Soap films and minimal surfaces

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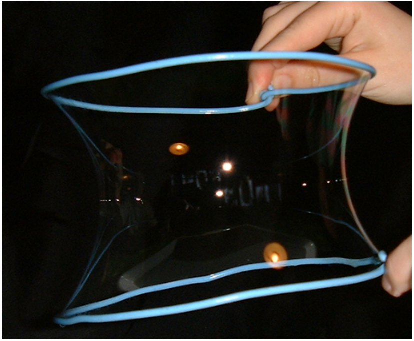
Soap films

Look at the surface the soap film makes as it is streched between the two loops. That is a minimal surface, a topic of interest in the subject of differential geometry.



To understand what this means, let's introduce some key concepts.

Concepts

1. Curvature
2. Minimal Surfaces and Surfaces of Revolution
3. Catenoid

Curvature

The word "curvature" corresponds to what you generally think of when the word comes to mind. A ball is something with a lot of curvature, while a floor (usually) corresponds to something that does not.

Definition

If γ is a unit-speed curve with parameter t, its *curvature* $\kappa(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$.

In other words, the curvature at a point t on the curve is the rate with which direction is changing at t.

Shape Operator

Definition

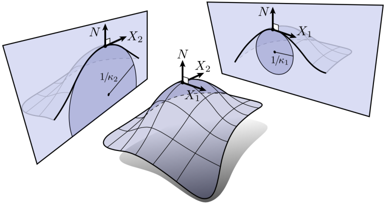
For a given surface, let $\mathbf{N}$ be the surface normal and $\mathbf{v}$ be tangent to the surface, then the shape operator is defined as

$$S_v = \nabla_{\mathbf{v}} \mathbf{N},$$

which is a linear map.

Or, the shape operator describes the rate at which the surface normal changes along a direction $\mathbf{v}$.

Principal Curvatures



In the picture we see X_1 and X_2 point in the *principal directions* which are the eigenvectors of the shape operator. The *principal curvatures*, κ_1 and κ_2 are the eigenvalues of the shape operator.

Remark

The curvature of a surface is completely characterized by the principal curvatures.

Mean Curvature

The mean curvature is an extrinsic measure of curvature that locally describes the curvature of a surface $\mathcal{S} \subset \mathbb{R}^3$.

Definition

The *mean curvature* H at $p \in \mathcal{S}$ is defined to be

$$H = \frac{1}{2}\mathrm{tr}(S_v) = \frac{1}{2}(\kappa_1 + \kappa_2).$$

The mean curvature is a property of the surface being embedded into an ambient Euclidean space (ambient meaning the space surround an object).

Minimal Surfaces

If a surface minimizes area for a given boundary, then it is a minimal surface.

Theorem

If a surface has minimal surface area for a given boundary, then

$$H = 0.$$

Minimal surfaces tend to be saddle-*like* surfaces, since the principal curvatures must have equal magnitude but opposite sign.

Surface of Revolution

A *surface of revolution* is the surface obtained by rotating a plane curve about a straight line in the plane.

Let's suppose we are rotating a curve around the z-axis, and the curve is given by

$$x = f(t), \quad z = t,$$

then the surface of revolution is the set of points

$$\mathcal{S}_r = \{(f(t)\cos\theta, f(t)\sin\theta, t) : \; t \in (\alpha, \beta), \, \theta \in [0, 2\pi)\}.$$

Minimizing the area of a surface of revolution

We can find the surface area of a surface of revolution for a plane curve as defined before, by the following equation:

$$\mathcal{A}(f) = 2\pi \int_{\alpha}^{\beta} f\sqrt{1+f'^2}dt.$$

Then to minimize the area, we take the derivative of the area functional with respect to an ϵ scaling of a variation ϕ of the surface, and set the quantity equal to zero:

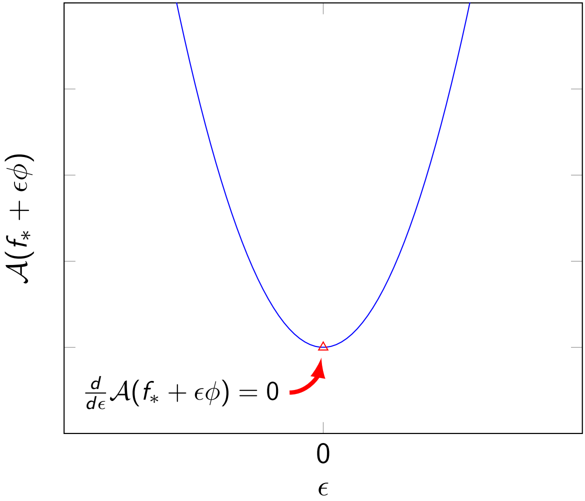
$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{A}(f+\epsilon\phi) = \frac{d}{d\epsilon}\int_{\alpha}^{\beta}(f+\epsilon\phi)\sqrt{1+(f+\epsilon\phi)'^2}\,dt$$

$$\vdots$$

$$= \int_{\alpha}^{\beta} \underbrace{\left[\sqrt{1+f'^2} - \left(\frac{f\cdot f'}{\sqrt{1+f'^2}}\right)'\right]}_{\text{mean curvature of a surface of revolution}} \phi\,dt = 0$$

What?

Let's say f_* is a function that minimizes surface area, then



Catenoid

The curve defined by f which solves

$$\sqrt{1+f'^2} - \left(\frac{f \cdot f'}{\sqrt{1+f'^2}}\right)' = 0$$

is the caternary. If we want to find the caternary that goes through the points in the plane: (x_1, y_1) and (x_2, y_2), this is described by

$$f(t) = a \cosh\left(\frac{t-b}{a}\right)$$

where a and b are constants determined by

$$y_1 = a \cosh\left(\frac{x_1 - b}{a}\right) \qquad y_2 = a \cosh\left(\frac{x_2 - b}{a}\right).$$

Remark

Catenoids are, apart from the plane, the only minimal surface of revolution.

So let's make a soap film!

or maybe just graph a catenoid in Mathematica

References

Andrew Pressley
Elementary Differential Geometry Second Edition
London: Springer, 2012

Peter Schröder, Keenan Crane
Discrete Differential Geometry - CS 177 (Fall 2011)

Wikipedia

The End