REAL ANALYSIS

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Contents

This is a short overview of a first semester course of real analysis using the book "Principles of Mathematical Analysis" by Walter Rudin [1]. Analysis is a branch of mathematics that uses inequalities and limits to study real and complex-valued continuous functions. In real analysis, we restrict ourselves to real numbers.

1. Real Numbers

The real number system extends the rationals with the least upper bound property, i.e. the supremum of a subset of $\mathbb{R}$ exists. This property provides solutions to equations such as

$$p^2 = 2,$$

since $\sqrt{2}$ is the $\sup\{x \in \mathbb{Q} \,|\, x^2 < 2\} \cup \mathbb{Q}_{\leq 0}$, i.e. the least upper bound property creates irrationals.

The real numbers are an ordered field. An ordering of a set is a relation, denoted by $<$, such that for a set S

(1) If $x \in S$, then exactly one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(2) For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

A field is a set F that is closed under operations of addition and multiplication, satisfies commutative and associative properties for addition and multiplication, and has both inverse and identity elements for both operations. A field also satisfies the distributive law.

2. Functions and Basic Topology

2.1. **Functions.**

Here are some images, courtesy of Wikipedia, that show what surjective, injective, and bijective functions are:

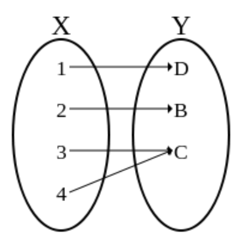


Figure 1. Sujective or onto function

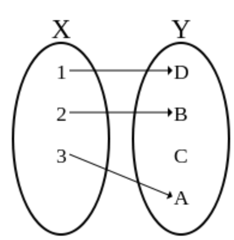


Figure 2. Injective or one-to-one function

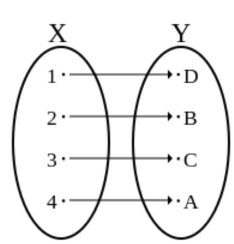


Figure 3. Bijective function or a one-to-one correspondence

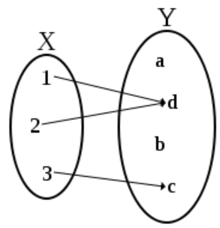


FIGURE 4. None of the above

A set is countable if there is an injection from the set to the positive integers, otherwise the set is called uncountable.

Definition 2.1. A *sequence* is an ordered collection of objects, i.e. a function $f : \mathbb{Z}^{\geq 0} \to X$ where X is an arbitrary set. If $f(n) = x_n$, for $n \in \mathbb{Z}^{\geq 0}$, we denote the sequence f with $\{x_n\}$.

Theorem 2.2. *Every infinite subset of a countable set is countable*

This means countable sets are the "smallest" infinity, since it implies that no uncountable set can be a subset of a countable set.

Theorem 2.3. *Let $\{E_n\}, n = \mathbb{Z}^{\geq 0}$ be a sequence fo countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Note that the rationals are countable, and the reals are uncountable.

2.2. **Metric Spaces.**

Definition 2.4. Let X be a set. A distance function on X is,

$$d : X \times X \to \mathbb{R}^{\geq 0}.$$

For $p, q, r \in X$, the distance function must satisfy three axioms:
(1) $d(p,q) = 0 \iff p = q$,
(2) $d(p,q) = d(q,p)$,
(3) $d(p,q) \leq d(p,r) + d(r,q)$.
The pair (X, d) is a metric space.

Now let's define a bunch of properties of metric spaces.

Definition 2.5. Let (X, d) be a metric space, $E \subset X$. For arbitrary $p \in X$ and $r \in \mathbb{R}^{>0}$:
(1) The *open ball* of radius r about p is $B_r(p) = \{q \in X : d(p,q) < r\}$.
(2) The *deleted ball* of radius r about p is $B'_r(p) = B_r(p) \setminus \{p\}$.
(3) E is *open* if for all $p \in X$ there exists $r \in \mathbb{R}^{>0}$ such that $B_r(p) \subset E$.
(4) E is *closed* if E^c is open.

(5) E is *bounded* if there is a ball such that $E \subset B_r(p)$.
(6) E is *dense* if for all $p \in X$ and $r \in \mathbb{R}^{>0}$, $B_r(p) \cap E \neq \varnothing$.
(7) A point $p \in X$ is
  (a) *interior* to E if there exists $B_r(p) \subset E$.
  (b) a *limit point* of E if for all $r > 0$, $B'_r(p) \cap E \neq \varnothing$.
  (c) an *isolated point* of E if there exists $r > 0$ such that $B_r(p) \cap E = \{p\}$.

Warning 2.6. A set does not have to be open or closed, e.g. $(a, b] \subset \mathbb{R}$. Also $\varnothing$ and $\mathbb{R}$ are both open and closed.

Proposition 2.7. *$E \subset X$ is closed if and only if E contains its limit points.*

Remark 2.8. Interior points are limit points.

Proposition 2.9. *If $E \subset X$, $p \in X$ is a limit point of E, then for any $r > 0$, $B_r(p) \cap E$ is infinite.*

Corollary 2.10. *A finite subset of a metric space has no limit points.*

Theorem 2.11.
(1) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.
(2) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
(3) For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
(4) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

2.2.1. *Compact Sets.*

Definition 2.12. An *open cover* of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.13. $K \subset X$ is compact if every open cover has a finite subcover, i.e. if $\{G_\alpha\}_{\alpha \in A}$ is an open cover of K, then there is a finite subset $F \subset A$ such that $K \subset \bigcup_{f \in F} G_f$.

Theorem 2.14. *Compact subsets of metric spaces are closed.*

Theorem 2.15. *Closed subsets of compact sets are compact.*

Theorem 2.16. *$K \subset X$ is compact if and only if every infinite subset $E \subset K$ has a limit point in K.*

Theorem 2.17. *(Heine-Borel) A set $K \subset \mathbb{E}^n$ is compact if and only if K is closed and bounded.*

2.2.2. *Connectedness.*

Definition 2.18. Let (X, d) be a metric space.
(1) Subsets $A, B \subset X$ are separated if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.
(2) X is connected if it is not the union of separated sets.
(3) A subset $E \subset X$ is connected if the metric space $(E, d|_{E \times E})$ is connected.

3. Sequences and Series

3.1. Sequences.

Definition 3.1. $\{p_n\} \subset X$ a sequence. Then $\{p_n\}$ converges to $p \in X$ if for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that for all $n \geq N$, $d(p_n, p) < \epsilon$ (or $p_n \in B_\epsilon(p)$).

Definition 3.2. $\{p_n\} \subset X$ a sequence. Then $\{p_n\}$ is *Cauchy* if for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$.

Note that the limit of a convergent sequence is unique, a convergent sequence is bounded, and a convergent sequence is Cauchy.

Theorem 3.3. *(X, d) metric space, $\{p_n\} \subset X$, $p \in X$, then $p_n \to p$, if and only if for every $\epsilon > 0$, $B_\epsilon(p) \cap \{p_n\}$ contains all but finitely many p_n.*

Theorem 3.4. *(X, d) compact metric space, $\{p_n\} \subset X$. Then $\{p_n\}$ has a subsequence which converges in X.*

Corollary 3.5. *If X compact and $\{p_n\}$ is Cauchy, then $\{p_n\}$ converges.*

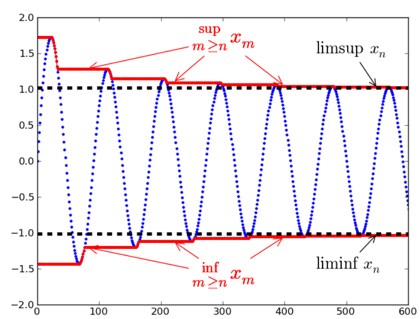
Corollary 3.6. *If $\{p_n\} \subset \mathbb{E}^k$ is Cauchy, then $\{p_n\}$ converges.*

Definition 3.7. (X, d) is *complete* if every Cauchy sequence in X converges.

Definition 3.8. $\{s_n\} \subset \mathbb{R}$ is monotonically increasing if $s_{n+1} \geq s_n$ for all n.

Theorem 3.9. *Let $\{s_n\} \subset \mathbb{R}$ be monotone. Then $\{s_n\}$ converges if and only if $\{s_n\}$ is bounded.*

Here is a picture that gives the idea of lim sup and lim inf, again, courtesy of Wikipedia:



3.2. Series.

Definition 3.10. Let $\{a_n\} \subset \mathbb{R}$. Then

(1) The partial sums of the series $\sum a_n$ are elements of the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_n = a_1 + a_2 + \cdots + a_n$.

(2) We say $\sum a_n$ converges if $\{s_n\}$ converges.

Proposition 3.11. *Let $\{a_n\} \subset \mathbb{R}$, then*

(1) $\sum a_n$ converges if and only if for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that $n, m \geq N$ implies $\left|\sum_{k=n}^{m} a_k\right| < \epsilon$.
(2) If $\sum a_n$ converges, then $a_n \to 0$.

Theorem 3.12. *Given a series $\sum a_n$ of real numbers, set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then*

(1) If $\alpha < 1$, then $\sum a_n$ converges.
(2) If $\alpha > 1$, then $\sum a_n$ diverges.
(3) If $\alpha = 1$, then the test is inconclusive.

Definition 3.13. Let $\{a_n\} \subset \mathbb{R}$. The *power series* with coefficients $\{a_n\}$ is $\sum_{n=1}^{\infty} a_n x^n$. Define $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, and set $R = \frac{1}{\alpha}$.

(1) $\sum a_n x^n$ converges if $|x| < R$.
(2) $\sum a_n x^n$ diverges if $|x| > R$.
(3) $\sum a_n x^n$ may or may not converge if $|x| = R$.

4. Limits and Continuity

For metric spaces X and Y, to say "the function $f : X \to Y$ is continuous" means:

(1) **Close enough points map to close points.** At each $p \in X$, for every $\epsilon > 0$, there is a $\delta > 0$ such that for $x \in X$, $d(x, p) < \delta$ implies $d(f(x), f(p)) < \epsilon$.

Another equivalent way to write this statement is as follows: Let $p \in X$. f continuous if for all $\epsilon > 0$ there is a $\delta > 0$ such that, for all p in X, $f(B_\delta(p)) \subset B_\epsilon(f(p))$.

(2) **The function preserves limits of sequences.** For every convergent sequence $\{x_n\}$ in X, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.
(3) **Inverse images of open sets are open.** If U is open in Y, then $f^{-1}(U)$ is open in X.
(4) **Inverse images of closed sets are closed.** If V is closed in Y, then $f^{-1}(V)$ is closed in X.

Theorem 4.1. *If f, g continuous, so is their composition.*

Proposition 4.2. *Suppose X has the discrete metric, Y is any metric space, and $f : X \to Y$, then f is continuous.*

Proof. The inverse image of any open set is a set in the discrete topology, and thus is open. □

Theorem 4.3. *X compact metric space, Y metric space, $f : X \to Y$, then $f(X) \subset Y$ compact.*

Corollary 4.4. *If f is compact metric space, $f : X \to \mathbb{E}^k$ continuous, then $f(X) \subset \mathbb{E}^k$ is closed and bounded.*

Corollary 4.5. *If X compact metric space, $f : X \to \mathbb{R}$ continuous, then f realizes its supremum and infimum.*

The above corollary is relevant to optimization problems.

Theorem 4.6. *X compact metric space, Y metric space, $f : X \to Y$ continuous and bijective. Then $f^{-1} : Y \to X$ is continuous. (This also means f is a* homeomorphism*).*

Definition 4.7. Call $f : X \to Y$ uniformly continuous on X if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $p \in X$, $f(B_\delta(p)) \subset B_\epsilon(f(p))$.

Theorem 4.8. *X compact metric space, Y metric space, $f : X \to Y$ continuous, then f is uniformly continuous.*

Theorem 4.9. *X is a connected metric space, Y metric space, $f : X \to Y$ continuous. Then $f(X)$ is connected.*

Theorem 4.10. *(Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ continuous, $c \in \mathbb{R}$, and $f(a) < c < f(b)$, then there is an $x \in [a, b]$ such that $f(x) = c$.*

Here is a picture from Wikipedia to show what is going on in the theorem directly above this text.

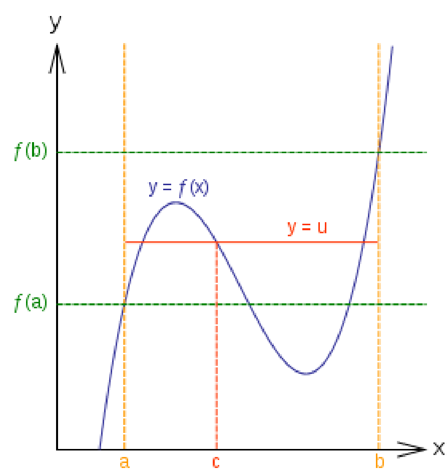


FIGURE 5. The intermediate value theorem

5. DIFFERENTIATION

Let's define something called the difference quotient:

$$\phi : (a, b) \setminus \{p\} \to \mathbb{R}$$
$$x \mapsto \frac{f(x) - f(p)}{x - p}.$$

Definition 5.1. f is differentiable at p if $\lim_{x \to p} \phi(x)$ exists. In that case, we set $f'(x) = \lim_{x \to p} \phi(x)$.

Theorem 5.2. *If f is differentiable at p, then f is continuous at p.*

Proof. Note $f(x) = f(p) + \phi(x)(x - p)$. Then

$$\lim_{x \to p} f(x) = f(p) + \lim_{x \to p} \phi(x)(x - p) = f(p)$$

exists. $\square$

Recall that differentiation is a linear operation, the product/quotient rule, and chain rule still exist.

Definition 5.3. X metric space, $f : X \to \mathbb{R}$, and $x \in X$. Then f has a local minimum at x if there exists a $\delta > 0$ such that $f(t) \geq f(x)$ for all $t \in B_\delta(x)$.

The definition for local maximum is similarly defined (exchange $\geq$ for $\leq$).

Theorem 5.4. *$f : (a, b) \to \mathbb{R}$ has a local minimum or local maximum at $x \in (a, b)$ and suppose f is differentiable at x. Then $f'(x) = 0$.*

5.1. **Mean Value Theorem.**

Theorem 5.5. *(Mean Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous, assume the restriction to (a, b) is differentiable. Then there is an $x \in (a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

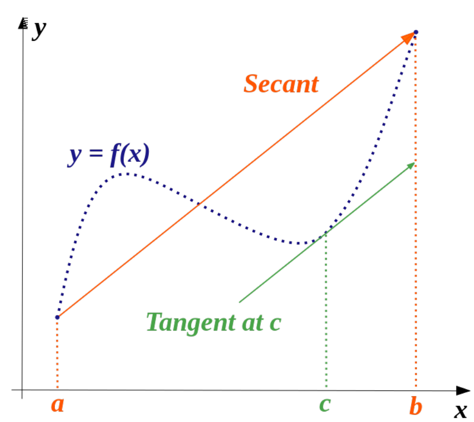


Figure 6. The mean value theorem (from Wikipedia)

Theorem 5.6. *(Generalized Mean Value Theorem) $a < b$ real numbers, $f, g : [a, b] \to \mathbb{R}$, and f, g continuous on $[a, b]$ and differentiable on (a, b). Then there is an $x \in (a, b)$ such that $f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)]$.*

Warning 5.7. The mean value theorem does not work in higher dimensions.

Note that all functions that result from differentiation of other functions have the intermediate value property, that is, the image of an interval is also an interval. It follows from this that if f is differentiable on $[a, b]$, then f' cannot have and simple discontinuities on $[a, b]$.

5.2. **L'Hôpital's Rule.**

Theorem 5.8. *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a$$

or if

$$g(x) \to +\infty \text{ as } x \to a,$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

5.3. **Taylor's Theorem.**

Theorem 5.9. $f : (a, b) \to \mathbb{R}$, $n \in \mathbb{Z}^{>0}$. *Assume f is n times differentiable. Fix $\alpha, \beta \in (a, b)$ with $\alpha \neq \beta$. Set*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists x between α, β such that

$$f(\beta) = P(\beta) + \underbrace{\frac{f^{(n)}}{n!}(\beta - \alpha)^n}_{\text{error term}}.$$

6. Integration

Definition 6.1. A *partition* P of $[a, b]$ is a finite set $\{x_1, x_2, \ldots, x_{n-1}\}$ of real numbers such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.

Remark 6.2. The partitions of $[a, b]$ form a set: $\mathcal{P}$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, i.e. there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for $x \in [a, b]$. Let $P = \{x_1 \leq x_2 \leq \cdots \leq x_{n-1}\}$ be a partition. For $i = 1, \ldots, n$ set $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. Define $L(P, f) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ and $U(P, f) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ (That is: $L, U : \mathcal{P} \to \mathbb{R}$). Since $m \leq m_i \leq M_i \leq M$ we have $m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$. In particular, $L(\cdot, f)$ is bounded above, and $U(\cdot, f)$ is bounded below.

Definition 6.3.

$$\underline{\int_a^b} f\,dx = \sup_{P\in\mathcal{P}} L(P,f) \qquad \overline{\int_a^b} f\,dx = \inf_{P\in\mathcal{P}} U(P,f)$$

Definition 6.4. If $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$, then we say that f is *Riemann-integrable* on $[a,b]$. We denote the common value by

$$\int_a^b f\,dx.$$

Definition 6.5. P_1, P_2 are partitions of $[a,b]$.

(1) A partition P^* is a refinement of P if $P \subset P^*$.
(2) P^* is a common refinement of P_1, P_2 if $P_1 \subseteq P^*$ and $P_2 \subseteq P^*$. Any two partitions have a minimal common refinement $P^* = P_1 \cup P_2$.

Theorem 6.6. *If P^* is a refinement of P, then*

$$L(P,f) \le L(P^*,f) \le U(P^*,f) \le U(P,f).$$

Corollary 6.7.

$$\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$$

Theorem 6.8. *f is Riemann-integrable if for all $\epsilon > 0$ there is a partition $P \in \mathcal{P}$ such that $U(P,f) - L(P,f) < \epsilon$.*

Theorem 6.9. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function*

(1) If f is continuous, it is Riemann-integrable.
(2) If f is monotone, it is Riemann-integrable.

Recall that integration is a linear operation.

Theorem 6.10. *Let $f : [a,b] \to \mathbb{R}$ be Riemann-integrable, denoted by $\mathscr{R}$. Then*

(1) $|f| \in \mathscr{R}$ and $\left|\int_a^b f\right| \le \int_a^b |f|$.
(2) If $|f(x)| \le M$ for all $x \in [a,b]$ and some $M \in \mathbb{R}$, then $\left|\int_a^b f\right| \le M(b-a)$.

6.1. Relation between Differentiation and Integration.

Theorem 6.11. *Suppose $f : [a,b] \to \mathbb{R}$ is Riemann-integrable. Define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,dt$$

for $x \in [a,b]$. Then

(1) F is uniformly continuous.
(2) If f is continuous at $x \in [a,b]$, then F is differentiable at x and $F'(x) = f(x)$.

Theorem 6.12. *(Fundamental Theorem of Calculus) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann-integrable and $F : [a, b] \to \mathbb{R}$ is continuous, $f|_{(a,b)}$ is differentiable, and $F'(x) = f(x)$ for $x \in (a, b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

7. Sequences and Series of Functions

Definition 7.1. $f : X \to \mathbb{R}$, $f_n \to f$ *pointwise* if for all $x \in X$ and for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that $n \geq N$ implies $\left|f(x) - f_n(x)\right| < \epsilon$.

Definition 7.2. $f : X \to \mathbb{R}$, $f_n \to f$ *uniformly* if for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that for all $x \in X$, $n \geq N$ implies $\left|f(x) - f_n(x)\right| < \epsilon$.

Note that we are switching the position of "ϵ condition" in these two definitions.

Remark 7.3. $f_n \to f$ uniformly implies $f_n \to f$ pointwise.

Theorem 7.4. *(X, d) metric space. $f_n, f : X \to \mathbb{R}$, $n = 1, 2, \ldots$, f_n continuous, and $f_n \to f$ uniformly, then f is continuous.*

Theorem 7.5. *$f_n \to f$ uniformly if and only if $\{f_n\}$ is "uniformly Cauchy," i.e. for all $\epsilon > 0$ there is an $N \in \mathbb{Z}^{>0}$ such that for all $x \in X$, $n, m \geq N$ implies $\left|f_n(x) - f_m(x)\right| < \epsilon$.*

Define a metric space $\mathcal{C}(X)$ of functions:

$$\mathcal{C}(X) = \{f : X \to \mathbb{R} : f \text{ is bounded and continuous.}\}$$

Define a function (called "norm"):

$$\begin{aligned} \|\cdot\| &: \mathcal{C}(X) \to \mathbb{R}^{>0} \\ f &\mapsto \sup_{x\in X}\left|f(x)\right| = \underbrace{\|f\|}_{\text{sup norm}} \end{aligned}$$

Theorem 7.6. *For $f, f_1, f_2 \in \mathcal{C}(X)$,*

(1) $\|f\| = 0$ *if and only if* $f = 0$.
(2) $\|cf\| = |c|\|f\|$.
(3) $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$.

Define a metric

$$\begin{aligned} d_{\mathcal{C}} &: \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}^{\geq 0} \\ f_1, f_2 &\mapsto \|f_1 - f_2\|. \end{aligned}$$

Corollary 7.7. *Let $\{f_n\} \subset \mathcal{C}(X)$,*

(1) $f_n \to f$ uniformly if and only if $f_n \to f$ in $(\mathcal{C}(X), d_{\mathcal{C}})$.
(2) $\mathcal{C}(X)$ is complete.

Theorem 7.8. *$f_n : [a,b] \to \mathbb{R}$ Riemann-integrable, $f_n \to f$ uniformly on $[a,b]$, then f is Riemann-integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Corollary 7.9.

$$\int_a^b : \mathcal{C}([a,b]) \to \mathbb{R}$$

$$f \mapsto \int_a^b f \text{ is continuous.}$$

References

[1] W. Rudin, Principles of Mathematical Analysis. New York: McGraw-Hill, 1964.
[2] D. Freed, 'Real Analysis I', The University of Texas at Austin, 2016.
[3] F. Su, 'Real Analysis', Harvey Mudd College, 2010.