At the endpoint (X, Y), $\beta = 0$:

$$X = \frac{r_0 \alpha}{\alpha^2+1}$$

$$Y = \frac{r_0}{\alpha^2+1}$$

$$\Rightarrow \quad X = \alpha Y$$

The range D is thus :

$$D = \sqrt{X^2+Y^2} = \sqrt{(1+\alpha^2)\,Y^2} = \sqrt{\alpha^2+1}\;Y = \frac{r_0}{\sqrt{\alpha^2+1}}$$

Path length is given as:

$$S = \sqrt{1+\frac{1}{\alpha^2}}\;D$$

$$\tan\theta = \frac{1}{\alpha} \quad \Rightarrow \quad \theta \text{ is independent of } v_0$$

Therefore any bird must be launched at the same angle to the line of sight, regardless of the desired range D

Additional analysis

Typical bowl has $m = 1.3kg$
$R = 0.06m$

Model this bowl for varying v_0 and d

$$D = \frac{r_0}{\sqrt{\alpha^2+1}} = \frac{\alpha v_0^2}{2\mu g\sqrt{\alpha^2+1}}$$

$$\therefore \quad D \propto v_0^2$$

From computational data, produce log plot to verify

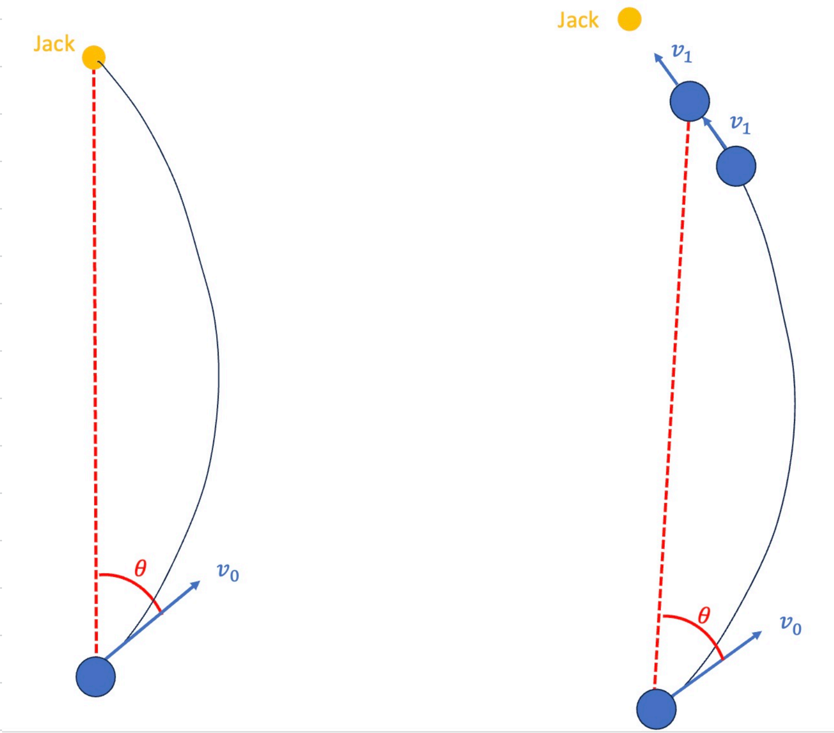
$$\ln D = \ln \frac{\alpha v_0^2}{2\mu g \sqrt{\alpha^2+1}}$$

$$\ln D = 2\ln v_0 + \ln \frac{\alpha}{2\mu g\sqrt{\alpha^2+1}}$$

$$y = m x + c$$

Collisions

In the game of flat green bowling there is not always a clear path to the Jack. It is often required to knock opposition bowls out of the way or push one of your own bowls closer.

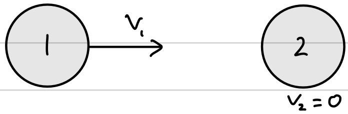


Assume all bowls have equal bias and mass

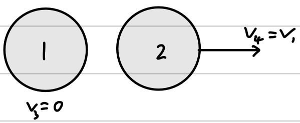
Assume elastic collision where the target bowl (bowl 2) is at rest : $v_2 = 0$

$m_1 = m_2$

Before Collision



After Collision



Using conservation of momentum and kinetic energy

$$v_3 = \frac{m_1 - m_2}{m_1 + m_2} v_1 + 0 = 0$$

$$v_4 = \frac{2m_1}{m_1 + m_2} v_1 + 0 = v_1$$

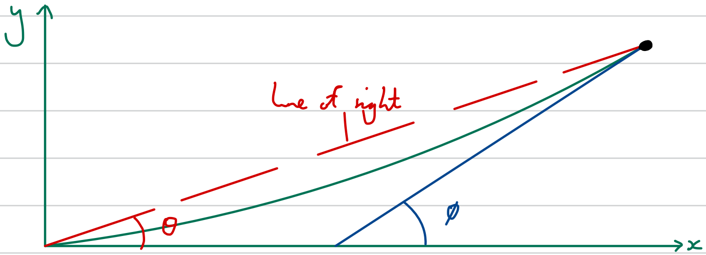
Distance to rest for ball 2:

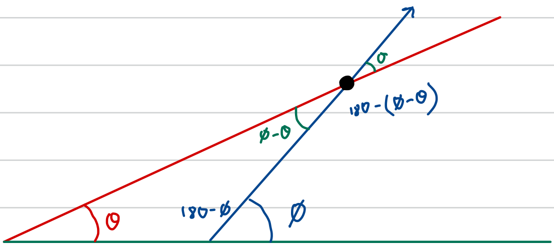
$$S = \frac{v_1^2}{2\mu g}$$

Therefore for a required ball projection distance the impact velocity is found as

$$v_1 = \sqrt{2\mu g S}$$

The collision angle to the line of sight can be found as





$$\Rightarrow \sigma = \phi - \theta$$