8. COMBINATIONS OF MODELS

- Assume that observations are made in two groups, with the second group consisting of one or several observations. Both groups have a common set of parameters, i.e.

- **Given:** 2 sets of observations collected at different times for the same parameters

i.e. $\overbrace{\boldsymbol{l}_1 \ and \ \boldsymbol{C}_{\boldsymbol{l}_1}}^{n_1} \ and \ \overbrace{\boldsymbol{l}_2 \ and \ \boldsymbol{C}_{\boldsymbol{l}_2}}^{n_2}$

- **Required:** $\mathbf{X}_{u,1}$ The best estimate for a group of parameters from a group of measurements that have been captured at two different times (e.g. $\mathbf{l_1}$ at t_1 and $\mathbf{l_2}$ at t_2)

- **Functional model:**

$$f_1(x, l_1) = 0 \qquad A_{1_{m1,u}} \delta_{u,1} + B_{11_{m1,n1}} v_{1_{n1,1}} + w_{1_{m1,1}} = \underline{0}$$

$$f_2(x, l_2) = 0 \qquad A_{m_2,u} \delta_{u,1} + B_{22_{m2,n2}} v_{2_{n2,1}} + w_{2_{m2,1}} = \underline{0}$$

- Variation function using Lagrange Multipliers

$$\varphi = v_1^T P_1 v_1 + v_2^T P_2 v_2 + 2k_1^T (A_1 \delta + B_{11} v_1 + w_1) + 2k_2^T (A_2 \delta + B_{22} v_2 + w_2)$$

- Note: For each group of observations, there’s a quadratic form and a Lagrange Multiplier

- To minimize φ, differentiate φ w.r.t. all variables (**δ, v_1, v_2, k_1, k_2**) and equate to zero.

- Then arrange in hyper-matrix notation and solve by elimination.

$$\frac{\partial \varphi}{\partial \delta} = 2\, k_1^T A_1 + 2\, k_2^T A_2 = 0$$

$$\frac{\partial \varphi}{\partial v_1} = 2\, v_1^T P_1 + 2\, k_1^T B_{11} = 0$$

$$\frac{\partial \varphi}{\partial v_2} = 2\, v_2^T P_2 + 2\, k_2^T B_{22} = 0$$

$$\frac{\partial \varphi}{\partial k_1} = 2\, \delta^T A_1^T + 2 v_1^T B_{11}^T + 2 w_1^T = 0$$

$$\frac{\partial \varphi}{\partial k_2} = 2 \delta^T A_2^T + 2 v_2^T B_{22}^T + 2 w_2^T = 0$$

- Transpose all equations and divide by 2 and arrange in hyper matrix

$$\begin{pmatrix} P_1 & 0 & B_{11}^T & 0 & 0 \\ 0 & P_2 & 0 & B_{22}^T & 0 \\ B_{11} & 0 & 0 & 0 & A_1 \\ 0 & B_{22} & 0 & 0 & A_2 \\ 0 & 0 & A_1^T & A_2^T & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ k_1 \\ k_2 \\ \delta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w_1 \\ w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- To solve this system algebraically, partition and eliminate $\mathbf{v_1}$
- Then partition and eliminate $\mathbf{v_2}$
- Perform the elimination until a solution for $\boldsymbol{\delta}$ is reached
- Then perform back substitution to get expressions for all other variables ($\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{k_1}$, and $\mathbf{k_2}$)
- Then perform the Law of propagation of covariance to estimate the v-c matrices

8.1. Summation of Normals – Parametric Models

- Given: two group of measurements, ($\mathbf{l_1}$ and $\mathbf{C_{l_1}}$) and ($\mathbf{l_2}$ and $\mathbf{C_{l_2}}$), that could have been collected either at different times or by different instruments, or by different observer. Furthermore, there is no correlation between $\mathbf{l_1}$ and $\mathbf{l_2}$

- Required: $\mathbf{X}$ by using the two sets of observation in one solution

- Functional Models:

$$\hat{l}_1 = f_1(\hat{x}) \xrightarrow{\text{with the linear model}} A_{1_{n1,u}}\delta_{u,1} + w_{1_{n1,1}} = v_{1_{n1,1}}$$

$$\hat{l}_2 = f_2(\hat{x}) \xrightarrow{\text{with the linear model}} A_{2_{n2,u}}\delta_{u,1} + w_{2_{n2,1}} = v_{2_{n2,1}}$$

- Variation function

$$\varphi = v_1^T\ P_1\, v_1 + v_2^T\ P_2\, v_2 \quad = \quad min$$

$$= \left(\delta^T A_1^T + w_1^T\right)P_1\left(A_1\,\delta + w_1\right) + \left(\delta^T A_2^T + w_2^T\right)P_2\left(A_2\,\delta + w_2\right)$$

$$= \delta^T A_1^T P_1 A_1\,\delta + \underbrace{\delta^T A_1^T P_1\, w_1}_{\substack{equivelant\\ scalers}} + \underbrace{w_1^T P_1 A_1\,\delta}_{\substack{equivelant\\ scalers}} + w_1^T P_1\, w_1$$

$$+ \delta_T\ A_2\ P_2\ A_1\,\delta + \underbrace{\delta^T A_2\ P_2\ \ w_2}_{\substack{equivelant\\ scalers \longrightarrow}} + \underbrace{w_1^T P_1 A_1\,\delta}_{\substack{equivelant\\ \longleftarrow scalers}} + w_2^T P_2\ w_2 = min$$

$$\varphi = \delta^T A_1^T P_1\ A_1\,\delta + 2\delta^T A_1^T P_1\ w_1 + w_1^T P_1\ w_1$$

$$+ \delta^T A_2^T P_2 A_2\,\delta + 2\delta^T A_2\ P_2\ w_2 + W_2^T P_2\ w_2 = min$$

- Now minimize φ (φ is only function of δ)

$$\varphi = \delta^T A_1^T P_1\ A_1\,\delta + 2\delta^T A_1^T P_1\ w_1 + \underbrace{w_1^T P_1\ w_1}_{Constant}$$

$$+ \delta^T A_2^T P_2 A_2\,\delta + 2\delta^T A_2\ P_2\ w_2 + \underbrace{w_2^T P_2\ w_2}_{Constant} = min$$

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\delta}} = 2\ \boldsymbol{\delta}^T \boldsymbol{A}_1^T \boldsymbol{P}_1\ \boldsymbol{A}_1 + 2\ \boldsymbol{w}_1^T \boldsymbol{P}_1 \boldsymbol{A}_1 \qquad + \ \ \boldsymbol{0}$$

$$2\ \boldsymbol{\delta}^T \boldsymbol{A}_2^T \boldsymbol{P}_2\ \boldsymbol{A}_2 + 2\ \boldsymbol{w}_2^T \boldsymbol{P}_2 \boldsymbol{A}_2 \qquad + \boldsymbol{0} \qquad = \boldsymbol{0}$$

- Transpose and divide by 2

$$\Big(\underbrace{\underbrace{\boldsymbol{A}_1^T \boldsymbol{P}_1 \boldsymbol{A}_1}_{u,u}}_{\boldsymbol{N}_1} + \underbrace{\underbrace{\boldsymbol{A}_2^T \boldsymbol{P}_2 \boldsymbol{A}_2}_{u,u}}_{\boldsymbol{N}_2} \Big) \ \boldsymbol{\delta} + \ \Big(\underbrace{\underbrace{\boldsymbol{A}_1^T \boldsymbol{P}_1 \boldsymbol{w}_1}_{u,1}}_{\boldsymbol{u}_1} + \underbrace{\underbrace{\boldsymbol{A}_2^T \boldsymbol{P}_2 \boldsymbol{w}_2}_{u,1}}_{\boldsymbol{u}_2} \Big) = \boldsymbol{0}$$

$$\therefore \boldsymbol{\delta} = -\left(\boldsymbol{N}_1 + \boldsymbol{N}_2\right)^{-1}\left(\boldsymbol{u}_1 + \boldsymbol{u}_2\right)$$

- Note: The δ vector involves addition of the normal equation matrices and vectors corresponding to each set of observations.
- The same procedure can be applied for 3 (or more) set of observations with the combined
- For n group of observation, the problem of summation of normal can be formulated as:

$$\delta = -\sum_{i=1}^{n} (N_i)^{-1} (u_i)$$

8.1.1. Variance propagation to estimate C_δ and $C_{\hat{x}} \left(C_l^{-1} = P \right)$

$$\boldsymbol{\delta} = -\left(\boldsymbol{N}_1 + \boldsymbol{N}_2\right)^{-1} \Big(\underbrace{\boldsymbol{A}_1^T \boldsymbol{C}_{l_1}^{-1} \left(\boldsymbol{f}\left(\boldsymbol{x}^o\right) - \boldsymbol{l}_1 \right)}_{\boldsymbol{u}_1} + \underbrace{\boldsymbol{A}_2^T \boldsymbol{C}_{l_2}^{-1} \left(\boldsymbol{f}\left(\boldsymbol{x}^o\right) - \boldsymbol{l}_2 \right)}_{\boldsymbol{u}_2} \Big)$$

$\boldsymbol{x}^o$: Non-stochastic and therefore, $\boldsymbol{\delta} = \boldsymbol{f(l)}$

$$\therefore C_\delta = J \ \ C_l \ \ J^T$$

$$\therefore C_\delta = \frac{\partial \delta}{\partial l} C_l \left(\frac{\partial \delta}{\partial l} \right)^T$$

Where:

$$l_{(n_1+n_2)} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \qquad C_{l_{(n_1+n_2)(n_1+n_2)}} = \begin{pmatrix} C_{l_1} & 0 \\ 0 & C_{l_2} \end{pmatrix}$$

$$\frac{\partial \delta}{\partial l} = \begin{pmatrix} \frac{\partial \delta}{\partial l_1} & \frac{\partial \delta}{\partial l_2} \end{pmatrix}$$

$$\frac{\partial \boldsymbol{\delta}}{\partial \boldsymbol{l_1}} = (\boldsymbol{N_1} + \boldsymbol{N_2})^{-1} \boldsymbol{A_1^T C_l^{-1}} = \boldsymbol{N^{-1} A_1^T C_{l_1}^{-1}}$$

$$\frac{\partial \boldsymbol{\delta}}{\partial \boldsymbol{l_2}} = (\boldsymbol{N_1} + \boldsymbol{N_2})^{-1} \boldsymbol{A_2^T C_l^{-2}} = \boldsymbol{N^{-1} A_2^T C_{l_2}^{-1}}$$

$$\therefore \boldsymbol{C_\delta} = \underbrace{\begin{pmatrix} \boldsymbol{N^{-1} A_1^T C_{l_1}^{-1}} & \boldsymbol{N^{-1} A_2^T C_{l_2}^{-1}} \end{pmatrix}}_{\downarrow} \underbrace{\begin{pmatrix} \boldsymbol{C_{l_1}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C_{l_2}} \end{pmatrix} \begin{pmatrix} \boldsymbol{C_{l_1}^{-1}} & \boldsymbol{A_1} & \boldsymbol{N^{-1}} \\ \boldsymbol{C_{l_2}^{-1}} & \boldsymbol{A_2} & \boldsymbol{N^{-1}} \end{pmatrix}}_{\begin{pmatrix} \boldsymbol{C_{l_1} C_{l_1}^{-1}} & \boldsymbol{A_1 N^{-1}} \\ \underbrace{\boldsymbol{C_{l_2} C_{l_2}^{-1}}}_{\boldsymbol{I}} & \boldsymbol{A_2 N^{-1}} \end{pmatrix}}$$

$$= \boldsymbol{N^{-1} A_1^T C_{l_1}^{-1} A_1 N^{-1}} + \boldsymbol{N^{-1} A_2^T C_{l_2}^{-1} A_2 N^{-1}}$$

$$= \boldsymbol{N^{-1}} \left(\boldsymbol{A_1^T C_{l_1}^{-1} A_1} + \boldsymbol{A_2^T C_{l_2}^{-1} A_2} \right) \boldsymbol{N^{-1}}$$

$$= \boldsymbol{N^{-1}} (\boldsymbol{N_1} + \boldsymbol{N_2}) \boldsymbol{N^{-1}} = \boldsymbol{N^{-1} N N^{-1}}$$

$$\therefore \boldsymbol{C_\delta} = \boldsymbol{N^{-1}} = (\boldsymbol{N_1} + \boldsymbol{N_2})^{-1}$$

$$\therefore \hat{\boldsymbol{x}} = \boldsymbol{x^o} + \boldsymbol{\delta}$$

$$\therefore \boldsymbol{C_{\hat{x}}} = \boldsymbol{C_\delta}$$

8.2. Sequential Least Squares-Parametric Model

- In the previous section (summation of normals), it has been shown that two (or more) sets of observations (L_1 and L_2) for the same set of parameters can be combined to get a new solution

$$\hat{\boldsymbol{\delta}} = -\left(\boldsymbol{N_1} + \boldsymbol{N_2}\right)^{-1}\left(\boldsymbol{u_1} + \boldsymbol{u_2}\right), \textit{where}$$

$$\boldsymbol{N_1} = \underbrace{\boldsymbol{A^T_{1_{u,n1}}}\,\boldsymbol{P_{1_{n1,n1}}}\,\boldsymbol{A_{1_{n1,u}}}}_{u,u}, \textit{and} \quad \boldsymbol{N_2} = \underbrace{\boldsymbol{A^T_{2_{un2}}}\,\boldsymbol{P_{2_{n2,n2}}}\,\boldsymbol{A_{2_{n2,u}}}}_{u,u}$$

- What if n_2 << u (e.g. n_2 = 1 and u= 4). With the summation of normals method, a (u,u) matrix must be inverted again to add the new (single) observation (the assumption here the solution has been obtained already for the 1st group of observations)

- This can be a significant computational burden especially when observations are being added at a regular interval in real time (as for example in GPS positioning).

- Sequential Least squares provides a method where only (n_2, n_2) inversion is required for updating the solution with new n_2 observations.

<u>Derivation</u>

To derive the sequential expressions, the summation of normal solution is re-written as:

$$\boldsymbol{\delta} = -\left(\boldsymbol{N_1} + \boldsymbol{A_2^T C_{l_2}^{-1} A_2}\right)^{-1}\left(\boldsymbol{u_1} + \boldsymbol{A_2^T C_{l_2}^{-1} w_2}\right)$$

$$\boldsymbol{\delta} = -\left(\boldsymbol{N_1} + \boldsymbol{A_2^T C_{l_2}^{-1} A_2}\right)^{-1}\boldsymbol{u_1} - \left(\boldsymbol{N_1} + \boldsymbol{A_2^T C_{l_2}^{-1} A_2}\right)^{-1}\boldsymbol{A_2^T C_{l_2}^{-1} w_2}$$

To proceed, two-matrix inversion lemmas are utilized.

$$\left(S^{-1} + T^T R^{-1} T\right)^{-1} = S - ST^T\left(R + TST^T\right)^{-1} TS \ldots\ldots\ldots\ldots(i)$$

$$\left(S^{-1} + T^T R^{-1} T\right)^{-1} T^T R^{-1} = ST^T\left(R + TST^T\right)^{-1} \ldots\ldots\ldots\ldots(ii)$$

Apply lemma (i) to the 1st term and lemma (ii) to the 2nd term of the δ equation with:

$N_1^{-1} = S \qquad T = A_2 \qquad R = C_{l_2}$

1st term:

$$-\underbrace{\left(N_1 + A_2^T C_{l_2}^{-1} A_2\right)^{-1}}_{LHS\,of\ Lemma(i)} \quad u_1 = -\underbrace{\left[N_1^{-1} - N_1^{-1} A_2^T \left(C_{L_2} + A_2 N_1^{-1} A_2^T\right)^{-1} A_2 N_1^{-1}\right]}_{RHS\,of\ Lemma(i)} \quad u_1$$

2nd term:

$$-\underbrace{\left(N_1 + A_2^T C_{l_2}^{-1} A_2\right)^{-1} A_2^T C_{l_2}^{-1}}_{LHS\ of\ Lemma(ii)} \quad w_2 = -\underbrace{N_1^{-1} A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1}}_{RHS\ of\ Lemma(ii)} \quad w_2$$

Combining term 1 and 2

$$\delta = -\left[N_1^{-1} - N_1^{-1} A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1} A_2 N_1^{-1}\right] u_1 - N_1 A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1} w_2$$

$$\delta = -N_1^{-1} u_1 + N_1^{-1} A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1} A_2 N_1^{-1} u_1 \quad - N_1^{-1} A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1} w_2$$

$$\delta = \underbrace{-N_1^{-1} u_1}_{\delta(-)} + N_1^{-1} A_2^T \left(C_{l_2} + A_2 N_1^{-1} A_2^T\right)^{-1} \left(A_2 \underbrace{N_1^{-1} u_1}_{-\delta(-)} - w_2\right)$$

Now set:

$-N_1^{-1} u_1 = \delta(-)$ Solution before update with new observation (s)

$\delta = \delta(+)$ Updated solution with new observation(s)

$$\therefore \underset{u,1}{\delta(+)} = \underset{u,1}{\delta(-)} - \underset{u,u}{N_1^{-1}} \underset{u,n2}{A_2^T} \underbrace{\left(\underset{n_2,n_2}{C_{l_2}} + \underset{n_2,u}{A_2} \underset{u,u}{N_1^{-1}} \underset{u,n_2}{A_2^T}\right)^{-1}}_{n_2,n_2} \left(\underset{n_2,u}{A_2} \underset{u,1}{\delta(-)} + \underset{n_2 1}{w_2}\right)$$

$$\boxed{\begin{aligned}&\delta(+) = \delta_{u,1}(-) - K_{u,n_2}\left(A_2\delta(-) + w_2\right)\\ &K = N_1^{-1}A_2^T\left(C_{l_2} + A_2N_1^{-1}A_2^T\right)^{-1}\end{aligned}}$$

K is known as the Gain Matrix, which quantifies how much each new observation will contribute to the corrections to the parameters

Note: 1.) Only an (n_2 x n_2) inversion is required
2.) The parameters before update area

$$\hat{\boldsymbol{x}}(-) = \boldsymbol{x^o} + \hat{\boldsymbol{\delta}}(-)$$

3.) The updated parameters are

$$\hat{\boldsymbol{x}}(+) = \boldsymbol{x^o} + \hat{\boldsymbol{\delta}}(+)$$

Note: that $\mathbf{x^0}$ (The POE) is the same in both cases

<u>Understanding the Gain Matrix:</u>

$$K = N_1^{-1}A_2^T\left(C_{l_2} + A_2N_1^{-1}A_2^T\right)^{-1}$$

- If the 2nd set of observations, l_2, is imprecise, i.e. C_{l_2} has large elements. The Gain Matrix will generally have small elements. Thus the new observations will not greatly contribute to solution update.

- As the precision of l_2 increases, C_{l_2} element decreases, l_2 tends to contribute more the solution update.

- $\boldsymbol{\delta}(+) = \boldsymbol{\delta}(-) - \boldsymbol{K}\ \underbrace{\left(\boldsymbol{A_2\delta}(-) + \boldsymbol{w_2}\right)}_{of\ \ the\ \ form\quad A\delta+w=v}$

That is:

- $v_2(-) = A_2\delta(-) + w_2$ The predicted residuals of innovations vector. Hence, the estimated residual after update is given by

- $v_2(+) = A_2\delta(+) + w_2$

Covariance Matrices

$$C_{x(-)} = C_{\delta(-)} = N_1^{-1} = \left(A_1^T C_{l_1}^{-1} A_1^{-1}\right)$$
$$C_{\delta(+)} = C_{\delta(-)} - KA_2 C_{\delta(-)}$$

Note: the $C_{\delta(+)} < C_{\delta(-)}$ due to the subtraction of $KA_2C_{\delta(-)}$ and thus the covariance matrix is improved by adding observations.

8.3. Sequential Solution of Linear parametric Models

Step 1:
Model: $\mathbf{l_1 = f_1(x)}$

- Linearized Model:

$$\underset{n_1,u}{A_1}\ \underset{u,1}{\delta} + \underset{n_1,1}{w_1} = \underset{n_1,1}{v_1}$$

- P.O.E. : x^0
- Initial solution with n_1 observations $(n_1 \geq u)$,

$$\delta(-) = -\left(A_1^T P_1 A_1\right)^{-1} A_1^T P_1 w_1$$
$$= -(N_1)^{-1} u_1$$

where

$$P = \sigma_0^2 C_{l_1}^{-1}$$
$$C_{\delta(-)} = N_1^{-1}$$

Step 2:

Model: $\mathbf{l_2} = \mathbf{f_2}(\mathbf{x})$

Linearized model: $\underset{n_2,u}{A_2}\ \underset{u,1}{\delta} + \underset{n_2,1}{w_2} = \underset{n_2,1}{w_2}$

P.O.E.: x^0 (same as 1st model)

Update solution with n_2 observations, $(n_2 \geq 1)$

$$\delta(+) = \delta(-) - K\left(A_2\delta_{(-)} + w_2\right)$$

Note: $\delta(-)$ is the solution from step 1 (i.e. from the 1st group of observations)
Final Parameter estimates

$$\hat{x} = x^0 + \delta(+)$$

Updated covariance matrix

$$C_{\delta(+)} = C_{\hat{x}} = C_{\delta(-)} - KA_2C_{\delta(-)}$$

<u>Step 3:</u>

Addition of a 3rd set of observations

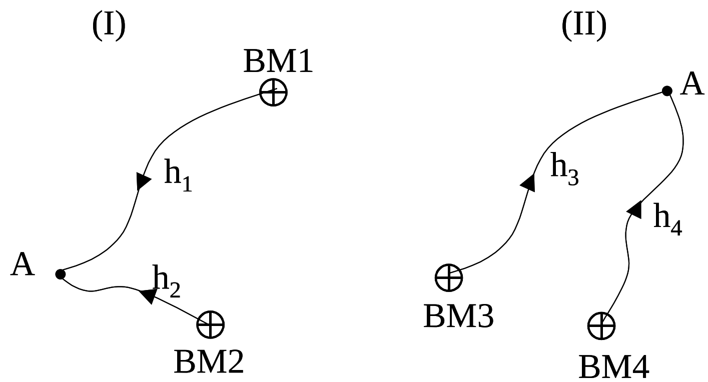
$$\mathbf{l_3} = \mathbf{f_3}(\mathbf{x})$$

$\delta(+)$ and $C_{\delta(+)}$ from step (2) become $\delta(-)$ and $C_{\delta(-)}$ respectively, for step (3).

Note: if n_2 >> u, summation of normals is the preferred method

Solved Example: ENGO 316 Final Exam - Question 4

The opposite figure shows a layout of two leveling networks (I and II) established for the determination of the height of point (A) from four different BM. You are given the following information:



For Leveling Network (I):

$$L_I = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3.2 \\ 0.9 \end{bmatrix} m \quad and\ the\ \ elevation\ of\ \begin{bmatrix} BM1 \\ BM2 \end{bmatrix} = \begin{bmatrix} 7.0 \\ 9.0 \end{bmatrix} m$$

$$C_{L(I)} = I \quad cm^2$$

For Leveling Network (II):

$$L_{II} = \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} 4.1 \\ 1.8 \end{bmatrix} m \quad and\ the\ \ elevation\ of\ \begin{bmatrix} BM3 \\ BM4 \end{bmatrix} = \begin{bmatrix} 6.0 \\ 8.0 \end{bmatrix} m$$

$$C_{L(II)} = I \quad cm^2$$

Calculate the elevation of point A (call it H) for the following cases:

a) Using all observations from Leveling Network I (i.e. L_I) and Leveling Network II (i.e. L_{II}) at once using the method of parametric least squares
b) Summation of normals using all observations from Leveling Network I (i.e. L_I) as first solution followed by using all observations from Leveling Network II (i.e. L_{II})
c) Using all observations from Leveling Network I (i.e. L_I) followed by sequential solution using all observations from Leveling Network II (i.e. L_{II})
d) Calculate the variance of the final solution [i.e. C_δ for questions (a) and (b) and $C_{\delta(+)}$ for question (c)] obtained in all the above cases
e) Are the results in (a), (b), (c) and (d) all the same? Why or why not?

<u>a) All observation – Parametric</u>

$L = [3.2 \quad 0.9 \quad 4.1 \quad 1.8]^T$ m $\qquad$ C = I $\qquad$ n = 4 $\qquad$ u = 1

Constants = BM elevations = [7 9 6 8]m

Functional Model: $\hat{L} = F(\hat{X}, C)$ $\qquad$ Linearized Model $\rightarrow$ $A_{4,1}\ \delta_{1,1} + W_{4,1} = V_{4,1}$

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w = F\left(\overset{\circ}{x}\right) - L \;\; m$$

Functional Model, $h_1 = H - BM_1$

$h_2 = H - BM_2$

$h_3 = H - BM_3$

$h_4 = H - BM_4$

assume $\overset{\circ}{x} = 0$ (i.e. H = 0)

$$\therefore \qquad W = -\begin{bmatrix} BM_1 + h_1 \\ BM_2 + h_2 \\ BM_3 + h_3 \\ BM_4 + h_4 \end{bmatrix} = -\begin{bmatrix} 10.2 \\ 9.9 \\ 10.1 \\ 9.8 \end{bmatrix} m$$

$\therefore \qquad \delta = -\left(A^T PA\right)^{-1} A^T \; PW \; = +\dfrac{\Sigma W_i}{4} \quad = 10$ m $\quad \rightarrow\rightarrow\rightarrow\rightarrow \quad x = \overset{\circ}{x}$

$+ \delta = 10$ m

b) <u>Summation of Normals</u>

δ = - $(N_1 + N_2)^{-1}$ $(U_1 + U_2)$ = $(2 + 2)^{-1}$ (20.1 + 19.9) = 10

where

$$N_1 = A_1^T \; A_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \; = N_2$$

$$U_1 = A_1^T \; W_1 = - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10.2 \\ 9.9 \end{bmatrix} = -20.1$$

$$U_2 = A_2^T \; W_2 = - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10.1 \\ 9.8 \end{bmatrix} = -19.9$$

$$\hat{x} = \overset{\circ}{x} + \delta = 10$$

c) Sequential

$$\delta(-) = N_1^{-1}u_1 = \frac{1}{2}[10.2+9.9] = 10.05$$

$$K = N_1^{-1}A_2^T\left(C_{L_2} + A_2 N_1^{-1} A_2^T\right)^{-1}$$

$$= \frac{1}{2}[1 \quad 1]\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\frac{1}{2}[1 \quad 1]\right)^{-1}$$

$$= \frac{1}{2}[1 \quad 1]\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)^{-1}$$

$$= \frac{1}{2}[1 \quad 1]\begin{pmatrix}1.5 & 0.5\\-0.5 & 1.5\end{pmatrix}^{-1}$$

$$= \frac{1}{2}[1 \quad 1]\frac{1}{2}\begin{pmatrix}1.5 & -0.5\\-0.5 & 1.5\end{pmatrix}$$

$$= \frac{1}{4}[1 \quad 1]$$

$$\therefore\ \delta(+) = \delta(-) - K\left(A_2\,\delta(-) + W_2\right)$$

$$=10.05 - \frac{1}{4}[1 \quad 1]\left(\begin{bmatrix}1\\1\end{bmatrix}10.05 + \begin{bmatrix}-10.1\\-9.8\end{bmatrix}\right)$$

$$=10.05 - \frac{1}{4}[1 \quad 1]\begin{pmatrix}10.05 & -10.1\\10.05 & -9.8\end{pmatrix}$$

$$=10.05 - \frac{1}{4}[1 \quad 1]\begin{pmatrix}-0.05\\0.25\end{pmatrix}$$

$$=10.05 - \frac{1}{4}[0.2] = 10$$

d)

for a) $C_\delta = N^{-1} = ¼$

for b) $C_\delta = (N_1 + N_2)^{-1} = ¼$

for c) $C_{\delta(+)} = C_{\delta(-)} - KA_2\, C_{\delta(-)}$

$$= \frac{1}{2} - \frac{1}{4}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^{1/2}$$

$$= \frac{1}{2} - \frac{1}{4}\cdot 2 \cdot \frac{1}{2} = \frac{1}{4}$$

e) The final results should all be the same since all methods are using the total number of information

8.4. Summation of Normals and Sequential LS for the implicit Models:

The combination of models discussed so far have mainly derived for the parametric model

The derivation of the summation of normals and sequential LS for the implicit models follow the same scheme as the parametric case.

PARAMETRIC	IMPLICIT
$\delta = -(N_1 + N_2)^{-1}(u_1 + u_2)$ $N_i = A_i^T P_i A_i$ $U_i = A_i^T P_i w_i$ $C_\delta = C_{\hat{x}} = (N_i + N_2)^{-1}$ N_i as before	$\delta = -(N_1' + N_2')^{-1}(u_1' + u_2')$ $N_i' = A_i^T (B_i P_i^{-1} B_i^T)^{-1} A_i$ $U_i' = A_i^T \underbrace{(B_i P_i^{-1} B_i^T)^{-1}}_{M_i} w_i$ $P_i = \sigma_0^2 C_{l_i}^{-1}$ & $C_{l_i} = \sigma_0^2 P_i^{-1}$ $C_\delta = C_{\hat{x}} = (N_i' + N_2')^{-1}$
$\delta(+) = \delta(-) - K(A_2\delta(-) + w_2)$ $\delta(-) = -\underbrace{(A_1^T P_1 A_1)^{-1}}_{N_1}\underbrace{(A_1^T P_1 w_1)}_{u_1}$ $K = N_1^{-1} A_2^T (C_{l_2} + A_2 N_1^{-1} A_2^T)^{-1}$	Change: P_1 by $(B_i P_1^{-1} B_1^T)^{-1}$ or $(B_i C_{l_1} B_1^T)^{-1}$ and $C_{l_2} \rightarrow B_2 C_{l_2} B_2^T$

8.5. Parameter Observations

- Parameter observation is a method which can be used in cases where a priori information about the parameters is available.
- For example, station coordinates (or elevations) and their covariance matrix $(\hat{x}$ and $C_{\hat{x}})$ and may be available from a previous adjustment.
- In this case $\hat{x}$ can be considered as direct observations (with $C_{\hat{x}}$) along with some observations vector to estimate better estimate of $\hat{x}$.

Functional Model:

$$\hat{x}^{obs}_{u,1} = \hat{x}_{u,1}$$

Linearized Functional Model:

$$x^{obs} = v_x = x^0 + \delta$$

$$v_x = \delta + \left(x^0 - x^{obs}\right)$$

$$v_{x_{u,1}} = \delta_{u,1} + w_{x_{u,1}} \qquad \left(v = A\delta + w\right)$$

with the stochastic model $C_{x_{u,u}} = P^{-1}_{x_{u,u}}$

- Note: a variable (observation or parameter) that has infinite variance, $\sigma^2 \rightarrow \infty$, has a corresponding weight of $P = 1/\sigma^2 = 0$. In this case, the variable becomes an unknown parameters.
- A variable with zero variance $\sigma^2 = 0$, has infinite weight, $P \rightarrow \infty$, and therefore is regarded as a constant.
- In between these two extreme cases there are an infinite number of possibilities for weighting parameters.

Functional Models (Linearized)

$$A_{n,u}\delta_{u,1} + w_{n,1} = v_{n,1}$$

$$I_{u,u}\delta_{u,1} + w_{x_{n,1}} = v_{u,1}$$

with the stochastic models:

$$C_l = P^{-1} \qquad \left(P = C_l^{-1}\right)$$

$$C_x = P_x^{-1} \qquad \left(P_x = C_x^{-1}\right)$$

Variation function:

$$\varphi = v^T P v + v_x^T\, P_x\, v_x \qquad = \min$$

$$= \left(\delta^T A^T + w^T\right) P\left(A\delta + w\right) + \left(\delta^T + w_x^T\right) P_x\left(\delta + w_x\right) \qquad = \min$$

$$\begin{aligned} &= \delta^T A^T P\, A\,\delta + \delta^T A^T P\, w + w^T P\, A\,\delta + w^T P\, w \\ &+ \delta^T P_x\,\delta + \delta^T P_x\, w_x + w_x^T\, P_x\,\delta + w_x^T P_x\, w_x \qquad = \min \end{aligned}$$

$$\begin{aligned} &= \delta^T A^T P A\,\delta + 2\,\delta^T A^T P\, w + w^T P\, w \\ &+ \delta^T P_x\,\delta + 2\,\delta^T P_x\, w_x + w_x^T P_w\, w_w \qquad = \min \end{aligned}$$

Minimize ϕ

$$\frac{\partial\varphi}{\partial\delta} = 2\delta^T A^T P A + 2w^T P A + 2\delta^T P_x + 2w_x^T P_x = 0$$

Transpose and divide by 2

$$A^T P A\delta + P_x\delta + A^T P w + P_x w_x = 0$$

$$\left(A^T P A + P_x\right)\delta + \left(A^T P w + P_x w_x\right) = 0$$

$$\boxed{\delta_{u,1} = -\left(A_{u,n}^T P_{n,n} A_{n,u} + P_{x_{u,u}}\right)^{-1}\left(A_{u,n}^T P_{n,n} w_{n,1} + P_{x_{u,u}}\, w_{x_{u,1}}\right)}$$

Variance-Covariance matrix of Estimated Correction C_δ:

Functional Model

$$\delta = -\left(A^T PA + P_x\right)^{-1}\left(A^T P\left(f(\ ) - l\right) + P_x\left(x^0 - x^{obs}\right)\right)$$

$$\therefore C_\delta = \left(\frac{\partial\delta}{\partial l}\right) C_l \left(\frac{\partial\delta}{\partial l}\right)^T + \left(\frac{\partial\delta}{\partial x^{obs}}\right) C_x \left(\frac{\partial\delta}{\partial x^{obs}}\right)^T$$

$$C_\delta = \left(\underbrace{A^T PA}_{N} + P_x\right)^{-1}$$

Analysis of the equation

$$\delta = -\left(A^T C_l^{-1} A + C_x^{-1}\right)\left(A^T C_l^{-1} w + C_l^{-1} w_x\right)$$

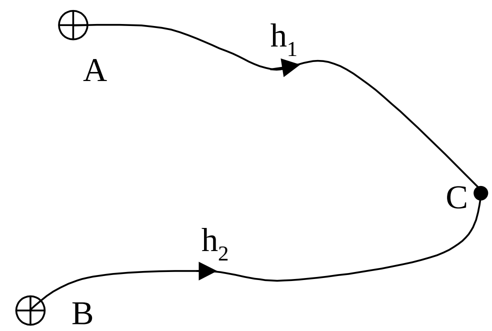
- If the observed parameters are very precise, i.e. $\mathbf{C_x}$ has small elements, therefore $\mathbf{C_x^{-1}}$ will have large elements, so as $\left(\mathbf{A^T C_l^{-1} A} + \mathbf{C_x^{-1}}\right)$ and therefore its inverse will be very small. Finally $\hat{\delta}$ will be smaller than if it is calculated without the additional parameter observations.

- If the parameters observations are not precise, $\mathbf{C_x}$ will have large elements, $\mathbf{C_x^{-1}}$ will have small elements. Thus the parameters observation will have little contribution to the solution vector.

Solved Problem On Parameter Observation (From Previous Final Exam):

$$L = \begin{bmatrix} h_1 \\ h_1 \end{bmatrix} = \begin{bmatrix} 1.74 \\ 2.76 \end{bmatrix} m \qquad \text{and} \qquad \begin{bmatrix} H_A \\ H_B \end{bmatrix} = \begin{bmatrix} 5.0 \\ 4.0 \end{bmatrix} m$$

$$C_L = 10^{-4} I_2 m^2$$



Required: Hc (i.e. X = [HC])

1) Using the parametric Model $L = f(x)$

$h_1 = H_C - H_A$ $\qquad \rightarrow H_C = H_A + \overset{0}{h_i} = 5 + 1.74 = 6.74$

$h_2 = H_C - H_B$

2) Linearized Model : $\therefore A\delta + W = V$, where $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$W = F\left(\overset{0}{X}\right) - L = \begin{bmatrix} \overset{0}{H}_C & -H_A & -h_1 \\ \overset{0}{H}_C & -H_B & -h_2 \end{bmatrix} \quad m$$

$$= \begin{bmatrix} 6.74 & -5 & -1.74 \\ 6.74 & -4 & 2.75 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}$$

3) Estimate the solution vector

$$\delta = -\left(A^T P A\right)^{-1} A^T P W \qquad\qquad P = C_L^{-1} = 10^4 I_2$$

$$A^T P A = \begin{bmatrix} 1 & 1 \end{bmatrix} 10^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \mathbf{x} 10^4$$

$$\left(A^T P A\right)^{-1} = \left(10^4 .2\right)^{-1} = 0.5 \mathbf{x} 10^{-4}$$

$$A^T PW = \begin{bmatrix}1 & 1\end{bmatrix} 10^4 \begin{bmatrix}0 \\ -0.02\end{bmatrix}$$

$$= -0.02\mathbf{x}10^4$$

$$\therefore \delta = -\left[A^T PA\right]^{-1} A^T PW$$

$$= -\left[0.5.10^{-4}\right]\left(-0.02.10^4\right) = 0.01m$$

$$\therefore \hat{X} = \overset{0}{X} + \delta = 6.74 + 0.01 = 6.75m$$

Assume further that an a priori value of the elevation of point C is 6.70m with variance 0.01m². Compute the elevation of point (C) considering this extra information.

This is a problem of parameter observation because we have new information about the parameter we are trying to estimate. Therefore we have to use the parametric model with parameter observation equations, that is

$$\delta' = -\left(A^T PA + P_X\right)^{-1}\left(A^T PW + P_X W_X\right)$$

where A^T, P, W has been defined before

$$P_X = C_X^{-1} = \frac{1}{0.01} = 100$$

$$W_X = \overset{0}{X} - X^{obs} = 6.74 - 6.70 = 0.04m$$

$$\left(A^T PA + P_X\right) = 2\,x10^4 + 100 = 20100 = 2.01\mathbf{x}10^4$$

$$\left(A^T PA + P_x\right)^{-1} = \frac{1}{2.01}10^{-4}$$

$$\left(A^T PW + P_X W_X\right) = \left(-0.02\,x 10^4\right) + 100(0.04) = -196$$

$$= -0.0196\mathbf{x}10^4$$

$$\delta' = -\left(A^T PA + P_X\right)^{-1}\left(A^T PW + P_X W_X\right)$$

$$= -\left(\frac{1}{2.01}\,\mathbf{x}\,10^{-4}\right)\left(-0.0196\,\mathbf{x}\,10^4\right) = 0.0098$$

$$\therefore \hat{X}' = \overset{0}{X} + \delta' = 6.74 + 0.0098 = 6.749m$$

Recall the Variance Covariance Matrix of the adjusted parameter:

$$C_{\hat{X}} = (N)^{-1} = \left(A^T PA\right)^{-1}$$

$$= 0.50 \quad \mathbf{x} \;\; 10^{-4} m^2$$

$$C_{\hat{X}'} = \left(A^T PA + P_X\right)^{-1}$$

$$= \frac{1}{2.01} \quad 10^{-4} m^2$$

$$= 0.49\,\mathbf{x}\;10^{-4} m^2$$

Having extra information about the elevation of point C increases the accuracy of the final estimate.