CHAPTER 4

NON-IDEALIZED
Trusses, Beams, Frames Using Complementary Virtual Work

4.1 Statically Determinate Trusses---Deflections

Static Determinacy of 2-D Trusses

A structure is *statically determinate* if the unknown reactions can be obtained by using only the equations of statics $(\sum F = 0\ ,\ \sum M = 0)$.

A *truss* is a structure that consists of only 2-force members.

A *truss is statically determinate* if all of the external reactions at the connections plus all of the internal forces in the members can be found by using only the equations of statics.

In addition, it must be determined
1. if the truss configuration can support the loads without collapsing and
2. that it is constrained so that the truss as a whole will not have either translational rigid body motion or rotational rigid body motion.

These are checked by observation.

If the truss is composed of all triangular regions, then it would be able to support the loads. Sometimes a truss that is not composed of all triangular regions may be able to support the loads. This must be determined by inspection.

For plane trusses, once items 1 and 2 above have been verified and if the equality below is true, then the truss is Statically Determinate.

$$\overbrace{\underbrace{2j}_{\text{Two Equations per Joint}}}^{\text{Number of Independent Static Equilibrium Equations}} = \overbrace{\underbrace{m}_{\text{One Unknown P per Member}} + \underbrace{r}_{\text{Number of Unknown External Reactions}}}^{\text{Number of Unknowns}} \qquad (4.1.1)$$

where

j = number of joints (pins, nodes),

m = number of members,

r = number of external reactions

All possibilities for Static Determinacy of Trusses are listed below.

$$\boxed{\begin{array}{lll} 2j = m + r & \Rightarrow & \text{Statically Determinate} \\ 2j < m + r & \Rightarrow & \text{Statically Indeterminate} \\ 2j > m + r & \Rightarrow & \text{Not Stable} \end{array}} \tag{4.1.2}$$

Trusses consist of only 2-force members. 2-Force members can only resist axial forces. They cannot bend. Thus, when a 2-force member is cut open, the only internal reaction is an axial force P [see Page 1.15].

The only stress produced in a 2-force member is [see page 1.15]

$$\sigma_x = \frac{P}{A}$$

where P is the internal axial force in the 2-force member.

$$\Rightarrow \qquad \varepsilon_x = \frac{P}{AE}$$

Now imagine a virtual change δP in an internal member.
<u>What could cause this</u>?

1. It could be caused by applying external virtual forces δB_i to the truss. These would cause some δP_n in each individual 2-force member.
2. We can imagine a δP to occur in a single interior member, or multiple interior members, with some mysterious origin; i.e., it may occur even though there is no external virtual force δB_i applied to the truss.

For a single 2-Force member,

$$\Rightarrow \qquad \delta\sigma_x = \frac{\delta P}{A}$$

Equation 3.5.3 states

$$\delta U^* = \iiint_V \begin{pmatrix} \varepsilon_x\,\delta\sigma_x + \varepsilon_y\,\delta\sigma_y + \varepsilon_z\,\delta\sigma_z \\ \\ + \gamma_{xy}\,\delta\tau_{xy} + \gamma_{xz}\,\delta\tau_{xz} + \gamma_{yz}\,\delta\tau_{yz} \end{pmatrix} dV \qquad (3.5.3)$$

Thus, for a single 2-force member

$$\delta U^* = \iiint_V \left(\frac{P}{AE}\,\frac{\delta P}{A} \right) dV$$

The notation below is added to keep the concepts straight.

$$\delta U^*\Big|_{\text{2-Force Member}} = \iiint_V \left(\frac{P\Big|_{\substack{\text{In Member Due}\\ \text{Actual Loads}}}}{AE}\,\frac{\delta P}{A} \right) dV$$

4.4

For a constant cross-section member with loads applied only at the ends, P and A would be constants.

$$\Rightarrow \qquad \delta U^*\Big|_{\text{2-Force Member}} = \left(\frac{P\Big|_{\substack{\text{In Member Due}\\ \text{Actual Loads}}}}{AE} \frac{\delta P}{A} \right) AL$$

$$\Rightarrow \qquad \boxed{\delta U^*\Big|_{\text{2-Force Member}} = \frac{\left(P\Big|_{\substack{\text{In Member Due}\\ \text{Actual Loads}}}\right) L}{AE} \delta P} \qquad (4.1.3)$$

[2-Force Member, Constant C-S, Loads at Ends]

For a truss, the total δU^* stored in the entire truss is obtained by summing the δU^*'s for each member of the truss. Thus,

$$\boxed{\delta U^*\Big|_{\text{Truss}} = \sum_i \frac{\left(P\Big|_{\substack{\text{In Member Due}\\ \text{Actual Loads}}}\right)_i L_i}{A_i E_i} (\delta P)_i} \qquad (4.1.4)$$

The Principle of Complementary Virtual Work (Equation 3.5.6) for trusses becomes

$$\boxed{\sum_i \underbrace{\left(b_i\Big|_{\substack{\text{At Location i on Original Structure}\\ \text{Caused by Actual Loads}\\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}}\right]} = \sum_i \frac{\left(P\Big|_{\substack{\text{In Member Due}\\ \text{Actual Loads}}}\right)_i L_i}{A_i E_i} (\delta P)_i} \qquad (4.1.5)$$

Example 4.1.1 **[Statically Determinate Truss—Displacements, Rotations]**

For the statically determinate truss shown in the figure below, determine, using the Principle of Complementary Virtual Work,

(a) the vertical component of displacement v_3 at node 3, and

(b) the rotation θ_{3-4} of member 3-4.

Material: Aluminum	Cross-Sections: Hollow Circular
$E = 10 \times 10^6$ psi	$d_{outer} = 1\frac{1}{4}$"
	$t = \frac{1}{8}$"

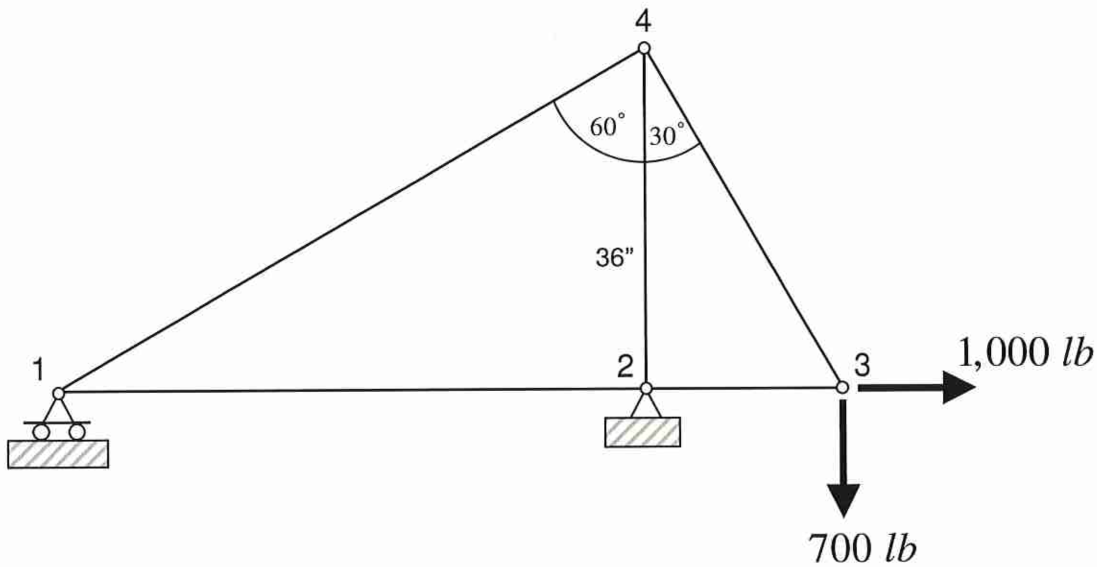


Figure 4.1.1

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals------Positive if Tension

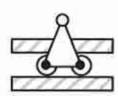
2-Force Members

Why is every member in Figure 4.1.1 a two-force member?

Roller Support Symbols

Typical symbols for a roller supports are

Their actual meaning is



Usually the details of the lower figure are not drawn.

First, let's check to see if the truss is statically determinate.

We must check [see Page 4.1]

1. if the truss configuration can support the loads without collapsing and
2. that it is constrained so that the truss as a whole will not have either translational rigid body motion or rotational rigid body motion.

We observe that

1. It can support the loads without collapsing since it is composed of all triangular regions.
2. It cannot have any rigid body motion with the given constraints.

Now we can apply Equation 4.1.1.

$$2j = m + r \tag{4.1.1}$$

$$\Rightarrow \qquad 2(4 \text{ joints}) = (5 \text{ members}) + (3 \text{ external reactions})$$

$$\Rightarrow \qquad 8 = 8$$

Checks

It is statically determinate.

We notice that, from static equilibrium, the

Maximum Number of Independent Equations = 2 (4 joints) = 8.

(a) To determine the vertical component of displacement v_3 at node 3 using the Principle of Complementary Virtual Work,

1. the actual loads of 1,000 lb and 700 lb will be removed, and
2. a vertical virtual force δQ will be applied at node 3 as shown in Figure 4.1.2 below.

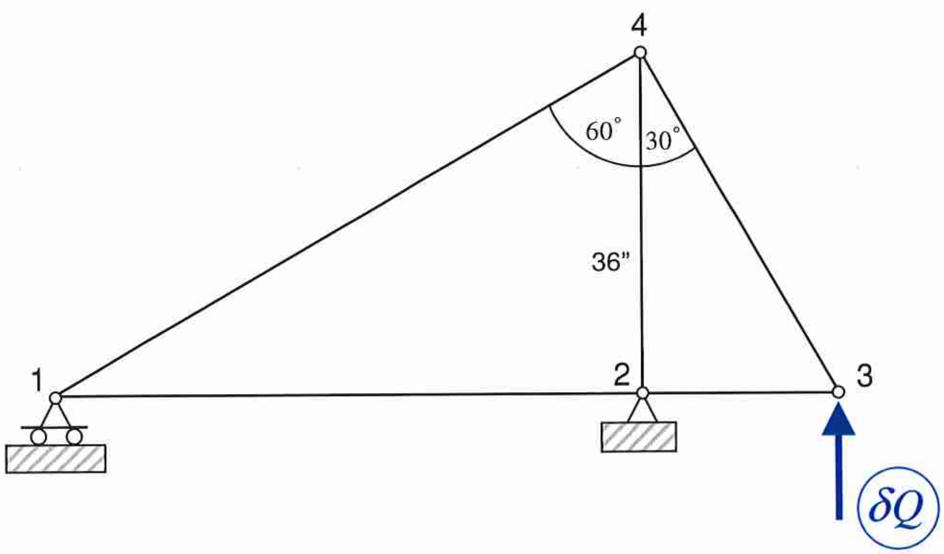


Figure 4.1.2

Circled Virtuals on Figures

* The circled virtuals on figures represent *applied* virtual loads, as opposed to virtual *reactions.*

(b) To determine the rotation θ_{3-4} of member 3-4 using the Principle of Complementary Virtual Work, the actual loads of 1,000 lb and 700 lb will be removed and a virtual couple will be applied to member 3-4.

However, since member 3-4 is a two-force member, the virtual couple cannot be applied directly on the member because it would then bend and no longer be a two-force member. The virtual couple must be applied as two equal and opposite virtual forces applied at the two end points of member 3-4 as shown in Figure 4.1.3 below.

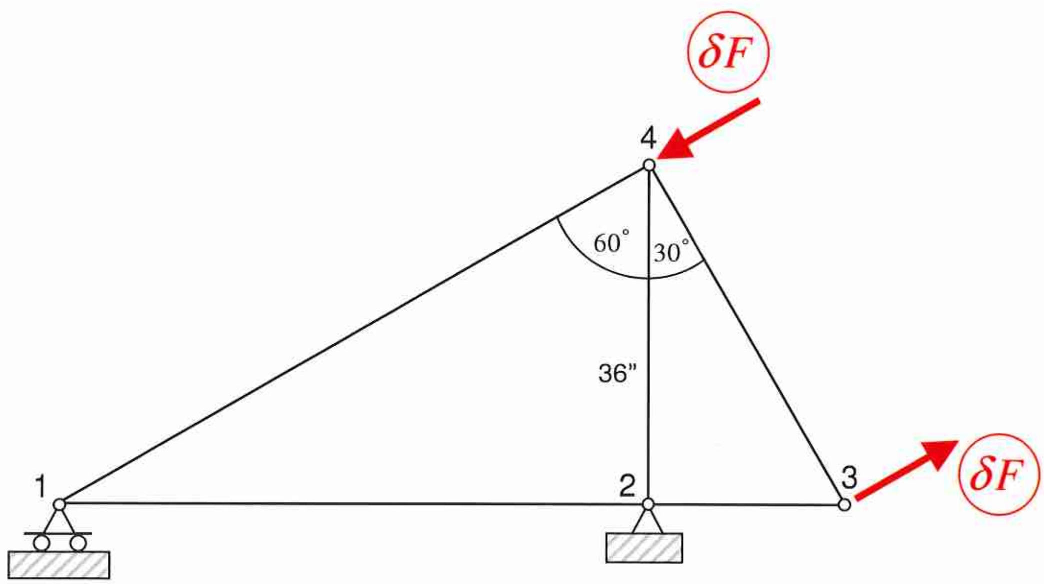


Figure 4.1.3

The magnitude of the virtual couple is

$$\delta M = \left(\frac{36''}{\cos 30^{\circ}} \right) \delta F$$

or

$$\boxed{\delta M = (41.57)\, \delta F}$$

Figures 4.1.2 and 4.1.3 could be combined to have a single figure with all virtual loads as below.

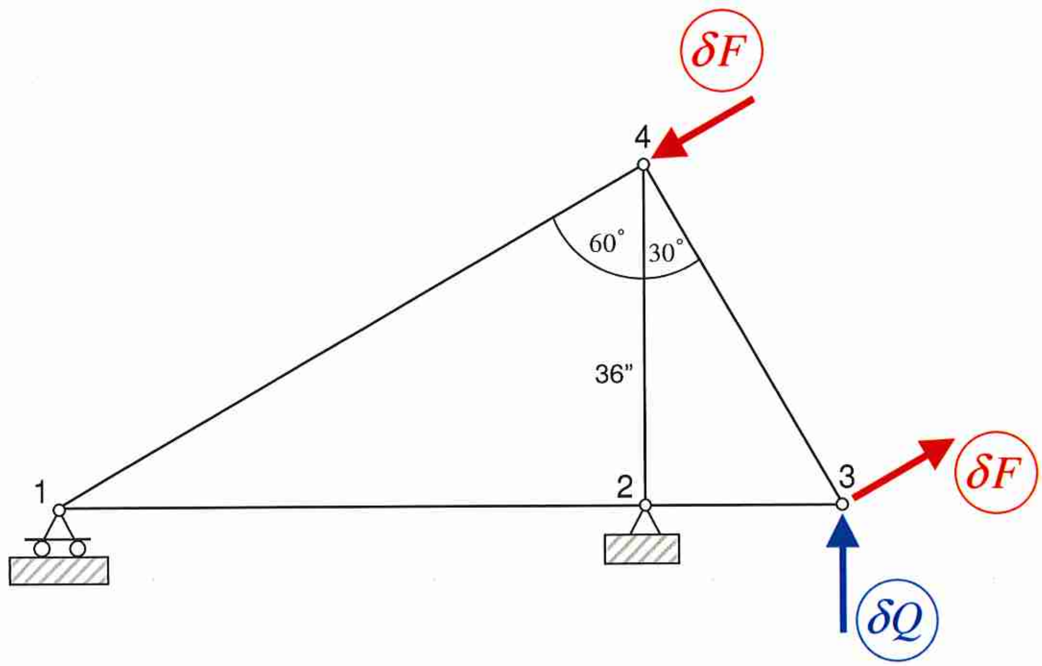


Figure 4.1.4

4.10

FBDs + STATICS

The FBD of the truss with the original loads of 1,000 lb and 700 lb [See Figure 4.1.1] is shown below in Figure 4.1.5.

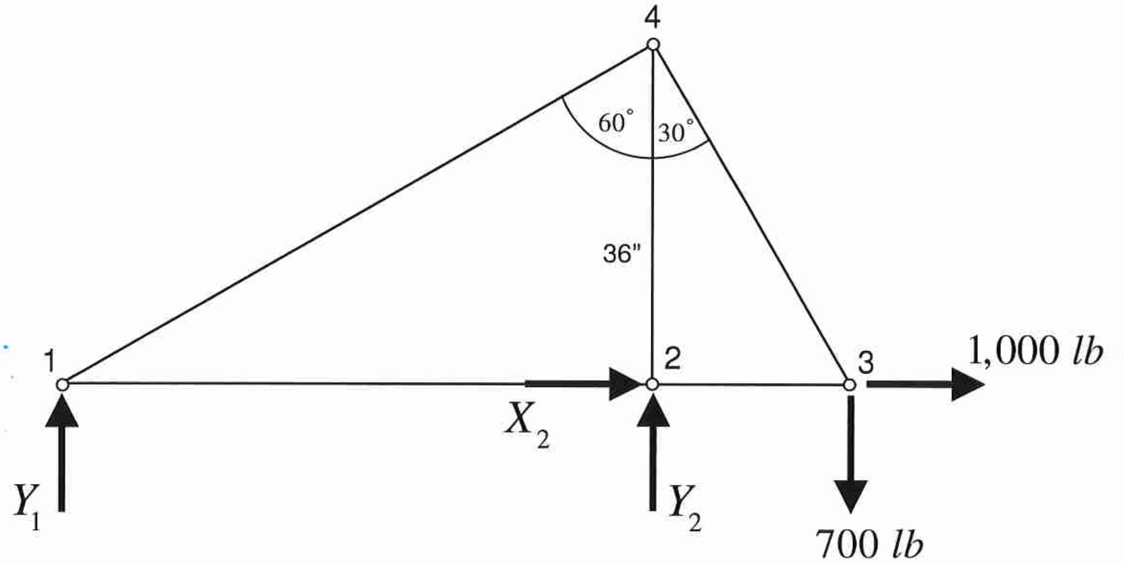


Figure 4.1.5

The FBD of the truss with only the virtual loads [See Figure 4.1.4] is shown below in Figure 4.1.6.

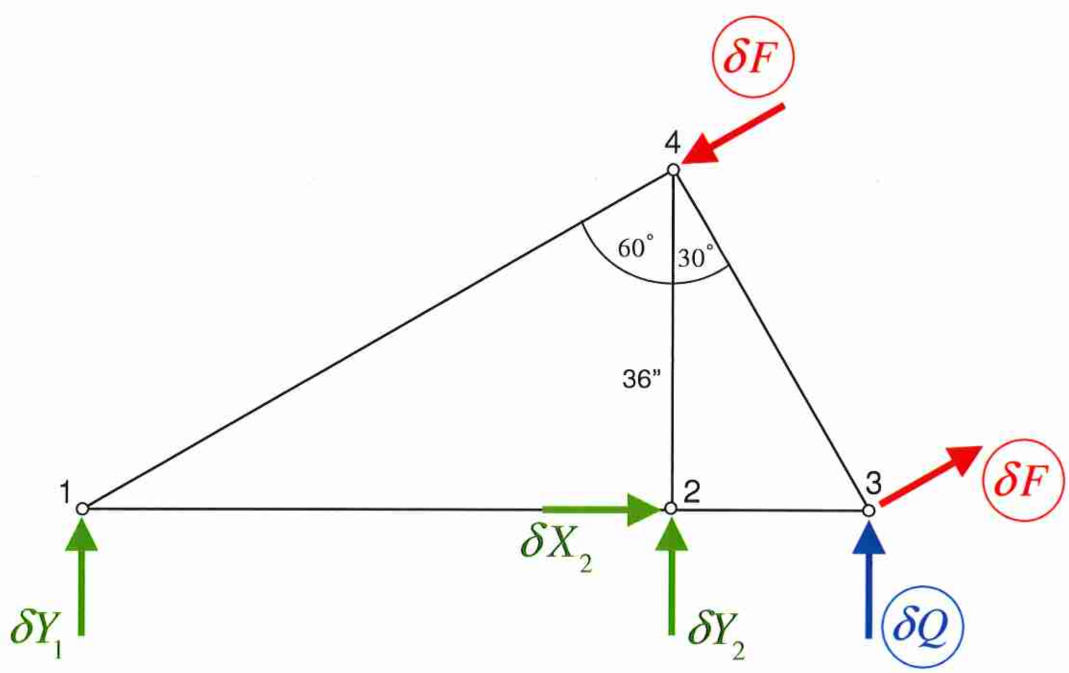


Figure 4.1.6

Uncircled Virtuals on Figures

* The uncircled virtuals on figures represent virtual *reactions* to the *applied* virtual loads.

The *internal axial force* in each member will have to be determined

1. for the actual loads of 1,000 lb and 700 lb [see Figure 4.1.5], and
2. for the virtual loads [see Figure 4.1.6].

These can be determined by using the equilibrium equations of the entire structure + using the method of joints for both Figures 4.1.5 and 4.1.6.

Short Cut Method

[This method was suggested by Edmund Charlton, a student in AE 418, Spring 2014]

Rather than doing equilibrium of the entire structure + the method of joints twice (once for the actual loads of 1,000 lb and 700 lb and again for the virtual loads), it could be done one single time by combining Figures 4.1.5 and 4.1.6 as below in Figure 4.1.7.

Technically, *Figure 4.1.7 is illegal* since the virtual loads are only applied when the actual loads are removed.

However, it will be useful temporarily, as a Short Cut Method, to reduce the work involved.

The results for the actual loads and those for the virtual loads will have to be separated when the Principle of Complementary Virtual Work is employed.

4.12

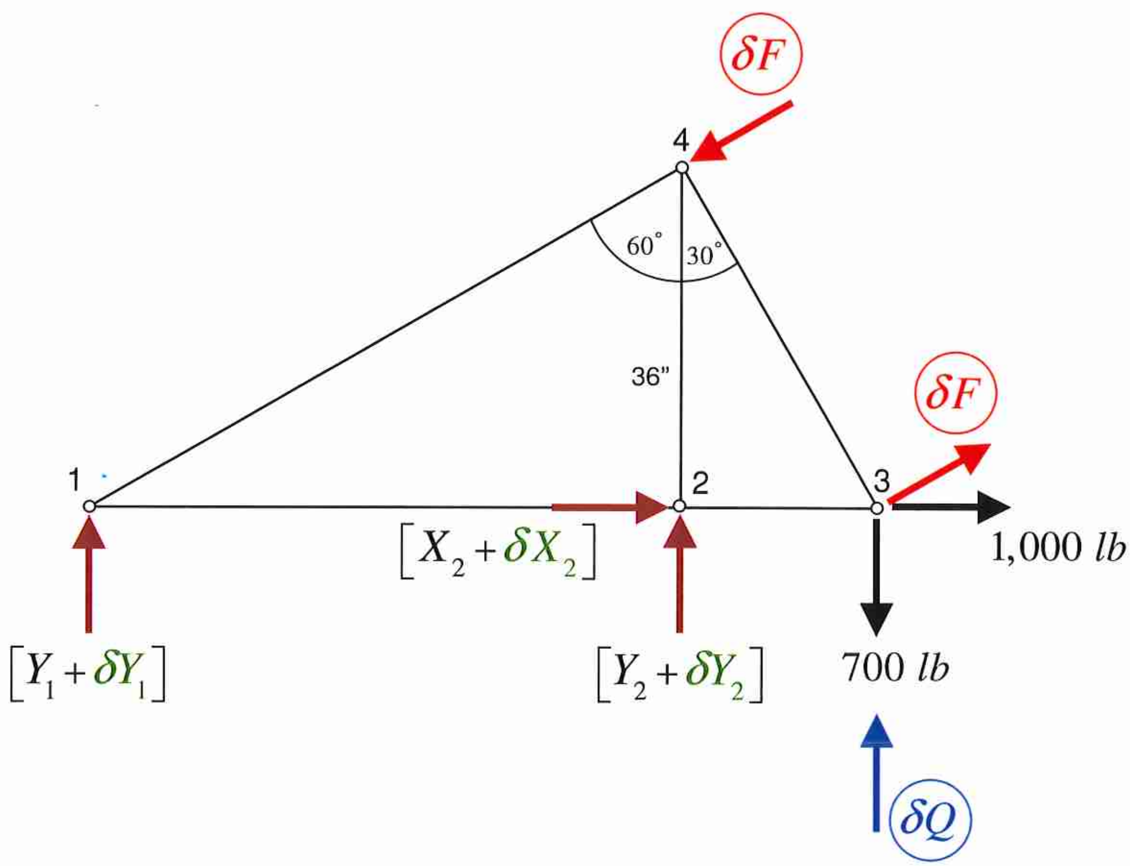


Figure 4.1.7

External Reactions
[Figure 4.1.7]

$$\sum F_x = 0$$

$\Rightarrow$ $$[X_2 + \delta X_2] - \delta F \cos 30° + \delta F \cos 30° + 1{,}000\ lb = 0$$

$\Rightarrow$ $$[X_2 + \delta X_2] = -1{,}000$$

$\Rightarrow$ $$\boxed{X_2 = -1{,}000} \qquad \boxed{\delta X_2 = 0}$$

Indep.
Eq.
1

$$\circlearrowleft + \quad \sum M_2 = 0$$

$$\Rightarrow -\left[Y_1 + \delta Y_1\right]\overbrace{\left(36''\tan 60^\circ\right)}^{62.35''} + \delta Q\overbrace{\left(36''\tan 30^\circ\right)}^{20.78''} + \overbrace{\left[(41.57)\delta F\right]}^{\delta M} - (700\ lb)(20.78'') = 0$$

$$\Rightarrow \quad \left[Y_1 + \delta Y_1\right] = (0.3333)\delta Q + (0.6667)\delta F - 233.3$$

$$\Rightarrow \quad \boxed{Y_1 = -233.3} \qquad \boxed{\delta Y_1 = (0.3333)\delta Q + (0.6667)\delta F}$$

Indep.
Eq.
2

$$\sum F_y = 0$$

$$\Rightarrow \overbrace{\begin{bmatrix} (0.3333)\delta Q \\ +(0.6667)\delta F - 233.3 \end{bmatrix}}^{\left[Y_1 + \delta Y_1\right]} + \left[Y_2 + \delta Y_2\right] + \delta Q - \delta F \sin 30^\circ + \delta F \sin 30^\circ - 700\ lb = 0$$

$$\Rightarrow \quad \left[Y_2 + \delta Y_2\right] = -(1.333)\delta Q - (0.6667)\delta F + 933.3$$

$$\Rightarrow \quad \boxed{Y_2 = 933.3} \qquad \boxed{\delta Y_2 = -(1.333)\delta Q - (0.6667)\delta F}$$

Indep.
Eq.
3

Notice

Numerical values for the external reactions were able to be found from the FBD of the structure as a whole.

Acceptable Equation Format in this Course

- The procedure presented on the previous page is the only equation format acceptable in this course.
- What does this mean?
  1. All equations, beyond the initial equation, must be immediately converted to decimals.
  2. All decimals shown on the page must be rounded to 4 significant digits (no more, no less).
  3. No fractions are to be carried through beyond the initial equation.
  4. Trig functions are not to be carried through beyond the initial equation.
  5. Symbols like π are not to be carried through beyond the initial equation.
- The error incurred by following the above procedure should be less than 1%.
- In practice, engineers may accept errors between 5% to 15%, depending on the application.
- Factors of safety are used to compensate for the % error.

> * According to Equation 4.1.5, in order to use the Principle of Complementary Virtual Work, the internal P and δP in each member must be found.
>
> * This will be accomplished by taking the necessary cuts through the members at each joint.

Internals
(Force in Each Member)

Using the method of joints,
Joint 1

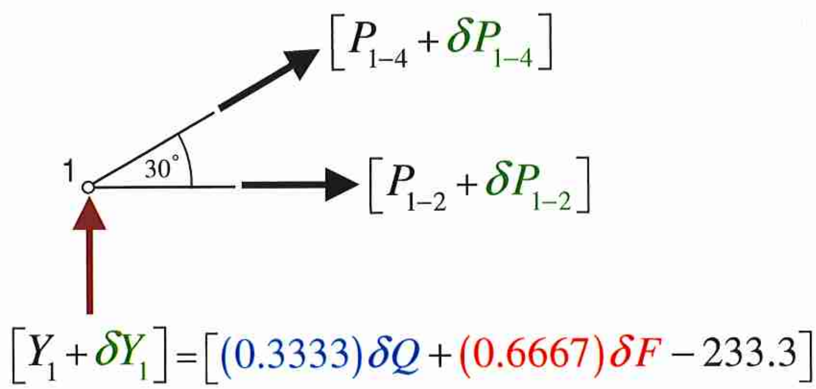


Figure 4.1.8

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[(0.3333)\delta Q+(0.6667)\delta F-233.3\right]+\left[P_{1-4}+\delta P_{1-4}\right]\sin 30^\circ = 0$$

$$\Rightarrow \quad \left[P_{1-4}+\delta P_{1-4}\right]=-(0.6666)\delta Q-(1.333)\delta F+466.6$$

$$\Rightarrow \quad \boxed{P_{1-4}=466.6} \qquad \boxed{\delta P_{1-4}=-(0.6666)\delta Q-(1.333)\delta F}$$

Indep.
Eq.
4

4.16

$$\sum F_x = 0$$

$$\Rightarrow \qquad \left[P_{1-2}+\delta P_{1-2}\right]+\overbrace{\left[-(0.6666)\delta Q-(1.333)\delta F+466.6\right]}^{\left[P_{1-4}+\delta P_{1-4}\right]}\cos 30^\circ = 0$$

$$\Rightarrow \qquad \left[P_{1-2}+\delta P_{1-2}\right]=(0.5773)\delta Q+(1.154)\delta F-404.1$$

$$\Rightarrow \qquad \boxed{P_{1-2}=-404.1} \qquad \boxed{\delta P_{1-2}=(0.5773)\delta Q+(1.154)\delta F}$$

Indep. Eq. 5

Joint 2

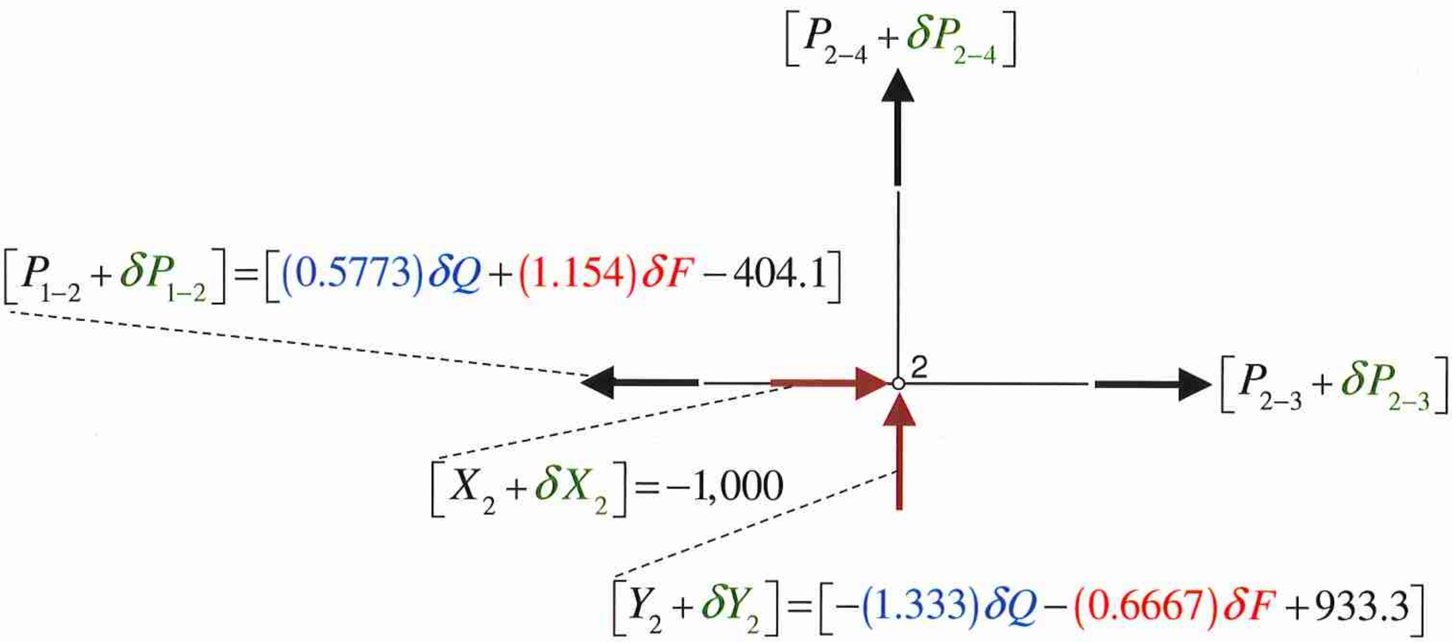


Figure 4.1.9

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[(0.5773)\delta Q + (1.154)\delta F - 404.1\right] + (-1{,}000) + \left[P_{2-3} + \delta P_{2-3}\right] = 0$$

$$\Rightarrow \quad \left[P_{2-3} + \delta P_{2-3}\right] = (0.5773)\delta Q + (1.154)\delta F + 595.9$$

$$\Rightarrow \quad \boxed{P_{2-3} = 595.9} \qquad \boxed{\delta P_{2-3} = (0.5773)\delta Q + (1.154)\delta F}$$

Indep. Eq. 6

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[P_{2-4} + \delta P_{2-4}\right] + \left[-(1.333)\delta Q - (0.6667)\delta F + 933.3\right] = 0$$

$$\Rightarrow \quad \left[P_{2-4} + \delta P_{2-4}\right] = \left[(1.333)\delta Q + (0.6667)\delta F - 933.3\right]$$

$$\Rightarrow \quad \boxed{P_{2-4} = -933.3} \qquad \boxed{\delta P_{2-4} = (1.333)\delta Q + (0.6667)\delta F}$$

Indep. Eq. 7

4.18

Joint 3

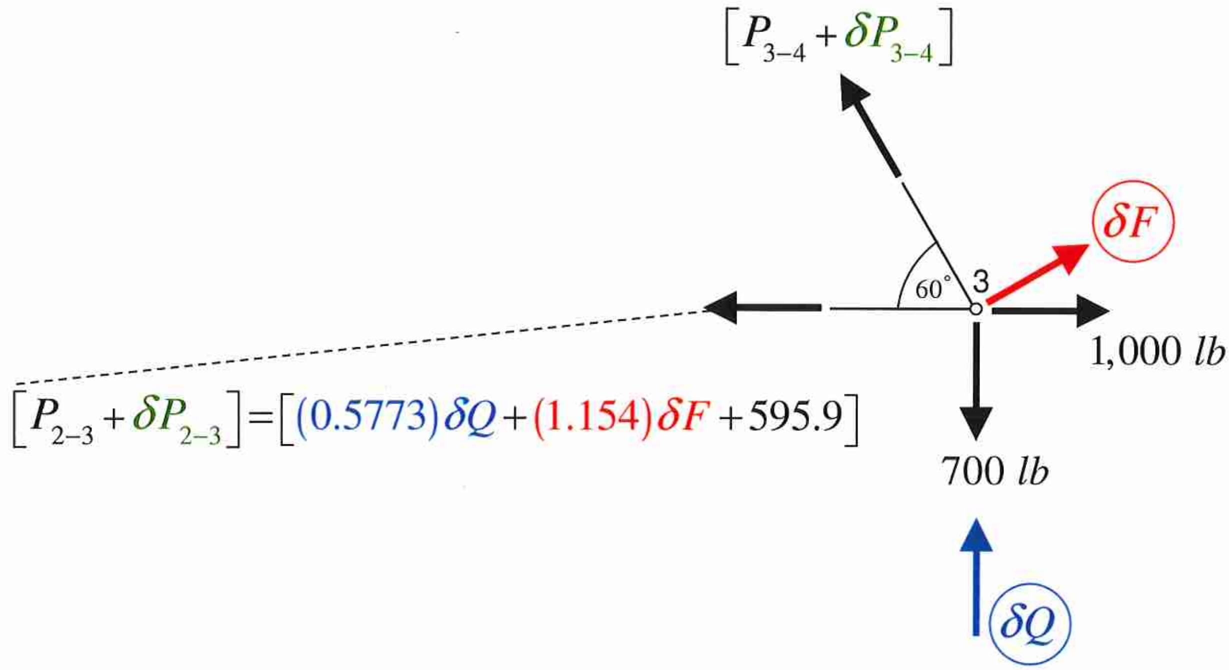


Figure 4.1.10

$$\sum F_y = 0$$

$\Rightarrow$ $$\left[P_{3-4} + \delta P_{3-4}\right]\sin 60^\circ + \delta Q + \delta F \sin 30^\circ - 700\ lb = 0$$

$\Rightarrow$ $$\left[P_{3-4} + \delta P_{3-4}\right] = -(1.155)\delta Q - (0.5774)\delta F + 808.3$$

$\Rightarrow$ $$\boxed{P_{3-4} = 808.3} \qquad \boxed{\delta P_{3-4} = -(1.155)\delta Q - (0.5774)\delta F}$$

Indep. Eq. 8

We now have our 8 independent static equations.

- Notice that we have totally used up all 8 of the static equilibrium equations.
- All of the internal forces, as well as all of the external reactions, have been found since this is a statically determinate truss.
- $\sum F_x = 0$ at Joint 3 will not give any new information.

  It is not independent of the 8 equations that we have obtained.

Check

Let's do $\sum F_x = 0$ for Joint 3.

$$\sum F_x = 0$$

$$\Rightarrow \quad -\begin{bmatrix} (0.5773)\delta Q \\ +(1.154)\delta F \\ +595.9 \end{bmatrix} - \overbrace{\begin{bmatrix} -(1.155)\delta Q \\ -(0.5774)\delta F \\ +808.8 \end{bmatrix}}^{\left[P_{3-4}+\delta P_{3-4}\right]} \cos 60^\circ + \delta F \cos 30^\circ + 1{,}000\ lb = 0$$

$$\Rightarrow \quad (0.0002)\delta Q + (0.0007)\delta F - 0.05 = 0$$

$$\Rightarrow \quad 0 = 0$$

(The LHS is not exactly = 0 due to roundoff)

INDEPENDENT EQUATIONS

Maximum Number of Independent Equations from Static Equlibrium for Trusses

* The maximum number of *independent* equations that can be obtained from static equilibrium for trusses is 2j. In our case, 8 independent static equations.

* Thus if we would have written the 2 equilibrium equations at every joint, that would have been the maximum number of independent equations that could have been obtained from static equilibrium.

* In that situation, the 3 equilibrium equations for the truss as a whole would not have been independent of the other equations, & hence would have given no new information.

* In our approach, we wrote the 3 equilibrium equations for the truss as a whole first.

* That means that we have used up 3 of the equilibrium equations that we might have obtained from the joints.

* Thus, in our approach, we can use all of the joint equilibrium equations except 3.

* The 3 joint equilibrium equations that we have skipped are the two at Joint 4 and $\sum F_x = 0$ at Joint 3.

Our results for the Reals are shown in Figure 4.1.11 below.

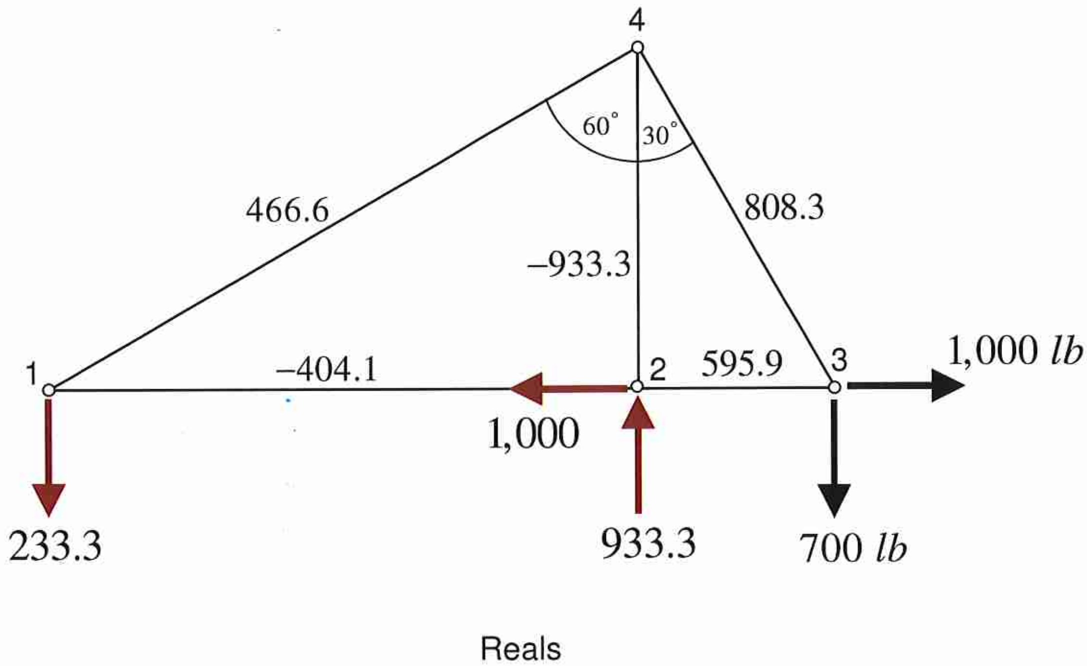
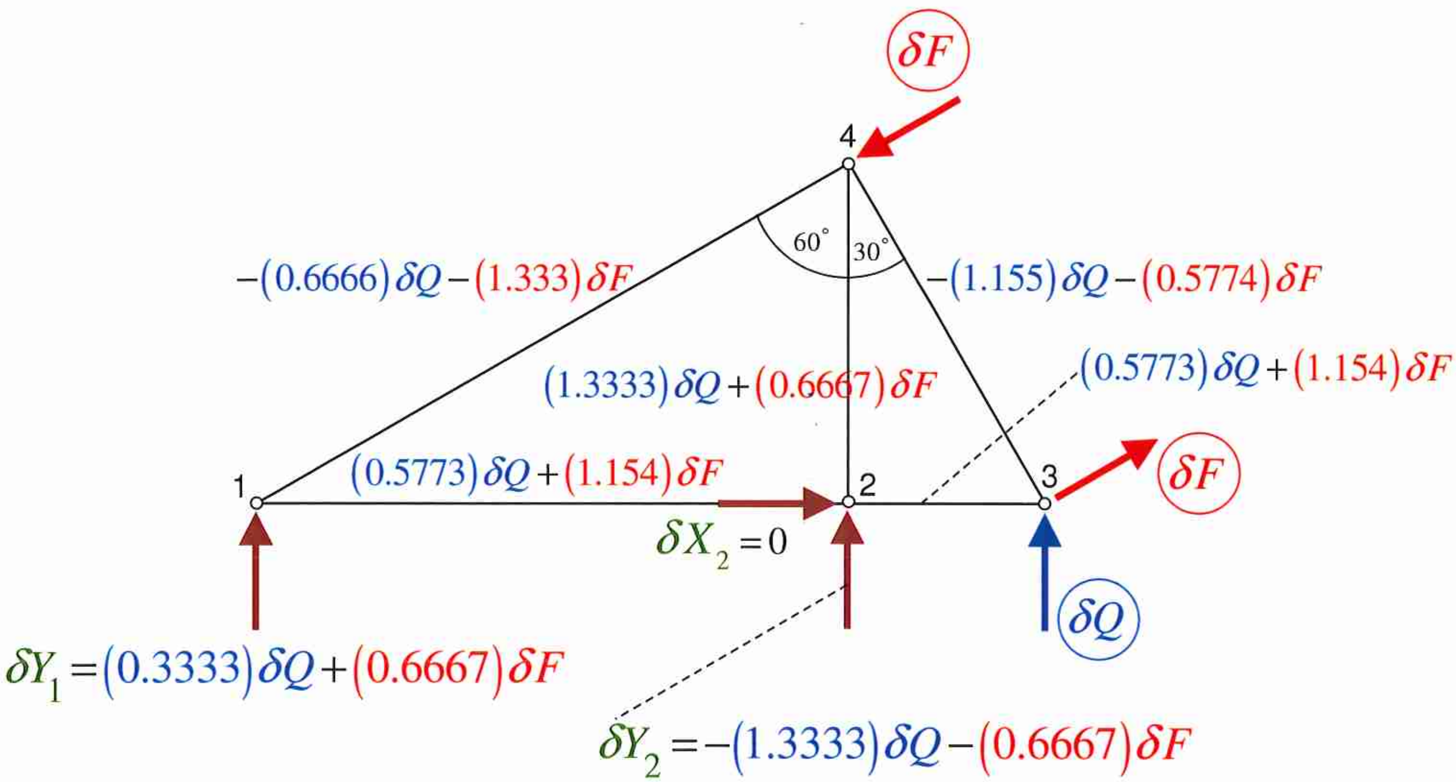


Figure 4.1.11

As mentioned previously, all of the internal forces, as well as all of the external reactions, have been found using only the static equilibrium equations since this is a statically determinate truss.

4.22

Our results for the Virtuals are shown in Figure 4.1.12 below.



Virtuals

Figure 4.1.12

* We do not have values for the virtuals and we will not obtain values for them.

* Remember that virtuals have an infinitesimal magnitude.

* The Principle of Complementary Virtual Work will be employed next.

* That will yield the Real Displacement and Real Rotation values.

The vertical component of displacement v_3 at node 3 and the rotation θ_{3-4} of member 3-4 will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.1.5) for the entire truss system states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \cdot \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \frac{\left(P \Big|_{\substack{\text{In Member Due} \\ \text{Actual Loads}}} \right)_i L_i}{A_i E_i} (\delta P)_i \qquad (4.1.5)$$

The term

$$\underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]}$$

* Refers to all external virtual forces or couples in Figure 4.1.12.

* This means
  1. all virtual applied forces or couples $[\delta Q,\ \delta M]$, and
  2. all external virtual reaction forces or couples $[\delta Y_1,\ \delta X_2,\ \delta Y_2]$.

<u>4.24</u>

$$\Rightarrow \left\{\begin{array}{l} v_3\,(\delta Q) \\ +\theta_{3-4}\overbrace{\left[(41.57)(\delta F)\right]}^{\delta M} \\ +\overbrace{v_1}^{0}\,(\delta Y_1) \\ +\overbrace{u_2}^{0}\,(\delta X_2)+\overbrace{v_2}^{0}\,(\delta Y_2) \end{array}\right\} = \left\{\begin{array}{l} \overbrace{\dfrac{(-404.1\ lb)(62.35")}{\underbrace{\left[\pi\underbrace{\left(\dfrac{1.25"}{2}\right)}_{0.625"}^2-\pi\left(0.625"-\dfrac{1}{8}"\right)^2\right]}_{0.4418\ in^2}(10x10^6\ psi)}\left[\begin{array}{l}(0.5773)\delta Q\\+(1.154)\delta F\end{array}\right]}^{\text{Member 1-2}} \\ \overbrace{+\dfrac{(466.6\ lb)\overbrace{\sqrt{(62.35")^2+(36")^2}}^{72"}}{(0.4418\ in^2)(10x10^6\ psi)}\left[-(0.6666)\delta Q-(1.333)\delta F\right]}^{\text{Member 1-4}} \\ \overbrace{+\dfrac{(595.9\ lb)(20.78")}{(0.4418\ in^2)(10x10^6\ psi)}\left[(0.5773)\delta Q+(1.154)\delta F\right]}^{\text{Member 2-3}} \\ \overbrace{+\dfrac{(-933.3\ lb)(36")}{(0.4418\ in^2)(10x10^6\ psi)}\left[(1.333)\delta Q+(0.6667)\delta F\right]}^{\text{Member 2-4}} \\ \overbrace{+\dfrac{(808.3\ lb)\overbrace{\sqrt{(20.78")^2+(36")^2}}^{41.57"}}{(0.4418\ in^2)(10x10^6\ psi)}\left[-(1.155)\delta Q-(0.5774)\delta F\right]}^{\text{Member 3-4}} \end{array}\right\}$$

<u>Sign Conventions</u>

* u, v, θ ---- Positive if in the Positive Global Axes Directions

* Work of Externals ---- Positive if Force or Couple is Helping the Motion

$$\Rightarrow \quad \left\{ v_3(\delta Q) + \left[(41.57)\theta_{3-4}\right](\delta F) \right\} = \left\{ \begin{array}{l} \overbrace{\left[-(0.003292)\delta Q - (0.006581)\delta F\right]}^{\text{Member 1-2}} \\ +\overbrace{\left[-(0.005069)\delta Q - (0.01014)\delta F\right]}^{\text{Member 1-4}} \\ +\overbrace{\left[(0.001618)\delta Q + (0.003234)\delta F\right]}^{\text{Member 2-3}} \\ +\overbrace{\left[-(0.01014)\delta Q - (0.005070)\delta F\right]}^{\text{Member 2-4}} \\ +\overbrace{\left[-(0.008763)\delta Q - (0.004381)\delta F\right]}^{\text{Member 3-4}} \end{array} \right\}$$

$$\Rightarrow \qquad v_3\,\delta Q + \left[(41.57)\theta_{3-4}\right]\delta F = -(0.02565)\delta Q - (0.02294)\delta F$$

$$\Rightarrow \qquad \boxed{\begin{array}{l} v_3 = -0.02565'' \\ \theta_{3-4} = -5.518\,x10^{-4}\,rad \doteq -0.03^\circ \end{array}}$$

* The negatives mean that they point in the negative global axes directions.

You should now have the ability to do Problems 4.1, 4.2

Homework: Do Problem 4.1

4.2 Statically Indeterminate Trusses---Forces and Deflections

Statically indeterminate trusses will have either additional members or additional external supports beyond those needed for static equilibrium. Additional members can reduce the amount of the load going to each member. The members can then be made thinner or of a weaker material.

Statically indeterminate trusses must still be checked to ensure

1. that the truss configuration can support the loads without collapsing and
2. that it is constrained so that the truss as a whole will not have either translational rigid body motion or rotational rigid body motion.

These types of trusses cannot be solved by using only the equations of static equilibrium. If there are n additional unknowns beyond those needed for the truss to be statically determinate, then n additional equations will be needed beyond those of static equilibrium.

The n additional equations will be obtained by applying n virtual loads & using the Principle of Complementary Virtual Work. Recall that in the previous example, the 2 virtual loads; namely, the virtual force δQ and the virtual couple $\left[(41.57)\delta F\right]$, led to 2 equations.

Example 4.2.1 **[Statically Indeterminate Truss—Forces in Members, Reactions]**

For the statically indeterminate truss shown in Figure 4.2.1,
(a) determine the number of redundancies,
(b) draw a FBD of the truss in Figure 4.2.1, showing both the reals and the virtuals for utilizing the short cut method as described previously,
(c) find the external reactions and the force in each member using the Principle of Complementary Virtual Work.

Material: Steel	Cross-Sections: Hollow Circular
$E = 30 \times 10^6$ psi	$d_{outer} = 1\frac{1}{8}$"
	$t = \frac{1}{16}$"

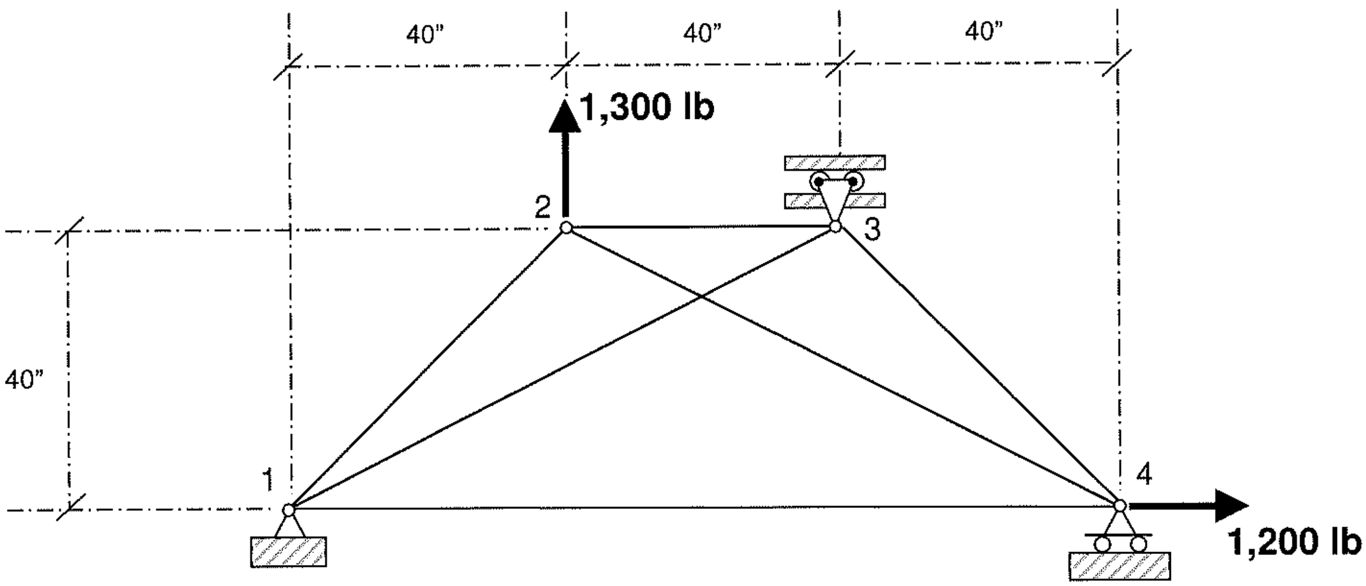


Figure 4.2.1

****

2-Force Members

Why is every member in Figure 4.2.1 a two-force member?

(a) determine the number of redundancies

We must check [see Page 4.1]

1. if the truss configuration can support the loads without collapsing and
2. that it is constrained so that the truss as a whole will not have either translational rigid body motion or rotational rigid body motion.

We observe that
1. It can support the loads without collapsing since it is composed of all triangular regions.
2. It cannot have any rigid body motion with the given constraints.

Now we can apply Equation 4.1.1.

$$2j = m + r \tag{4.1.1}$$

Equations 4.1.2 state

$$\begin{aligned} 2j = m + r \quad &\Rightarrow \quad \text{Statically Determinate} \\ 2j < m + r \quad &\Rightarrow \quad \text{Statically Indeterminate} \\ 2j > m + r \quad &\Rightarrow \quad \text{Not Stable} \end{aligned} \tag{4.1.2}$$

In our case,

$$2j < m + r \quad \Rightarrow \quad \text{Statically Indeterminate}$$

* Therefore, we have a Static Indeterminacy = 2.

* That means that we will be short by 2 equations after we have completed using the static equilibrium equations.

* The additional 2 equations will come from the Principle of Complementary Virtual Work.

* We will have to apply 2 virtual forces in order to get 2 equations from the Principle of Complementary Virtual Work.

We notice that, from static equilibrium, the

Maximum Number of Independent Equations = 2 (4 joints) = 8.

(b) draw a FBD of the truss in Figure 4.2.1, showing both the reals and the virtuals for utilizing the short cut method as described previously

2 Redundants

* The Principle of Complementary Virtual Work will be used to obtain the 2 additional equations needed beyond those of static equilibrium.

* This means that we will have to apply 2 virtual forces.

* Let us imagine applying 2 virtual forces in the same location and in the same direction as 2 of the external reactions.

* These applied virtual forces will cause
  (i) virtual internal reactions inside the truss, as well as
  (ii) virtual external reactions at the other connections.

* Of all the resulting virtuals, which will be treated as the applied virtuals & which will be treated as the reaction virtuals?

* The applied virtuals could be any 2 of the resulting virtuals; namely, any combination of the external virtuals and/or internal virtuals.

* There is no need to distinguish between them.

At some point, we must decide which of the real forces we will select as our 2 redundants.

Choice of Redundants

Two Methods for Choosing Redundants

Method 1

Best method when there are a large number of simultaneous equations produced. Trusses generally fit into this category.

* To choose the redundants

1. write all of the static equilibrium equations in terms of all of the variables

2. then determine which choice of redundants
   i. would make sense mathematically and
   ii. would make the equations easiest to solve.

* As a general rule, for trusses, the algebra will be reduced if P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work equation.

Method 2

Quite often this is the best method for simple beams and frames.

* To choose the redundants

1. visibly identify which reactions could be removed so that the structure would
   i. be statically determinate and
   ii. still support the loads without the entire structure having any rigid body motions.

2. Use the reactions that could be removed as the redundants.

To determine the reactions at all connections using the Principle of Complementary Virtual Work,

1. the actual loads of $1{,}300\ lb$ and $1{,}200\ lb$, as well as all externals, will be removed, and

2. two virtual loads will be applied at two of the supports in the same directions as two of the support reactions.

As mentioned above, this will cause virtual reactions everywhere else.

The virtual forces are to be applied with all externals removed.

However, for the Short Cut Method as described in the previous example, a FBD of the structure with both reals, and virtuals will be drawn [see Figure 4.2.2 below].

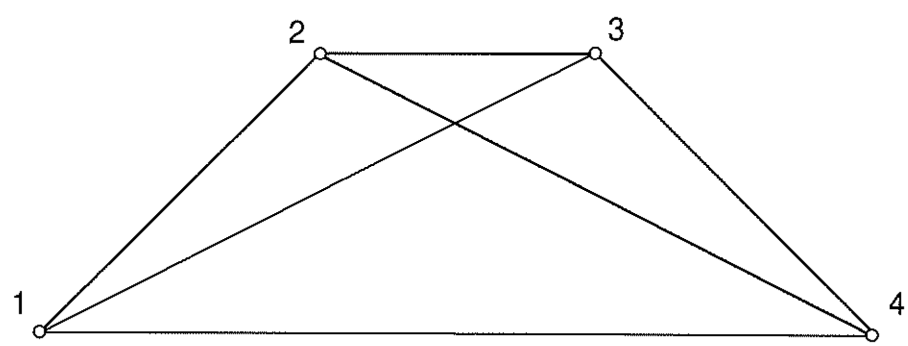


Figure 4.2.2

(c) find the external reactions and the force in each member using the Principle of Complementary Virtual Work.

Let's proceed in the same way as in the previous example.

FBDs + STATICS

External Reactions
[Figure 4.2.2]

$$\sum F_x = 0$$

$$\Rightarrow \quad \left[X_1 + \delta X_1\right] + 1{,}200\ lb = 0$$

$$\Rightarrow \quad \boxed{\left[X_1 + \delta X_1\right] = -1{,}200}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[Y_1 + \delta Y_1\right] + \left[Y_3 + \delta Y_3\right] + \left[Y_4 + \delta Y_4\right] + 1{,}300\ lb = 0$$

$$\Rightarrow \quad \boxed{\left[Y_1 + \delta Y_1\right] + \left[Y_3 + \delta Y_3\right] + \left[Y_4 + \delta Y_4\right] = -1{,}300}$$

$$\circlearrowleft + \quad \sum M_1 = 0$$

$$\Rightarrow \quad (1{,}300\ lb)(40") + \left[Y_3 + \delta Y_3\right](80") + \left[Y_4 + \delta Y_4\right](120") = 0$$

$$\Rightarrow \quad \boxed{-(2)\left[Y_3 + \delta Y_3\right] - (3)\left[Y_4 + \delta Y_4\right] = 1{,}300}$$

INDEPENDENT EQUATIONS

Notice

Numerical values for the external reactions were **NOT** able to be found from the FBD of the structure as a whole.

Problem

* When numerical values for the external reactions cannot be found from the FBD of the structure as a whole, then, if those 3 equations were used, they would have to be counted in the total number of simultaneous equations to eventually be solved.

* This will cause a problem as we move from joint to joint to obtain the other simultaneous equations.

* The problem is that it may be difficult (although not impossible) to determine which of the joint equations are independent of the 3 equations from the FBD of the structure as a whole.

* The final group of simultaneous equations to be solved must be independent of each other.

Solution to the Problem

* When numerical values for the external reactions cannot be found from the FBD of the structure as a whole, **do not use the 3 equations from the FBD of the structure as a whole**.

* **Use only the equations obtained from the internal joints** since we know that the $2j$ number of equations obtained from the joints are all independent of each other [see Page 4.1].

* According to Equation 4.1.5, in order to use the Principle of Complementary Virtual Work, the internal P and δP in each member must be found.

* This will be accomplished by taking the necessary cuts through the members at each joint.

Internals
(Force in Each Member)

Using the method of joints,
Joint 1

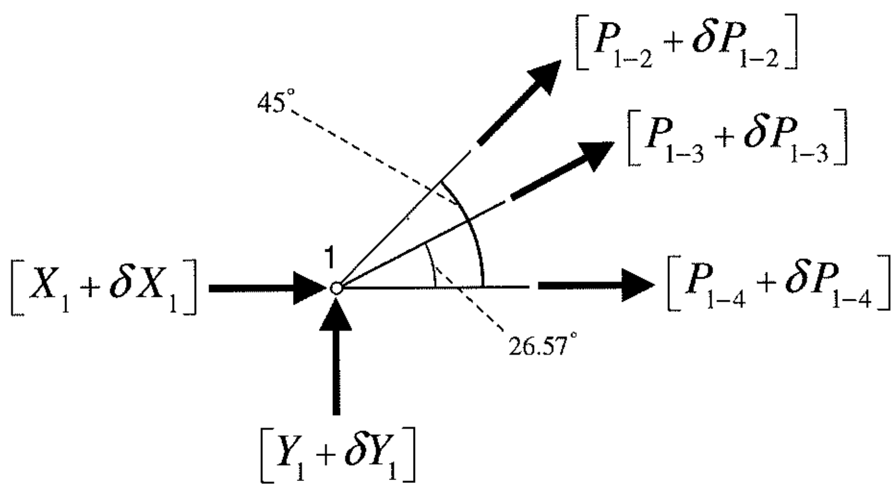


Figure 4.2.3

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{\begin{array}{l}\left[X_1+\delta X_1\right]+\left[P_{1-2}+\delta P_{1-2}\right]\cos 45^\circ \\ +\left[P_{1-3}+\delta P_{1-3}\right]\cos 26.57^\circ+\left[P_{1-4}+\delta P_{1-4}\right]\end{array}\right\}=0$$

Indep. Eq. 1

$$\Rightarrow \quad \left\{\begin{array}{l}\left[P_{1-2}+\delta P_{1-2}\right]+(1.265)\left[P_{1-3}+\delta P_{1-3}\right] \\ +(1.414)\left[P_{1-4}+\delta P_{1-4}\right]+(1.414)\left[X_1+\delta X_1\right]\end{array}\right\}=0 \qquad (4.2.1)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[Y_1 + \delta Y_1\right] + \left[P_{1-2} + \delta P_{1-2}\right]\sin 45^\circ + \left[P_{1-3} + \delta P_{1-3}\right]\sin 26.57^\circ = 0$$

Indep.
Eq.
2

$$\Rightarrow \quad \boxed{\left[P_{1-2} + \delta P_{1-2}\right] + (0.6326)\left[P_{1-3} + \delta P_{1-3}\right] + (1.414)\left[Y_1 + \delta Y_1\right] = 0} \qquad (4.2.2)$$

Joint 2

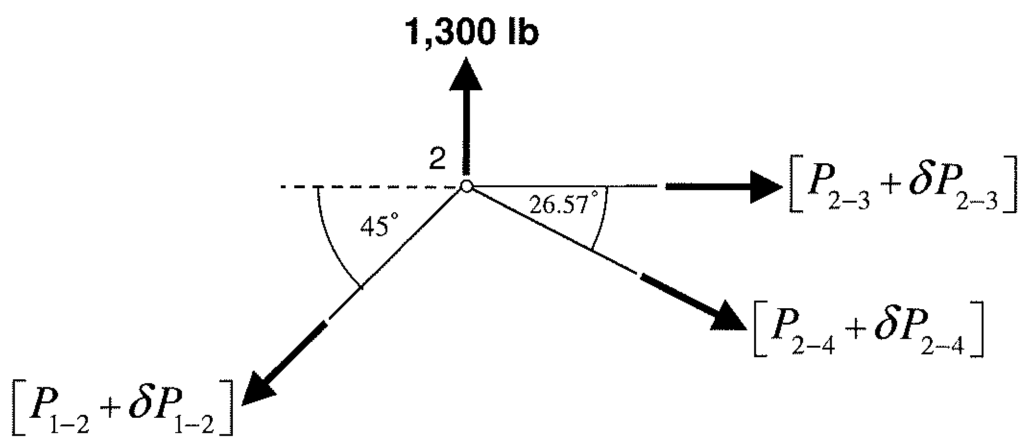


Figure 4.2.4

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{1-2} + \delta P_{1-2}\right]\cos 45^\circ + \left[P_{2-3} + \delta P_{2-3}\right] + \left[P_{2-4} + \delta P_{2-4}\right]\cos 26.57^\circ = 0$$

$$\Rightarrow \quad \boxed{\left[P_{1-2} + \delta P_{1-2}\right] - (1.414)\left[P_{2-3} + \delta P_{2-3}\right] - (1.265)\left[P_{2-4} + \delta P_{2-4}\right] = 0}$$

Indep.
Eq.
3

$$\sum F_y = 0$$

Indep. Eq. 4

$$\Rightarrow \quad -\left[P_{1-2} + \delta P_{1-2}\right]\sin 45^\circ - \left[P_{2-4} + \delta P_{2-4}\right]\sin 26.57^\circ + 1{,}300\ lb = 0$$

$$\Rightarrow \quad \boxed{\left[P_{1-2} + \delta P_{1-2}\right] + (0.6326)\left[P_{2-4} + \delta P_{2-4}\right] = 1{,}838} \qquad (4.2.4)$$

Joint 3

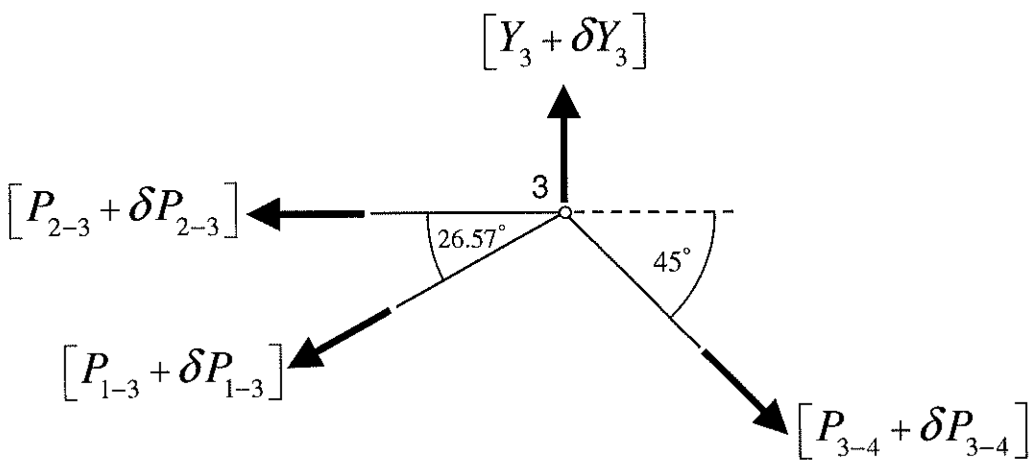


Figure 4.2.5

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{2-3} + \delta P_{2-3}\right] + \left[P_{3-4} + \delta P_{3-4}\right]\cos 45^\circ - \left[P_{1-3} + \delta P_{1-3}\right]\cos 26.57^\circ = 0$$

$$\Rightarrow \quad \boxed{\left\{\begin{array}{l} \left[P_{1-3} + \delta P_{1-3}\right] + (1.118)\left[P_{2-3} + \delta P_{2-3}\right] \\ -(0.7906)\left[P_{3-4} + \delta P_{3-4}\right] \end{array}\right\} = 0} \qquad (4.2.5)$$

Indep. Eq. 5

4.38

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[Y_3 + \delta Y_3\right] - \left[P_{1-3} + \delta P_{1-3}\right]\sin 26.57^\circ - \left[P_{3-4} + \delta P_{3-4}\right]\sin 45^\circ = 0$$

$$\Rightarrow \quad \boxed{\left[P_{1-3} + \delta P_{1-3}\right] + (1.581)\left[P_{3-4} + \delta P_{3-4}\right] - (2.236)\left[Y_3 + \delta Y_3\right] = 0} \quad (4.2.6)$$

Indep.
Eq.
6

Joint 4

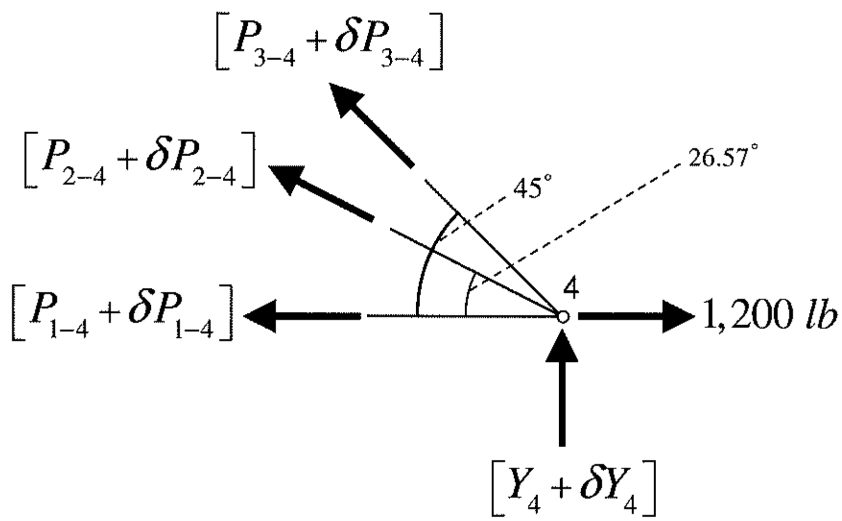


Figure 4.2.6

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{\begin{array}{l} -\left[P_{1-4} + \delta P_{1-4}\right] - \left[P_{2-4} + \delta P_{2-4}\right]\cos 26.57^\circ \\ \\ -\left[P_{3-4} + \delta P_{3-4}\right]\cos 45^\circ + 1{,}200\ lb \end{array}\right\} = 0$$

$$\Rightarrow \boxed{\left\{\begin{aligned}&\left[P_{1-4}+\delta P_{1-4}\right]+(0.8944)\left[P_{2-4}+\delta P_{2-4}\right]\\&+(0.7071)\left[P_{3-4}+\delta P_{3-4}\right]\end{aligned}\right\}=1,200} \quad (4.2.7)$$

Indep.
Eq.
7

$$\sum F_y = 0$$

$$\Rightarrow \left[P_{2-4}+\delta P_{2-4}\right]\sin 26.57^\circ + \left[P_{3-4}+\delta P_{3-4}\right]\sin 45^\circ + \left[Y_4+\delta Y_4\right]=0$$

$$\Rightarrow \boxed{\left[P_{2-4}+\delta P_{2-4}\right]+(1.581)\left[P_{3-4}+\delta P_{3-4}\right]+(2.236)\left[Y_4+\delta Y_4\right]=0} \quad (4.2.8)$$

Indep.
Eq.
8

We now have our 8 independent static equations.

Thus, our 8 Independent Equations from Static Equilibrium are

$$\left\{\begin{array}{l}\left[P_{1-2}+\delta P_{1-2}\right]+(1.265)\left[P_{1-3}+\delta P_{1-3}\right] \\ +(1.414)\left[P_{1-4}+\delta P_{1-4}\right]+(1.414)\left[X_1+\delta X_1\right]\end{array}\right\}=0 \tag{4.2.1}$$

$$\left[P_{1-2}+\delta P_{1-2}\right]+(0.6326)\left[P_{1-3}+\delta P_{1-3}\right]+(1.414)\left[Y_1+\delta Y_1\right]=0 \tag{4.2.2}$$

$$\left[P_{1-2}+\delta P_{1-2}\right]-(1.414)\left[P_{2-3}+\delta P_{2-3}\right]-(1.265)\left[P_{2-4}+\delta P_{2-4}\right]=0 \tag{4.2.3}$$

$$\left[P_{1-2}+\delta P_{1-2}\right]+(0.6326)\left[P_{2-4}+\delta P_{2-4}\right]=1{,}838 \tag{4.2.4}$$

$$\left\{\begin{array}{l}\left[P_{1-3}+\delta P_{1-3}\right]+(1.118)\left[P_{2-3}+\delta P_{2-3}\right] \\ -(0.7906)\left[P_{3-4}+\delta P_{3-4}\right]\end{array}\right\}=0 \tag{4.2.5}$$

$$\left[P_{1-3}+\delta P_{1-3}\right]+(1.581)\left[P_{3-4}+\delta P_{3-4}\right]-(2.236)\left[Y_3+\delta Y_3\right]=0 \tag{4.2.6}$$

$$\left\{\begin{array}{l}\left[P_{1-4}+\delta P_{1-4}\right]+(0.8944)\left[P_{2-4}+\delta P_{2-4}\right] \\ +(0.7071)\left[P_{3-4}+\delta P_{3-4}\right]\end{array}\right\}=1{,}200 \tag{4.2.7}$$

$$\left[P_{2-4}+\delta P_{2-4}\right]+(1.581)\left[P_{3-4}+\delta P_{3-4}\right]+(2.236)\left[Y_4+\delta Y_4\right]=0 \tag{4.2.8}$$

* We have 8 equations, but 10 unknowns.

* The additional 2 equations will come from the Principle of Complementary Virtual Work.

* Before beginning the Principle of Complementary Virtual Work, we must decide which unknowns will be considered as our 2 redundants.

* We will use Method 1 as outlined on Page 4.31.

Choice of Redundants for This Truss

Step 1 Frequency Table

The table below shows the frequency in which each of the 10 variables appear in the 8 independent static equilibrium equations.

Variable Frequency Table

		1	2	3	4	5	6	7	8	
	Eq. Nos.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Freq**
1	**P-1-2**	X	X	X	X					**4**
2	**P-1-3**	X	X			X	X			**4**
3	**P-1-4**	X						X		2
4	**P-2-3**			X		X				2
5	**P-2-4**			X	X			X	X	**4**
6	**P-3-4**					X	X	X	X	**4**
7	**X-1**	X								1
8	**Y-1**		X							1
9	**Y-3**						X			1
10	**Y-4**								X	1

* In the Frequency Table on Page 4.41, the unknowns are listed in the usual order in the 1st column.

Step 2 Move Rows So That the X's in Each Column are as Close as Possible
(Beginning with the 1st Column with X's, then the 2nd, etc.)

		1	2	3	4	5	6	7	8	
	Eq. Nos.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Freq**
1	P-1-2	X	X	X	X					4
2	P-1-3	X	X			X	X			4
3	P-1-4	X						X		2
4	X-1	X								1
5	Y-1		X							1
6	P-2-3			X		X				2
7	P-2-4			X	X			X	X	4
8	P-3-4					X	X	X	X	4
9	Y-3						X			1
10	Y-4								X	1

Step 3 Move Columns So That the X's in Each Row are as Close as Possible
(Beginning with the 1st Row with X's, then the 2nd, etc.)

		1	2	3	4	5	6	7	8	
	Eq. Nos.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Freq**
1	P-1-2	X	X	X	X					4
2	P-1-3	X	X			X	X			4
3	P-1-4	X						X		2
4	X-1	X								1
5	Y-1		X							1
6	P-2-3			X		X				2
7	P-2-4			X	X			X	X	4
8	P-3-4					X	X	X	X	4
9	Y-3						X			1
10	Y-4								X	1

* In this particular problem, Step 3 caused no change in the table from that of Step 2.

Step 4 Identify Unknowns That Can Be Solved for Numerical Values

It does not appear that any of the 10 variables can be solved for numerical values using these 8 equations.

[See Page 6.41 for a statically indeterminate case in which some numerical values can be found using the static equilibrium equations.]

Any unknowns in which a numerical value can be found from the static equilibrium equations cannot be chosen as a redundant.

* If they would be chosen as a redundant, the equations obtained, in terms of the redundants, would not all be independent.

The bottom two stars in the box below were brought to light by Nicholas Fernandez, a student in AE 418, Fall 2018

Rule Set #1 for Choosing Redundants

* Any unknowns in which a numerical value can be found from the static equilibrium equations cannot be chosen as a redundant.

* Redundants must be independent of each other.

* Thus if any of the equation columns in the Frequency Table obtained from Step 3 have only 2 X's in the column, then those 2 unknowns cannot be used together as redundants, since they can be related to each other & would not be independent.

* All unknowns in a column of the table cannot be used together as redundants.

* If all of the values in a column are used as the redundants, that equation becomes unusable since there would be no unknown left in that equation to solve in terms of a redundant.
Hence there would be a shortage of one equation when trying to solve for the 8 unknowns in terms of redundants.

Following these rules,

$$P_{1-2} \text{ and } P_{2-4}$$

cannot be chosen together as redundants.

Rule Set #2 for Choosing Redundants

We expect that the algebra will be reduced if

(1) P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work Equation (Equation 4.1.5),

and

(2) P values are chosen that have the greatest number of equation frequencies (rows).

Using the Frequency Table in Step 3 on Page 4.41,

* We note that P_{1-2}, P_{1-3}, P_{2-4}, and P_{3-4} have the greatest equation frequencies (rows) with a value of 4 each.

* However, based on our conclusion on the previous page, P_{1-2} and P_{2-4} cannot be chosen together as redundants.

P_{1-2} and P_{3-4} will be chosen as the 2 redundants

This means that, in a sense, they will be treated in the same way as the numerical values in the 8 static equilibrium equations on page 4.40; i.e., they will be placed on the right hand side of the equations as below.

$$\left\{\begin{array}{l}\left[P_{1-3}+\delta P_{1-3}\right]+(1.118)\left[P_{1-4}+\delta P_{1-4}\right] \\ +(1.118)\left[X_1+\delta X_1\right]\end{array}\right\}=-(0.7905)\left[P_{1-2}+\delta P_{1-2}\right] \tag{4.2.9}$$

$$\left[P_{1-3}+\delta P_{1-3}\right]+(2.235)\left[Y_1+\delta Y_1\right]=-(1.581)\left[P_{1-2}+\delta P_{1-2}\right] \tag{4.2.10}$$

$$\left[P_{2-3}+\delta P_{2-3}\right]+(0.8946)\left[P_{2-4}+\delta P_{2-4}\right]=(0.7072)\left[P_{1-2}+\delta P_{1-2}\right] \tag{4.2.11}$$

$$\left[P_{2-4}+\delta P_{2-4}\right]=-(1.581)\left[P_{1-2}+\delta P_{1-2}\right]+2,905 \tag{4.2.12}$$

$$\left[P_{1-3}+\delta P_{1-3}\right]+(1.118)\left[P_{2-3}+\delta P_{2-3}\right]=(0.7906)\left[P_{3-4}+\delta P_{3-4}\right] \tag{4.2.13}$$

$$\left[P_{1-3}+\delta P_{1-3}\right]-(2.236)\left[Y_3+\delta Y_3\right]=-(1.581)\left[P_{3-4}+\delta P_{3-4}\right] \tag{4.2.14}$$

$$\left\{\begin{array}{l}\left[P_{1-4}+\delta P_{1-4}\right] \\ +(0.8944)\left[P_{2-4}+\delta P_{2-4}\right]\end{array}\right\}=-(0.7071)\left[P_{3-4}+\delta P_{3-4}\right]+1,200 \tag{4.2.15}$$

$$\left[P_{2-4}+\delta P_{2-4}\right]+(2.236)\left[Y_4+\delta Y_4\right]=-(1.581)\left[P_{3-4}+\delta P_{3-4}\right] \tag{4.2.16}$$

Equations 4.2.9 - 4.2.16 must now be solved for the 8 unknowns on the left, in terms of the quantities on the right.

This could be done in MAPLE, by hand, or **.....**

The results are

$$\left[P_{1-3}+\delta P_{1-3}\right]=-(2.372)\left[P_{1-2}+\delta P_{1-2}\right]+(0.7906)\left[P_{3-4}+\delta P_{3-4}\right]+2{,}905 \quad (4.2.17)$$

$$\left[P_{1-4}+\delta P_{1-4}\right]=(1.414)\left[P_{1-2}+\delta P_{1-2}\right]-(0.7071)\left[P_{3-4}+\delta P_{3-4}\right]-1{,}398 \quad (4.2.18)$$

$$\left[P_{2-3}+\delta P_{2-3}\right]=(2.122)\left[P_{1-2}+\delta P_{1-2}\right]-2{,}599 \quad (4.2.19)$$

$$\left[P_{2-4}+\delta P_{2-4}\right]=-(1.581)\left[P_{1-2}+\delta P_{1-2}\right]+2{,}905 \quad (4.2.20)$$

$$\left[X_1+\delta X_1\right]=\left\{\begin{matrix}(0.0005796)\left[P_{1-2}+\delta P_{1-2}\right]\\ -(0.00005564)\left[P_{3-4}+\delta P_{3-4}\right]-1{,}200\end{matrix}\right\} \quad (4.2.21)$$

$$\left[Y_1+\delta Y_1\right]=(0.3539)\left[P_{1-2}+\delta P_{1-2}\right]-(0.3537)\left[P_{3-4}+\delta P_{3-4}\right]-1{,}300 \quad (4.2.22)$$

$$\left[Y_3+\delta Y_3\right]=-(1.061)\left[P_{1-2}+\delta P_{1-2}\right]+(1.061)\left[P_{3-4}+\delta P_{3-4}\right]+1{,}299 \quad (4.2.23)$$

$$\left[Y_4+\delta Y_4\right]=(0.7071)\left[P_{1-2}+\delta P_{1-2}\right]-(0.7071)\left[P_{3-4}+\delta P_{3-4}\right]-1{,}299 \quad (4.2.24)$$

Equations 4.2.17 - 4.2.24 have been rearranged as below, separating the reals and the virtuals.

$P_{1-3} = -(2.372)P_{1-2} + (0.7906)P_{3-4} + 2{,}905$	$\delta P_{1-3} = -(2.372)\delta P_{1-2} + (0.7906)\delta P_{3-4}$	(4.2.25)
$P_{1-4} = (1.414)P_{1-2} - (0.7071)P_{3-4} - 1{,}398$	$\delta P_{1-4} = (1.414)\delta P_{1-2} - (0.7071)\delta P_{3-4}$	(4.2.26)
$P_{2-3} = (2.122)P_{1-2} - 2{,}599$	$\delta P_{2-3} = (2.122)\delta P_{1-2}$	(4.2.27)
$P_{2-4} = -(1.581)P_{1-2} + 2{,}905$	$\delta P_{2-4} = -(1.581)\delta P_{1-2}$	(4.2.28)
$X_1 = \left\{\begin{matrix}(0.0005796)P_{1-2} \\ -(0.00005564)P_{3-4} - 1{,}200\end{matrix}\right\}$	$\delta X_1 = \left\{\begin{matrix}(0.0005796)\delta P_{1-2} \\ -(0.00005564)\delta P_{3-4}\end{matrix}\right\}$	(4.2.29)
$Y_1 = (0.3539)P_{1-2} - (0.3537)P_{3-4} - 1{,}300$	$\delta Y_1 = (0.3539)\delta P_{1-2} - (0.3537)\delta P_{3-4}$	(4.2.30)
$Y_3 = -(1.061)P_{1-2} + (1.061)P_{3-4} + 1{,}299$	$\delta Y_3 = -(1.061)\delta P_{1-2} + (1.061)\delta P_{3-4}$	(4.2.31)
$Y_4 = (0.7071)P_{1-2} - (0.7071)P_{3-4} - 1{,}299$	$\delta Y_4 = (0.7071)\delta P_{1-2} - (0.7071)\delta P_{3-4}$	(4.2.32)

Note that in all cases "δ" acts like a differential operator "d".

Recall that, since we have 2 redundants, we need 2 additional equations beyond those of static equilibrium. All of the static equilibrium equations have been used up to get the answers above.

The 2 additional equations will come from the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.1.5) for the entire truss system states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \frac{\left(P\Big|_{\substack{\text{In Member Due} \\ \text{Actual Loads}}} \right)_i L_i}{A_i E_i} (\delta P)_i \qquad (4.1.5)$$

$$\Rightarrow \left\{ \begin{aligned} &\overbrace{\mathrm{u}_1}^{0} (\delta X_1) + \overbrace{\mathrm{v}_1}^{0} (\delta Y_1) \\ &+ \overbrace{\mathrm{v}_3}^{0} (\delta Y_3) + \overbrace{\mathrm{v}_4}^{0} (\delta Y_4) \end{aligned} \right\} = \left[\begin{array}{c} \overbrace{\dfrac{[P_{1-2}] \overbrace{\sqrt{(40")^2 + (40")^2}}^{56.57"}}{\underbrace{\left[\pi \underbrace{\left(\dfrac{1.125"}{2} \right)}_{0.5625"}^2 - \pi \left(0.5625" - \dfrac{1}{16}" \right)^2 \right]}_{0.2086\, in^2} (30\, x 10^6\, psi)} [\delta P_{1-2}]}^{\text{Member 1-2}} \\ \\ \bullet\bullet\bullet \end{array} \right]$$

•••

$$+\overbrace{\frac{\begin{bmatrix}-(2.372)P_{1-2}\\+(0.7906)P_{3-4}\\+2{,}905\end{bmatrix}\overbrace{\sqrt{(40")^2+(80")^2}}^{89.44"}}{(0.2086\ in^2)(30\,x10^6\ psi)}\begin{bmatrix}-(2.372)\delta P_{1-2}\\+(0.7906)\delta P_{3-4}\end{bmatrix}}^{\text{Member 1-3}}$$

$$+\overbrace{\frac{\begin{bmatrix}(1.414)P_{1-2}\\-(0.7071)P_{3-4}\\-1{,}398\end{bmatrix}(120")}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[(1.414)\delta P_{1-2}-(0.7071)\delta P_{3-4}\right]}^{\text{Member 1-4}}$$

$$+\overbrace{\frac{\begin{bmatrix}(2.122)P_{1-2}\\-2{,}599\end{bmatrix}(40")}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[(2.122)\delta P_{1-2}\right]}^{\text{Member 2-3}}$$

$$+\overbrace{\frac{\begin{bmatrix}-(1.581)P_{1-2}\\+2{,}905\end{bmatrix}\overbrace{\sqrt{(40")^2+(80")^2}}^{89.44"}}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[-(1.581)\delta P_{1-2}\right]}^{\text{Member 2-4}}$$

$$+\overbrace{\frac{\left[P_{3-4}\right](56.57")}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[\delta P_{3-4}\right]}^{\text{Member 3-4}}$$

4.50

$$\Rightarrow \quad 0=\left\{\begin{array}{l}
\overbrace{\left(9.040\,x10^{-6}\right)\left[P_{1-2}\right]\left[\delta P_{1-2}\right]}^{\text{Member 1-2}}+\overbrace{\begin{bmatrix}-\left(3.390\,x10^{-5}\right)P_{1-2}\\+\left(1.130\,x10^{-5}\right)P_{3-4}\\+\left(4.152\,x10^{-2}\right)\end{bmatrix}\begin{bmatrix}-\left(2.372\right)\delta P_{1-2}\\+\left(0.7906\right)\delta P_{3-4}\end{bmatrix}}^{\text{Member 1-3}}\\
+\overbrace{\begin{bmatrix}\left(2.711\,x10^{-5}\right)P_{1-2}-\left(1.356\,x10^{-5}\right)P_{3-4}\\-\left(2.681\,x10^{-2}\right)\end{bmatrix}\left[\left(1.414\right)\delta P_{1-2}-\left(0.7071\right)\delta P_{3-4}\right]}^{\text{Member 1-4}}\\
+\overbrace{\left[\left(1.356\,x10^{-5}\right)P_{1-2}-\left(1.661\,x10^{-2}\right)\right]\left[\left(2.121\right)\delta P_{1-2}\right]}^{\text{Member 2-3}}\\
+\overbrace{\begin{bmatrix}-\left(2.260\,x10^{-5}\right)P_{1-2}\\+\left(4.152\,x10^{-2}\right)\end{bmatrix}\left[-\left(1.581\right)\delta P_{1-2}\right]}^{\text{Member 2-4}}+\overbrace{\left(9.040\,x10^{-6}\right)\left[P_{3-4}\right]\left[\delta P_{3-4}\right]}^{\text{Member 3-4}}
\end{array}\right\}$$

$$\Rightarrow \quad 0=\left\{\begin{array}{l}
\overbrace{\left(9.040x10^{-6}\right)\left[P_{1-2}\right]\left[\delta P_{1-2}\right]}^{\text{Member 1-2}} \\[2ex]
+\overbrace{\begin{array}{l}\left[\left(8.041x10^{-5}\right)P_{1-2}-\left(2.680x10^{-5}\right)P_{3-4}-\left(9.849x10^{-2}\right)\right]\left[\delta P_{1-2}\right] \\ +\left[\left(-2.680x10^{-5}\right)P_{1-2}+\left(8.934x10^{-6}\right)P_{3-4}+\left(3.283x10^{-2}\right)\right]\left[\delta P_{3-4}\right]\end{array}}^{\text{Member 1-3}} \\[2ex]
+\overbrace{\left\{\begin{array}{l}\left[\left(3.833x10^{-5}\right)P_{1-2}-\left(1.917x10^{-5}\right)P_{3-4}-\left(3.791x10^{-2}\right)\right]\left[\delta P_{1-2}\right] \\ +\left[\left(-1.917x10^{-5}\right)P_{1-2}+\left(9.588x10^{-6}\right)P_{3-4}+\left(1.896x10^{-2}\right)\right]\left[\delta P_{3-4}\right]\end{array}\right\}}^{\text{Member 1-4}} \\[2ex]
+\overbrace{\left[\left(2.876x10^{-5}\right)P_{1-2}-\left(3.523x10^{-2}\right)\right]\left[\delta P_{1-2}\right]}^{\text{Member 2-3}} \\[2ex]
+\overbrace{\left[\left(3.573x10^{-5}\right)P_{1-2}-\left(6.564x10^{-2}\right)\right]\left[\delta P_{1-2}\right]}^{\text{Member 2-4}} + \overbrace{\left(9.040x10^{-6}\right)\left[P_{3-4}\right]\left[\delta P_{3-4}\right]}^{\text{Member 3-4}}
\end{array}\right\}$$

$$\Rightarrow \quad 0=\left\{\begin{array}{l}
\left[\left(1.923x10^{-4}\right)P_{1-2}-\left(4.597x10^{-5}\right)P_{3-4}-\left(2.373x10^{-1}\right)\right]\left[\delta P_{1-2}\right] \\[2ex]
+\left[-\left(4.597x10^{-5}\right)P_{1-2}+\left(2.756x10^{-5}\right)P_{3-4}+\left(5.179x10^{-2}\right)\right]\left[\delta P_{3-4}\right]
\end{array}\right\}$$

$$\Rightarrow \quad \boxed{\begin{array}{r}
\left(1.923x10^{-4}\right)P_{1-2}-\left(4.597x10^{-5}\right)P_{3-4}-\left(2.373x10^{-1}\right)=0 \\[2ex]
-\left(4.597x10^{-5}\right)P_{1-2}+\left(2.756x10^{-5}\right)P_{3-4}+\left(5.179x10^{-2}\right)=0
\end{array}} \quad (4.2.33)$$

Equations 4.2.33 are the additional two equations needed to solve for our 10 unknowns.

Solving Equations 4.2.33 gives

$$\boxed{P_{1-2} = 1{,}305\ lb} \tag{4.2.34}$$

$\Rightarrow$

$$\boxed{P_{3-4} = 298.0\ lb} \tag{4.2.35}$$

Now substituting Equations 4.2.34 and 4.2.35 into Equations 4.2.25 through 4.2.32, we obtain the remaining unknowns as below.

$$P_{1-3} = 45.14\ lb \tag{4.2.36}$$

$$P_{1-4} = 236.6\ lb \tag{4.2.37}$$

$$P_{2-3} = 170.2\ lb \tag{4.2.38}$$

$$P_{2-4} = 841.8\ lb \tag{4.2.39}$$

$$X_1 = -1{,}200\ lb \tag{4.2.40}$$

$$Y_1 = -943.6\ lb \tag{4.2.41}$$

$$Y_3 = 230.9\ lb \tag{4.2.42}$$

$$Y_4 = -586.9\ lb \tag{4.2.43}$$

The answers are shown in Figure 4.2.6 below.

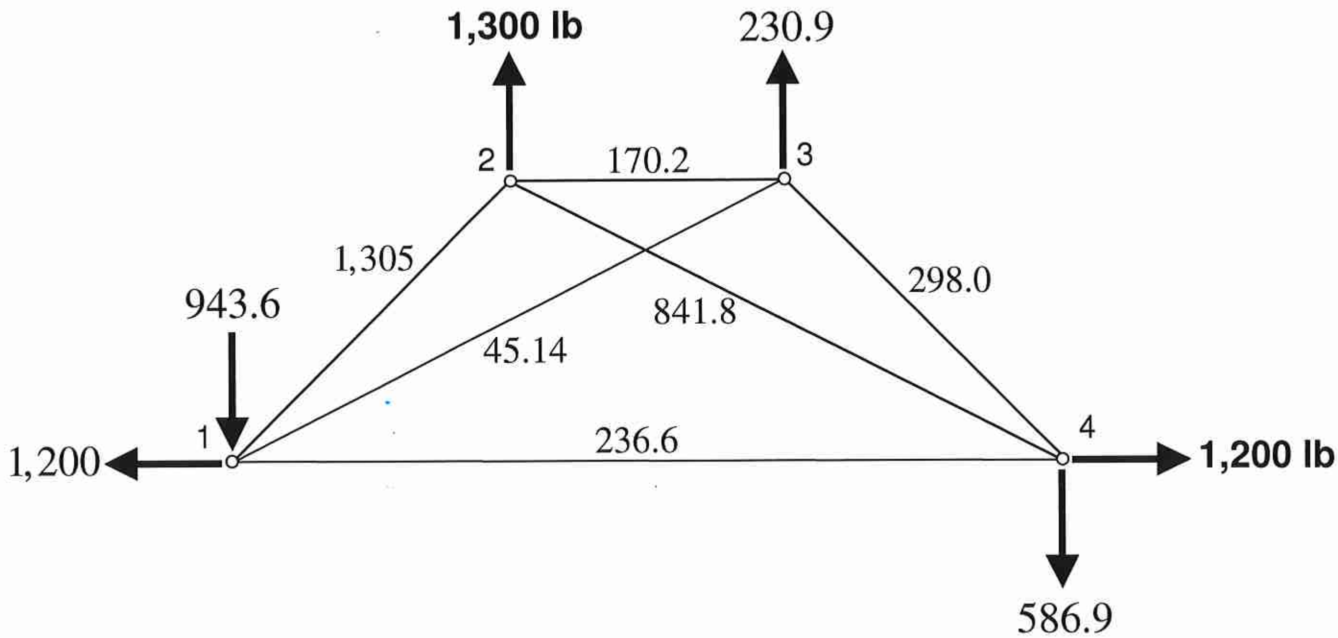


Figure 4.2.6

Suppose that you were now asked to find the vertical displacement of node 2. How would you do it?

Example 4.2.2 **[Statically Indeterminate Truss—Forces in Members, Reactions]**

For the statically indeterminate truss shown in Figure 4.2.7,
(a) determine the number of redundancies,
(b) draw a FBD of the truss in Figure 4.2.7, showing both the reals and the virtuals for utilizing the short cut method as described previously,
(c) find the external reactions and the force in each member using the Principle of Complementary Virtual Work.

Material: Steel	Cross-Sections: Hollow Circular
$E = 30 \times 10^6$ psi	$d_{outer} = 1\frac{1}{8}$"
	$t = \frac{1}{16}$"

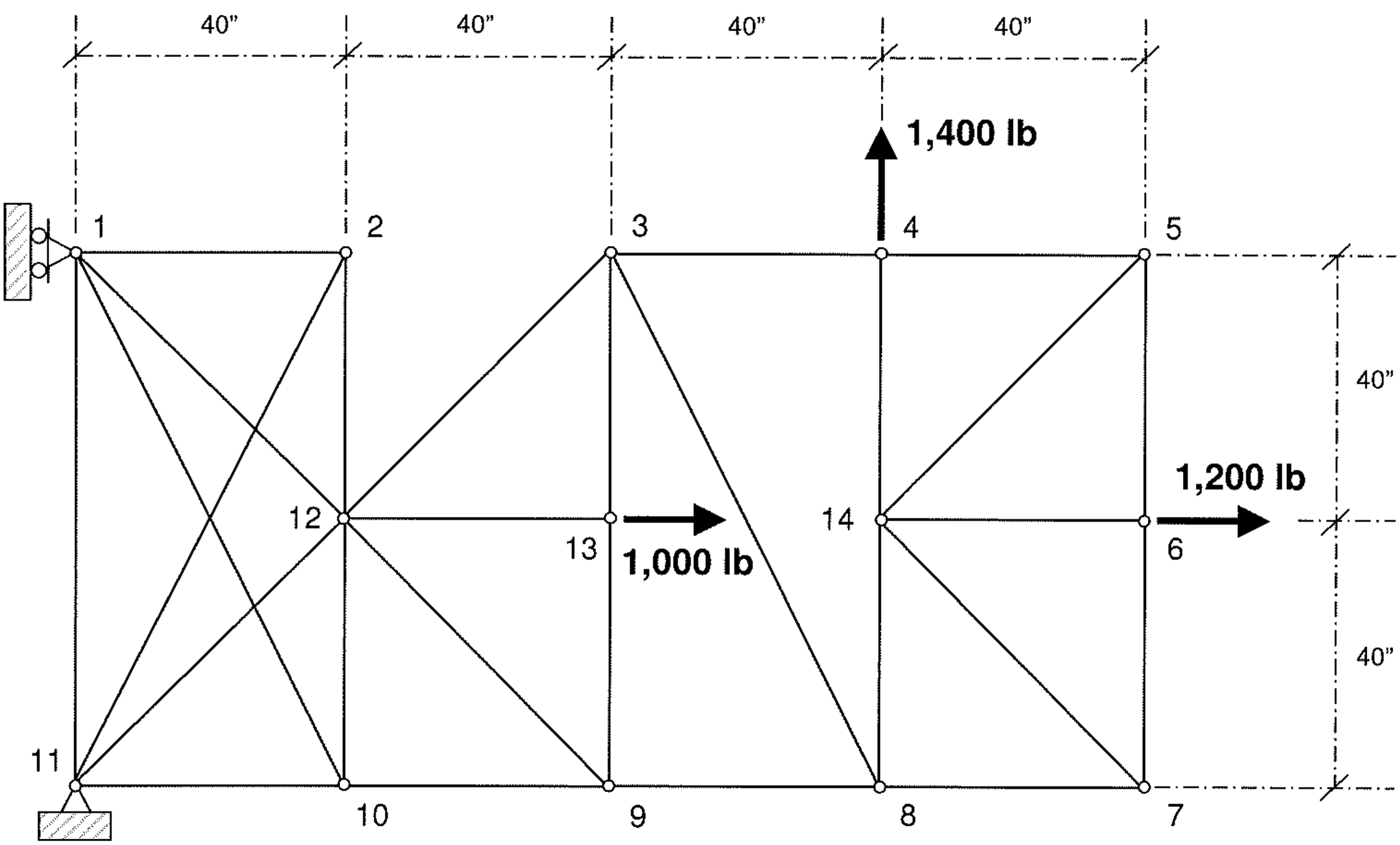


Figure 4.2.7

****

2-Force Members

Why is every member in Figure 4.2.7 a two-force member?

Because

1. there are no applied bending moments on any member, and

2. all forces on each member are applied at only two locations on the member.

(a) determine the number of redundancies

We must check [see Page 4.1]

1. if the truss configuration can support the loads without collapsing and
2. that it is constrained so that the truss as a whole will not have either translational rigid body motion or rotational rigid body motion.

We observe that
1. It can support the loads without collapsing since it is composed of all triangular regions.
2. It cannot have any rigid body motion with the given constraints.

Now we can apply Equation 4.1.1.

$$2j = m + r \qquad (4.1.1)$$

$$\Rightarrow \qquad 2(14 \text{ joints}) = (27 \text{ members}) + (3 \text{ external reactions})$$

$$\Rightarrow \qquad 28 \neq 30$$

It is <u>NOT</u> statically determinate.

Equations 4.1.2 state

$$\begin{aligned} 2j &= m + r \quad \Rightarrow \quad \text{Statically Determinate} \\ 2j &< m + r \quad \Rightarrow \quad \text{Statically Indeterminate} \qquad (4.1.2) \\ 2j &> m + r \quad \Rightarrow \quad \text{Not Stable} \end{aligned}$$

4.56

In our case,

$$2j < m + r \quad \Rightarrow \quad \text{Statically Indeterminate}$$

* Therefore, we have a Static Indeterminacy with 2 Redundants.

* That means that we will be short by 2 equations after we have completed using the static equilibrium equations.

* The additional 2 equations will come from the Principle of Complementary Virtual Work.

* We will have to apply 2 virtual forces in order to get 2 equations from the Principle of Complementary Virtual Work.

We notice that, from static equilibrium, the

Maximum Number of Independent Equations = 2 (14 joints) = 28.

(b) draw a FBD of the truss in Figure 4.2.7, showing both the reals and the virtuals for utilizing the short cut method as described previously

2 Redundants

* The Principle of Complementary Virtual Work will be used to obtain the 2 additional equations needed beyond those of static equilibrium.

* This means that we will have to apply 2 virtual forces.

* Let us imagine applying 2 virtual forces in the same location and in the same direction as 2 of the external reactions.

* These applied virtual forces will cause
  (i) virtual internal reactions inside the truss, as well as
  (ii) virtual external reactions at the other connections.

* Of all the resulting virtuals, which will be treated as the applied virtuals & which will be treated as the reaction virtuals?

* The applied virtuals could be any 2 of the resulting virtuals; namely, any combination of the external virtuals and/or internal virtuals.

* There is no need to distinguish between them.

At some point, we must decide which of the real forces we will select as our 2 redundants.

Choice of Redundants

Two Methods for Choosing Redundants

Method 1

Best method when there are a large number of simultaneous equations produced. Trusses generally fit into this category.

* To choose the redundants

1. write all of the static equilibrium equations in terms of all of the variables
2. then determine which choice of redundants
   - i. would make sense mathematically and
   - ii. would make the equations easiest to solve.

* As a general rule, for trusses, the algebra will be reduced if P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work equation.

Method 2

Quite often this is the best method for simple beams and frames.

* To choose the redundants

1. visibly identify which reactions could be removed so that the structure would
   - i. be statically determinate and
   - ii. still support the loads without the entire structure having any rigid body motions.
2. Use the reactions that could be removed as the redundants.

To determine the reactions at all connections using the Principle of Complementary Virtual Work,

1. the actual loads of $1{,}400\ lb$ and $1{,}200\ lb$, as well as all externals, will be removed, and

2. two virtual loads will be applied at two of the supports in the same directions as two of the support reactions.

As mentioned above, this will cause virtual reactions everywhere else.

The virtual forces are to be applied with all externals removed.

However, for the Short Cut Method as described in the previous example, a FBD of the structure with both reals, and virtuals will be drawn [see Figure 4.2.8 below].

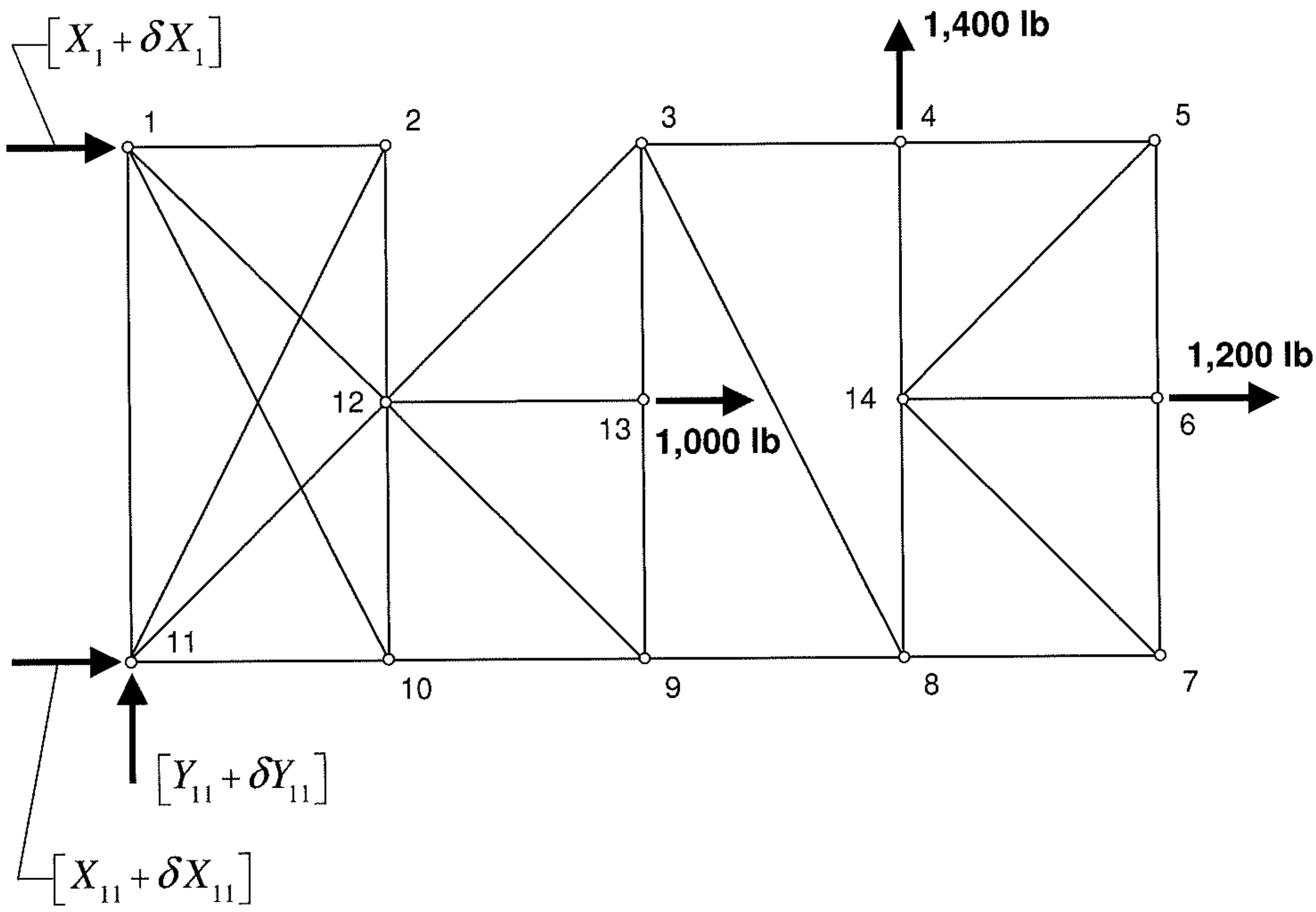


Figure 4.2.8

(c) find the external reactions and the force in each member using the Principle of Complementary Virtual Work.

Let's proceed in the same way as in the previous example.

FBDs + STATICS

External Reactions
[Figure 4.2.8]

$$\sum F_x = 0$$

$$\Rightarrow \quad \left[X_1 + \delta X_1\right] + \left[X_{11} + \delta X_{11}\right] + 1{,}200\ lb + 1{,}000\ lb = 0$$

$$\Rightarrow \quad \boxed{\left[X_1 + \delta X_1\right] + \left[X_{11} + \delta X_{11}\right] = -2{,}200}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[Y_{11} + \delta Y_{11}\right] + 1{,}400\ lb = 0$$

$$\Rightarrow \quad \boxed{\left[Y_{11} + \delta Y_{11}\right] = -1{,}400}$$

$$\sum M_{11} = 0$$

$$\Rightarrow \quad (1{,}400\ lb)(120") - \left[X_1 + \delta X_1\right](80") - (1{,}200\ lb)(40") - (1{,}000\ lb)(40") = 0$$

$$\Rightarrow \quad \boxed{\left[X_1 + \delta X_1\right] = 1{,}000}$$

INDEPENDENT EQUATIONS

Notice

Numerical values for the external reactions were **NOT** able to be found from the FBD of the structure as a whole.

Problem

* When numerical values for the external reactions cannot be found from the FBD of the structure as a whole, then, if those 3 equations were used, they would have to be counted in the total number of simultaneous equations to eventually be solved.

* This will cause a problem as we move from joint to joint to obtain the other simultaneous equations.

* The problem is that it may be difficult (although not impossible) to determine which of the joint equations are independent of the 3 equations from the FBD of the structure as a whole.

* The final group of simultaneous equations to be solved must be independent of each other.

Solution to the Problem

* When numerical values for the external reactions cannot be found from the FBD of the structure as a whole, **do not use the 3 equations from the FBD of the structure as a whole**.

* **Use only the equations obtained from the internal joints** since we know that the $2j$ number of equations obtained from the joints are all independent of each other [see Page 4.1].

* According to Equation 4.1.5, in order to use the Principle of Complementary Virtual Work, the internal P and δP in each member must be found.

* This will be accomplished by taking the necessary cuts through the members at each joint.

Internals
(Force in Each Member)

Using the method of joints,
Joint 1

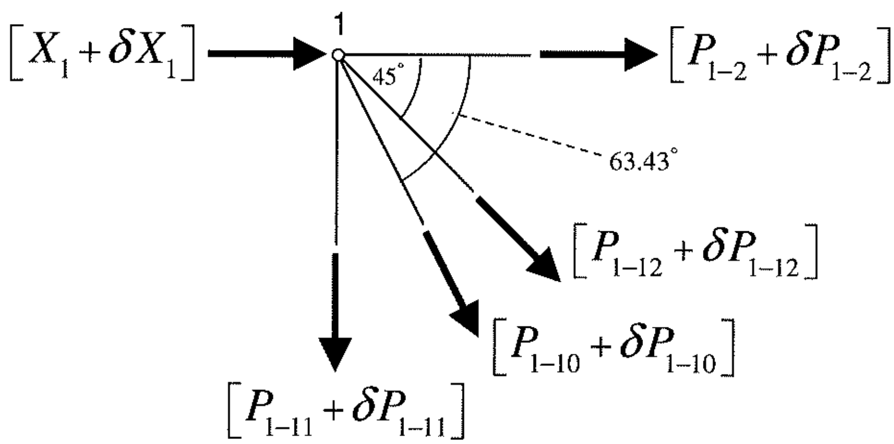


Figure 4.2.9

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[X_1+\delta X_1\right]+\left[P_{1-2}+\delta P_{1-2}\right]+\left[P_{1-12}+\delta P_{1-12}\right]\cos 45^\circ \\ \\ +\left[P_{1-10}+\delta P_{1-10}\right]\cos 63.43^\circ \end{array} \right\} = 0$$

$$\boxed{P_{1-2}+(0.4473)P_{1-10}+(0.7071)P_{1-12}+X_1=0}$$

$\Rightarrow$ (4.2.44)

$$\boxed{\delta P_{1-2}+(0.4473)\delta P_{1-10}+(0.7071)\delta P_{1-12}+\delta X_1=0}$$

Indep. Eq. 1

$$\sum F_y=0$$

$$\Rightarrow\ -\left[P_{1-12}+\delta P_{1-12}\right]\sin 45^\circ-\left[P_{1-10}+\delta P_{1-10}\right]\sin 63.43^\circ-\left[P_{1-11}+\delta P_{1-11}\right]=0$$

$$\boxed{P_{1-10}+(1.118)P_{1-11}+(0.7906)P_{1-12}=0}$$

$\Rightarrow$ (4.2.45)

$$\boxed{\delta P_{1-10}+(1.118)\delta P_{1-11}+(0.7906)\delta P_{1-12}=0}$$

Indep. Eq. 2

Joint 2

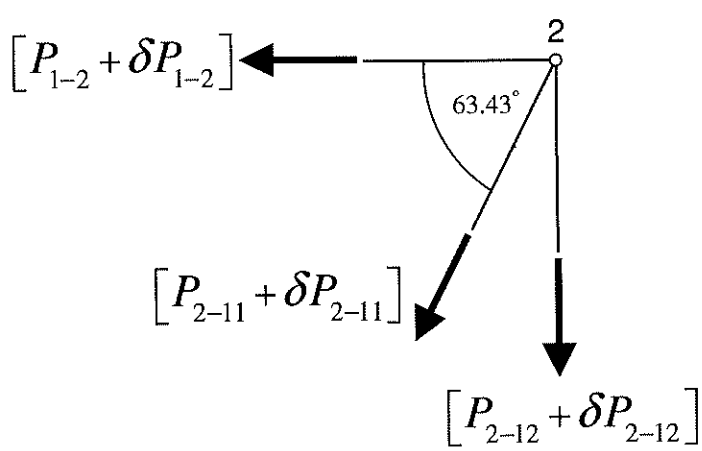


Figure 4.2.10

4.64

$$\sum F_x = 0$$

Indep. Eq. 3

$$\Rightarrow \quad -\left[P_{1-2} + \delta P_{1-2}\right] - \left[P_{2-11} + \delta P_{2-11}\right]\cos 63.43^\circ = 0$$

$$\Rightarrow \quad \boxed{P_{1-2} + (0.4473)P_{2-11} = 0} \qquad \boxed{\delta P_{1-2} + (0.4473)\delta P_{2-11} = 0} \qquad (4.2.46)$$

$$\sum F_y = 0$$

Indep. Eq. 4

$$\Rightarrow \quad -\left[P_{2-11} + \delta P_{2-11}\right]\sin 63.43^\circ - \left[P_{2-12} + \delta P_{2-12}\right] = 0$$

$$\Rightarrow \quad \boxed{P_{2-11} + (1.118)P_{2-12} = 0} \qquad \boxed{\delta P_{2-11} + (1.118)\delta P_{2-12} = 0} \qquad (4.2.47)$$

Joint 3

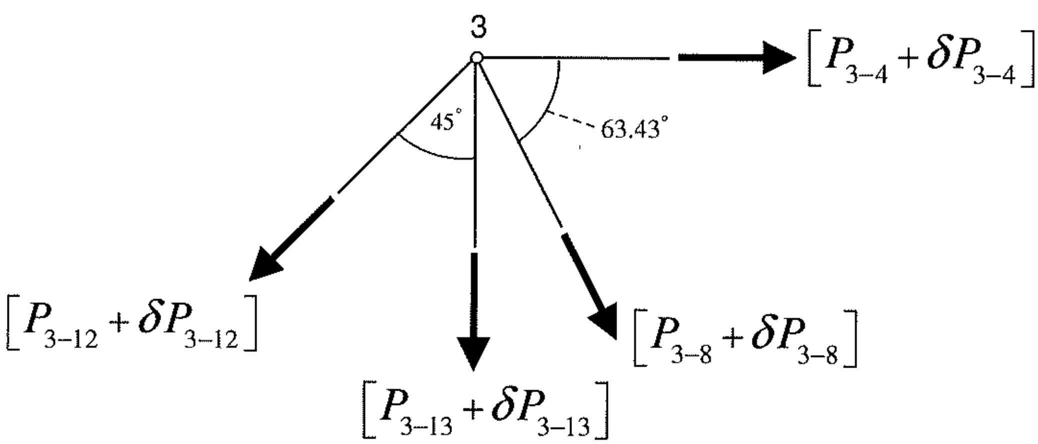


Figure 4.2.11

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -\left[P_{3-12} + \delta P_{3-12}\right]\sin 45^\circ + \left[P_{3-8} + \delta P_{3-8}\right]\cos 63.43^\circ \\ +\left[P_{3-4} + \delta P_{3-4}\right] \end{array} \right\} = 0$$

Indep. Eq. 5

$$\boxed{P_{3-4} + (0.4473)P_{3-8} - (0.7071)P_{3-12} = 0}$$

$$\Rightarrow \qquad (4.2.48)$$

$$\boxed{\delta P_{3-4} + (0.4473)\delta P_{3-8} - (0.7071)\delta P_{3-12} = 0}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -\left[P_{3-12} + \delta P_{3-12}\right]\cos 45^\circ - \left[P_{3-13} + \delta P_{3-13}\right] \\ -\left[P_{3-8} + \delta P_{3-8}\right]\sin 63.43^\circ \end{array} \right\} = 0$$

Indep. Eq. 6

$$\boxed{P_{3-8} + (0.7906)P_{3-12} + (1.118)P_{3-13} = 0}$$

$$\Rightarrow \qquad (4.2.49)$$

$$\boxed{\delta P_{3-8} + (0.7906)\delta P_{3-12} + (1.118)\delta P_{3-13} = 0}$$

4.66

Joint 4

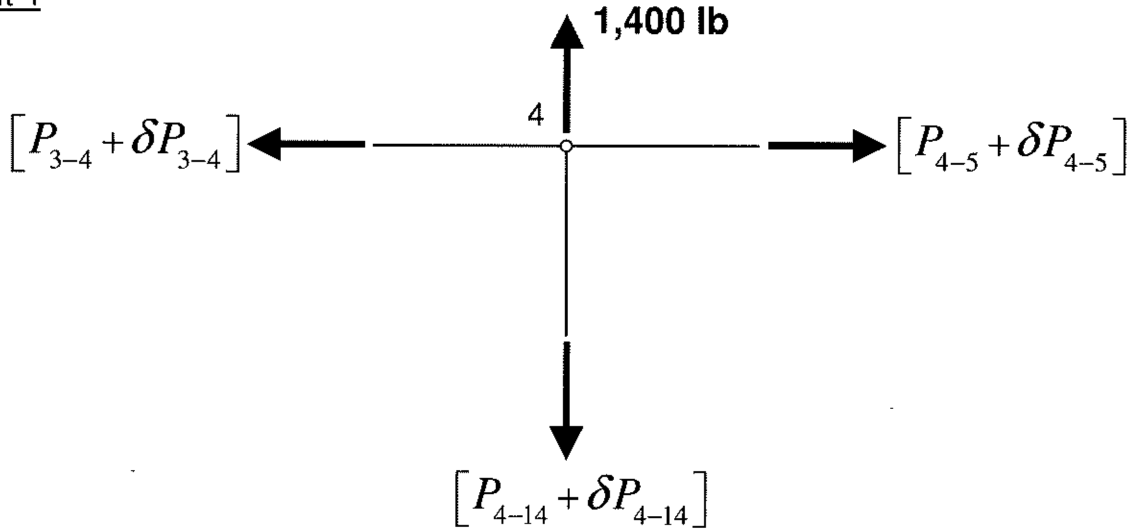


Figure 4.2.12

$$\sum F_x = 0$$

$\Rightarrow$ $$-\left[P_{3-4}+\delta P_{3-4}\right]+\left[P_{4-5}+\delta P_{4-5}\right]=0$$

Indep. Eq. 7

$\Rightarrow$ $$\boxed{P_{3-4}-P_{4-5}=0} \qquad \boxed{\delta P_{3-4}-\delta P_{4-5}=0} \qquad (4.2.50)$$

Indep. Eq. 8

$$\sum F_y = 0$$

$\Rightarrow$ $$\boxed{P_{4-14}=1{,}400\ lb} \qquad \boxed{\delta P_{4-14}=0} \qquad (4.2.51)$$

Joint 5

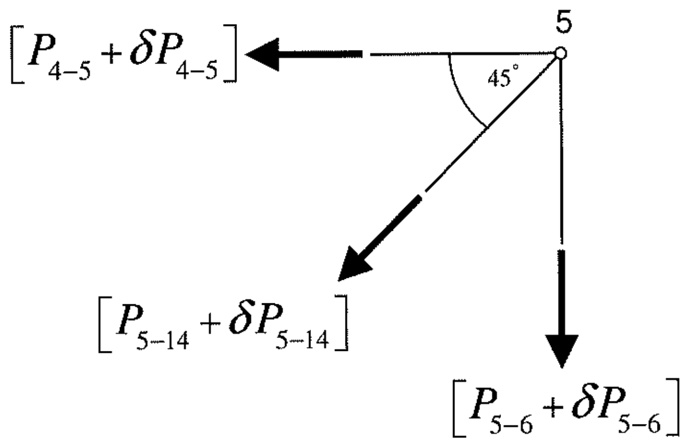


Figure 4.2.13

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{4-5}+\delta P_{4-5}\right]-\left[P_{5-14}+\delta P_{5-14}\right]\cos 45^\circ = 0$$

Indep. Eq. 9

$$\Rightarrow \quad \boxed{P_{4-5}+(0.7071)P_{5-14}=0} \qquad \boxed{\delta P_{4-5}+(0.7071)\delta P_{5-14}=0} \qquad (4.2.52)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad -\left[P_{5-14}+\delta P_{5-14}\right]\sin 45^\circ - \left[P_{5-6}+\delta P_{5-6}\right]=0$$

Indep. Eq. 10

$$\Rightarrow \quad \boxed{P_{5-6}+(0.7071)P_{5-14}=0} \qquad \boxed{\delta P_{5-6}+(0.7071)\delta P_{5-14}=0} \qquad (4.2.53)$$

Joint 6

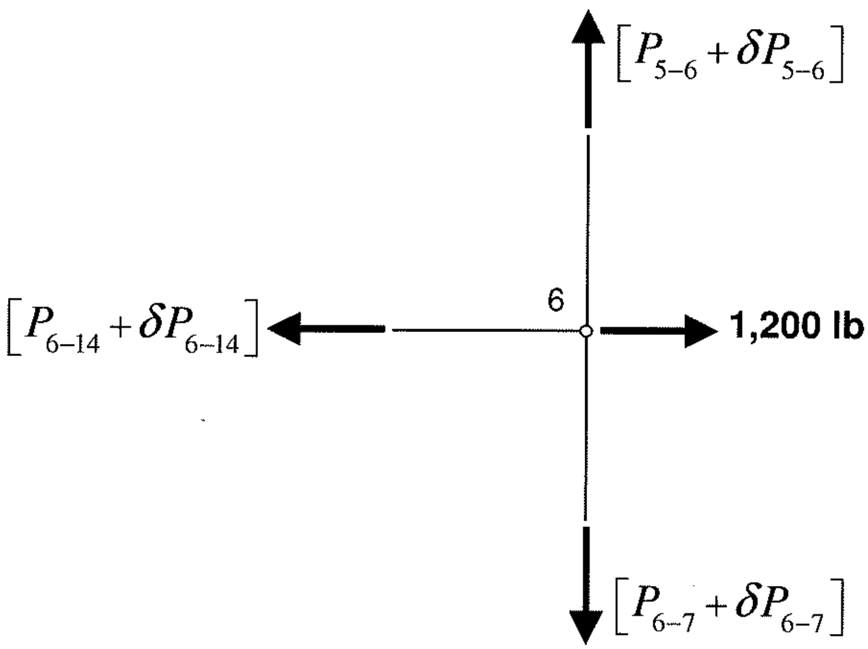


Figure 4.2.14

Indep. Eq. 11

$$\sum F_x = 0$$

$\Rightarrow$ $\boxed{P_{6-14} = 1{,}200\ lb}$ $\boxed{\delta P_{6-14} = 0}$ (4.2.54)

Indep. Eq. 12

$$\sum F_y = 0$$

$\Rightarrow$ $\boxed{P_{5-6} - P_{6-7} = 0}$ $\boxed{\delta P_{5-6} - \delta P_{6-7} = 0}$ (4.2.55)

Joint 7

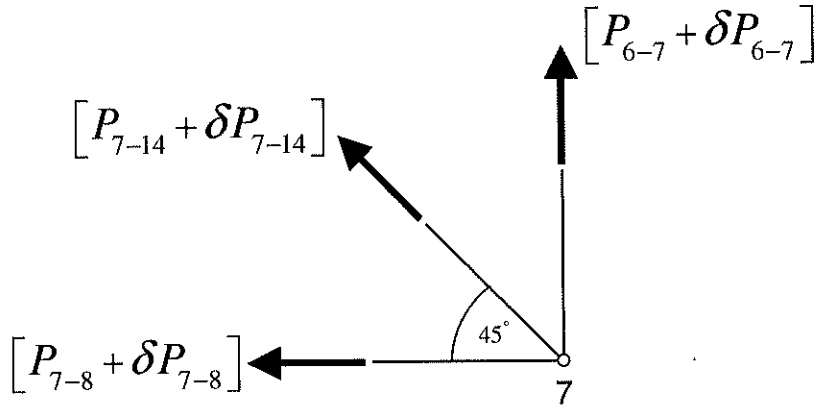


Figure 4.2.15

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{7-8}+\delta P_{7-8}\right]-\left[P_{7-14}+\delta P_{7-14}\right]\cos 45^\circ = 0$$

Indep. Eq. 13

$$\Rightarrow \quad \boxed{P_{7-8}+(0.7071)P_{7-14}=0} \qquad \boxed{\delta P_{7-8}+(0.7071)\delta P_{7-14}=0} \qquad (4.2.56)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[P_{7-14}+\delta P_{7-14}\right]\sin 45^\circ + \left[P_{6-7}+\delta P_{6-7}\right] = 0$$

Indep. Eq. 14

$$\Rightarrow \quad \boxed{P_{6-7}+(0.7071)P_{7-14}=0} \qquad \boxed{\delta P_{6-7}+(0.7071)\delta P_{7-14}=0} \qquad (4.2.57)$$

4.70

Joint 8

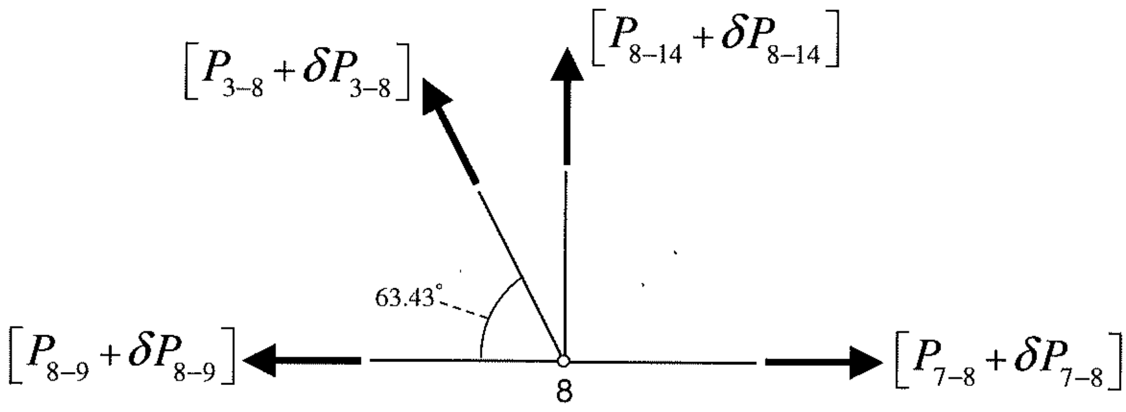


Figure 4.2.16

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{8-9}+\delta P_{8-9}\right]-\left[P_{3-8}+\delta P_{3-8}\right]\cos 63.43^\circ+\left[P_{7-8}+\delta P_{7-8}\right]=0$$

Indep. Eq. 15

$$\Rightarrow \quad \boxed{P_{3-8}-(2.236)P_{7-8}+(2.236)P_{8-9}=0} \qquad \boxed{\delta P_{3-8}-(2.236)\delta P_{7-8}+(2.236)\delta P_{8-9}=0} \qquad (4.2.58)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[P_{3-8}+\delta P_{3-8}\right]\sin 63.43^\circ+\left[P_{8-14}+\delta P_{8-14}\right]=0$$

Indep. Eq. 16

$$\Rightarrow \quad \boxed{P_{3-8}+(1.118)P_{8-14}=0} \qquad \boxed{\delta P_{3-8}+(1.118)\delta P_{8-14}=0} \qquad (4.2.59)$$

Joint 9

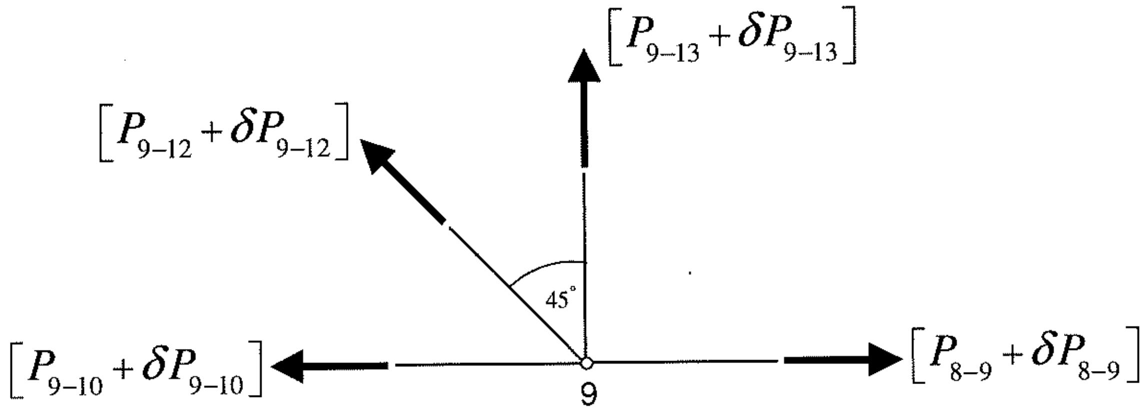


Figure 4.2.17

$$\sum F_x = 0$$

$$\Rightarrow \quad -\left[P_{9-10}+\delta P_{9-10}\right]-\left[P_{9-12}+\delta P_{9-12}\right]\sin 45^\circ+\left[P_{8-9}+\delta P_{8-9}\right]=0$$

Indep. Eq. 17

$$\boxed{P_{8-9}-P_{9-10}-(0.7071)P_{9-12}=0}$$

$$\Rightarrow \qquad (4.2.60)$$

$$\boxed{\delta P_{8-9}-\delta P_{9-10}-(0.7071)\delta P_{9-12}=0}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[P_{9-12}+\delta P_{9-12}\right]\cos 45^\circ+\left[P_{9-13}+\delta P_{9-13}\right]=0$$

Indep. Eq. 18

$$\Rightarrow \quad \boxed{P_{9-12}+(1.414)P_{9-13}=0} \qquad \boxed{\delta P_{9-12}+(1.414)\delta P_{9-13}=0} \qquad (4.2.61)$$

Joint 10

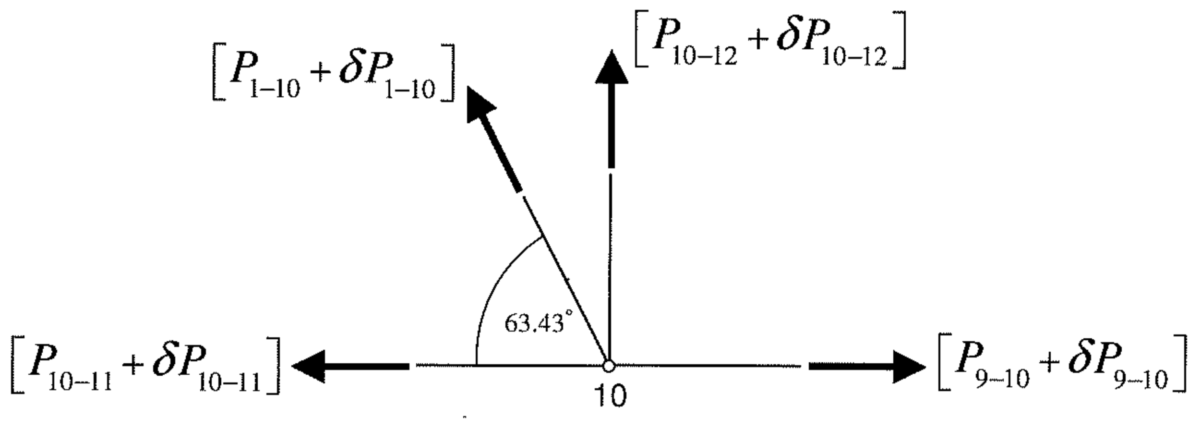


Figure 4.2.18

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -\left[P_{10-11}+\delta P_{10-11}\right]-\left[P_{1-10}+\delta P_{1-10}\right]\cos 63.43^\circ \\ +\left[P_{9-10}+\delta P_{9-10}\right] \end{array} \right\} = 0$$

Indep. Eq. 19

$$\Rightarrow \quad \boxed{P_{1-10}-(2.236)P_{9-10}+(2.236)P_{10-11}=0} \qquad (4.2.62)$$

$$\boxed{\delta P_{1-10}-(2.236)\delta P_{9-10}+(2.236)\delta P_{10-11}=0}$$

$$\sum F_y = 0$$

Indep. Eq. 20

$$\Rightarrow \quad \left[P_{1-10}+\delta P_{1-10}\right]\sin 63.43^\circ+\left[P_{10-12}+\delta P_{10-12}\right]=0$$

$$\Rightarrow \quad \boxed{P_{1-10}+(1.118)P_{10-12}=0} \qquad \boxed{\delta P_{1-10}+(1.118)\delta P_{10-12}=0} \qquad (4.2.63)$$

Joint 11

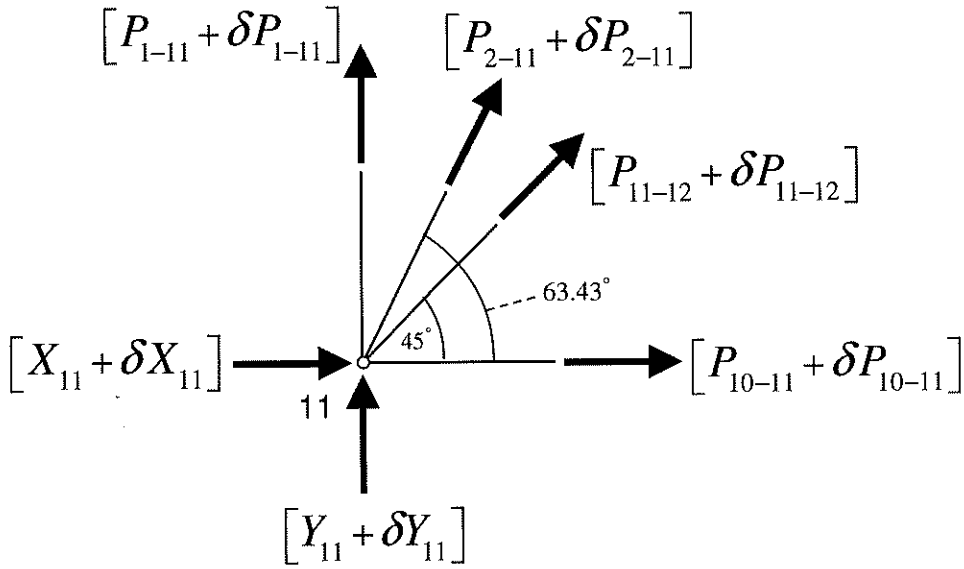


Figure 4.2.19

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[X_{11}+\delta X_{11}\right]+\left[P_{2-11}+\delta P_{2-11}\right]\cos 63.43^\circ \\ +\left[P_{11-12}+\delta P_{11-12}\right]\cos 45^\circ+\left[P_{10-11}+\delta P_{10-11}\right] \end{array} \right\} = 0$$

Indep. Eq. 21

$$\boxed{P_{2-11}+(2.236)P_{10-11}+(1.581)P_{11-12}+(2.236)X_{11}=0}$$

$$\Rightarrow \qquad (4.2.64)$$

$$\boxed{\delta P_{2-11}+(2.236)\delta P_{10-11}+(1.581)\delta P_{11-12}+(2.236)\delta X_{11}=0}$$

4.74

$$\sum F_y = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[Y_{11} + \delta Y_{11}\right] + \left[P_{1-11} + \delta P_{1-11}\right] \\ + \left[P_{2-11} + \delta P_{2-11}\right] \sin 63.43^\circ + \left[P_{11-12} + \delta P_{11-12}\right] \sin 45^\circ \end{array} \right\} = 0$$

Indep. Eq. 22

$$\Rightarrow \quad \boxed{P_{1-11} + (0.8944) P_{2-11} + (0.7071) P_{11-12} + Y_{11} = 0}$$

(4.2.65)

$$\boxed{\delta P_{1-11} + (0.8944) \delta P_{2-11} + (0.7071) \delta P_{11-12} + \delta Y_{11} = 0}$$

Joint 12

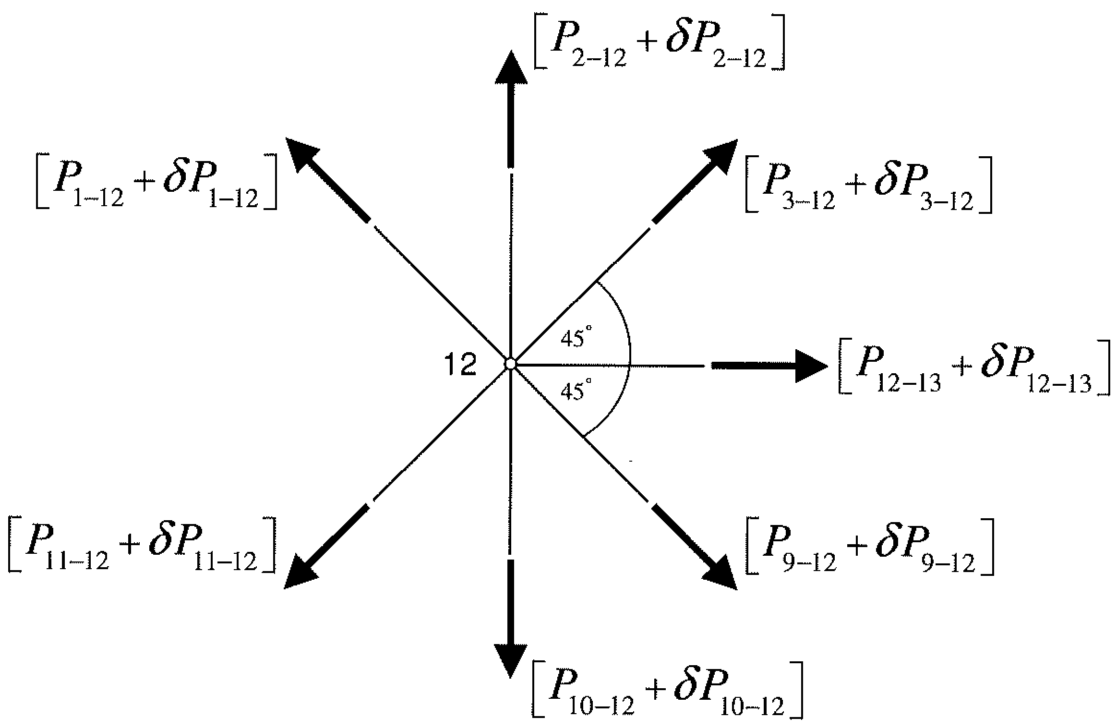


Figure 4.2.20

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -\left[P_{1-12}+\delta P_{1-12}\right]\cos 45^\circ + \left[P_{3-12}+\delta P_{3-12}\right]\cos 45^\circ \\ +\left[P_{12-13}+\delta P_{12-13}\right] + \left[P_{9-12}+\delta P_{9-12}\right]\cos 45^\circ \\ -\left[P_{11-12}+\delta P_{11-12}\right]\cos 45^\circ \end{array} \right\} = 0$$

Indep.
Eq.
23

$$\boxed{P_{1-12} - P_{3-12} - P_{9-12} + P_{11-12} - (1.414)P_{12-13} = 0}$$

$$\Rightarrow$$

$$\boxed{\delta P_{1-12} - \delta P_{3-12} - \delta P_{9-12} + \delta P_{11-12} - (1.414)\delta P_{12-13} = 0} \qquad (4.2.66)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[P_{1-12}+\delta P_{1-12}\right]\sin 45^\circ + \left[P_{2-12}+\delta P_{2-12}\right] \\ +\left[P_{3-12}+\delta P_{3-12}\right]\sin 45^\circ - \left[P_{9-12}+\delta P_{9-12}\right]\sin 45^\circ \\ -\left[P_{10-12}+\delta P_{10-12}\right] - \left[P_{11-12}+\delta P_{11-12}\right]\sin 45^\circ \end{array} \right\} = 0$$

Indep.
Eq.
24

$$\boxed{\left\{ \begin{array}{l} P_{1-12} + (1.414)P_{2-12} + P_{3-12} - P_{9-12} \\ -(1.414)P_{10-12} - P_{11-12} \end{array} \right\} = 0}$$

$$\Rightarrow$$

$$\boxed{\left\{ \begin{array}{l} \delta P_{1-12} + (1.414)\delta P_{2-12} + \delta P_{3-12} - \delta P_{9-12} \\ -(1.414)\delta P_{10-12} - \delta P_{11-12} \end{array} \right\} = 0} \qquad (4.2.67)$$

Joint 13

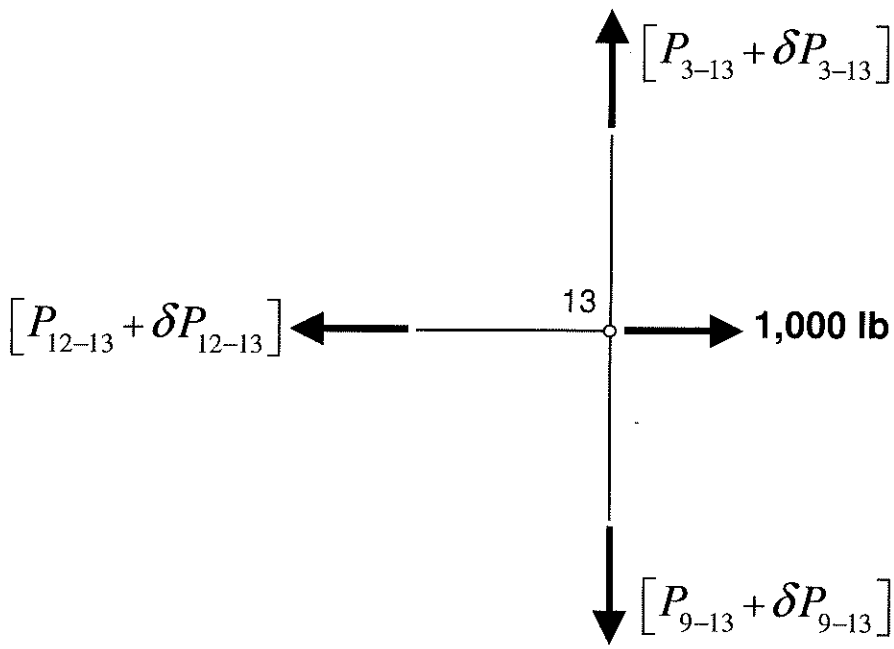


Figure 4.2.21

Indep. Eq. 25

$$\sum F_x = 0$$

$\Rightarrow$ $\boxed{P_{12-13} = 1{,}000\ lb}$ $\boxed{\delta P_{12-13} = 0}$ (4.2.68)

Indep. Eq. 26

$$\sum F_y = 0$$

$\Rightarrow$ $\boxed{P_{3-13} - P_{9-13} = 0}$ $\boxed{\delta P_{3-13} - \delta P_{9-13} = 0}$ (4.2.69)

Joint 14

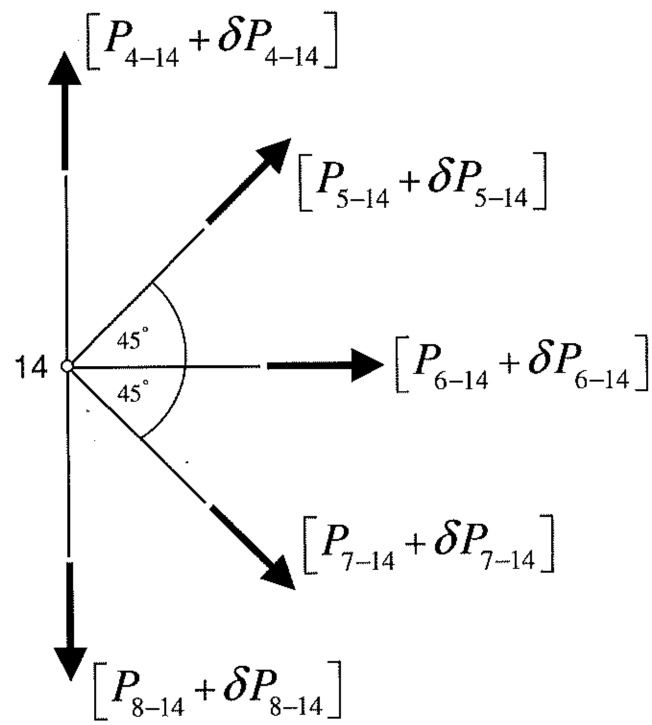


Figure 4.2.22

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[P_{5-14}+\delta P_{5-14}\right]\cos 45^\circ + \left[P_{6-14}+\delta P_{6-14}\right] \\ +\left[P_{7-14}+\delta P_{7-14}\right]\cos 45^\circ \end{array} \right\} = 0$$

$$\boxed{P_{5-14} + (1.414) P_{6-14} + P_{7-14} = 0}$$

Indep. Eq. 27

$$\Rightarrow \quad \boxed{\delta P_{5-14} + (1.414)\delta P_{6-14} + \delta P_{7-14} = 0} \qquad (4.2.70)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[P_{4-14} + \delta P_{4-14} \right] + \left[P_{5-14} + \delta P_{5-14} \right] \sin 45^\circ \\ - \left[P_{7-14} + \delta P_{7-14} \right] \sin 45^\circ - \left[P_{8-14} + \delta P_{8-14} \right] \end{array} \right\} = 0$$

Indep.
Eq.
28

$$\boxed{P_{4-14} + (0.7071) P_{5-14} - (0.7071) P_{7-14} - P_{8-14} = 0}$$

$$\Rightarrow \qquad (4.2.71)$$

$$\boxed{\delta P_{4-14} + (0.7071) \delta P_{5-14} - (0.7071) \delta P_{7-14} - \delta P_{8-14} = 0}$$

We now have our 28 independent static equilibrium equations.

Note that all of the equations for the virtuals are simply the differential of the equations for the reals.

* **The 3 equations below yielded numerical values for some of the unknowns.**

* **These values will now be substituted into any other equations where they occur.**

1 $$P_{4-14} = 1{,}400\ lb \qquad (4.2.51)$$

2 $$P_{6-14} = 1{,}200\ lb \qquad (4.2.54)$$

3 $$P_{12-13} = 1{,}000\ lb \qquad (4.2.68)$$

SUMMARY #1

The 25 remaining independent equations from static equilibrium are listed below.

1 $$P_{1-2}+(0.4473)P_{1-10}+(0.7071)P_{1-12}+X_1=0 \quad (4.2.44)$$

2 $$P_{1-2}+(0.4473)P_{2-11}=0 \quad (4.2.46)$$

3 $$P_{1-10}+(1.118)P_{1-11}+(0.7906)P_{1-12}=0 \quad (4.2.45)$$

4 $$P_{1-10}+(2.236)P_{10-11}=0 \quad (4.2.62)$$

5 $$P_{1-10}+(1.118)P_{10-12}=0 \quad (4.2.63)$$

6 $$P_{1-11}+(0.8944)P_{2-11}+(0.7071)P_{11-12}+Y_{11}=0 \quad (4.2.65)$$

7 $$P_{1-12}+(1.414)P_{2-12}+P_{3-12}-P_{9-12}-(1.414)P_{10-12}-P_{11-12}=0 \quad (4.2.67)$$

8 $$P_{1-12}-P_{3-12}-P_{9-12}+P_{11-12}=1{,}414 \quad (4.2.66)$$

9 $$P_{2-11}+(1.118)P_{2-12}=0 \quad (4.2.47)$$

10 $$P_{2-11}+(2.236)P_{10-11}+(1.581)P_{11-12}+(2.236)X_{11}=0 \quad (4.2.64)$$

11 $$P_{3-4}+(0.4473)P_{3-8}-(0.7071)P_{3-12}=0 \quad (4.2.48)$$

12 $$P_{3-4}-P_{4-5}=0 \quad (4.2.50)$$

13 $$P_{3-8}+(0.7906)P_{3-12}+(1.118)P_{3-13}=0 \quad (4.2.49)$$

4.80

14 $$P_{3-8}-(2.236)P_{7-8}+(2.236)P_{8-9}=0 \quad (4.2.58)$$

15 $$P_{3-8}+(1.118)P_{8-14}=0 \quad (4.2.59)$$

16 $$P_{3-13}-P_{9-13}=0 \quad (4.2.69)$$

17 $$P_{4-5}+(0.7071)P_{5-14}=0 \quad (4.2.52)$$

18 $$P_{5-6}+(0.7071)P_{5-14}=0 \quad (4.2.53)$$

19 $$P_{5-6}-P_{6-7}=0 \quad (4.2.55)$$

20 $$P_{5-14}+P_{7-14}=-1{,}697 \quad (4.2.70)$$

21 $$P_{5-14}-P_{7-14}-(1.414)P_{8-14}=-1{,}980 \quad (4.2.71)$$

22 $$P_{6-7}+(0.7071)P_{7-14}=0 \quad (4.2.57)$$

23 $$P_{7-8}+(0.7071)P_{7-14}=0 \quad (4.2.56)$$

24 $$P_{8-9}-P_{9-10}-(0.7071)P_{9-12}=0 \quad (4.2.60)$$

25 $$P_{9-12}+(1.414)P_{9-13}=0 \quad (4.2.61)$$

* From Page 4.56 we learned that this problem has a redundancy of 2.

* From Page 4.56, we also learned that this problem yields a total of 28 independent static equilibrium equations, but there are a total of 30 unknowns.

* Of our 30 unknowns, 3 values have already been found from the static equilibrium equations.

* There are 27 unknowns yet to be found.

* The equations on Pages 4.79 and 4.80 show our 28 static equilibrium equations.

* 3 of the 28 static equilibrium equations have yielded values for 3 of the unknowns.

* We have 25 of our static equilibrium equations remaining to help us determine values for the 27 unknowns yet to be found.

* The additional 2 equations needed to obtain values for the remaining 27 unknowns will come from the Principle of Complementary Virtual Work.

* Before beginning the Principle of Complementary Virtual Work, we must decide which of the unknowns will be considered as our 2 redundants.

* To choose our redundants we will use Method 1 as outlined on Page 4.57.

4.82

Choice of Redundants for This Truss

Step 1 Frequency Table

The table below shows the frequency in which each of the 27 variables appear in the 25 independent static equilibrium equations.

Variable Frequency Table

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
	Eq. Nos.	44	45	46	47	48	49	50	52	53	55	56	57	58	59	60	61	62	63	64	65	66	67	69	70	71	**Freq**
1	**P-1-2**	X		X																							2
2	**P-1-10**	X	X															X	X								4
3	**P-1-11**		X																		X						2
4	**P-1-12**	X	X																			X	X				4
5	**P-2-11**			X	X															X	X						4
6	**P-2-12**				X																		X				2
7	**P-3-4**					X		X																			2
8	**P-3-8**					X	X							X	X												4
9	**P-3-12**					X	X															X	X				4
10	**P-3-13**						X																	X			2
11	**P-4-5**							X	X																		2
12	**P-5-6**									X	X																2
13	**P-5-14**								X	X															X	X	4
14	**P-6-7**										X		X														2
15	**P-7-8**											X		X													2
16	**P-7-14**											X	X												X	X	4
17	**P-8-9**													X		X											2
18	**P-8-14**														X											X	2
19	**P-9-10**															X		X									2
20	**P-9-12**															X	X					X	X				4
21	**P-9-13**																X							X			2
22	**P-10-11**																	X		X							2
23	**P-10-12**																		X				X				2
24	**P-11-12**																			X	X	X	X				4
25	**X-1**	X																									1
26	**X-11**																			X							1
27	**Y-11**																				X						1

* In the Frequency Table on Page 4.82, the unknowns are listed in the usual order in the 1st column.

Step 2 Move Rows So That the X's in Each Column are as Close as Possible
(Beginning with the 1st Column with X's, then the 2nd, etc.)

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
	Eq. Nos.	44	45	46	47	48	49	50	52	53	55	56	57	58	59	60	61	62	63	64	65	66	67	69	70	71	**Freq**
1	**P**$_{-1-2}$	X		X																							2
2	**P**$_{-1-10}$	X	X															X	X								4
3	**P**$_{-1-12}$	X	X																			X	X				4
4	**X**$_{-1}$	X																									1
5	**P**$_{-1-11}$		X																		X						2
6	**P**$_{-2-11}$			X	X															X	X						4
7	**P**$_{-2-12}$				X																		X				2
8	**P**$_{-3-4}$					X		X																			2
9	**P**$_{-3-8}$					X	X							X	X												4
10	**P**$_{-3-12}$					X	X															X	X				4
11	**P**$_{-3-13}$						X																	X			2
12	**P**$_{-4-5}$							X	X																		2
13	**P**$_{-5-14}$								X	X															X	X	4
14	**P**$_{-5-6}$									X	X																2
15	**P**$_{-6-7}$										X		X														2
16	**P**$_{-7-8}$											X		X													2
17	**P**$_{-7-14}$											X	X												X	X	4
18	**P**$_{-8-9}$													X		X											2
19	**P**$_{-8-14}$														X											X	2
20	**P**$_{-9-10}$															X		X									2
21	**P**$_{-9-12}$															X	X					X	X				4
22	**P**$_{-9-13}$																X							X			2
23	**P**$_{-10-11}$																	X		X							2
24	**P**$_{-10-12}$																		X				X				2
25	**P**$_{-11-12}$																			X	X	X	X				4
26	**X**$_{-11}$																			X							1
27	**Y**$_{-11}$																				X						1

4.84

Step 3 Move Columns So That the X's in Each Row are as Close as Possible

(Beginning with the 1st Row with X's, then the 2nd, etc.)

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
	Eq. Nos.	44	46	45	62	63	66	67	65	47	64	48	50	49	58	59	69	52	53	70	71	55	57	56	60	61	**Freq**
1	**P-1-2**	X	X																								2
2	**P-1-10**	X		X	X	X																					**4**
3	**P-1-12**	X		X			X	X																			**4**
4	**X-1**	X																									1
5	**P-1-11**			X					X																		2
6	**P-2-11**		X						X	X	X																**4**
7	**P-2-12**							X		X																	2
8	**P-3-4**											X	X														2
9	**P-3-8**											X		X	X	X											**4**
10	**P-3-12**						X	X				X		X													**4**
11	**P-3-13**													X			X										2
12	**P-4-5**												X					X									2
13	**P-5-14**																	X	X	X	X						**4**
14	**P-5-6**																		X			X					2
15	**P-6-7**																					X	X				2
16	**P-7-8**														X									X			2
17	**P-7-14**																			X	X		X	X			**4**
18	**P-8-9**														X										X		2
19	**P-8-14**															X					X						2
20	**P-9-10**				X																				X		2
21	**P-9-12**						X	X																	X	X	**4**
22	**P-9-13**																X									X	2
23	**P-10-11**				X						X																2
24	**P-10-12**					X		X																			2
25	**P-11-12**						X	X	X		X																**4**
26	**X-11**										X																1
27	**Y-11**								X																		1

Step 4 Identify Unknowns That Can Be Solved for Numerical Values

												1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25		
	Eq. Nos.	44	46	45	62	63	66	67	65	47	64	48	50	49	58	59	69	52	53	70	71	55	57	56	60	61	Freq	
1	**P-1-2**	X	X																								2	
2	**P-1-10**	X		X	X	X																					**4**	
3	**P-1-12**	X		X			X	X																			**4**	
4	**X-1**	X																									1	
5	**P-1-11**			X					X																		2	
6	**P-2-11**		X						X	X	X																**4**	
7	**P-2-12**							X		X																	2	
8	**P-3-4**											X	X														2	1
9	**P-3-8**											X		X	X	X											**4**	2
10	**P-3-12**						X	X				X		X													**4**	3
11	**P-3-13**													X			X										2	4
12	**P-4-5**												X					X									2	5
13	**P-5-14**																	X	X	X	X						**4**	6
14	**P-5-6**																		X			X					2	7
15	**P-6-7**																					X	X				2	8
16	**P-7-8**														X									X			2	9
17	**P-7-14**																			X	X		X	X			**4**	10
18	**P-8-9**														X										X		2	11
19	**P-8-14**															X					X						2	12
20	**P-9-10**				X																				X		2	13
21	**P-9-12**						X	X																	X	X	**4**	14
22	**P-9-13**																X									X	2	15
23	**P-10-11**				X						X																2	
24	**P-10-12**					X		X																			2	
25	**P-11-12**						X	X	X		X																**4**	
26	**X-11**										X																1	
27	**Y-11**								X																		1	

* In the Frequency Table on Page 4.85 the yellow cells show 15 unknowns with 15 equations to solve for them. Hence we can get numerical values for these unknowns.

Step 5 Place Green Columns Together (if possible)

												1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25		
	Eq. Nos.	44	46	45	62	63	66	67	65	47	64	48	50	49	58	59	69	52	53	70	71	55	57	56	60	61	Freq	
1	P-1-2	X	X																								2	
2	P-1-10	X		X	X	X																					4	
3	P-1-12	X		X			X	X																			4	
4	X-1	X																									1	
5	P-1-11			X					X																		2	
6	P-2-11		X						X	X	X																4	
7	P-2-12							X		X																	2	
8	P-10-11				X						X																2	
9	P-10-12					X		X																			2	
10	P-11-12						X	X	X		X																4	
11	X-11										X																1	
12	Y-11								X																		1	
13	P-3-4											X	X														2	1
14	P-3-8											X		X	X	X											4	2
15	P-3-12						X	X				X		X													4	3
16	P-3-13													X			X										2	4
17	P-4-5												X					X									2	5
18	P-5-14																	X	X	X	X						4	6
19	P-5-6																		X			X					2	7
20	P-6-7																					X	X				2	8
21	P-7-8														X									X			2	9
22	P-7-14																			X	X		X	X			4	10
23	P-8-9														X										X		2	11
24	P-8-14															X					X						2	12
25	P-9-10				X																				X		2	13
26	P-9-12						X	X																	X	X	4	14
27	P-9-13																X									X	2	15

* In the Frequency Table on Page 5.86 the yellow cells show 15 unknowns with 15 equations to solve for them. Hence we can get numerical values for these unknowns.

None of these 15 can be used as a redundant.

* If any of these 15 had been chosen as a redundant prior to implementing Steps 1-4, the 25 equations obtained would not have all been independent.

* Thus, we would not have been able to solve for the 25 unknowns in terms of the redundants.

Note:

- The movement of the rows and columns in the frequency table should help to decipher which unknowns numerical values could be obtained for from the static equilibrium equations alone.
- In this particular example the 15 equations with 15 unknowns were shifted to the lower right portion of the frequency table as shown in the table on Page 4.85.
- The frequency table on Page 6.41 shows a situation in which they were shifted to the upper left portion of the table.
- To check for unknowns in which numerical values could be obtained from the static equilibrium equations alone, the upper left portion of the table and the lower right portion of the table should both be checked.

4.88

The 15 equations to solve for the 15 unknowns are listed below.

1	$P_{3-4}+(0.4473)P_{3-8}-(0.7071)P_{3-12}=0$	(4.2.48)
2	$P_{3-4}-P_{4-5}=0$	(4.2.50)
3	$P_{3-8}+(0.7906)P_{3-12}+(1.118)P_{3-13}=0$	(4.2.49)
4	$P_{3-8}-(2.236)P_{7-8}+(2.236)P_{8-9}=0$	(4.2.58)
5	$P_{3-8}+(1.118)P_{8-14}=0$	(4.2.59)
6	$P_{3-13}-P_{9-13}=0$	(4.2.69)
7	$P_{4-5}+(0.7071)P_{5-14}=0$	(4.2.52)
8	$P_{4-14}+(0.7071)P_{5-14}-(0.7071)P_{7-14}-P_{8-14}=0$	(4.2.71)
9	$P_{5-6}+(0.7071)P_{5-14}=0$	(4.2.53)
10	$P_{5-6}-P_{6-7}=0$	(4.2.55)
11	$P_{5-14}+(1.414)P_{6-14}+P_{7-14}=0$	(4.2.70)
12	$P_{6-7}+(0.7071)P_{7-14}=0$	(4.2.57)
13	$P_{7-8}+(0.7071)P_{7-14}=0$	(4.2.56)
14	$P_{8-9}-P_{9-10}-(0.7071)P_{9-12}=0$	(4.2.60)
15	$P_{9-12}+(1.414)P_{9-13}=0$	(4.2.61)

The solutions to these 15 equations are given below.

1 $$P_{3-4} = 600\ lb \tag{4.2.72}$$

2 $$P_{3-8} = -1{,}565\ lb \tag{4.2.73}$$

3 $$P_{3-12} = -141.7\ lb \tag{4.2.74}$$

4 $$P_{3-13} = 1{,}500\ lb \tag{4.2.75}$$

5 $$P_{4-5} = 600\ lb \tag{4.2.76}$$

6 $$P_{5-6} = 600\ lb \tag{4.2.77}$$

7 $$P_{5-14} = -848.4\ lb \tag{4.2.78}$$

8 $$P_{6-7} = -600\ lb \tag{4.2.79}$$

9 $$P_{7-8} = 600\ lb \tag{4.2.80}$$

10 $$P_{7-14} = -848.4\ lb \tag{4.2.81}$$

11 $$P_{8-9} = 1{,}300\ lb \tag{4.2.82}$$

12 $$P_{8-14} = 1{,}400\ lb \tag{4.2.83}$$

13 $$P_{9-10} = 2{,}800\ lb \tag{4.2.84}$$

14 $$P_{9-12} = -2{,}121\ lb \tag{4.2.85}$$

15 $$P_{9-13} = 1{,}500\ lb \tag{4.2.86}$$

SUMMARY #2

The revised 28 independent equations from static equilibrium are listed below.

1	$P_{3-4} = 600\ lb$	(4.2.72)
2	$P_{3-8} = -1{,}565\ lb$	(4.2.73)
3	$P_{3-12} = -141.7\ lb$	(4.2.74)
4	$P_{3-13} = 1{,}500\ lb$	(4.2.75)
5	$P_{4-5} = 600\ lb$	(4.2.76)
6	$P_{4-14} = 1{,}400\ lb$	(4.2.51)
7	$P_{5-6} = 600\ lb$	(4.2.77)
8	$P_{5-14} = -848.4\ lb$	(4.2.78)
9	$P_{6-7} = -600\ lb$	(4.2.79)
10	$P_{6-14} = 1{,}200\ lb$	(4.2.54)
11	$P_{7-8} = 600\ lb$	(4.2.80)
12	$P_{7-14} = -848.4\ lb$	(4.2.81)
13	$P_{8-9} = 1{,}300\ lb$	(4.2.82)
14	$P_{8-14} = 1{,}400\ lb$	(4.2.83)
15	$P_{9-10} = 2{,}800\ lb$	(4.2.84)
16	$P_{9-12} = -2{,}121\ lb$	(4.2.85)

17 $$P_{9-13} = 1{,}500\ lb$$ (4.2.86)

18 $$P_{12-13} = 1{,}000\ lb$$ (4.2.68)

19 $$P_{1-2} + (0.4473)P_{1-10} + (0.7071)P_{1-12} + X_1 = 0$$ (4.2.44)

20 $$P_{1-2} + (0.4473)P_{2-11} = 0$$ (4.2.46)

21 $$P_{1-10} + (1.118)P_{1-11} + (0.7906)P_{1-12} = 0$$ (4.2.45)

22 $$P_{1-10} - (2.236)P_{9-10} + (2.236)P_{10-11} = 0$$ (4.2.62)

23 $$P_{1-10} + (1.118)P_{10-12} = 0$$ (4.2.63)

24 $$P_{1-11} + (0.8944)P_{2-11} + (0.7071)P_{11-12} + Y_{11} = 0$$ (4.2.65)

25 $$P_{1-12} + (1.414)P_{2-12} + P_{3-12} - P_{9-12} - (1.414)P_{10-12} - P_{11-12} = 0$$ (4.2.67)

26 $$P_{1-12} - P_{3-12} - P_{9-12} + P_{11-12} - (1.414)P_{12-13} = 0$$ (4.2.66)

27 $$P_{2-11} + (1.118)P_{2-12} = 0$$ (4.2.47)

28 $$P_{2-11} + (2.236)P_{10-11} + (1.581)P_{11-12} + (2.236)X_{11} = 0$$ (4.2.64)

* From Page 4.56 we learned that this problem has a redundancy of 2.

* From Page 4.56, we also learned that this problem yields a total of 28 independent static equilibrium equations, but there are a total of 30 unknowns.

* Of our 30 unknowns, 18 values have already been found from the static equilibrium equations.

* There are 12 unknowns yet to be found.

* The equations on Pages 4.79 and 4.80 show our 28 static equilibrium equations.

* 18 of the 28 static equilibrium equations have yielded values for 18 of the unknowns.

* We have 10 of our static equilibrium equations remaining to help us determine values for the 12 unknowns yet to be found.

* The additional 2 equations needed to obtain values for the remaining 12 unknowns will come from the Principle of Complementary Virtual Work.

* Before beginning the Principle of Complementary Virtual Work, we must decide which of the unknowns will be considered as our 2 redundants.

* To choose our redundants we will use Method 1 as outlined on Page 4.57.

The redundants must be chosen from the 12 unknowns 1 through 12 as indicated on the Frequency Table on Page 4.86.

The bottom two stars in the box below were brought to light by Nicholas Fernandez, a student in AE 418, Fall 2018

Rule Set #1 for Choosing Redundants

* Any unknowns in which a numerical value can be found from the static equilibrium equations cannot be chosen as a redundant.
* Redundants must be independent of each other.
* Thus if any of the equation columns in the Frequency Table obtained from Step 3 have only 2 X's in the column, then those 2 unknowns cannot be used together as redundants, since they can be related to each other & would not be independent.
* All unknowns in a column of the table cannot be used together as redundants.
* If all of the values in a column are used as the redundants, that equation becomes unusable since there would be no unknown left in that equation to solve in terms of a redundant.
  Hence there would be a shortage of one equation when trying to solve for the 10 unknowns in terms of the 2 redundants.

All the unknowns in each of the 12 columns in the green portion of the table are listed below. [The column that each appears in is shown in brackets.]

P_{1-2} -- P_{1-10} -- P_{1-12} -- X_1 [1]

P_{1-2} -- P_{2-11} [2]

P_{1-10} -- P_{1-12} -- P_{1-11} [3]

P_{1-10} -- P_{10-11} [4]

P_{1-10} -- P_{10-12} [5]

P_{1-12} -- P_{11-12} [6]

P_{1-12} -- P_{2-12} -- P_{10-12} -- P_{11-12} [7]

P_{1-11} -- P_{2-11} -- P_{11-12} -- Y_{11} [8]

P_{2-11} -- P_{2-12} [9]

P_{2-11} -- P_{10-11} -- P_{11-12} -- X_{11} [10]

Rearranging the order,

P_{1-2} -- P_{2-11} [2] P_{1-10} -- P_{10-11} [4] P_{1-10} -- P_{10-12} [5]

P_{1-12} -- P_{11-12} [6] P_{2-11} -- P_{2-12} [9]

P_{1-10} -- P_{1-12} -- P_{1-11} [3] P_{1-12} -- P_{2-12} -- P_{10-12} -- P_{11-12} [7]

P_{1-2} -- P_{1-10} -- P_{1-12} -- X_1 [1] P_{1-11} -- P_{2-11} -- P_{11-12} -- Y_{11} [8]

P_{2-11} -- P_{10-11} -- P_{11-12} -- X_{11} [10]

Rule Set #2 for Choosing Redundants

We expect that the algebra will be reduced if

(1) P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work Equation (Equation 4.1.5),

and

(2) P values are chosen that have the greatest number of equation frequencies (rows).

Using Rule Set #2 (1)

The unknowns in the last two rows; namely,

P_{1-2} -- P_{1-10} -- P_{1-12} -- X_1 [1]

P_{1-11} -- P_{2-11} -- P_{11-12} -- Y_{11} [8]

P_{2-11} -- P_{10-11} -- P_{11-12} -- X_{11} [10]

can be disregarded since they will not be considered as possible redundants because they all involve external reactions.

Using Rule Set #1, 3rd *

For the 3rd row at the top of Page 4.53-AO; namely,

P_{1-10} -- P_{1-12} -- P_{1-11} [3] P_{1-12} -- P_{2-12} -- P_{10-12} -- P_{11-12} [7]

all of the terms in a particular group cannot be chosen together as the redundants. However, this will not affect us here since we are only going to choose 2 redundants.

Using Rule Set #1, 2nd *

For the 1st and 2nd rows at the top of Page 4.53-AO; namely,

P_{1-2} -- P_{2-11} [2] P_{1-10} -- P_{10-11} [4] P_{1-10} -- P_{10-12} [5]

P_{1-12} -- P_{11-12} [6] P_{2-11} -- P_{2-12} [9]

each of the two terms in a particular group cannot be used together as the two redundants because they can be directly related to each other.

The paths below can be deduced from the relationships above.

P_{1-2} and P_{2-11} [2]

P_{2-11} and P_{2-12} [9]

P_{10-11} and P_{1-10} [4]

P_{1-10} and P_{10-12} [5]

P_{1-12} -- P_{11-12} [6]

The 1st row above indicates that P_{1-2} could be written directly in terms of P_{2-11}.
Plugging this into the 2nd row allows P_{1-2} to be written directly in terms of P_{2-12}.
This process can also be repeated for the 3rd and 4th rows above.

Conclusions:

- None of the these unknowns can be listed together as redundants.

$$P_{1-2} \;,\; P_{2-11} \;,\; P_{2-12}$$

- None of the these unknowns can be listed together as redundants.

$$P_{1-10} \;,\; P_{10-11} \;,\; P_{10-12}$$

- None of the these unknowns can be listed together as redundants.

$$P_{1-12} \;,\; P_{11-12}$$

Using Rule Set #2 (2)

-- The Frequency Table on Page 4.53-AG shows that

$$P_{1-10},\; P_{1-12},\; P_{2-11},\; P_{11-12}$$

have the greatest equation frequencies (rows) with a value of 4 each.

-- However, based on our conclusions above,

$$P_{1-12} \text{ and } P_{11-12}$$

cannot be chosen together as redundants.

-- Hence the best choice for our 2 redundants should be any two unknowns from

$$P_{1-10},\; P_{1-12},\; P_{2-11}$$

or any two unknowns from

$$P_{1-10},\; P_{2-11},\; P_{11-12}.$$

P_{1-10} and P_{1-12} will be chosen as the 2 redundants

The 10 green highlighted equations on Page 4.91 must be solved for the 10 unknowns in terms of P_{1-10}, P_{1-12}.

The results are given below.

1 $$P_{1-2} = -(0.4473)P_{1-10} - (0.7073)P_{1-12} - 1{,}000 \tag{4.2.64}$$

2 $$P_{1-11} = -(0.8945)P_{1-10} - (0.7072)P_{1-12} \tag{4.2.65}$$

3 $$P_{2-11} = P_{1-10} + (1.581)P_{1-12} + 2{,}236 \tag{4.2.66}$$

4 $$P_{2-12} = -(0.8945)P_{1-10} - (1.414)P_{1-12} - 2{,}000 \tag{4.2.67}$$

5 $$P_{10-11} = -(0.4472)P_{1-10} + 2{,}800 \tag{4.2.68}$$

6 $$P_{10-12} = -(0.8945)P_{1-10} \tag{4.2.69}$$

7 $$P_{11-12} = -P_{1-12} - 848.7 \tag{4.2.70}$$

8 $$X_1 = (0.0002287)P_{1-12} + 1{,}000 \tag{4.2.71}$$

9 $$X_{11} = -(0.0001474)P_{1-12} - 3{,}200 \tag{4.2.72}$$

10 $$Y_{11} = (0.00005438)P_{1-10} - (0.00008552)P_{1-12} - 1{,}400 \tag{4.2.73}$$

Recall that, since we have 2 redundants, we need 2 additional equations beyond those of static equilibrium.

The 2 additional equations will come from the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.1.5) for the entire truss system states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \frac{\left(P \Big|_{\substack{\text{In Member Due} \\ \text{Actual Loads}}} \right)_i L_i}{A_i E_i} (\delta P)_i \qquad (4.1.5)$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \overbrace{u_1}^{0} (\delta X_1) \\ \\ + \overbrace{u_{11}}^{0} (\delta X_{11}) \\ \\ + \overbrace{v_{11}}^{0} (\delta Y_{11}) \end{array} \right\} = \left[\overbrace{\frac{\begin{bmatrix} -(0.4473) P_{1-10} \\ -(0.7073) P_{1-12} \\ -1{,}000 \end{bmatrix} (40")}{\underbrace{\left[\pi \underbrace{\left(\frac{1.125"}{2} \right)}_{0.5625"}^2 - \pi \left(0.5625" - \frac{1}{16}" \right)^2 \right]}_{0.2086\ in^2} (30 x 10^6\ psi)} \begin{bmatrix} -(0.4473) \delta P_{1-10} \\ -(0.7073) \delta P_{1-12} \end{bmatrix}}^{\text{Member 1-2}} \quad \cdots \right]$$

•••

$$+\frac{\overbrace{[P_{1-10}]\overbrace{\sqrt{(40")^2+(80")^2}}^{89.44"}}^{\text{Member 1-10}}}{(0.2086\ in^2)(30\,x10^6\,psi)}[\delta P_{1-10}]$$

$$+\overbrace{\frac{\begin{bmatrix}-(0.8945)P_{1-10}\\-(0.7072)P_{1-12}\end{bmatrix}(80")}{(0.2086\ in^2)(30\,x10^6\,psi)}\left[-(0.8945)\delta P_{1-10}-(0.7072)\delta P_{1-12}\right]}^{\text{Member 1-11}}$$

$$+\frac{\overbrace{[P_{1-12}]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}^{\text{Member 1-12}}}{(0.2086\ in^2)(30\,x10^6\,psi)}[\delta P_{1-12}]$$

$$+\overbrace{\frac{\begin{bmatrix}P_{1-10}\\+(1.581)P_{1-12}\\+2,236\end{bmatrix}\overbrace{\sqrt{(40")^2+(80")^2}}^{89.44"}}{(0.2086\ in^2)(30\,x10^6\,psi)}\left[\delta P_{1-10}+(1.581)\delta P_{1-12}\right]}^{\text{Member 2-11}}$$

•••

•••

$$+\overbrace{\frac{\begin{bmatrix}-(0.8945)P_{1-10}\\-(1.414)P_{1-12}\\-2{,}000\end{bmatrix}(40")}{(0.2086\;in^2)(30\,x10^6\,psi)}\left[-(0.8945)\delta P_{1-10}-(1.414)\delta P_{1-12}\right]}^{\text{Member 2-12}}$$

$$+\overbrace{\frac{[600\;lb](40")}{(0.2086\;in^2)(30\,x10^6\,psi)}[0]}^{\text{Member 3-4}}$$

$$+\overbrace{\frac{[-1{,}565\;lb]\overbrace{\sqrt{(40")^2+(80")^2}}^{89.44"}}{(0.2086\;in^2)(30\,x10^6\,psi)}[0]}^{\text{Member 3-8}}$$

$$+\overbrace{\frac{[-141.7\;lb]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}{(0.2086\;in^2)(30\,x10^6\,psi)}[0]}^{\text{Member 3-12}}$$

$$+\overbrace{\frac{[1{,}500\;lb](40")}{(0.2086\;in^2)(30\,x10^6\,psi)}[0]}^{\text{Member 3-13}}+\overbrace{\frac{[600\;lb](40")}{(0.2086\;in^2)(30\,x10^6\,psi)}[0]}^{\text{Member 4-5}}$$

•••

•••

$$+\overbrace{\frac{[1{,}400\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 4-14}} + \overbrace{\frac{[600\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 5-6}}$$

$$+\overbrace{\frac{[-848.4\ lb]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 5-14}}$$

$$+\overbrace{\frac{[-600\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 6-7}} + \overbrace{\frac{[1{,}200\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 6-14}}$$

$$+\overbrace{\frac{[600\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 7-8}}$$

$$+\overbrace{\frac{[-848.4\ lb]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 7-14}}$$

$$+\overbrace{\frac{[1{,}300\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 8-9}} + \overbrace{\frac{[1{,}400\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 8-14}}$$

•••

$$\bullet\bullet\bullet$$

$$+\overbrace{\frac{[2{,}800\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 9-10}} + \overbrace{\frac{[-2{,}121\ lb]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 9-12}}$$

$$+\overbrace{\frac{[1{,}500\ lb](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}[0]}^{\text{Member 9-13}}$$

$$+\overbrace{\frac{\left[-(0.4472)P_{1-10}+2{,}800\right](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[-(0.4472)\delta P_{1-10}\right]}^{\text{Member 10-11}}$$

$$+\overbrace{\frac{\left[-(0.8945)P_{1-10}\right](40")}{(0.2086\ in^2)(30\,x10^6\ psi)}\left[-(0.8945)\delta P_{1-10}\right]}^{\text{Member 10-12}}$$

$$+\overbrace{\frac{[-P_{1-12}-848.7]\overbrace{\sqrt{(40")^2+(40")^2}}^{56.57"}}{(0.2086\ in^2)(30\,x10^6\ psi)}[-\delta P_{1-12}]}^{\text{Member 11-12}}$$

$$\bullet\bullet\bullet$$

$$\left. \cdots + \overbrace{\frac{[1{,}000\ lb](40'')}{(0.2086\ in^2)(30\,x10^6\ psi)}}^{\text{Member 12-13}}[0] \right]$$

4.104

$$\Rightarrow\ 0=\left\{\begin{array}{l}
\overbrace{(40")\begin{bmatrix}-(0.4473)P_{1-10}\\-(0.7073)P_{1-12}\\-1{,}000\end{bmatrix}\begin{bmatrix}-(0.4473)\delta P_{1-10}\\-(0.7073)\delta P_{1-12}\end{bmatrix}}^{\text{Member 1-2}}+\overbrace{(89.44")[P_{1-10}][\delta P_{1-10}]}^{\text{Member 1-10}}\\[2em]
+\overbrace{(80")\begin{bmatrix}-(0.8945)P_{1-10}\\-(0.7072)P_{1-12}\end{bmatrix}\begin{bmatrix}-(0.8945)\delta P_{1-10}\\-(0.7072)\delta P_{1-12}\end{bmatrix}}^{\text{Member 1-11}}+\overbrace{(56.57")[P_{1-12}][\delta P_{1-12}]}^{\text{Member 1-12}}\\[2em]
+\overbrace{(89.44")\left[P_{1-10}+(1.581)P_{1-12}+2{,}236\right]\left[\delta P_{1-10}+(1.581)\delta P_{1-12}\right]}^{\text{Member 2-11}}\\[2em]
+\overbrace{(40")\begin{bmatrix}-(0.8945)P_{1-10}\\-(1.414)P_{1-12}\\-2{,}000\end{bmatrix}\left[-(0.8945)\delta P_{1-10}-(1.414)\delta P_{1-12}\right]}^{\text{Member 2-12}}\\[2em]
+\overbrace{(40")\left[-(0.4472)P_{1-10}+2{,}800\right]\left[-(0.4472)\delta P_{1-10}\right]}^{\text{Member 10-11}}\\[2em]
+\overbrace{(40")\left[-(0.8945)P_{1-10}\right]\left[-(0.8945)\delta P_{1-10}\right]}^{\text{Member 10-12}}\\[2em]
+\overbrace{(56.57")[-P_{1-12}-848.7][-\delta P_{1-12}]}^{\text{Member 11-12}}
\end{array}\right\}$$

$$\Rightarrow \quad 0 = \left\{ \begin{array}{l}
\overbrace{\left\{ \begin{array}{l} \left[(8.003)P_{1-10} + (12.66)P_{1-12} + 17{,}890\right]\left[\delta P_{1-10}\right] \\ + \left[(12.66)P_{1-10} + (20.01)P_{1-12} + 28{,}290\right]\left[\delta P_{1-12}\right] \end{array} \right\}}^{\text{Member 1-2}} \\
\\
+ \overbrace{(89.44")\left[P_{1-10}\right]\left[\delta P_{1-10}\right]}^{\text{Member 1-10}} \\
\\
+ \overbrace{\left\{ \begin{array}{l} \left[(64.01)P_{1-10} + (50.60)P_{1-12}\right]\left[\delta P_{1-10}\right] \\ + \left[(50.60)P_{1-10} + (40)P_{1-12}\right]\left[\delta P_{1-12}\right] \end{array} \right\}}^{\text{Member 1-11}} + \overbrace{(56.57")\left[P_{1-12}\right]\left[\delta P_{1-12}\right]}^{\text{Member 1-12}} \\
\\
+ \overbrace{\left\{ \begin{array}{l} \left[(89.44)P_{1-10} + (141.4)P_{1-12} + 200{,}000\right]\left[\delta P_{1-10}\right] \\ + \left[(141.4)P_{1-10} + (223.6)P_{1-12} + 316{,}200\right]\left[\delta P_{1-12}\right] \end{array} \right\}}^{\text{Member 2-11}} \\
\\
+ \overbrace{\left\{ \begin{array}{l} \left[(32.00)P_{1-10} + (50.59)P_{1-12} + 71{,}550\right]\left[\delta P_{1-10}\right] \\ + \left[(50.59)P_{1-10} + (79.98)P_{1-12} + 113{,}100\right]\left[\delta P_{1-12}\right] \end{array} \right\}}^{\text{Member 2-12}} \\
\\
+ \overbrace{(8.000)\left[P_{1-10}\right]\left[\delta P_{1-10}\right]}^{\text{Member 10-11}} + \overbrace{(32.01)\left[P_{1-10}\right]\left[\delta P_{1-10}\right]}^{\text{Member 10-12}} \\
\\
+ \overbrace{\left[(56.57)P_{1-12} + 48{,}010\right]\left[\delta P_{1-12}\right]}^{\text{Member 11-12}}
\end{array} \right\}$$

$$\Rightarrow \quad 0 = \left\{ \begin{array}{l} \left[(322.9)P_{1-10} + (255.3)P_{1-12} + (239,400)\right]\left[\delta P_{1-10}\right] \\ \\ +\left[(255.3)P_{1-10} + (476.7)P_{1-12} + (505,600)\right]\left[\delta P_{1-12}\right] \end{array} \right\}$$

$$\Rightarrow \quad \begin{array}{|c|} \hline (322.9)P_{1-10} + (255.3)P_{1-12} + (239,400) = 0 \\ \\ (255.3)P_{1-10} + (476.7)P_{1-12} + (505,600) = 0 \\ \hline \end{array} \tag{4.2.74}$$

Equations 4.2.74 are the additional two equations needed to solve for our 12 unknowns.

Solving Equations 4.2.74 gives

$$\boxed{P_{1-10} = 168.5\ lb} \tag{4.2.75}$$

$$\Rightarrow$$

$$\boxed{P_{1-12} = -1{,}151\ lb} \tag{4.2.76}$$

Now substituting Equations 4.2.75 and 4.2.76 into Equations 4.2.6 through 4.2.73, we obtain the remaining unknowns as below.

$$\boxed{\begin{aligned} &P_{1-2} = -261.3\ lb \\ &P_{1-11} = 663.3\ lb \\ &P_{2-11} = 584.8\ lb \\ &P_{2-12} = -523.2\ lb \\ &P_{10-11} = 2{,}725\ lb \\ &P_{10-12} = -150.7\ lb \\ &P_{11-12} = 302.3\ lb \\ &X_1 = 1{,}000\ lb \\ &X_{11} = -3{,}200\ lb \\ &Y_{11} = -1{,}400\ lb \end{aligned}}$$

(4.2.77)
(4.2.78)
(4.279)
(4.2.80)
(4.2.81)
(4.2.82)
(4.2.83)
(4.2.84)
(4.2.85)
(4.2.86)

4.108

All answers are shown in Figure 4.2.23 below.

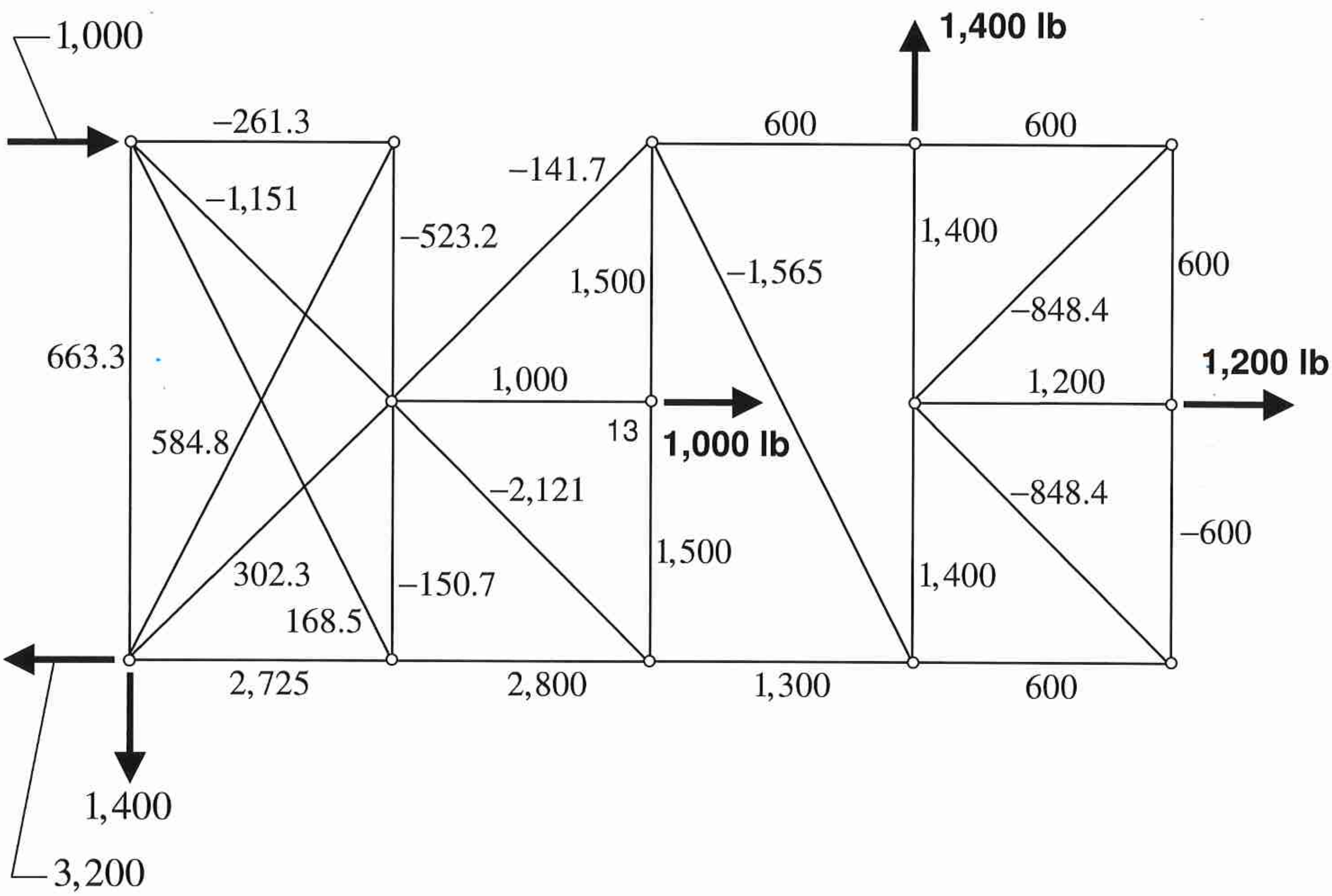


Figure 4.2.23

You should now have the ability to do Problems 4.3

Homework: Do Problem 4.3

4.3 Two-Dimensional Beams Under Both Transverse Loading and Axial Loading

For a long, slender, untapered 2-D beam (with symmetric cross-sections), under both transverse loading and axial loading, the stresses produced are [see Pages 1.15 - 1.16]

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z}$$

$$\tau_{xy} = \frac{V_y Q_z}{I_z b} \tag{4.3.1}$$

where

$$Q_z = \bar{y}' A' = \iint_{A'} y \, dA \tag{4.3.2}$$

$$\varepsilon_x = \frac{1}{E}\left(\frac{P}{A} - \frac{M_z y}{I_z}\right)$$

$$\Rightarrow \qquad \gamma_{xy} = \frac{1}{G}\frac{V_y Q_z}{I_z b} \tag{4.3.3}$$

The negative in front of the term $\dfrac{M_z y}{I_z}$ in the above formulas is correct if the outward normal sign convention is being used for internals.

Now imagine virtual changes δP, δM_z, and δV_y in the beam interior.

What could cause this?

1. It could be caused by applying external virtual forces or couples δB_i to the beam. These will cause some δP, δM_z, and δV_y in the beam interior.

2. We can imagine δP, δM_z, and δV_y to occur in the beam interior with some mysterious origin; i.e., they may occur even though there are no external virtual forces or couples δB_i applied to the beam.

The virtual stresses produced will be

$$\delta\sigma_x = \left[\frac{\delta P}{A} - \frac{(\delta M_z)\,y}{I_z}\right]$$

$$\delta\tau_{xy} = \frac{(\delta V_y)\,Q_z}{I_z b} \tag{4.3.4}$$

Equation 3.5.3 states

$$\delta U^* = \iiint_V \begin{pmatrix} \varepsilon_x\,\delta\sigma_x \;+\; \varepsilon_y\,\delta\sigma_y \;+\; \varepsilon_z\,\delta\sigma_z \\ \\ +\;\gamma_{xy}\,\delta\tau_{xy} \;+\; \gamma_{xz}\,\delta\tau_{xz} \;+\; \gamma_{yz}\,\delta\tau_{yz} \end{pmatrix} dV \tag{3.5.3}$$

For our beam,

$$\varepsilon_y = 0,\quad \delta\sigma_y = 0,\quad \varepsilon_z = 0,\quad \delta\sigma_z = 0$$

$$\gamma_{xz} = 0,\quad \delta\tau_{xz} = 0,\quad \gamma_{yz} = 0,\quad \delta\tau_{yz} = 0 \tag{4.3.5}$$

Using Equations 4.3.3 - 4.3.5, Equation 3.5.3 becomes

$$\delta U^* = \iiint_V \left\{ \left[\frac{1}{E}\left(\frac{P}{A} - \frac{M_z y}{I_z} \right) \right] \left[\frac{\delta P}{A} - \frac{(\delta M_z) y}{I_z} \right] + \left(\frac{1}{G} \frac{V_y Q_z}{I_z b} \right) \left[\frac{(\delta V_y) Q_z}{I_z b} \right] \right\} dV$$

$$= \left\langle \begin{array}{l} \displaystyle\int_0^L \left\{ \iint_A \left[\frac{1}{E}\left(\frac{P}{A} - \frac{M_z y}{I_z} \right) \right] \left[\frac{\delta P}{A} - \frac{(\delta M_z) y}{I_z} \right] dA \right\} dx \\ \displaystyle + \int_0^L \left\{ \iint_A \left(\frac{V_y (Q_z)^2}{G (I_z)^2 b^2} \right) (\delta V_y) dA \right\} dx \end{array} \right\rangle$$

$$= \left\{ \begin{array}{l} \displaystyle\int_0^L \left(\frac{P}{EA^2} \delta P \right) \left[\iint_A dA \right] dx - \int_0^L \left[\frac{P}{EAI_z} \delta M_z \right] \overbrace{\left[\iint_A y\, dA \right]}^{\bar{y}=0} dx \\ \displaystyle - \int_0^L \left[\frac{M_z}{EAI_z} \delta P \right] \overbrace{\left[\iint_A y\, dA \right]}^{\bar{y}=0} dx + \int_0^L \left[\frac{M_z}{E(I_z)^2} \delta M_z \right] \overbrace{\left[\iint_A y^2\, dA \right]}^{I_z} dx \\ \displaystyle + \int_0^L \left[\frac{V_y}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\}$$

$[\bar{y} = 0$ because our axes are at the centroid$]$

$$\Rightarrow \qquad \boxed{\delta U^* = \left\{ \begin{aligned} &\int_0^L \left[\frac{P}{EA} \delta P \right] dx + \int_0^L \left[\frac{M_z}{EI_z} \delta M_z \right] dx \\ &+ \int_0^L \left[\frac{V_y}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{aligned} \right\}} \qquad (4.3.6)$$

The Principle of Complementary Virtual Work (Equation 3.5.6) is repeated below.

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \iiint_V \left(\begin{aligned} &\varepsilon_x \, \delta\sigma_x + \varepsilon_y \, \delta\sigma_y + \varepsilon_z \, \delta\sigma_z \\ &+ \gamma_{xy} \, \delta\tau_{xy} + \gamma_{xz} \, \delta\tau_{xz} + \gamma_{yz} \, \delta\tau_{yz} \end{aligned} \right) dV \qquad (3.5.6)$$

Using Equation 4.3.6 in Equation 3.5.6,

$$\Rightarrow \qquad \sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{aligned} &\int_0^L \left[\frac{P}{EA} \delta P \right] dx + \int_0^L \left[\frac{M_z}{EI_z} \delta M_z \right] dx \\ &+ \int_0^L \left[\frac{V_y}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{aligned} \right\}$$

This is the Principle of Complementary Virtual Work for a single beam.

Adding some additional subscripts as reminders, we have below in Equation 4.3.7 the official *Principle of Complementary Virtual Work for a Single Untapered 2-D Beam with a Symmetric Cross-Section Under Both Transverse and Axial Loading.*

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{G\left(I_z\right)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \tag{4.3.7}$$

4.4 Statically Determinate Beams and Frames---Deflections

Frames

1. Composed of two or more members.
2. Unlike a truss, all members are not 2-force members.
3. Members can consist of 2-force members and beams (including curved beams), or possibly all beams.

Beam members of a frame may be under both transverse loading and axial loading.

Therefore, the RHS of Equation 4.3.7 would be appropriate for each beam member, as well as each two-force member, of a frame.

For the *Frame as a Whole*, we would sum each of the quantities on the RHS of Equation 4.3.7 as in Equation 4.4.1 below.

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on}\\ \text{Original Structure}\\ \text{Caused by Actual}\\ \text{Loads with Actual}\\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}}\right]} = \sum_i \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{\left(P \Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(M_z \Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(V_y \Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{G_i (I_z)_i^2} (\delta V_y)_i \right] \left[\iint_A \left(\frac{(Q_z)_i}{b_i} \right)^2 dA \right] dx \end{array} \right\} \quad (4.4.1)$$

Example 4.4.1 **[Statically Determinate Straight Beam—Displacements, Rotations]**

For the statically determinate beam shown in the figure below, determine, using the Principle of Complementary Virtual Work,

(a) the vertical component of displacement v at the free end, and
(b) the rotation θ at the free end.

Material: Titanium (Ti-6Al-4V)	Cross-Section: Hollow Rectangular
$E = 16.5 \times 10^6$ psi	Height = 5"
$G = 6.38 \times 10^6$ psi	Width = 3"
$\nu = 0.2375$	$t = \frac{1}{2}$"

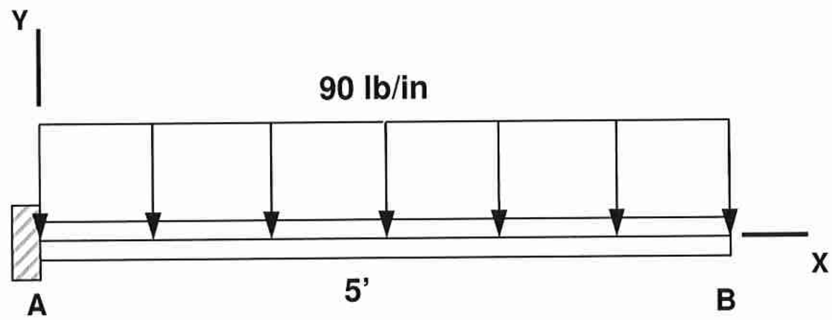


Figure 4.4.1

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

4.116

2-Force Members

Why is the structure in Figure 4.4.1 NOT a two-force member?

(a) To determine the vertical component of displacement v at the free end using the Principle of Complementary Virtual Work, the actual load of 90 lb/in plus the external reactions due to the actual load will be removed and a vertical virtual force δQ will be applied at the free end.

(b) To determine the rotation θ at the free end using the Principle of Complementary Virtual Work, the actual load of 90 lb/in plus the external reactions due to the actual load will be removed and a virtual couple δC will be applied at the free end.

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

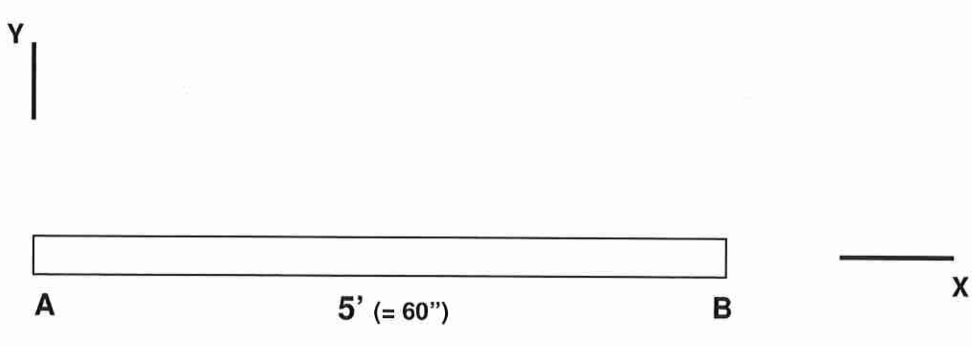


Figure 4.4.2

Sign Conventions

* Externals ---- Positive if in the Positive Global Axes Directions

* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

The FBD of the left portion of the beam and a FBD of the right portion of the beam at an arbitrary x-location are shown below in Figure 4.4.3. This exposes the internals.

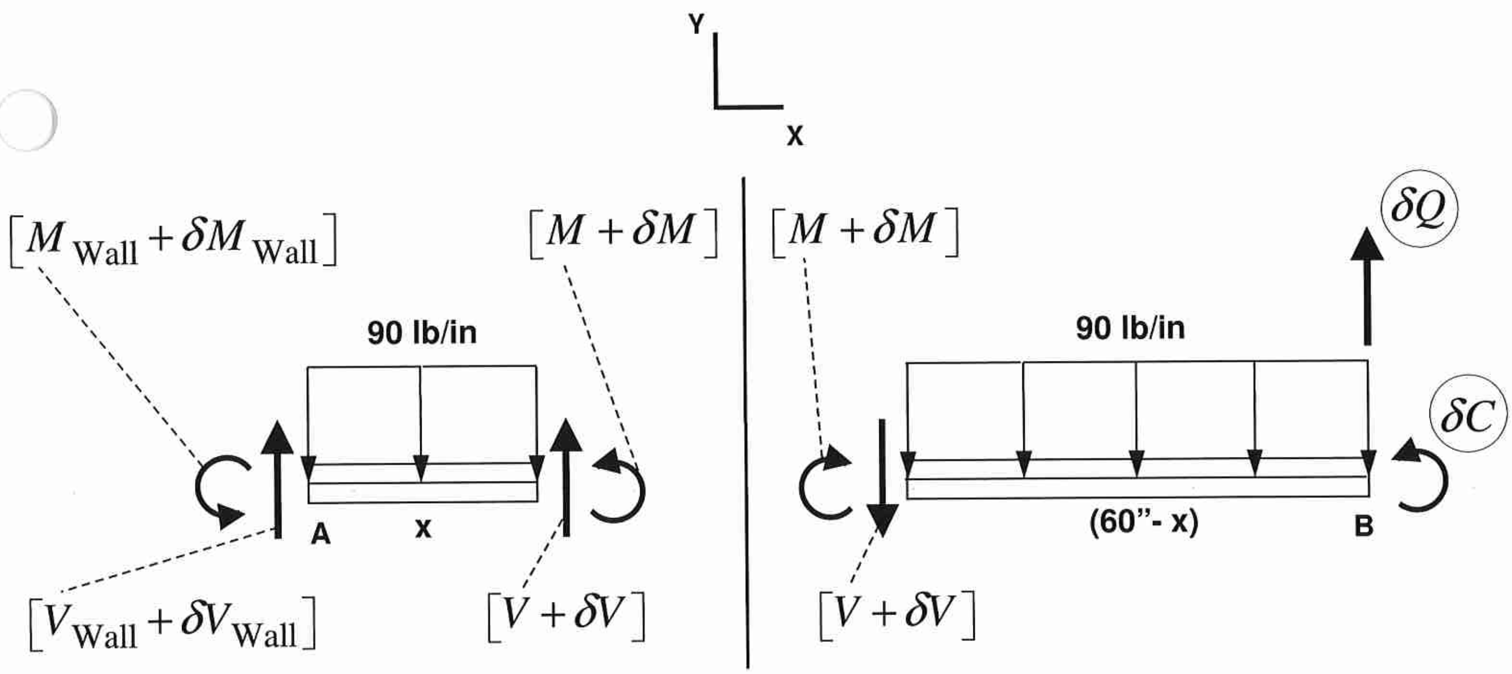


Figure 4.4.3

Sign Conventions

* Externals ---- Positive if in the Positive Global Axes Directions

* Internals ---- Positive based on the Outward Normal Sign Convention

* Figure 4.4.3 shows both portions of the beam after the cut has been made.

* The cut must be made at an arbitrary x-location so that the internals can be found as functions of x for the integrals in the Principle of Complementary Virtual Work.

* It is only necessary to show one portion; either the left portion or the right portion.

* Static equilibrium equations can be used on either portion, but you cannot use both.

* Static equilibrium equations on either portion will give the exact same results.

* In this case, it would be easier to use the right portion since that would eliminate the necessity of finding the reactions at the wall.

Cuts

Why does the structure in Figure 4.4.2 only need one cut?

Because there are no concentrated loads along the span.

External Reactions
[Figure 4.4.2]

There is no need to find the external reactions since we will be using the right portion of the cut in Figure 4.4.3.

Internals
[Figure 4.4.3, Right Figure]

$$\sum F_y = 0$$

$$\Rightarrow \quad -[V + \delta V] - \left[\left(90\,\frac{lb}{in}\right)(60'' - x)\right] + \delta Q = 0$$

$$\Rightarrow \quad [V + \delta V] = 90x - 5{,}400 + \delta Q$$

$$\Rightarrow \quad \boxed{V = 90x - 5{,}400} \qquad \boxed{\delta V = \delta Q} \qquad (4.4.2)$$

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad -[M + \delta M] - \left[\left(90\,\frac{lb}{in}\right)(60'' - x)\right]\left[\frac{(60'' - x)}{2}\right] + (\delta Q)(60'' - x) + \delta C = 0$$

$$\Rightarrow \quad [M + \delta M] = -(45)(60 - x)^2 + (\delta Q)(60 - x) + \delta C$$

$$\Rightarrow \quad \boxed{M = -(45)(60 - x)^2} \qquad \boxed{\delta M = (\delta Q)(60 - x) + \delta C} \qquad (4.4.3)$$

The vertical component of displacement v at the free end and the rotation θ at the free end will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.3.7) for a single beam states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \tag{4.3.7}$$

For our beam,

1. $P = 0$, and
2. the effects of shear deformation are neglected.

These will cause Equation 4.3.7 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \tag{4.4.4}$$

Plugging Equations 4.4.3 into Equation 4.4.4,

$$\Rightarrow \left\{\begin{array}{l} v_{\text{Free End}}\left(\delta Q\right) \\ +\theta_{\text{Free End}}\left(\delta C\right) \\ +\overbrace{v_{\text{Wall}}}^{0}\left(\delta V_{\text{Wall}}\right) \\ +\overbrace{\theta_{\text{Wall}}}^{0}\left(\delta M_{\text{Wall}}\right) \end{array}\right\} = \frac{1}{\left(16.5\,x10^{6}\,psi\right)\underbrace{\left(20.58\,in^{4}\right)}_{I_z}}\int_{0}^{60}\left[-(45)(60-x)^{2}\right]\left[(\delta Q)(60-x)+\delta C\right]dx$$

I_z Computation

$$I_z = \left.\left(\frac{bh^3}{12}\right)\right|_{\text{Outer}} - \left.\left(\frac{bh^3}{12}\right)\right|_{\text{Inner}}$$

$$\Rightarrow \qquad I_z = \left[\frac{(3")(5")^3}{12}\right] - \left[\frac{(3"-1")(5"-1")^3}{12}\right]$$

$$\Rightarrow \qquad \boxed{I_z = 20.58\ in^4}$$

$$\Rightarrow \left\{\begin{array}{l} v_{\text{Free End}}\left(\delta Q\right) \\ +\theta_{\text{Free End}}\left(\delta C\right) \end{array}\right\} = -\left(1.325\,x10^{-7}\right)\left\{\begin{array}{l} \int_{0}^{60}(60-x)^{3}(\delta Q)\,dx \\ +\int_{0}^{60}(60-x)^{2}(\delta C)\,dx \end{array}\right\}$$

$$\Rightarrow \qquad \begin{Bmatrix} \mathrm{v}_{\text{Free End}}\left(\delta Q\right) \\ \\ +\theta_{\text{Free End}}\left(\delta C\right) \end{Bmatrix} = -\left(1.325\,x10^{-7}\right)\begin{Bmatrix} \left[-\dfrac{\left(60-x\right)^4}{4}\right]_0^{60}\left(\delta Q\right) \\ +\left[-\dfrac{\left(60-x\right)^3}{3}\right]_0^{60}\left(\delta C\right) \end{Bmatrix}$$

$$\Rightarrow \qquad \left(\mathrm{v}_{\text{Free End}}\right)\delta Q + \left(\theta_{\text{Free End}}\right)\delta C = \left(-0.4294\right)\delta Q + \left(-0.009542\right)\delta C$$

$$\Rightarrow \qquad \boxed{\mathrm{v}_{\text{Free End}} = -0.4294"} \qquad \boxed{\theta_{\text{Free End}} = -0.009542\ rad \doteq -0.55^\circ} \qquad (4.4.5)$$

Sign Conventions

* u & v ---- Positive if in the Positive Global Axes Directions
* Work of Externals ---- Positive if Helping the Motion

Notes

* We see that Equation 4.4.2 was never used.
* The reason is that the effects of shear deformation were neglected.
* Hence, the internal V functions were not needed for the Principle of Complementary Virtual Work.
* Whenever the effects of shear deformation are neglected, there will be no need to write the equation to determine the internal V functions.
* However, the external V_{Wall}, in general, will still have to be found.

Example 4.4.2 **[Statically Determinate Straight Beam—Displacements, Rotations]**

For the statically determinate beam shown in the figure below, determine, using the Principle of Complementary Virtual Work,

(a) the reactions at the wall,
(b) the vertical component of displacement v at the *center*, and
(c) the rotation θ at the *center*.

Material: Titanium (Ti-6Al-4V)	Cross-Section: Hollow Rectangular
$E = 16.5 \times 10^6$ psi	Height = 5"
$G = 6.38 \times 10^6$ psi	Width = 3"
$\nu = 0.2375$	$t = \frac{1}{2}$"

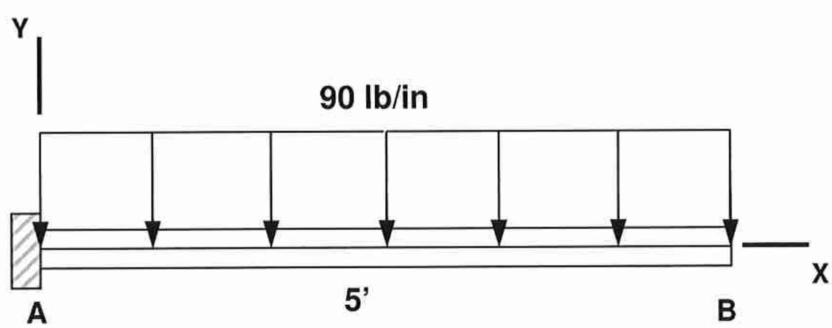


Figure 4.4.4

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

4.124

This beam is exactly the same as that of Example 4.4.1.

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

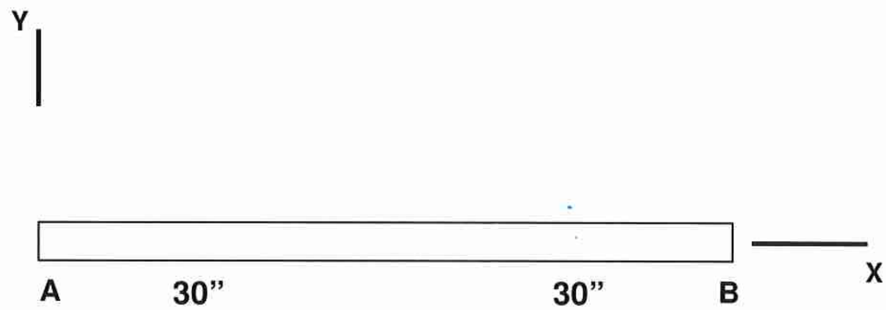


Figure 4.4.5

> **Sign Conventions**
>
> * Externals-----Positive if in the Positive Global Axes Directions

External Reactions
[Figure 4.4.5]

$$\sum F_y = 0$$

$$\Rightarrow \qquad \left[V_{\text{Wall}} + \delta V_{\text{Wall}}\right] - \left[\left(90\ \frac{lb}{in}\right)(60")\right] + \delta Q = 0$$

$$\Rightarrow \qquad \left[V_{\text{Wall}} + \delta V_{\text{Wall}}\right] = 5{,}400 - \delta Q$$

$$\Rightarrow \qquad \boxed{V_{\text{Wall}} = 5,400} \qquad \boxed{\delta V_{\text{Wall}} = -\delta Q} \tag{4.4.6}$$

$$\circlearrowleft + \quad \sum M_{\text{Wall}} = 0$$

$$\Rightarrow \quad \left[M_{\text{Wall}} + \delta M_{\text{Wall}}\right] - \left[\left(90\,\frac{lb}{in}\right)(60")\right](30") + (\delta Q)(30") + \delta C = 0$$

$$\Rightarrow \quad \left[M_{\text{Wall}} + \delta M_{\text{Wall}}\right] = 162,000 - (30)\,\delta Q - \delta C$$

$$\Rightarrow \qquad \boxed{M_{\text{Wall}} = 162,000} \qquad \boxed{\delta M_{\text{Wall}} = -(30)\,\delta Q - \delta C} \tag{4.4.7}$$

The answer to Part (a); namely, determine the reactions at the wall, is

$$\boxed{V_{\text{Wall}} = 5,400\ lb\ \uparrow}$$

$$\boxed{M_{\text{Wall}} = 162,000\ in \cdot lb\ \circlearrowleft} \tag{4.4.8}$$

* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

4.126

This beam has concentrated loads at the *center* of the beam, thus

(i) a cut must be made at an arbitrary x-location to the left of the center, and

(ii) a cut must be made at an arbitrary x-location to the right of the center.

Cuts

Why does the structure in Figure 4.4.5 need two cuts?
Because there are concentrated loads at one location along the span of the beam.

Cut #1

For Cut #1, $0 \leq x < 30"$

* When Cut #1 is made, we must choose either the portion of the beam to the left of the cut or the portion of the beam to the right of the cut for analysis.

* Both portions will give the exact same result.

The Left Portion of the Beam for Cut #1 is shown below in Figure 4.4.6.

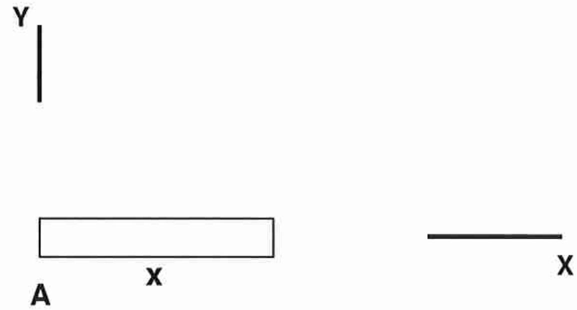


Figure 4.4.6

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention

Internals for Cut #1
[Figure 4.4.6]

$$\sum F_y = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \left[162{,}000 - (30)\delta Q - \delta C\right] - \left[5{,}400 - \delta Q\right]x + \left[\left(90\,\frac{lb}{in}\right)x\right]\left(\frac{x}{2}\right) + \left[M + \delta M\right] = 0$$

$$\Rightarrow \quad \left[M + \delta M\right] = -(45)x^2 + (5{,}400)x - 162{,}000 - (\delta Q)x + (30)\delta Q + \delta C$$

$$\Rightarrow \boxed{M = -(45)x^2 + (5{,}400)x - 162{,}000} \qquad \boxed{\delta M = -(\delta Q)x + (30)\delta Q + \delta C} \qquad (4.4.9)$$

Cut #2

For Cut #2, $30'' < x \leq 60''$

* When Cut #2 is made, we must choose either the portion of the beam to the left of the cut or the portion of the beam to the right of the cut for analysis.

* Both portions will give the exact same result.

The Right Portion of the Beam for Cut #2 is shown below in Figure 4.4.7.

Figure 4.4.7

Internals for Cut #2
[Figure 4.4.7]

$$\sum F_y = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \quad \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \qquad -[M + \delta M] - \left[\left(90\,\frac{lb}{in}\right)(60'' - x)\right]\left[\frac{(60'' - x)}{2}\right] = 0$$

$$\Rightarrow \qquad [M + \delta M] = -(45)x^2 + (5{,}400)x - 162{,}000$$

$$\Rightarrow \qquad \boxed{M = -(45)x^2 + (5{,}400)x - 162{,}000} \qquad \boxed{\delta M = 0} \qquad (4.4.10)$$

The vertical component of displacement v at the center and the rotation θ at the center will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.3.7) for a single beam states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \qquad (4.3.7)$$

4.130

For our beam,

1. $P = 0$, and

2. the effects of shear deformation are neglected.

These will cause Equation 4.3.7 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \int_0^L \left[\frac{M_z \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \qquad (4.4.11)$$

Plugging Equations 4.4.9 and 4.4.10 into Equation 4.4.11,

$$\Rightarrow \left\{ \begin{array}{l} v_{\text{Center}}(\delta Q) \\ +\theta_{\text{Center}}(\delta C) \\ +\overbrace{v_{\text{Wall}}}^{0}(\delta V_{\text{Wall}}) \\ +\overbrace{\theta_{\text{Wall}}}^{0}(\delta M_{\text{Wall}}) \end{array} \right\} = \frac{1}{(16.5\,x10^6\,psi)\underbrace{(20.58\ in^4)}_{I_z}} \left\{ \begin{array}{l} \overbrace{\int_0^{30} \begin{bmatrix} -(45)x^2 \\ +(5{,}400)x \\ -162{,}000 \end{bmatrix} \begin{bmatrix} -(\delta Q)x \\ +(30)\delta Q \\ +\delta C \end{bmatrix} dx}^{\text{Cut \#1}} \\ \overbrace{+ \int_{30}^{60} \begin{bmatrix} -(45)x^2 \\ +(5{,}400)x \\ -162{,}000 \end{bmatrix} \underbrace{[0]}_{\delta M} dx}^{\text{Cut \#2}} \end{array} \right\}$$

$$\Rightarrow \quad \begin{Bmatrix} v_{\text{Center}}(\delta Q) \\ \\ +\theta_{\text{Center}}(\delta C) \end{Bmatrix} = \left(2.945\,x10^{-9}\right) \left\{ \begin{gathered} \int_0^{30} \begin{bmatrix} (45)x^3 \\ -(5,400)x^2 \\ +(162,000)x \end{bmatrix} (\delta Q)\,dx \\ \\ + \int_0^{30} \begin{bmatrix} -(45)x^2 \\ +(5,400)x \\ -162,000 \end{bmatrix} \left[(30)\delta Q + \delta C\right] dx \end{gathered} \right\}$$

$$\Rightarrow \quad (v_{\text{Center}})\,\delta Q + (\theta_{\text{Center}})\,\delta C = (-0.1521)\,\delta Q + (-0.008349)\,\delta C$$

$$\Rightarrow \quad \boxed{v_{\text{Center}} = -0.1521''} \qquad \boxed{\theta_{\text{Center}} = -0.008349\ rad \doteq -0.48^\circ} \qquad (4.4.12)$$

Sign Conventions

* u & v ---- Positive if in the Positive Global Axes Directions

* Work of Externals ---- Positive if Helping the Motion

Comparing the results of Examples 4.4.1 and 4.4.2 (Equations 4.4.5 and 4.4.12),

$$\boxed{\begin{gathered} v_{\text{Center}} = (0.35)\, v_{\text{Free End}} \\ \theta_{\text{Center}} = (0.87)\, \theta_{\text{Free End}} \end{gathered}}$$

You should now have the ability to do Problem 4.4

Homework: Do Problem 4.4

Example 4.4.3 **[Statically Determinate Curved Beam—Displacements]**

For the statically determinate curved beam shown in Figure 4.4.8 below, determine, using the Principle of Complementary Virtual Work, the vertical component of displacement v at the *free end.*

Material: Aluminum
$E = 9.6 \times 10^6$ psi

Cross-Section: Solid Circular
Diameter = 2"

Diameter of Beam Curvature = 6' (= 72")
(Measured to the Cross-Section Centerline)

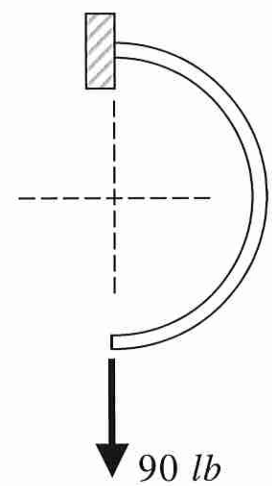


Figure 4.4.8

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

2-Force Members

Is the structure in Figure 4.4.8 a two-force member?

The initial observation is that it is not a 2-Force member because there will be a bending moment reaction from the wall onto the member.

However, we will see that with the given loading, the bending moment at the wall is zero. Hence, technically it would be a 2-force member.

To determine the vertical component of displacement v at the free end using the Principle of Complementary Virtual Work, the actual load of 90 lb and all external actual reactions will be removed and a vertical virtual force δQ will be applied at the free end. Using the short cut method, we get the diagrams below.

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

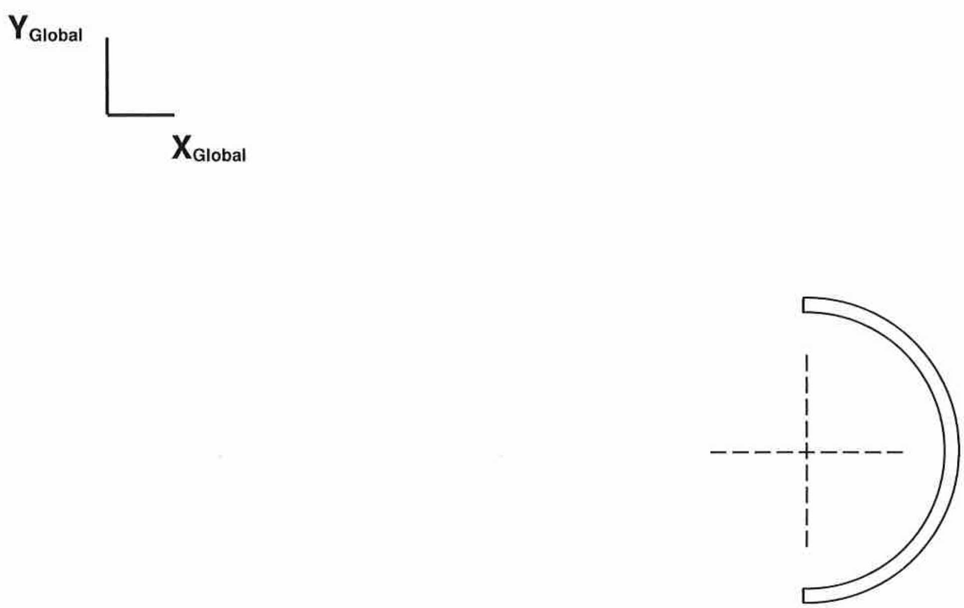


Figure 4.4.9

Sign Conventions

* Externals ---- Positive if in the Positive Global Axes Directions

Why is this curved beam statically determinate?
Because it has only 3 unknown external reactions & we can get 3 equilibrium equations from statics.

* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary θ- locations.

Cuts

Why does the structure in Figure 4.4.9 only need one cut?
Because there are no concentrated loads along the beam length.

4.136

* When the beam is cut, we must choose either the portion of the beam above the cut or the portion of the beam below the cut for analysis.

* Both portions will give the exact same result.

The Lower Portion of the Beam is shown below in Figure 4.4.10.

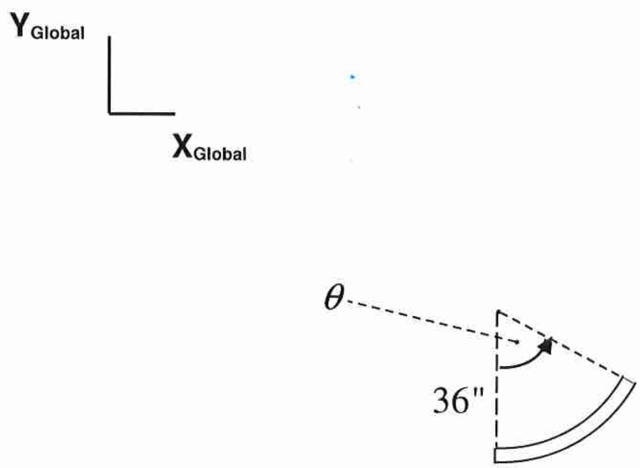


Figure 4.4.10

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

Temporary Axes

* All of the formulas in Chapter 1 are derived for standard $x-y$ axes along the length of the member.

* When using Chapter 1 formulas in problems where the axes are not the standard $x-y$ axes, it is best to set up a temporary set of $x_{\text{Temp}} - y_{\text{Temp}}$ axes for using the Chapter 1 formulas.

* The x_{Temp} axis must be lined up along the beam axis.

External Reactions
[Figure 4.4.9]

There is no need to find the external reactions for the cut that we are utilizing here.

Internals
[Figure 4.4.10]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad [P+\delta P]-(90\ lb)\sin\theta+(\delta Q)\sin\theta=0$$

$$\Rightarrow \quad [P+\delta P]=(90)\sin\theta-(\delta Q)\sin\theta$$

$$\Rightarrow \quad \boxed{P=(90)\sin\theta} \qquad \boxed{\delta P=-(\delta Q)\sin\theta} \tag{4.4.13}$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad [M + \delta M] + (90\ lb)\left[(36")\sin\theta\right] - (\delta Q)\left[(36")\sin\theta\right] = 0$$

$$\Rightarrow \quad [M + \delta M] = -(3{,}240)\sin\theta + (36)(\delta Q)\sin\theta$$

$$\Rightarrow \quad \boxed{M = -(3{,}240)\sin\theta} \qquad \boxed{\delta M = (36)(\delta Q)\sin\theta} \qquad (4.4.14)$$

The vertical component of displacement v at the free end will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.3.7) for a single beam states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \tag{4.3.7}$$

For our beam,

1. $P \neq 0$, and
2. the effects of shear deformation are neglected.

These will cause Equation 4.3.7 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \end{array} \right\} \tag{4.4.15}$$

Plugging Equations 4.4.13 and 4.4.14 into Equation 4.4.15,

$$\Rightarrow \left\{ \begin{array}{l} \mathrm{v}_{\text{Free End}}\left(\delta Q\right) \\ + \overbrace{\mathrm{u}_{\text{Wall}}}^{0}\left(\delta P_{\text{Wall}}\right) \\ + \overbrace{\mathrm{v}_{\text{Wall}}}^{0}\left(\delta V_{\text{Wall}}\right) \\ + \overbrace{\theta_{\text{Wall}}}^{0}\left(\delta M_{\text{Wall}}\right) \end{array} \right\} = \left\{ \begin{array}{l} \overbrace{\dfrac{1}{\left(9.6x10^6\,psi\right)\underbrace{\left[\pi\left(1"\right)^2\right]}_{A=3.1416\ in^2}}\int_0^{\pi}\left[(90)\sin\theta\right]\left[-(\delta Q)\sin\theta\right]\overbrace{\left[(36")d\theta\right]}^{dx}}^{\text{[Due To Internal Axial Forces]}} \\ \overbrace{+\dfrac{1}{\left(9.6x10^6\,psi\right)\underbrace{\left[\dfrac{\pi\left(1"\right)^4}{4}\right]}_{I_z=0.7854\ in^4}}\int_0^{\pi}\left[-(3{,}240)\sin\theta\right]\left[(36)(\delta Q)\sin\theta\right]\overbrace{\left[(36")d\theta\right]}^{dx}}^{\text{[Due To Internal Bending Moments]}} \end{array} \right\}$$

$$\Rightarrow \mathrm{v}_{\text{Free End}}\left(\delta Q\right) = \left\{ \overbrace{-(0.0001074)(\delta Q)\int_0^{\pi}(\sin\theta)^2\,d\theta}^{\text{[Due To Internal Axial Forces]}} - \overbrace{(0.5570)(\delta Q)\int_0^{\pi}(\sin\theta)^2\,d\theta}^{\text{[Due To Internal Bending Moments]}} \right\}$$

$$\Rightarrow \qquad \mathrm{v}_{\text{Free End}} = \overbrace{(-0.0001687)}^{\text{[Deformation Due To Internal Axial Forces]}} + \overbrace{(-0.8749)}^{\text{[Deformation Due To Internal Moments]}}$$

$$\boxed{\mathrm{v}_{\text{Free End}} = -0.8751"}$$

Sign Conventions

* u & v -----Positive if in the Positive Global Axes Directions

* Work of Externals----- Positive if Helping the Motion

We can see that, in this particular curved beam problem, including the effect of axial deformation caused a minor change in the answer (out to 4 significant digits) for the deformation at the free end.

You should now have the ability to do Problems 4.5, 4.6

Homework: Do Problem 4.5

Example 4.4.4 **[Statically Determinate Frame Consisting of Straight Beams and 2-Force Members—Displacements]**

For the statically determinate frame shown in Figure 4.4.11 below, determine, using the Principle of Complementary Virtual Work,

(a) the horizontal component of displacement u at *Point A*, and
(b) the vertical component of displacement v at *Point A.*

Material: Titanium	Cross-Section: Solid Circular
E = 15.7 x 10^6 psi	Diameter = $1\frac{3}{4}$"

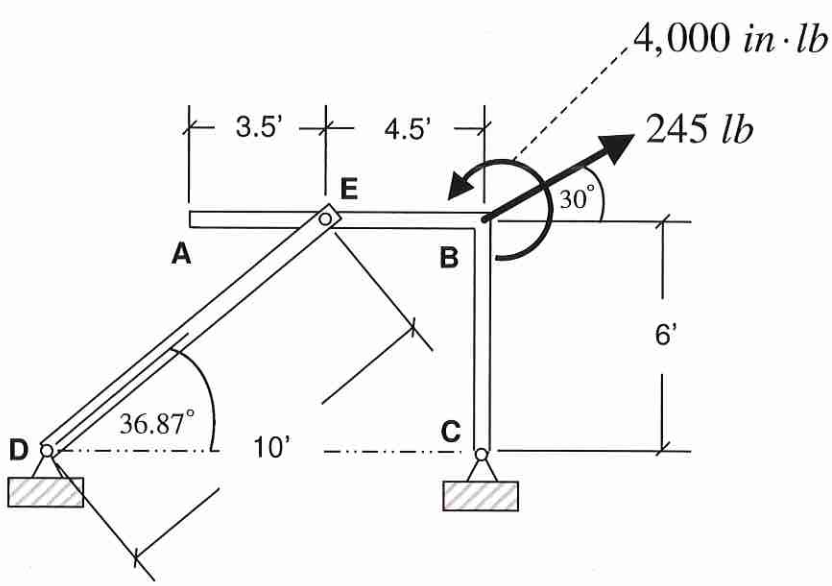


Figure 4.4.11

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

2-Force Members

Are there any two-force members in Figure 4.4.11?

Member DE

Member AB

Member BC

To determine the horizontal component of displacement u at *Point A* using the Principle of Complementary Virtual Work, the actual loads of $245\ lb$ and $4{,}000\ \mathrm{in} \cdot lb$, as well as all externals, will be removed, and a horizontal virtual force δR will be applied at *Point A*.

To determine the vertical component of displacement v at *Point A* using the Principle of Complementary Virtual Work, the actual loads of $245\ lb$ and $4{,}000\ \mathrm{in} \cdot lb$, as well as all externals, will be removed, and a vertical virtual force δQ will be applied at *Point A*. Using the short cut method, we get the diagrams below.

Using the short cut method, we get the diagrams below.

4.144

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

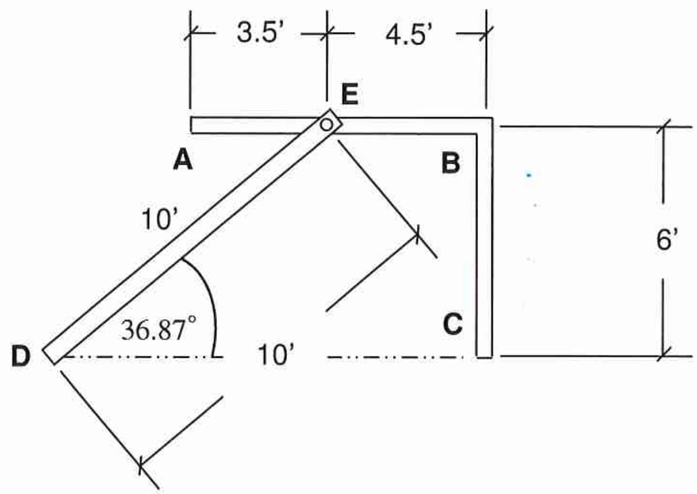


Figure 4.4.12

2-Force Members

Why is there one reaction at Pin D in Figure 4.4.12?

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

Why is this frame statically determinate?
Because it has only 3 unknown external reactions & we can get 3 equilibrium equations from statics.

External Reactions
[Figure 4.4.12]

$$\circlearrowleft + \quad \sum M_{\mathrm{D}} = 0$$

$$\Rightarrow \quad \left\{ \begin{aligned} & \left[Y_{\mathrm{C}} + \delta Y_{\mathrm{C}}\right]\overbrace{\left[\underbrace{54''}_{4.5'} + \underbrace{(120'')}_{10'}\cos 36.87^\circ\right]}^{150.0''} + 4{,}000\ in \cdot lb \\ & +\left[(245\ lb)\sin 30^\circ\right](150.0'') - \left[(245\ lb)\cos 30^\circ\right]\underbrace{(72'')}_{6'} \\ & +(\delta Q)\overbrace{\left[\underbrace{(120'')}_{10'}\cos 36.87^\circ - \underbrace{42''}_{3.5'}\right]}^{54.00''} - (\delta R)\underbrace{(72'')}_{6'} \end{aligned} \right\} = 0$$

$$\Rightarrow \quad \left[Y_{\mathrm{C}} + \delta Y_{\mathrm{C}}\right] = -47.32 - (0.36)\,\delta Q + (0.48)\,\delta R$$

$$\Rightarrow \quad \boxed{Y_{\mathrm{C}} = -47.32} \qquad \boxed{\delta Y_{\mathrm{C}} = -(0.36)\,\delta Q + (0.48)\,\delta R} \qquad (4.4.16)$$

<u>4.146</u>

$$\sum F_y = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \overbrace{\left[-47.32-(0.36)\delta Q+(0.48)\delta R\right]}^{\left[Y_C+\delta Y_C\right]}+(245\ lb)\sin 30^\circ \\ \\ +\delta Q-\left[P_{D-E}+\delta P_{D-E}\right]\sin 36.87^\circ \end{array} \right\} = 0$$

$$\Rightarrow \quad \left[P_{D-E}+\delta P_{D-E}\right]=125.3+(1.067)\delta Q+(0.8000)\delta R$$

$$\Rightarrow \quad \boxed{P_{D-E}=125.3} \qquad \boxed{\delta P_{D-E}=(1.067)\delta Q+(0.8000)\delta R} \qquad (4.4.17)$$

$$\sum F_x = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -\overbrace{\left[125.3+(1.067)\delta Q+(0.8000)\delta R\right]}^{\left[P_{D-E}+\delta P_{D-E}\right]}\cos 36.87^\circ+\left[X_C+\delta X_C\right] \\ \\ +(245\ lb)\cos 30^\circ+\delta R \end{array} \right\} = 0$$

$$\Rightarrow \quad \left[X_C+\delta X_C\right]=-111.9+(0.8536)\delta Q-(0.3600)\delta R$$

$$\Rightarrow \quad \boxed{X_C=-111.9} \qquad \boxed{\delta X_C=(0.8536)\delta Q-(0.3600)\delta R} \qquad (4.4.18)$$

* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

Cuts

How many cuts does the structure in Figure 4.4.12 need?

Cut #1

$$\boxed{\text{Cut \#1} \Rightarrow \text{Cut Between B and C}}$$

* When Cut #1 is made, we must choose either the portion of the frame above the cut or the portion of the frame below the cut for analysis.

* Both portions will give the exact same result.

The Lower Portion of the Frame for Cut #1 is shown below in Figure 4.4.13.

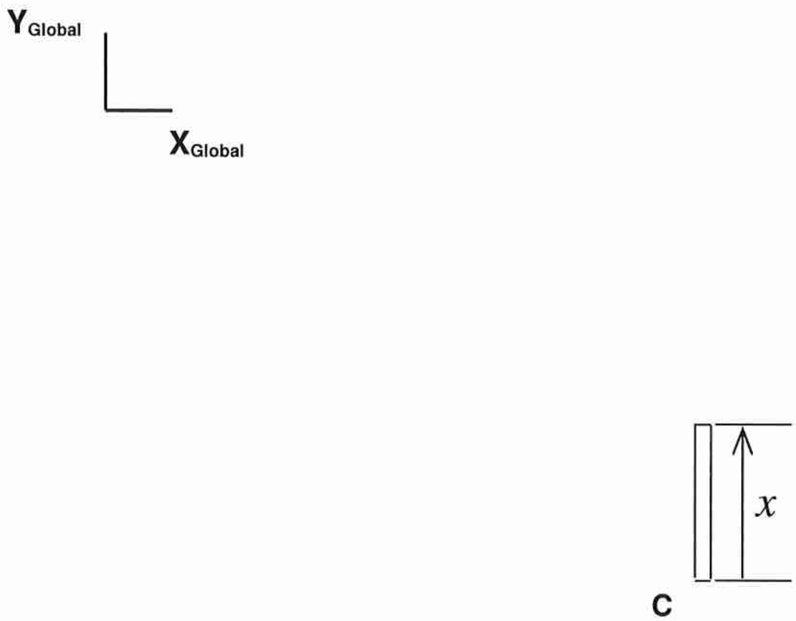


Figure 4.4.13

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

Temporary Axes

* All of the formulas in Chapter 1 are derived for standard $x-y$ axes along the length of the member.

* When using Chapter 1 formulas in problems where the axes are not the standard $x-y$ axes, it is best to set up a temporary set of $x_{\text{Temp}}-y_{\text{Temp}}$ axes for using the Chapter 1 formulas.

* The x_{Temp} axis must be lined up along the beam axis.

Internals for Cut #1
[Figure 4.4.13]

$$\sum F_{x_{\text{Temp}}}=0$$

$$\Rightarrow \qquad [P+\delta P]+\overbrace{\left[-47.32-(0.36)\delta Q+(0.48)\delta R\right]}^{\left[Y_C+\delta Y_C\right]}=0$$

$$\Rightarrow \qquad \boxed{P=47.32} \qquad \boxed{\delta P=(0.36)\delta Q-(0.48)\delta R} \qquad (4.4.19)$$

$$\sum F_{y_{\text{Temp}}}=0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft^{+} \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \qquad [M + \delta M] + \overbrace{\left[-111.9 + (0.8536)\,\delta Q - (0.3600)\,\delta R\right]}^{\left[X_C + \delta X_C\right]} x = 0$$

$$\Rightarrow \qquad [M + \delta M] = \left[111.9 - (0.8536)\,\delta Q + (0.3600)\,\delta R\right] x$$

$$\Rightarrow \qquad \boxed{M = (111.9)\,x} \qquad \boxed{\delta M = \left[-(0.8536)\,\delta Q + (0.3600)\,\delta R\right] x} \quad \text{(4.4.20)}$$

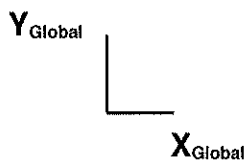
Cut #2

$$\boxed{\text{Cut \#2} \Rightarrow \text{Cut Between B and E}}$$

* When Cut #2 is made, we must choose either the portion of the frame to the right of the cut or the portion of the frame to the left of the cut for analysis.

* Both portions will give the exact same result.

The Right Portion of the Frame for Cut #2 is shown in Figure 4.4.14.



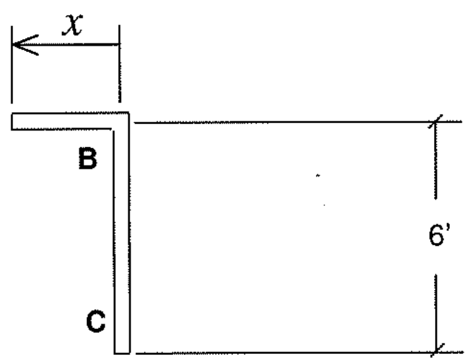


Figure 4.4.14

Internals for Cut #2
[Figure 4.4.14]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad [P+\delta P]-(245\ lb)\cos 30^\circ - \overbrace{\left[-111.9+(0.8536)\delta Q-(0.3600)\delta R\right]}^{[X_C+\delta X_C]}=0$$

$$\Rightarrow \quad [P+\delta P]=-100.3+(0.8536)\delta Q-(0.3600)\delta R$$

$$\Rightarrow \quad \boxed{P=-100.3} \qquad \boxed{\delta P=(0.8536)\delta Q-(0.3600)\delta R} \qquad (4.21)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad \left\{\begin{array}{l} [M+\delta M]+4{,}000\ in\cdot lb+\left[(245\ lb)\sin 30^\circ\right]x \\ +\overbrace{\left[-111.9+(0.8536)\delta Q-(0.3600)\delta R\right]}^{[X_C+\delta X_C]}(72") \\ +\overbrace{\left[-47.32-(0.36)\delta Q+(0.48)\delta R\right]}^{[Y_C+\delta Y_C]}x \end{array}\right\}=0$$

$$\Rightarrow \qquad [M+\delta M] = \begin{Bmatrix} \left[4{,}057-(61.46)\delta Q+(25.92)\delta R\right] \\ \\ +\left[-75.18+(0.36)\delta Q-(0.48)\delta R\right]x \end{Bmatrix}$$

$$\boxed{M = 4{,}057-(75.18)x}$$

$$\Rightarrow \qquad (4.4.22)$$

$$\boxed{\delta M = \left[-(61.46)\delta Q+(25.92)\delta R\right]+\left[(0.36)\delta Q-(0.48)\delta R\right]x}$$

Cut #3

$$\boxed{\text{Cut \#3} \Rightarrow \text{Cut Between A and E}}$$

* When Cut #3 is made, we must choose either the portion of the frame to the right of the cut or the portion of the frame to the left of the cut for analysis.

* Both portions will give the exact same result.

The Left Portion of the Frame for Cut #3 is shown in Figure 4.4.15.

4.154

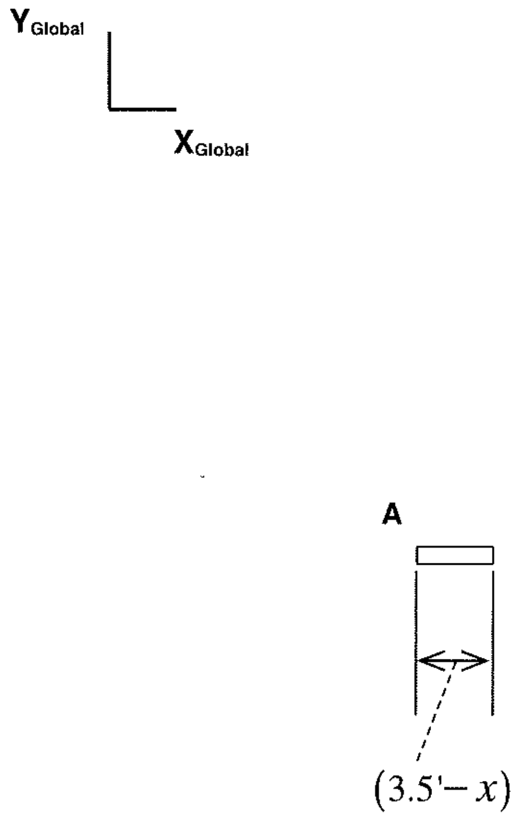


Figure 4.4.15

Internals for Cut #3
[Figure 4.4.15]

$$\sum F_{x_{\text{Temp}}} = 0$$

$\Rightarrow$ $$\delta R + [P + \delta P] = 0$$

$\Rightarrow$ $$\boxed{P = 0} \qquad \boxed{\delta P = -\delta R} \qquad (4.4.23)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$\Rightarrow$ $$-[M + \delta M] - \delta Q\left(\overbrace{42''}^{3.5'} - x\right) = 0$$

$\Rightarrow$ $$\boxed{M = 0} \qquad \boxed{\delta M = -\delta Q(42 - x)} \qquad (4.4.24)$$

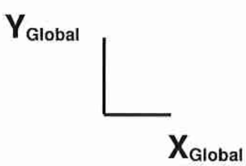
Cut #4

$$\boxed{\text{Cut \#4} \Rightarrow \text{Cut Between D and E}}$$

* When Cut #4 is made, we must choose either the portion of the frame above the cut or the portion of the frame below the cut for analysis.

* Both portions will give the exact same result.

The Lower Portion of the Frame for Cut #4 is shown in Figure 4.4.16.



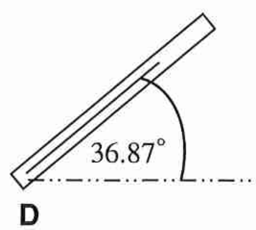


Figure 4.4.16

Internals for Cut #4
[Figure 4.4.16]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \qquad -\overbrace{\left[125.3 + (1.067)\delta Q + (0.8)\delta R\right]}^{\left[P_{D-E} + \delta P_{D-E}\right]} + \left[P + \delta P\right] = 0$$

$$\Rightarrow \qquad \boxed{P = 125.3} \qquad \boxed{\delta P = (1.067)\delta Q + (0.8)\delta R} \qquad (4.4.25)$$

The horizontal component of displacement u and the vertical component of displacement v at the free end will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.4.1) for a frame states,

$$\sum_i \underbrace{\left(b_i \Bigg|_{\substack{\text{At Location i on}\\ \text{Original Structure}\\ \text{Caused by Actual}\\ \text{Loads with Actual}\\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\substack{[\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}]}} = \sum_i \left\{ \begin{aligned} &\int_0^L \left[\frac{\left(P\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ &+ \int_0^L \left[\frac{\left(M_z\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \\ &+ \int_0^L \left[\frac{\left(V_y\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{G_i (I_z)_i^{\,2}} (\delta V_y)_i \right] \left[\iint_A \left(\frac{(Q_z)_i}{b_i} \right)^2 dA \right] dx \end{aligned} \right\} \tag{4.4.1}$$

For the two-force member and for the beams in our frame,

1. $P \neq 0$, and

2. the effects of shear deformation are neglected.

These will cause Equation 4.4.1 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on} \\ \text{Original Structure} \\ \text{Caused by Actual} \\ \text{Loads with Actual} \\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{\left(P \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(M_z \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \end{array} \right\} \tag{4.4.26}$$

Plugging Equations 4.4.16 - 4.4.25 into Equation 4.4.26,

$$\Rightarrow \left\{ \begin{array}{l} u_A (\delta R) \\ v_A (\delta Q) \\ + \overbrace{u_C}^{0} (\delta X_C) \\ + \overbrace{v_C}^{0} (\delta Y_C) \\ + \underbrace{\overbrace{b_D}^{0}}_{\left[\substack{\text{Displ.} \\ \text{of D}}\right]} (\delta P_{\text{D-E}}) \end{array} \right\} = \left[\frac{1}{(15.7\,x 10^6\, psi) \underbrace{\left[\pi (0.875")^2 \right]}_{A = 2.405\ in^2}} \left\{ \begin{array}{l} \overbrace{\int_0^{72} [47.32] \left[(0.36)\delta Q - (0.48)\delta R \right] dx}^{[\text{Cut \#1, Axial, Beam}]} \\ + \overbrace{\int_0^{54} [-100.3] \left[\begin{array}{l} (0.8536)\delta Q \\ -(0.3600)\delta R \end{array} \right] dx}^{[\text{Cut \#2, Axial, Beam}]} \\ + \overbrace{\int_0^{42} [0][-\delta R] dx}^{[\text{Cut \#3, Axial, Beam}]} \\ + \overbrace{\int_0^{120} [125.3] \left[\begin{array}{l} (1.067)\delta Q \\ +(0.8)\delta R \end{array} \right] dx}^{[\text{Cut \#4, Axial, 2- Force Member}]} \end{array} \right\} \right.$$

•••

4.160

$$\cdots$$

$$
\left[+ \frac{1}{\left(15.7\,x10^{6}\,psi\right)\underbrace{\left[\frac{\pi\left(0.875''\right)^{4}}{4}\right]}_{I_z=0.4604\ in^4}}
\left\{
\begin{array}{l}
\overbrace{\int_0^{72}\left[(111.9)x\right]\left\{\begin{bmatrix}-(0.8536)\,\delta Q\\ +(0.3600)\,\delta R\end{bmatrix}x\right\}dx}^{\left[\text{Cut \#1, Bending}\right]} \\[2ex]
\overbrace{+\int_0^{54}\begin{bmatrix}4{,}057\\ -(75.18)x\end{bmatrix}\left\{\begin{array}{l}\begin{bmatrix}-(61.46)\,\delta Q\\ +(25.92)\,\delta R\end{bmatrix}\\ +\begin{bmatrix}(0.36)\,\delta Q\\ -(0.48)\,\delta R\end{bmatrix}x\end{array}\right\}dx}^{\left[\text{Cut \#2, Bending}\right]} \\[2ex]
\overbrace{+\int_0^{42}[0]\left[-\delta Q(42-x)\right]dx}^{\left[\text{Cut \#3, Bending}\right]} \\[2ex]
+\overbrace{\int_0^{120}[0][0]dx}^{\left[\text{Cut \#4, Bending}\right]}
\end{array}
\right\} \right]
$$

$$\Rightarrow \begin{Bmatrix} u_A\,(\delta R) \\ v_A\,(\delta Q) \end{Bmatrix} = \left[\begin{aligned} &\left\{ \begin{aligned} &\overbrace{\int_0^{72} \left[\left(4.511\,x10^{-7}\right)\delta Q - \left(6.015\,x10^{-7}\right)\delta R \right] dx}^{\text{[Cut \#1, Axial]}} \\ &+ \overbrace{\int_0^{54} \left[-\left(2.267\,x10^{-6}\right)\delta Q + \left(9.561\,x10^{-7}\right)\delta R \right] dx}^{\text{[Cut \#2, Axial]}} \\ &+ \overbrace{\int_0^{120} \left[\left(3.540\,x10^{-6}\right)\delta Q + \left(2.654\,x10^{-6}\right)\delta R \right] dx}^{\text{[Cut \#4, Axial]}} \end{aligned} \right\} \\ &+ \left\{ \begin{aligned} &\overbrace{\int_0^{72} \left[-\left(1.321\,x10^{-5}\right)\delta Q + \left(5.571\,x10^{-6}\right)\delta R \right] x^2 dx}^{\text{[Cut \#1, Bending]}} \\ &+ \overbrace{\int_0^{54} \left\{ \begin{bmatrix} -\left(3.448\,x10^{-2}\right)\delta Q \\ +\left(1.454\,x10^{-2}\right)\delta R \end{bmatrix} + \begin{bmatrix} \left(2.020\,x10^{-4}\right)\delta Q \\ -\left(2.693\,x10^{-4}\right)\delta R \end{bmatrix} x \right\} dx}^{\text{[Cut \#2, Bending]}} \\ &+ \overbrace{\int_0^{54} \left\{ \begin{bmatrix} \left(6.390\,x10^{-4}\right)\delta Q \\ -\left(2.695\,x10^{-4}\right)\delta R \end{bmatrix} x + \begin{bmatrix} -\left(3.743\,x10^{-6}\right)\delta Q \\ +\left(4.991\,x10^{-6}\right)\delta R \end{bmatrix} x^2 \right\} dx}^{\text{[Cut \#2, Bending]}} \end{aligned} \right\} \end{aligned} \right]$$

4.162

$$\Rightarrow \begin{Bmatrix} u_A\,(\delta R) \\ v_A\,(\delta Q) \end{Bmatrix} = \left[\left\{ \begin{array}{l} \overbrace{\left[\left(3.248x10^{-5}\right)\delta Q - \left(4.331x10^{-5}\right)\delta R\right]}^{[\text{Cut \#1, Axial, Beam}]} \\ + \overbrace{\left[-\left(1.224x10^{-4}\right)\delta Q + \left(5.163x10^{-5}\right)\delta R\right]}^{[\text{Cut \#2, Axial, Beam}]} \\ + \overbrace{\left[\left(4.248x10^{-4}\right)\delta Q + \left(3.185x10^{-4}\right)\delta R\right]}^{[\text{Cut \#4, Axial, 2-Force Member}]} \end{array} \right\} + \left\{ \begin{array}{l} \overbrace{\left[-(1.644)\delta Q + \left(6.931x10^{-1}\right)\delta R\right]}^{[\text{Cut \#1, Bending}]} \\ + \overbrace{\left[-\left(8.322x10^{-1}\right)\delta Q + \left(2.616x10^{-1}\right)\delta R\right]}^{[\text{Cut \#2, Bending}]} \end{array} \right\} \right]$$

$$\Rightarrow \begin{Bmatrix} u_A\,(\delta R) \\ v_A\,(\delta Q) \end{Bmatrix} = \left\{ \begin{array}{l} \overbrace{\left[-\left(8.992x10^{-5}\right)\delta Q + \left(8.320x10^{-6}\right)\delta R\right]}^{[\text{Due To Internal Axial Forces in Beams}]} \\ + \overbrace{\left[\left(4.248x10^{-4}\right)\delta Q + \left(3.185x10^{-4}\right)\delta R\right]}^{[\text{Due To Internal Axial Force in 2-Force Member}]} \\ + \overbrace{\left[-(2.476)\delta Q + \left(9.556x10^{-1}\right)\delta R\right]}^{[\text{Due To Internal Bending Moments}]} \end{array} \right\}$$

$$\Rightarrow \qquad \begin{Bmatrix} u_A\,(\delta R) \\ \\ v_A\,(\delta Q) \end{Bmatrix} = \begin{Bmatrix} \begin{bmatrix} \overbrace{\left(8.320\,x10^{-6}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Forces in Beams}\end{bmatrix}} + \overbrace{\left(3.185\,x10^{-4}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Force in 2-Force Member}\end{bmatrix}} \\ + \overbrace{\left(9.556\,x10^{-1}\right)}^{\begin{bmatrix}\text{Due To Internal Bending}\\ \text{Moments}\end{bmatrix}} \end{bmatrix} \delta R \\ \\ \begin{bmatrix} \overbrace{-\left(8.992\,x10^{-5}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Forces in Beams}\end{bmatrix}} + \overbrace{\left(4.248\,x10^{-4}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Force in 2-Force Member}\end{bmatrix}} \\ + \overbrace{-(2.476)}^{\begin{bmatrix}\text{Due To Internal Bending}\\ \text{Moments}\end{bmatrix}} \end{bmatrix} \delta Q \end{Bmatrix}$$

Thus,

$$u_A = \overbrace{\left(8.320\,x10^{-6}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Forces in Beams}\end{bmatrix}} + \overbrace{\left(3.185\,x10^{-4}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Force in 2-Force Member}\end{bmatrix}} + \overbrace{\left(9.556\,x10^{-1}\right)}^{\begin{bmatrix}\text{Due To Internal Bending}\\ \text{Moments}\end{bmatrix}}$$

$$v_A = \overbrace{-\left(8.992\,x10^{-5}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Forces in Beams}\end{bmatrix}} + \overbrace{\left(4.248\,x10^{-4}\right)}^{\begin{bmatrix}\text{Due To Internal Axial}\\ \text{Force in 2-Force Member}\end{bmatrix}} + \overbrace{-(2.476)}^{\begin{bmatrix}\text{Due To Internal Bending}\\ \text{Moments}\end{bmatrix}}$$

4.164

$$\boxed{u_A = 0.9559''}$$

$$\boxed{v_A = -2.476''}$$

Sign Conventions

* u & v -----Positive if in the Positive Global Axes Directions

* Work of Externals----- Positive if Helping the Motion

We can see that, in this particular frame problem, the axial force in the 2-force member played a slightly more important role than the axial force in the beams. However, once again, including the effect of axial deformation caused a minor change in the answers (out to 4 significant digits) for the deformations of Point A.

You should now have the ability to do Problem 4.7, 4.8

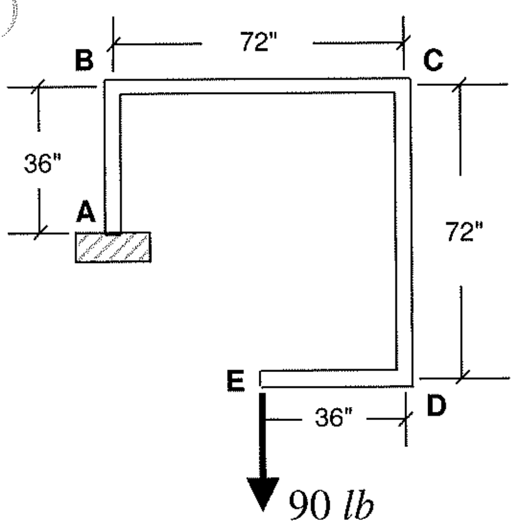
Homework: Do Problem 4.7

Example 4.4.5 **[Statically Determinate Frame Consisting of Straight Beams—Displacements]**

For the statically determinate frame shown in Figure 4.4.17 below, determine, using the Principle of Complementary Virtual Work, the vertical component of displacement v at the *free end.*

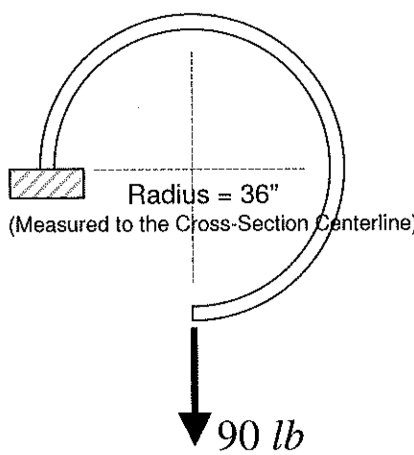
Material: Aluminum $E = 9.6 \times 10^6$ psi	Cross-Section: Solid Circular Diameter = 2"

* The frame below is an attempt to replace the curved beam of Problem 4.6 with a frame consisting of straight beams.
* The loads, material properties, cross-section properties, and overall dimensions are the same for both.
* The resulting deflections will then be compared to see which design gives the least deflection.



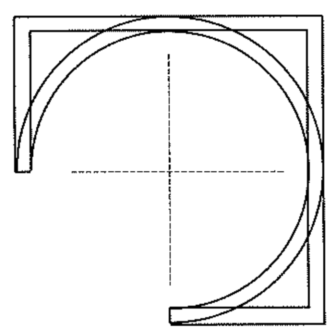
Frame Structure for This Example

Figure 4.4.17



Curved Beam from Problem 4.6

Figure 4.4.18



Figures Superimposed for Comparison

Figure 4.4.19

4.166

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

2-Force Members

Are there any two-force members in Figure 4.4.17?

There are no two-force members
because
1. there is a bending moment reaction at A due to the built-in connection, and
2. there are bending moment reactions at B, C, and D where the members connect to each other since the members are welded together and not pin-connected.

To determine the vertical component of displacement v at the free end using the Principle of Complementary Virtual Work, the actual load of 90 lb and all external actual reactions will be removed and a vertical virtual force δQ will be applied at the free end.

Using the short cut method, we get the diagrams below.

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

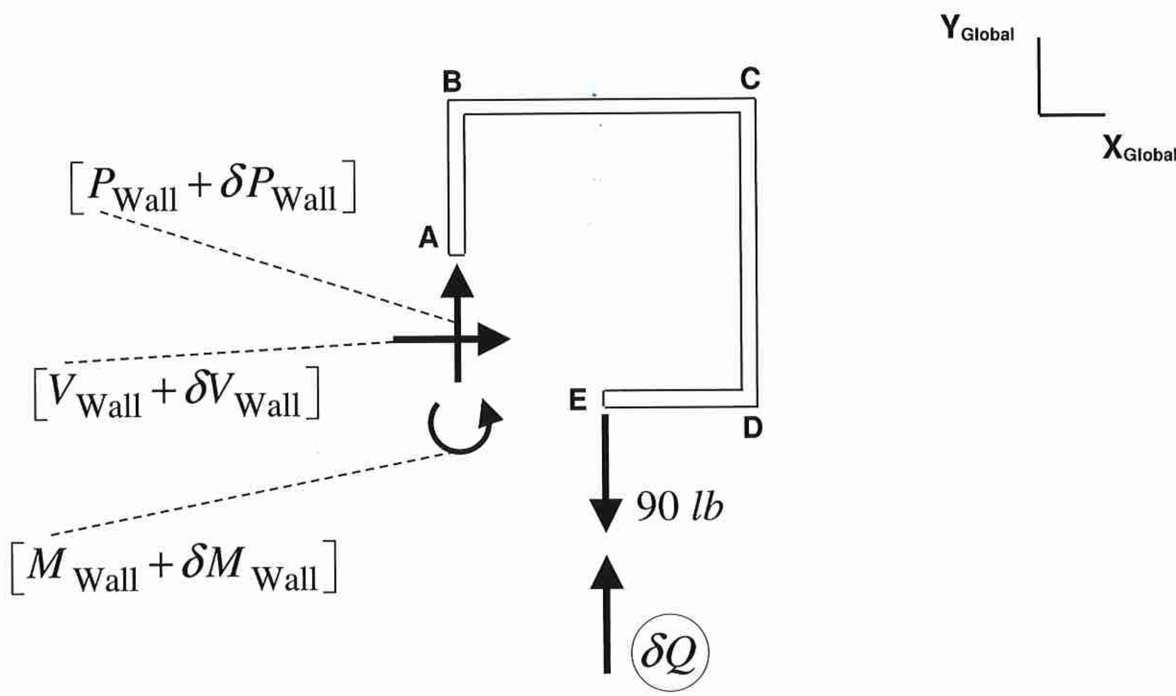


Figure 4.4.20

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

Why is this frame statically determinate?
Because it has only 3 unknown external reactions & we can get 3 equilibrium equations from statics.

External Reactions
[Figure 4.4.20]

There is no need to find the external reactions for the cuts that we are utilizing here.

* According to Equation 4.4.1, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

* This frame consists of 4 straight beams.

* None of the beams have a concentrated load along its span.

* Thus a single cut must be made at an arbitrary x-location along each beam.

Cuts

How many cuts does the structure in Figure 4.4.20 need?
It needs 4 cuts.
1. One between A and B
2. One between B and C
3. One between C and D
4. One between D and E

Cut #1

Cut #1 $\Rightarrow$ Cut Between D and E

* When Cut #1 is made, we must choose either the portion of the frame to the left of the cut or the portion of the frame to the right of the cut for analysis.

* Both portions will give the exact same result.

The Left Portion of the Frame for Cut #1 is shown below in Figure 4.4.21.

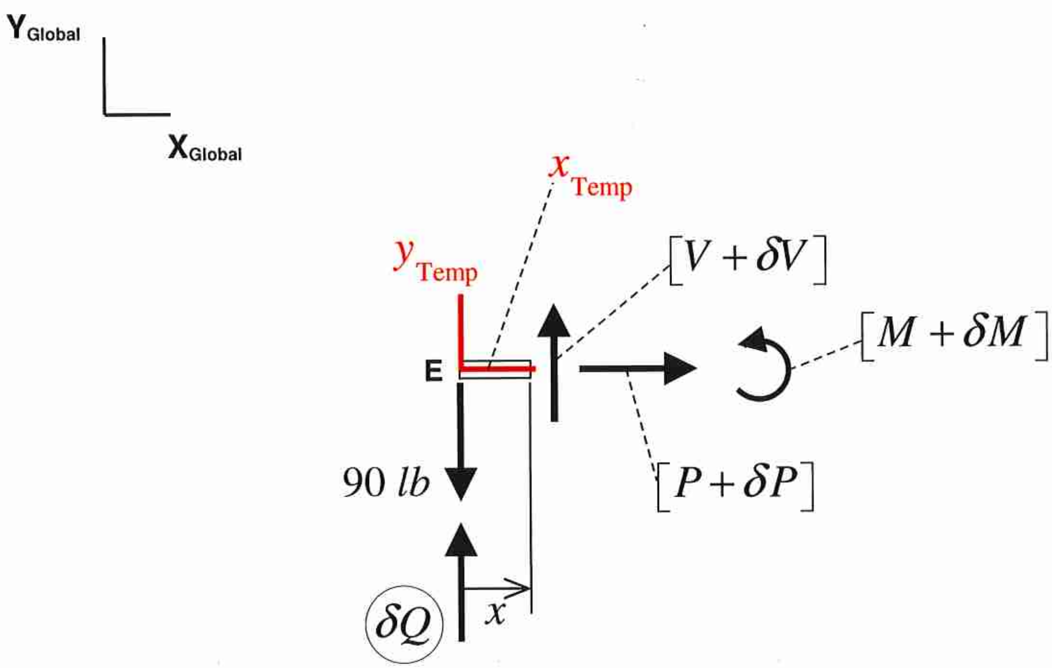


Figure 4.4.21.

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

Temporary Axes

* All of the formulas in Chapter 1 are derived for standard $x-y$ axes along the length of the member.

* When using Chapter 1 formulas in problems where the axes are not the standard $x-y$ axes, it is best to set up a temporary set of $x_{\text{Temp}} - y_{\text{Temp}}$ axes for using the Chapter 1 formulas.

* The x_{Temp} axis must be lined up along the beam axis.

Internals for Cut #1
[Figure 4.4.21]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \qquad [P + \delta P] = 0$$

$$\Rightarrow \qquad \boxed{P = 0} \qquad \boxed{\delta P = 0} \qquad (4.4.27)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\sum M_{\text{Cut}} = 0$$

$$\Rightarrow \qquad [M + \delta M] + (90\ lb)x - (\delta Q)x = 0$$

$$\Rightarrow \qquad [M + \delta M] = -(90)x + (\delta Q)x$$

$$\Rightarrow \qquad \boxed{M = -(90)x} \qquad \boxed{\delta M = (\delta Q)x} \tag{4.4.28}$$

Cut #2

Cut #2 $\Rightarrow$ Cut Between C and D

* When Cut #2 is made, we must choose either the portion of the frame above the cut or the portion of the frame below the cut for analysis.

* Both portions will give the exact same result.

The Portion of the Frame Below Cut #2 is shown in Figure 4.4.22.

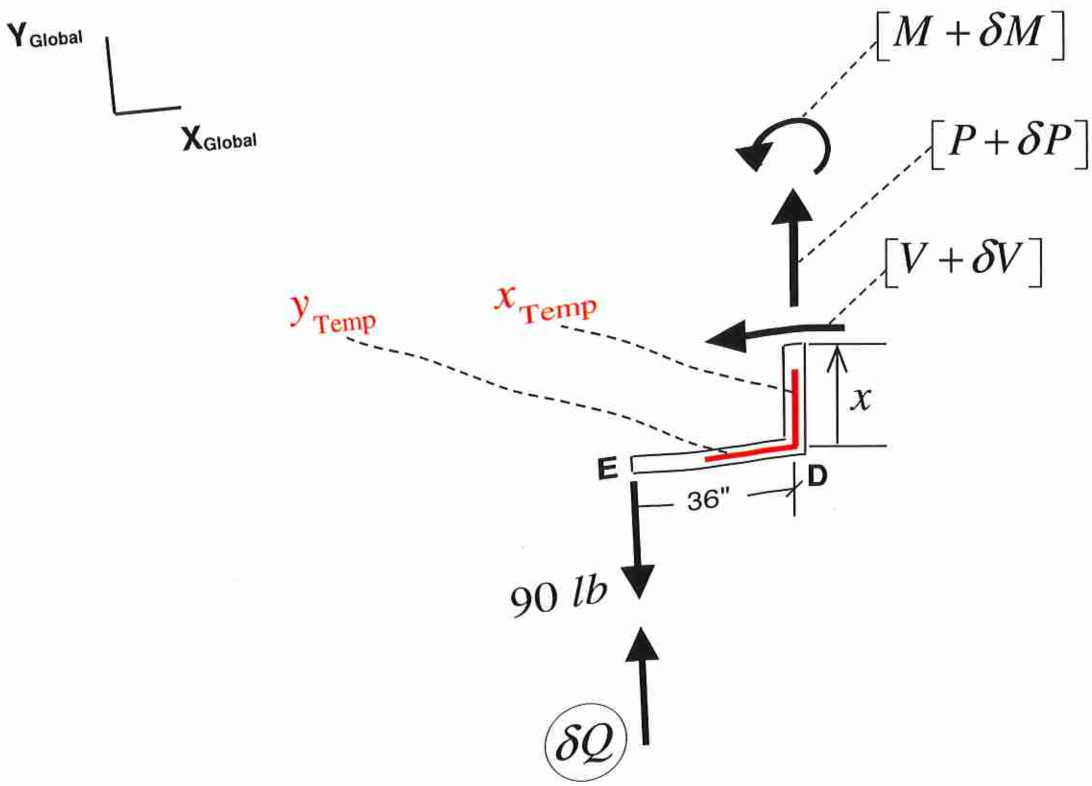


Figure 4.4.22

Internals for Cut #2
[Figure 4.4.22]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad [P + \delta P] - 90\ lb + \delta Q = 0$$

$$\Rightarrow \quad [P + \delta P] = 90 - \delta Q$$

$$\Rightarrow \quad \boxed{P = 90} \qquad \boxed{\delta P = -\delta Q} \tag{4.4.29}$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad [M + \delta M] + (90\ lb)(36") - (\delta Q)(36") = 0$$

$$\Rightarrow \quad [M + \delta M] = -3,240 + \delta Q$$

$$\Rightarrow \quad \boxed{M = -3,240} \qquad \boxed{\delta M = (36)\delta Q} \tag{4.4.30}$$

4.174

Cut #3

Cut #3 $\Rightarrow$ Cut Between B and C

* When Cut #3 is made, we must choose either the portion of the frame to the left of the cut or the portion of the frame to the right of the cut for analysis.

* Both portions will give the exact same result.

The Right Portion of the Frame for Cut #3 is shown below in Figure 4.4.23.

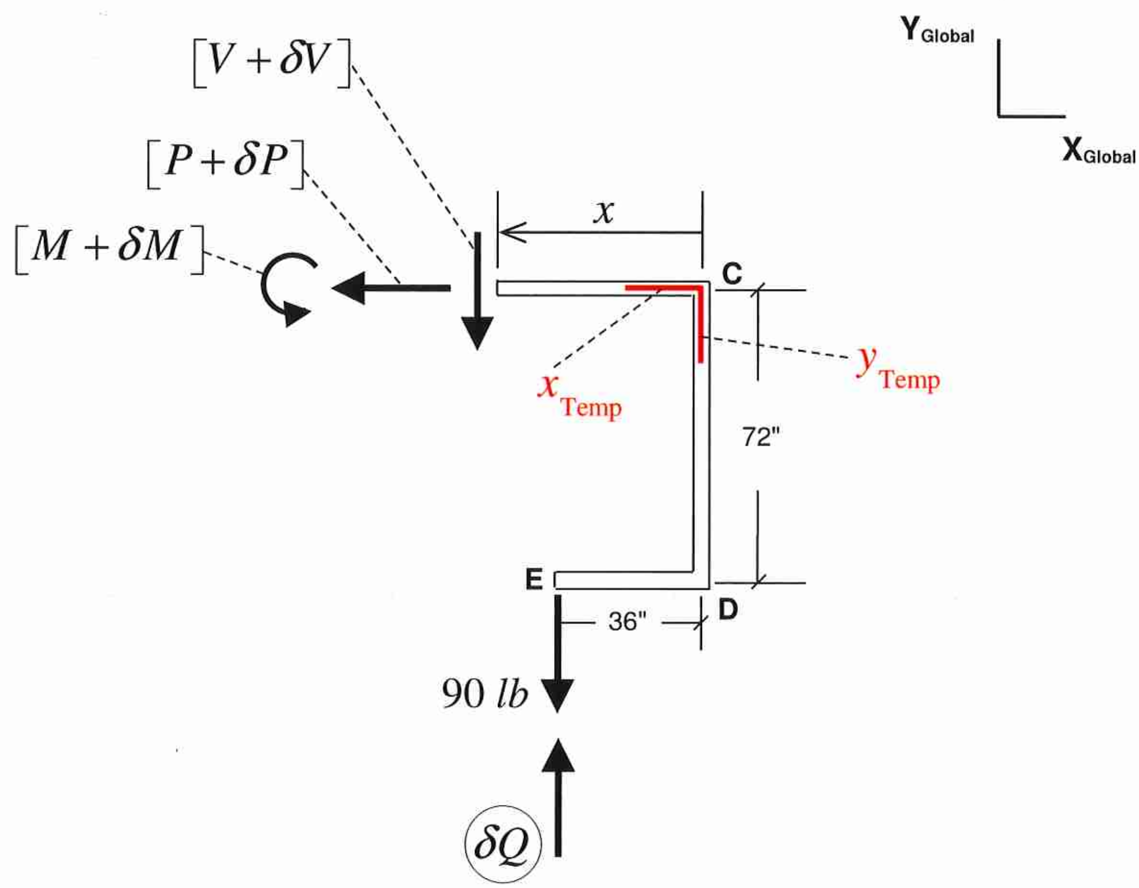


Figure 4.4.23

Internals for Cut #3
[Figure 4.4.23]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad [P + \delta P] = 0$$

$$\Rightarrow \quad \boxed{P = 0} \qquad \boxed{\delta P = 0} \qquad (4.4.31)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad [M + \delta M] - (90\ lb)(x - 36") + (\delta Q)(x - 36") = 0$$

$$\Rightarrow \quad [M + \delta M] = (90)(x - 36) - (\delta Q)(x - 36)$$

$$\Rightarrow \quad \boxed{M = (90)(x - 36)} \qquad \boxed{\delta M = -(\delta Q)(x - 36)} \qquad (4.4.32)$$

Cut #4

$$\boxed{\text{Cut \#4} \Rightarrow \text{Cut Between A and B}}$$

* When Cut #4 is made, we must choose either the portion of the frame above the cut or the portion of the frame below the cut for analysis.

* Both portions will give the exact same result.

The Portion of the Frame Above Cut #4 is shown in Figure 4.4.24.

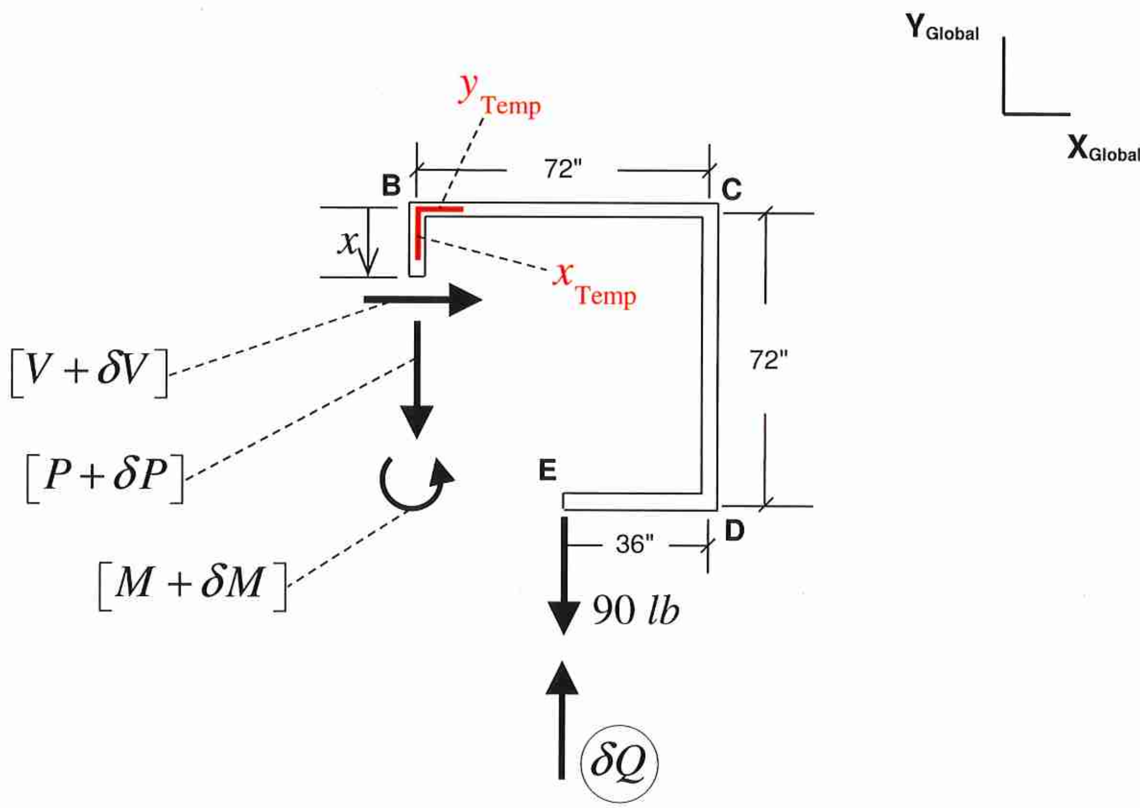


Figure 4.4.24

Internals for Cut #4
[Figure 4.4.24]

$$\sum F_{x_{\text{Temp}}} = 0$$

$\Rightarrow$ $$[P + \delta P] + 90\ lb - \delta Q = 0$$

$\Rightarrow$ $$[P + \delta P] = -90 + \delta Q$$

$\Rightarrow$ $$\boxed{P = -90} \qquad \boxed{\delta P = \delta Q} \tag{4.4.33}$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$\Rightarrow$ $$[M + \delta M] - (90\ lb)(36") + (\delta Q)(36") = 0$$

$\Rightarrow$ $$[M + \delta M] = 3,240 - (36)\delta Q$$

$\Rightarrow$ $$\boxed{M = 3,240} \qquad \boxed{\delta M = -(36)\delta Q} \tag{4.4.34}$$

The vertical component of displacement v at the free end will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.4.1) for a frame states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on} \\ \text{Original Structure} \\ \text{Caused by Actual} \\ \text{Loads with Actual} \\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{\left(P \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(M_z \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(V_y \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{G_i (I_z)_i^2} (\delta V_y)_i \right] \left[\iint_A \left(\frac{(Q_z)_i}{b_i} \right)^2 dA \right] dx \end{array} \right\} \tag{4.4.1}$$

For the beams in our frame, $P \neq 0$ and the effects of shear deformation are neglected.

These will cause Equation 4.4.1 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on} \\ \text{Original Structure} \\ \text{Caused by Actual} \\ \text{Loads with Actual} \\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \sum_i \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{\left(P \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ \displaystyle + \int_0^L \left[\frac{\left(M_z \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \end{array} \right\} \tag{4.4.35}$$

Plugging Equations 4.4.27 - 4.4.34 into Equation 4.4.35,

$$\Rightarrow \left\{ \begin{array}{l} v_{\text{Free End}}(\delta Q) \\ + \overbrace{u_{\text{Wall}}}^{0}(\delta P_{\text{Wall}}) \\ + \overbrace{v_{\text{Wall}}}^{0}(\delta V_{\text{Wall}}) \\ + \overbrace{\theta_{\text{Wall}}}^{0}(\delta M_{\text{Wall}}) \end{array} \right\} = \left\{ \begin{array}{l} \dfrac{1}{(9.6x10^6\,psi)\underbrace{\left[\pi(1")^2\right]}_{A=3.1416\ in^2}} \left\{ \begin{array}{l} \overbrace{\int_0^{36}[0][0]dx}^{[\text{Cut \#1, Axial}]} + \overbrace{\int_0^{72}[90]\left[-(\delta Q)\right]dx}^{[\text{Cut \#2, Axial}]} \\ + \overbrace{\int_0^{72}[0][0]dx}^{[\text{Cut \#3, Axial}]} + \overbrace{\int_0^{36}[-90][\delta Q]dx}^{[\text{Cut \#4, Axial}]} \end{array} \right\} \\ + \dfrac{1}{(9.6x10^6\,psi)\underbrace{\left[\dfrac{\pi(1")^4}{4}\right]}_{I_z=0.7854\ in^4}} \left\{ \begin{array}{l} \overbrace{\int_0^{36}\left[-(90)x\right]\left[(\delta Q)x\right]dx}^{[\text{Cut \#1, Bending}]} \\ + \overbrace{\int_0^{72}[-3,240]\left[(36)\delta Q\right]dx}^{[\text{Cut \#2, Bending}]} \\ + \overbrace{\int_0^{72}\left[(90)(x-36)\right]\left[-(\delta Q)(x-36)\right]dx}^{[\text{Cut \#3, Bending}]} \\ + \overbrace{\int_0^{36}[3,240]\left[-(36)\delta Q\right]dx}^{[\text{Cut \#4, Bending}]} \end{array} \right\} \end{array} \right\}$$

4.180

$$\Rightarrow \quad v_{\text{Free End}}(\delta Q) = \left\{ \begin{array}{l} \overbrace{\left(2.984\,x10^{-6}\right)(\delta Q)\left\{ -\int_0^{72} dx + \int_0^{36} dx \right\}}^{[\text{Due To Internal Axial Forces}]} \\ \\ \overbrace{-\left(1.326\,x10^{7}\right)(\delta Q)\left\{ \begin{array}{l} (90)\int_0^{36} x^2 dx + \left(1.296\,x10^5\right)\int_0^{72} dx \\ \\ +(90)\int_0^{72} (x-36)^2 dx + \left(1.296\,x10^5\right)\int_0^{36} dx \end{array} \right\}}^{[\text{Due To Internal Bending Moments}]} \end{array} \right\}$$

$$\Rightarrow \qquad v_{\text{Free End}} = \overbrace{(-0.0001074)}^{\left[\begin{array}{l}\text{Deformation Due To}\\ \text{Internal Axial Forces}\end{array}\right]} + \overbrace{(-2.227)}^{\left[\begin{array}{l}\text{Deformation Due To}\\ \text{Internal Moments}\end{array}\right]}$$

$$\Rightarrow \qquad \boxed{v_{\text{Free End}} = -2.227"}$$

Sign Conventions

* u & v -----Positive if in the Positive Global Axes Directions

* Work of Externals----- Positive if Helping the Motion

We can see that, in this particular frame problem, including the effect of axial deformation caused no change in the answer (out to 4 significant digits) for the deformation at the free end.

The deflection obtained from the curved beam of Problem 4.6 was

$$v_{\text{Free End}} = -1.312''$$

The deflection of the frame in this example is about 1.7 times that of the curved beam.

We can see that the curved beam can support the load with a significantly smaller deflection, even though the rectangular frame consists of more material.

Thus the rectangular frame will be heavier & weaker.

4.5 Statically Indeterminate Beams and Frames---Forces and Deflections

Example 4.5.1 **[Statically Indeterminate Straight Beam—Reactions, Displacements, Rotations]**

For the statically indeterminate beam shown in the figure below, determine, using the Principle of Complementary Virtual Work,

(a) the reactions at all connections,
(b) the vertical component of displacement v at the *center*, and
(c) the rotation θ at the *center*.

Material: Titanium (Ti-6Al-4V)	Cross-Section: Hollow Rectangular
$E = 16.5 \times 10^6$ psi	Height = 5"
$G = 6.38 \times 10^6$ psi	Width = 3"
$\nu = 0.2375$	$t = \frac{1}{2}$"

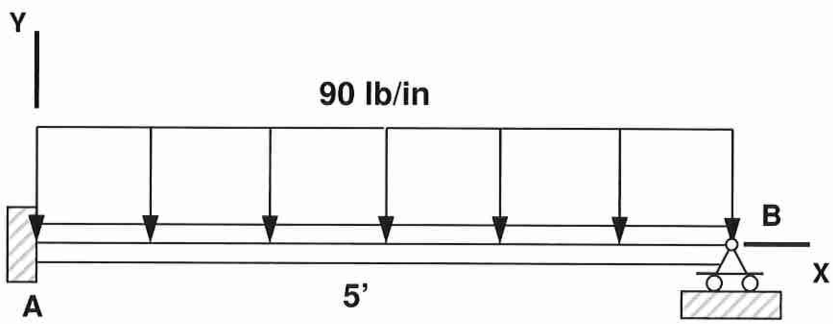


Figure 4.5.1

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

This beam is exactly the same as that of Example 4.4.2 *except* that a roller support has been added at B, where the free end used to be. It has the same loading, the same material, the same cross-sectional shape, and the same dimensions.

Why is this beam statically indeterminate?

Because it has 4 unknown external reactions & we can only get 3 equilibrium equations from statics.

This means that there is 1 Redundant.

1 Redundant

* The Principle of Complementary Virtual Work will be used to obtain the 1 additional equation needed beyond those of static equilibrium.

* This means that we will have to apply 1 virtual force.

* Let us imagine applying 1 virtual force in the same location and in the same direction as 1 of the external reactions.

* This applied virtual force will cause
  (i) virtual internal reactions inside the beam, as well as
  (ii) virtual external reactions at the external supports.

* Of all the resulting virtuals, which will be treated as the applied virtual & which will be treated as the reaction virtuals?

* The applied virtual could be any 1 of the resulting virtuals; namely, any of the external virtuals or any of the internal virtuals.

* There is no need to distinguish between them.

At some point, we must decide which of the real forces we will select as our redundant.

Choice of Redundants

Two Methods for Choosing Redundants

Method 1

Best method when there are a large number of simultaneous equations produced. Trusses generally fit into this category.

* To choose the redundants
  1. write all of the static equilibrium equations in terms of all of the variables
  2. then determine which choice of redundants
     i. would make sense mathematically and
     ii. would make the equations easiest to solve.

* As a general rule, for trusses, the algebra will be reduced if P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work equation.

Method 2

Quite often this is the best method for simple beams and frames.

* To choose the redundants
  1. visibly identify which reactions could be removed so that the structure would
     i. be statically determinate and
     ii. still support the loads without the entire structure having any rigid body motions.
  2. Use the reactions that could be removed as the redundants.

(a) To determine the reactions at all connections using the Principle of Complementary Virtual Work,

1. the actual load of 90 lb/in, as well as all externals, will be removed, and
2. a virtual load will be applied at one of the supports in the same direction as one of the support reactions.

(b) To determine the vertical component of displacement v at the center using the Principle of Complementary Virtual Work,

1. the actual load of 90 lb/in, as well as all externals, will be removed, and
2. a vertical virtual force δQ will be applied at the center.

(c) To determine the rotation θ at the center using the Principle of Complementary Virtual Work,

1. the actual load of 90 lb/in, as well as all externals, will be removed, and
2. a virtual couple δC will be applied at the center.

The 3 applied virtuals, as described in Parts a, b, and c above, can be applied simultaneously.

For the Short Cut Method a FBD of the structure with both reals, and virtuals is drawn in Figure 4.5.2 below.

FBDs + STATICS

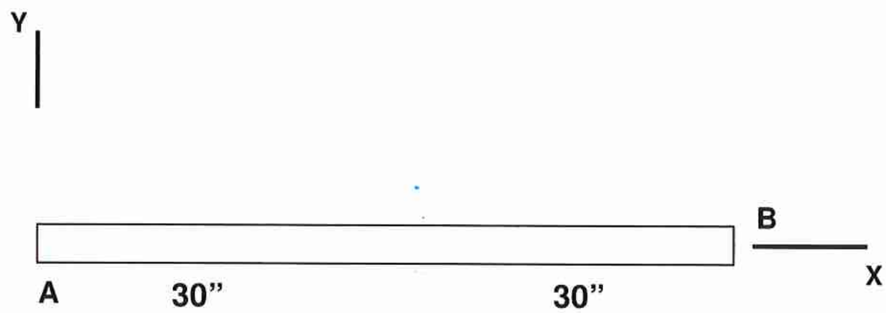


Figure 4.5.2

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

Choice of Redundant for This Beam

Following Method 2 on Page 4.184, an obvious choice here is to use Y_{Roller} for the redundant.

If the roller at B were removed, the beam
1. would be statically determinate and
2. would still be able to support the loads without the entire beam having any rigid body motions.

Y_{Roller} will be chosen as the redundant

External Reactions
[Figure 4.5.2]

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[V_{\text{Wall}} + \delta V_{\text{Wall}}\right] - \left[\left(90\,\frac{lb}{in}\right)(60")\right] + \delta Q + \left[Y_{\text{Roller}} + \delta Y_{\text{Roller}}\right] = 0$$

$$\Rightarrow \quad \left[V_{\text{Wall}} + \delta V_{\text{Wall}}\right] = \left[-Y_{\text{Roller}} + 5,400\right] + \left[-\delta Y_{\text{Roller}} - \delta Q\right]$$

$$\Rightarrow \quad \boxed{V_{\text{Wall}} = -Y_{\text{Roller}} + 5,400} \qquad \boxed{\delta V_{\text{Wall}} = -\delta Y_{\text{Roller}} - \delta Q} \tag{4.5.1}$$

$$\circlearrowleft + \sum M_{\text{Wall}} = 0$$

$$\Rightarrow \left[M_{\text{Wall}} + \delta M_{\text{Wall}}\right] - \left[\left(90\,\frac{lb}{in}\right)(60")\right](30") + (\delta Q)(30") + \delta C + \left[Y_{\text{Roller}} + \delta Y_{\text{Roller}}\right](60") = 0$$

$$\Rightarrow \quad \left[M_{\text{Wall}} + \delta M_{\text{Wall}}\right] = \left[-(60)Y_{\text{Roller}} + 162,000\right] + \left[-(60)\delta Y_{\text{Roller}} - (30)\delta Q - \delta C\right]$$

$$\Rightarrow \quad \boxed{M_{\text{Wall}} = -(60)Y_{\text{Roller}} + 162,000} \qquad \boxed{\delta M_{\text{Wall}} = -(60)\delta Y_{\text{Roller}} - (30)\delta Q - \delta C} \tag{4.5.2}$$

4.188

* Equations 4.5.1 and 4.5.2 are our 2 Static Equilibrium Equations.

* The third Static Equilibrium Equation for P was not used since it was recognized that it would simply give P =0.

* But there are 3 unknown external reactions.

* The Principle of Complementary Virtual Work will yield the third equation to find the redundant Y_{Roller}.

* The Principle of Complementary Virtual Work will also yield two more equations to find $\mathrm{v}_{\text{Center}}$ and θ_{Center}.

* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

Cut #1

For Cut #1, $0 \leq x < 30"$

* When Cut #1 is made, we must choose either the portion of the beam to the left of the cut or the portion of the beam to the right of the cut for analysis.

* Both portions will give the exact same result.

The Left Portion of the Beam for Cut #1 is shown below in Figure 4.5.3.

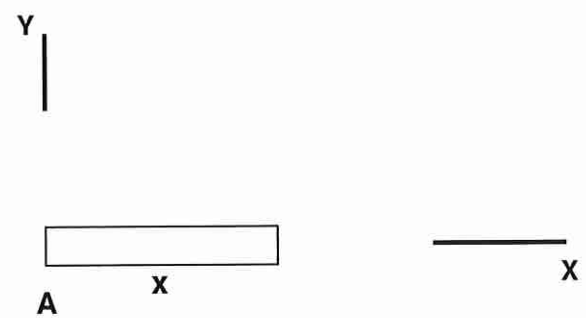


Figure 4.5.3

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

4.190

Internals for Cut #1
[Figure 4.5.3]

$$\sum F_y = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \begin{bmatrix} -(60)Y_{\text{Roller}} + 162{,}000 \\ -(60)\delta Y_{\text{Roller}} - (30)\delta Q - \delta C \end{bmatrix} - \begin{bmatrix} -Y_{\text{Roller}} + 5{,}400 \\ -\delta Y_{\text{Roller}} - \delta Q \end{bmatrix} x + \left[\left(90\,\frac{lb}{in}\right)x\right]\left(\frac{x}{2}\right) + [M + \delta M] = 0$$

$$\Rightarrow \quad [M + \delta M] = \begin{Bmatrix} -(45)x^2 + \left(-Y_{\text{Roller}} + 5{,}400\right)x \\ +\left[(60)Y_{\text{Roller}} - 162{,}000\right] \end{Bmatrix} + \begin{Bmatrix} \left(-\delta Y_{\text{Roller}} - \delta Q\right)x \\ +\left[(60)\delta Y_{\text{Roller}} + (30)\delta Q + \delta C\right] \end{Bmatrix}$$

$$\Rightarrow \quad \boxed{M = \begin{Bmatrix} -(45)x^2 \\ +\left(-Y_{\text{Roller}} + 5{,}400\right)x \\ +\left[(60)Y_{\text{Roller}} - 162{,}000\right] \end{Bmatrix}} \qquad \boxed{\delta M = \begin{Bmatrix} \left(-\delta Y_{\text{Roller}} - \delta Q\right)x \\ +\left[(60)\delta Y_{\text{Roller}} + (30)\delta Q + \delta C\right] \end{Bmatrix}} \quad (4.5.3)$$

Cut #2

For Cut #2, $30'' < x \leq 60''$

* When Cut #2 is made, we must choose either the portion of the beam to the left of the cut or the portion of the beam to the right of the cut for analysis.

* Both portions will give the exact same result.

The Right Portion of the Beam for Cut #2 is shown below in Figure 4.5.3.

Figure 4.5.4

Internals for Cut #2
[Figure 4.5.4]

$$\sum F_y = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

4.192

$$\circlearrowleft + \quad \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad -[M+\delta M]-\left[\left(90\,\frac{lb}{in}\right)(60''-x)\right]\left[\frac{(60''-x)}{2}\right]+\left[Y_{\text{Roller}}+\delta Y_{\text{Roller}}\right](60''-x)=0$$

$$\Rightarrow \quad [M+\delta M]=\begin{Bmatrix}-(45)x^2\\+\left(-Y_{\text{Roller}}+5,400\right)x\\+\left[(60)Y_{\text{Roller}}-162,000\right]\end{Bmatrix}+\left[-\left(\delta Y_{\text{Roller}}\right)x+(60)\,\delta Y_{\text{Roller}}\right]$$

$$\Rightarrow \quad \boxed{M=\begin{Bmatrix}-(45)x^2\\+\left(-Y_{\text{Roller}}+5,400\right)x\\+\left[(60)Y_{\text{Roller}}-162,000\right]\end{Bmatrix}} \qquad \boxed{\delta M=\left[-\left(\delta Y_{\text{Roller}}\right)x+(60)\,\delta Y_{\text{Roller}}\right]} \quad (4.5.4)$$

The redundant Y_{Roller}, the vertical component of displacement v at the center, and the rotation θ at the center will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.3.7) for a single beam states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure}\\ \text{Caused by Actual Loads}\\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}}}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \tag{4.3.7}$$

For our beam,

1. $P = 0$, and
2. the effects of shear deformation are neglected.

These will cause Equation 4.3.7 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \int_0^L \left[\frac{M_z \Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \qquad (4.5.5)$$

Plugging Equations 4.5.3 and 4.5.4 into Equation 4.5.5, gives

$$\left\{ \begin{array}{l} \text{v}_{\text{Center}} \left(\delta Q \right) \\ + \theta_{\text{Center}} \left(\delta C \right) \\ + \overbrace{\text{v}_{\text{Roller}}}^{0} \left(\delta Y_{\text{Roller}} \right) \\ + \overbrace{\text{v}_{\text{Wall}}}^{0} \left(\delta V_{\text{Wall}} \right) \\ + \overbrace{\theta_{\text{Wall}}}^{0} \left(\delta M_{\text{Wall}} \right) \end{array} \right\} = \frac{1}{\left(16.5\,x10^6\,psi\right)\underbrace{\left(20.58\,in^4\right)}_{I_z}} \left\{ \begin{array}{l} \overbrace{\int_0^{30} \left\{ \begin{array}{l} -(45)x^2 \\ +\left(-Y_{\text{Roller}} + 5{,}400\right)x \\ + \left[\begin{array}{l} (60)Y_{\text{Roller}} \\ -162{,}000 \end{array} \right] \end{array} \right\} \left\{ \begin{array}{l} \left(-\delta Y_{\text{Roller}} - \delta Q\right)x \\ + \left[\begin{array}{l} (60)\delta Y_{\text{Roller}} \\ +(30)\delta Q \\ +\delta C \end{array} \right] \end{array} \right\} dx}^{\text{Cut \#1}} \\ + \overbrace{\int_{30}^{60} \left\{ \begin{array}{l} -(45)x^2 \\ +\left(-Y_{\text{Roller}} + 5{,}400\right)x \\ + \left[\begin{array}{l} (60)Y_{\text{Roller}} \\ -162{,}000 \end{array} \right] \end{array} \right\} \left[\begin{array}{l} -\left(\delta Y_{\text{Roller}}\right)x \\ +(60)\delta Y_{\text{Roller}} \end{array} \right] dx}^{\text{Cut \#2}} \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{matrix} \mathrm{v}_{\text{Center}}\left(\delta Q\right) \\ \\ +\theta_{\text{Center}}\left(\delta C\right) \end{matrix} \right\} = \left(2.945\,x10^{-9}\right) \left\{ \begin{matrix} \displaystyle\int_0^{30} \left\{ \begin{matrix} (45)x^3 \\ +\left(Y_{\text{Roller}} - 5{,}400\right)x^2 \\ +\left[-(60)Y_{\text{Roller}} + 162{,}000\right]x \end{matrix} \right\} \left(\delta Y_{\text{Roller}} + \delta Q\right) dx \\ \\ +\displaystyle\int_0^{30} \left\{ \begin{matrix} -(45)x^2 \\ +\left(-Y_{\text{Roller}} + 5{,}400\right)x \\ +\left[(60)Y_{\text{Roller}} - 162{,}000\right] \end{matrix} \right\} \left[\begin{matrix} (60)\,\delta Y_{\text{Roller}} \\ +(30)\,\delta Q + \delta C \end{matrix} \right] dx \\ \\ +\displaystyle\int_{30}^{60} \left\{ \begin{matrix} (45)x^3 \\ +\left(Y_{\text{Roller}} - 5{,}400\right)x^2 \\ +\left[-(60)Y_{\text{Roller}} + 162{,}000\right]x \end{matrix} \right\} \left(\delta Y_{\text{Roller}}\right) dx \\ \\ +\displaystyle\int_{30}^{60} \left\{ \begin{matrix} -(2{,}700)x^2 \\ +\left[-(60)Y_{\text{Roller}} + 324{,}000\right]x \\ +\left[(3{,}600)Y_{\text{Roller}} - 9{,}720{,}000\right] \end{matrix} \right\} \left(\delta Y_{\text{Roller}}\right) dx \end{matrix} \right\}$$

$$\Rightarrow \qquad \left(\mathrm{v}_{\text{Center}}\right)\delta Q + \left(\theta_{\text{Center}}\right)\delta C = \left\{ \begin{matrix} \left[\left(2.120\,x10^{-4}\right)Y_{\text{Roller}} - 0.4294\right]\delta Y_{\text{Roller}} \\ +\left[\left(6.626\,x10^{-5}\right)Y_{\text{Roller}} - 0.1521\right]\delta Q \\ +\left[\left(3.976\,x10^{-6}\right)Y_{\text{Roller}} - 0.008349\right]\delta C \end{matrix} \right\}$$

$$0 = \left(2.120\,x10^{-4}\right)Y_{\text{Roller}} - 0.4294$$

$$\Rightarrow \qquad \mathrm{v}_{\text{Center}} = \left(6.626\,x10^{-5}\right)Y_{\text{Roller}} - 0.1521 \qquad (4.5.6)$$

$$\theta_{\text{Center}} = \left(3.976\,x10^{-6}\right)Y_{\text{Roller}} - 0.008349$$

We thus have 3 simultaneous equations to solve (Equations 4.5.6) for the 3 unknowns. The results are

$$\Rightarrow \quad \boxed{Y_{\text{Roller}} = 2{,}025\ lb} \quad \boxed{\text{v}_{\text{Center}} = -0.01792''} \quad \boxed{\theta_{\text{Center}} = -0.0002976\ rad \doteq -0.017^{\circ}} \qquad (4.5.7)$$

Sign Conventions

* u & v ---- Positive if in the Positive Global Axes Directions
* Work of Externals ---- Positive if Helping the Motion

Now that v_{Center} and θ_{Center} have been found, Parts (b) and (c) of this example have been answered.

The reactions at all connections now have to be found in order to answer Part (a). This is easily done by substituting the value of $Y_{\text{Roller}} = 2{,}025$ into Equations 4.5.1 and 4.5.2 to obtain V_{Wall} and M_{Wall}.

The values obtained are

$$V_{\text{Wall}} = 3{,}375$$

$$M_{\text{Wall}} = 40{,}500$$

Therefore, the reactions at all connections are

$$\boxed{\begin{array}{l} V_{\text{Wall}} = 3{,}375\ lb \uparrow \\ M_{\text{Wall}} = 40{,}500\ in \cdot lb \circlearrowleft \\ Y_{\text{Roller}} = 2{,}025\ lb \uparrow \end{array}} \tag{4.5.8}$$

Deflections ComparisonTable

	Beam	Deflection of Center (Inches)	Deflection of End B (Inches)	Rotation of Center (Degrees)	Rotation of End B (Degrees)
Example 4.4.1 Example 4.4.2	Cantilevered	-0.1521	-0.4294	-0.48	-0.55
Example 4.5.1	Cantilevered + Roller	-0.01792	0	-0.017	

Reactions ComparisonTable

	Beam	Force at Wall (Pounds)	Moment at Wall (In-Pounds)	Force at Roller (Pounds)
Example 4.4.1 Example 4.4.2	Cantilevered	5,400	162,000	
Example 4.5.1	Cantilevered + Roller	3,375	40,500	2,025

* From the above comparisons, it can easily be seen why someone may want to have additional supports beyond those that are absolutely necessary.

* The strength of each connection could be significantly less when additional members are added.

* Also, additional members could be included for the purpose of a Fail-Safe Design.

You should now have the ability to do Problem 4.9

Homework: Do Problem 4.9

Example 4.5.2 **[Statically Indeterminate Curved Beam—Reactions]**

For the statically indeterminate curved beam shown in the figure below, determine, using the Principle of Complementary Virtual Work, the reactions at all connections.

Material: Steel
$E = 29.1 \times 10^6$ psi

Cross-Section: Solid Circular
Diameter = 1.5"

Diameter of Beam Curvature = 8' (= 96")
(Measured to the Cross-Section Centerline)

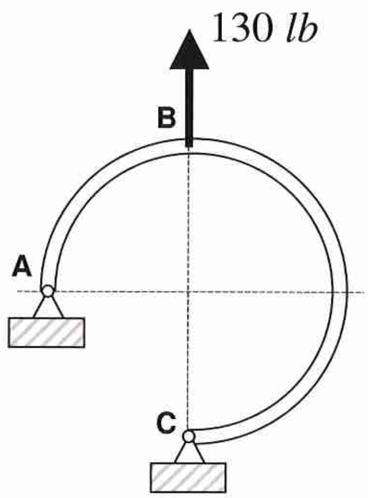


Figure 4.5.5

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

2-Force Members

Why is the structure in Figure 4.5.5 NOT a two-force member?

Why is this curved beam statically indeterminate?
Because it has 4 unknown external reactions & we can only get 3 equilibrium equations from statics.

This means that there is 1 Redundant.

1 Redundant

* The Principle of Complementary Virtual Work will be used to obtain the 1 additional equation needed beyond those of static equilibrium.
* This means that we will have to apply 1 virtual force.
* Let us imagine applying 1 virtual force in the same location and in the same direction as 1 of the external reactions.
* This applied virtual force will cause
  (i) virtual internal reactions inside the beam, as well as
  (ii) virtual external reactions at the external supports.
* Of all the resulting virtuals, which will be treated as the applied virtual & which will be treated as the reaction virtuals?
* The applied virtual could be any 1 of the resulting virtuals; namely, any of the external virtuals or any of the internal virtuals.
* There is no need to distinguish between them.

At some point, we must decide which of the real forces we will select as our redundant.

Choice of Redundants

Two Methods for Choosing Redundants

Method 1

Best method when there are a large number of simultaneous equations produced. Trusses generally fit into this category.

* To choose the redundants
  1. write all of the static equilibrium equations in terms of all of the variables
  2. then determine which choice of redundants
     i. would make sense mathematically and
     ii. would make the equations easiest to solve.

* As a general rule, for trusses, the algebra will be reduced if P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work equation.

Method 2

Quite often this is the best method for simple beams and frames.

* To choose the redundants
  1. visibly identify which reactions could be removed so that the structure would
     i. be statically determinate and
     ii. still support the loads without the entire structure having any rigid body motions.
  2. Use the reactions that could be removed as the redundants.

To determine the reactions at all connections using the Principle of Complementary Virtual Work,

1. the actual load of 130 lb, as well as all externals, will be removed, and
2. a virtual load will be applied at one of the supports in the same direction as one of the support reactions.

For the Short Cut Method a FBD of the structure with both reals, and virtuals is drawn in Figure 4.5.6 below.

FBDs + STATICS

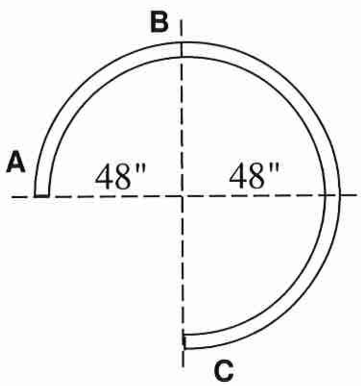


Figure 4.5.6

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

Choice of Redundant for This Curved Beam

One choice would be to use X_C for the redundant.

If the x-reaction at C were removed, it would be as if there was a roller at C.
The beam

1. would be statically determinate and
2. would still be able to support the loads without the entire beam having any rigid body motions.

X_C will be chosen as the redundant

External Reactions
[Figure 4.5.6]

$$\sum F_x = 0$$

$$\Rightarrow \quad \left[X_A + \delta X_A\right] + \left[X_C + \delta X_C\right] = 0$$

$$\Rightarrow \quad \left[X_A + \delta X_A\right] = -X_C - \delta X_C$$

$$\Rightarrow \quad \boxed{X_A = -X_C} \qquad \boxed{\delta X_A = -\delta X_C} \tag{4.5.9}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[Y_A + \delta Y_A\right] + \left[Y_C + \delta Y_C\right] + 130\ lb = 0$$

$$\Rightarrow \quad \boxed{Y_A + Y_C = -130} \qquad \boxed{\delta Y_A + \delta Y_C = 0} \tag{4.5.10}$$

$$\circlearrowleft + \quad \sum M_A = 0$$

$$\Rightarrow \quad \left[Y_C + \delta Y_C\right](48") + (130\ lb)(48") + \left[X_C + \delta X_C\right](48") = 0$$

$$\Rightarrow \quad \left[Y_C + \delta Y_C\right] = \left(-X_C - 130\right) - \delta X_C$$

$$\Rightarrow \quad \boxed{Y_C = -X_C - 130} \qquad \boxed{\delta Y_C = -\delta X_C} \qquad (4.5.11)$$

Plugging Equation 4.5.11 into Equation 4.5.10, allows us to write Y_A in terms of the redundant X_C. The process is shown below.

$$Y_A + Y_C = -130 \qquad \delta Y_A + \delta Y_C = 0 \qquad (4.5.10)$$

$$Y_A + \overbrace{\left(-X_C - 130\right)}^{Y_C} = -130 \qquad \delta Y_A + \overbrace{\left(-\delta X_C\right)}^{\delta Y_C} = 0$$

$$\boxed{Y_A = X_C} \qquad \boxed{\delta Y_A = \delta X_C} \qquad (4.5.12)$$

* Equations 4.5.9, 4.5.11, and 4.5.12 are our 3 Static Equilibrium Equations.
* But there are 4 unknowns.
* The fourth equation will come from the Principle of Complementary Virtual Work.
* According to Equation 4.3.7, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.
* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

Cut #1

For Cut #1, $0 \leq \theta < 180°$

* When Cut #1 is made, we must choose either the portion of the beam below the cut or the portion of the beam above the cut for analysis.

* Both portions will give the exact same result.

The Portion of the Beam Below Cut #1 is shown below in Figure 4.5.7.

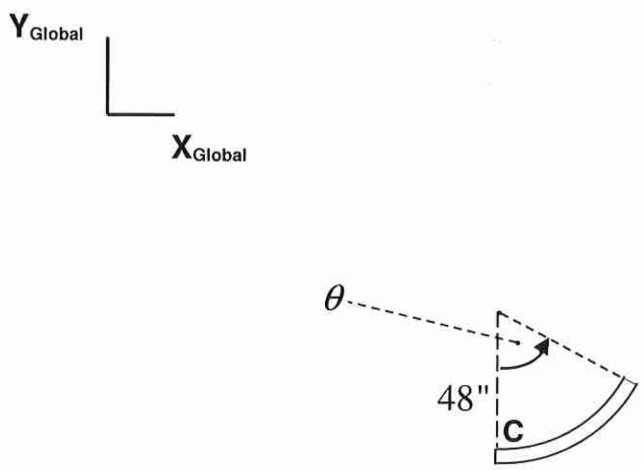


Figure 4.5.7

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

Internals for Cut #1
[Figure 4.5.7]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \qquad [P+\delta P]+\left[X_C+\delta X_C\right]\cos\theta+\overbrace{\left[\left(-X_C-130\right)-\delta X_C\right]}^{\left[Y_C+\delta Y_C\right]}\sin\theta=0$$

$$\Rightarrow \qquad [P+\delta P]=\left[X_C\left(\sin\theta-\cos\theta\right)+\left(130\right)\sin\theta\right]+\left[\delta X_C\left(\sin\theta-\cos\theta\right)\right]$$

$$\Rightarrow \qquad \boxed{P=X_C\left(\sin\theta-\cos\theta\right)+\left(130\right)\sin\theta} \qquad \boxed{\delta P=\delta X_C\left(\sin\theta-\cos\theta\right)} \qquad (4.5.13)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \quad \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \left[M+\delta M\right]+\left[X_C+\delta X_C\right]\left(48"-48"\cos\theta\right)-\overbrace{\left[\left(-X_C-130\right)-\delta X_C\right]}^{\left[Y_C+\delta Y_C\right]}\left(48"\sin\theta\right)=0$$

$$\Rightarrow \qquad \left[M+\delta M\right]=\begin{bmatrix}\left(48\right)X_C\left(-1-\sin\theta+\cos\theta\right)\\ -\left(6{,}240\right)\sin\theta\end{bmatrix}+\left[\left(48\right)\delta X_C\left(-1-\sin\theta+\cos\theta\right)\right]$$

$$\Rightarrow \qquad \boxed{M=\left(48\right)X_C\left(-1-\sin\theta+\cos\theta\right)-\left(6{,}240\right)\sin\theta} \qquad (4.5.14)$$

$$\boxed{\delta M=\left(48\right)\left(\delta X_C\right)\left(-1-\sin\theta+\cos\theta\right)}$$

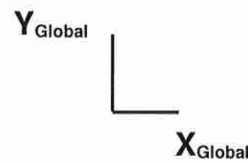
Cut #2

$$\boxed{\text{For Cut \#2, } 180^\circ \leq \theta < 270^\circ}$$

* When Cut #2 is made, we must choose either the portion of the beam below the cut or the portion of the beam above the cut for analysis.

* Both portions will give the exact same result.

The Portion of the Beam Below Cut #2 is shown below in Figure 4.5.8.



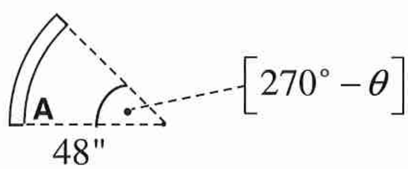


Figure 4.5.8

Internals for Cut #2
[Figure 4.5.8]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} [P+\delta P] + \overbrace{\left[-X_C - \delta X_C\right]}^{\left[X_A+\delta X_A\right]} \overbrace{\sin\left(270^\circ - \theta\right)}^{-\cos\theta} \\ \\ + \overbrace{\left[X_C + \delta X_C\right]}^{\left[Y_A+\delta Y_A\right]} \overbrace{\cos\left(270^\circ - \theta\right)}^{-\sin\theta} \end{array} \right\} = 0$$

$$\Rightarrow \quad [P+\delta P] = \left[X_C + \delta X_C\right](\sin\theta - \cos\theta)$$

$$\Rightarrow \quad \boxed{P = X_C(\sin\theta - \cos\theta)} \qquad \boxed{\delta P = (\delta X_C)(\sin\theta - \cos\theta)} \qquad (4.5.15)$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

4.208

$$\circlearrowleft^{+}\ \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} -[M+\delta M] + \overbrace{\left[-X_C - \delta X_C\right]}^{\left[X_A + \delta X_A\right]} \left[(48'') \overbrace{\sin\left(270^\circ - \theta\right)}^{-\cos\theta} \right] \\ \\ -\overbrace{\left[X_C + \delta X_C\right]}^{\left[Y_A + \delta Y_A\right]} \left[(48'') - (48'') \overbrace{\cos\left(270^\circ - \theta\right)}^{-\sin\theta} \right] \end{array} \right\} = 0$$

$$\Rightarrow \quad [M+\delta M] = (48)\left[X_C + \delta X_C\right]\left[-1 - \sin\theta + \cos\theta\right]$$

$$\Rightarrow \quad \boxed{M = (48) X_C \left(-1 - \sin\theta + \cos\theta\right)}$$

$$\boxed{\delta M = (48)\left(\delta X_C\right)\left(-1 - \sin\theta + \cos\theta\right)} \qquad (4.5.16)$$

The redundant X_C, will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.3.7) for a single beam states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \\ \displaystyle + \int_0^L \left[\frac{V_y\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{G(I_z)^2} \delta V_y \right] \left[\iint_A \left(\frac{Q_z}{b} \right)^2 dA \right] dx \end{array} \right\} \tag{4.3.7}$$

For our beam,

1. $P \neq 0$, and
2. the effects of shear deformation are neglected.

These will cause Equation 4.3.7 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \left\{ \begin{array}{l} \displaystyle\int_0^L \left[\frac{P\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EA} \delta P \right] dx \\ \displaystyle + \int_0^L \left[\frac{M_z\Big|_{\substack{\text{In Beam Due} \\ \text{to Actual Loads}}}}{EI_z} \delta M_z \right] dx \end{array} \right\} \tag{4.5.17}$$

4.210

Plugging Equations 4.5.13 - 4.5.16 into Equation 4.5.17, yields

$$\begin{Bmatrix} \overbrace{\mathrm{u}_A}^{0}\,(\delta X_A) \\ +\overbrace{\mathrm{v}_A}^{0}\,(\delta Y_A) \\ +\overbrace{\mathrm{u}_C}^{0}\,(\delta X_C) \\ +\overbrace{\mathrm{v}_C}^{0}\,(\delta Y_C) \end{Bmatrix} = \left\{ \begin{array}{l} \dfrac{1}{\left(29.1x10^6\,psi\right)\underbrace{\left[\pi\left(0.75"\right)^2\right]}_{A=1.767\ in^2}} \left\{ \begin{array}{l} \overbrace{\int_0^{\pi} \begin{bmatrix} X_C\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix} \\ +(130)\sin\theta \end{bmatrix} \left[\delta X_C\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}\right] \overbrace{\left[(48)\,d\theta\right]}^{dx}}^{[\text{Cut \#1, Axial}]} \\ + \overbrace{\int_{\pi}^{\frac{3\pi}{2}} \left[X_C\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}\right] \left[(\delta X_C)\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}\right] \overbrace{\left[(48)\,d\theta\right]}^{dx}}^{[\text{Cut \#2, Axial}]} \end{array} \right\} \\ \\ + \dfrac{1}{\left(29.1x10^6\,psi\right)\underbrace{\left[\dfrac{\pi\left(0.75"\right)^4}{4}\right]}_{I_z=0.2485\ in^4}} \left\{ \begin{array}{l} \overbrace{\int_0^{\pi} \begin{bmatrix} 48X_C\begin{pmatrix}-1 \\ -\sin\theta \\ +\cos\theta\end{pmatrix} \\ -(6,240)\sin\theta \end{bmatrix} \left[48(\delta X_C)\begin{pmatrix}-1 \\ -\sin\theta \\ +\cos\theta\end{pmatrix}\right] \overbrace{\left[(48)\,d\theta\right]}^{dx}}^{[\text{Cut \#1, Bending}]} \\ + \overbrace{\int_{\pi}^{\frac{3\pi}{2}} \left[48X_C\begin{pmatrix}-1 \\ -\sin\theta \\ +\cos\theta\end{pmatrix}\right] \left[48(\delta X_C)\begin{pmatrix}-1 \\ -\sin\theta \\ +\cos\theta\end{pmatrix}\right] \overbrace{\left[(48)\,d\theta\right]}^{dx}}^{[\text{Cut \#2, Bending}]} \end{array} \right\} \end{array} \right\}$$

$$\Rightarrow \qquad 0=(\delta X_C)\left\{\begin{array}{l}
\left(9.335\,x10^{-7}\right)\left\{\begin{array}{l}
\overbrace{\int_0^{\pi}\left[\begin{array}{l} X_C(\sin\theta-\cos\theta)^2 \\ +(130)(\sin\theta)(\sin\theta-\cos\theta)\end{array}\right]d\theta}^{[\text{Cut \#1, Axial}]} \\
+\overbrace{\int_{\pi}^{\frac{3\pi}{2}}\left[X_C(\sin\theta-\cos\theta)^2\right]d\theta}^{[\text{Cut \#2, Axial}]}
\end{array}\right\} \\
+\left(1.529\,x10^{-2}\right)\left\{\begin{array}{l}
\overbrace{\int_0^{\pi}\left[\begin{array}{l} X_C(-1-\sin\theta+\cos\theta)^2 \\ -(130)(\sin\theta)(-1-\sin\theta+\cos\theta)\end{array}\right]d\theta}^{[\text{Cut \#1, Bending}]} \\
+\overbrace{\int_{\pi}^{\frac{3\pi}{2}}\left[X_C(-1-\sin\theta+\cos\theta)^2\right]d\theta}^{[\text{Cut \#2, Bending}]}
\end{array}\right\}
\end{array}\right\}$$

4.212

$$\Rightarrow \quad 0=\left\{\begin{array}{l}\overbrace{\left[\begin{array}{l}X_C\left[\theta+(\cos\theta)^2\right]\\+\dfrac{(130)}{4}\left[2\theta-\sin 2\theta+\cos 2\theta+1\right]\end{array}\right]_0^{\pi}}^{[\text{Cut \#1, Axial}]} + \overbrace{\left[X_C\left[\theta+(\cos\theta)^2\right]\right]_{\pi}^{\frac{3\pi}{2}}}^{[\text{Cut \#2, Axial}]}\\[2ex] +\left(1.638\,x10^4\right)\left\{\begin{array}{l}\overbrace{\left[\begin{array}{l}X_C\left(2\theta-2\sin\theta-2\cos\theta+(0.5)\cos 2\theta\right)\\+(130)\left[\dfrac{\theta}{2}-\dfrac{\sin 2\theta}{4}+\dfrac{(\cos\theta)^2}{2}-\cos\theta\right]\end{array}\right]_0^{\pi}}^{[\text{Cut \#1, Bending}]}\\[2ex] \overbrace{+\left[X_C\left(2\theta-2\sin\theta-2\cos\theta+(0.5)\cos 2\theta\right)\right]_{\pi}^{\frac{3\pi}{2}}}^{[\text{Cut \#2, Bending}]}\end{array}\right\}\end{array}\right\}$$

(Integrated using Wolfram Mathematica Online Integrator)

$$\Rightarrow \quad 0=\left\{\begin{array}{l}\overbrace{\left[\begin{array}{l}(3.142)X_C\\+(204.2)\end{array}\right]}^{[\text{Cut \#1, Axial}]} + \overbrace{\left[(0.5708)X_C\right]}^{[\text{Cut \#2, Axial}]}\\[2ex] +\left(1.638\,x10^4\right)\left\{\overbrace{\left[+(10.28)X_C+(464.2)\right]}^{[\text{Cut \#1, Bending}]}+\overbrace{\left[(2.142)X_C\right]}^{[\text{Cut \#2, Bending}]}\right\}\end{array}\right\}$$

$$\Rightarrow \qquad 0=\overbrace{\left[(3.713)X_C+(204.2)\right]}^{[\text{Axial}]}+\overbrace{\left[\left(2.035\,x10^5\right)X_C+\left(7.604\,x10^6\right)\right]}^{[\text{Bending}]}$$

$$\Rightarrow \qquad 0=\left(2.035\,x10^5\right)X_C+\left(7.604\,x10^6\right)$$

$$\Rightarrow \qquad \boxed{X_C=-37.37} \qquad (4.5.18)$$

In this particular curved beam problem, including the effect of axial deformation caused no change in the answer for the redundant (out to 4 significant digits).

The reactions at all connections now have to be found.
This is easily done by substituting Equation 4.5.18 into Equations 4.5.9, 4.5.11, and 4.5.12.

Therefore, the reactions at all connections are

$$\boxed{\begin{array}{l} X_A=37.37\ lb\ \rightarrow \\ Y_A=37.37\ lb\ \downarrow \\ X_C=37.37\ lb\ \leftarrow \\ Y_C=92.63\ lb\ \downarrow \end{array}} \qquad (4.5.19)$$

You should now have the ability to do Problems 4.10, 4.11

Homework: Do Problem 4.10

Example 4.5.3 **[Statically Indeterminate Frame—Reactions]**

For the statically indeterminate frame shown in the figure below, determine, using the Principle of Complementary Virtual Work, the reactions at all connections.

Material: Titanium	Cross-Section: Solid Circular
$E = 15.7 \times 10^6$ psi	Diameter = $1\frac{3}{4}$"

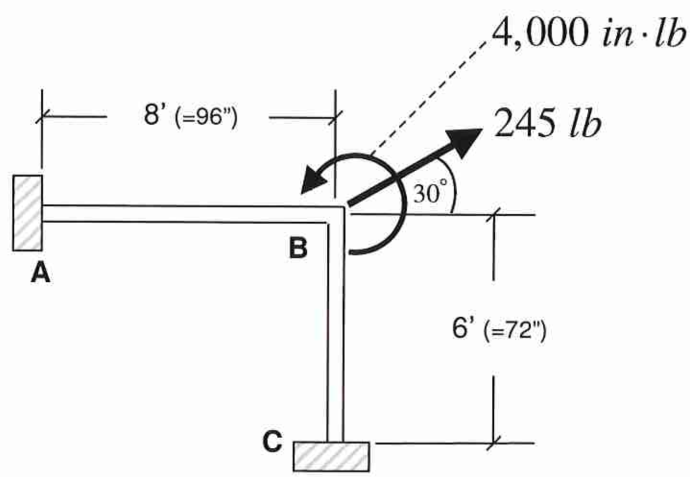


Figure 4.5.9

Neglect the deformation caused by $\tau_{xy} = \dfrac{V_y Q_z}{I_z b}$

****

Why is this frame statically indeterminate?

Because it has 6 unknown external reactions & we can only get 3 equilibrium equations from statics.

This means that there are 3 Redundants.

2-Force Members

Are there any two-force members in Figure 4.5.9?

3 Redundants

* The Principle of Complementary Virtual Work will be used to obtain the 3 additional equations needed beyond those of static equilibrium.

* This means that we will have to apply 3 virtual forces or couples.

* Let us imagine applying 3 virtual forces or couples in the same location and in the same direction as 3 of the external reactions.

* These applied virtual forces or couples will cause
  (i) virtual internal reactions inside the beam, as well as
  (ii) virtual external reactions at the external supports.

* Of all the resulting virtuals, which will be treated as the applied virtuals & which will be treated as the reaction virtuals?

* The applied virtual could be any 3 of the resulting virtuals; namely, any of the external virtuals or any of the internal virtuals.

* There is no need to distinguish between them.

At some point, we must decide which of the real forces we will select as our redundant.

Choice of Redundants

Two Methods for Choosing Redundants

Method 1
Best method when there are a large number of simultaneous equations produced. Trusses generally fit into this category.

* To choose the redundants
  1. write all of the static equilibrium equations in terms of all of the variables
  2. then determine which choice of redundants
     i. would make sense mathematically and
     ii. would make the equations easiest to solve.

* As a general rule, for trusses, the algebra will be reduced if P values are chosen as the redundants rather than external reactions because it is the P values which must be directly substituted on the RHS of the Complementary Virtual Work equation.

Method 2
Quite often this is the best method for simple beams and frames.

* To choose the redundants
  1. visibly identify which reactions could be removed so that the structure would
     i. be statically determinate and
     ii. still support the loads without the entire structure having any rigid body motions.
  2. Use the reactions that could be removed as the redundants.

To determine the reactions at all connections using the Principle of Complementary Virtual Work,

1. the actual loads of $245\ lb$ and $4{,}000\ \text{in} \cdot lb$, as well as all externals, will be removed, and
2. 3 virtual loads will be applied at the supports in the same direction as 3 of the support reactions.

FBDs + STATICS

The FBD of the entire beam using the shortcut method is shown below.

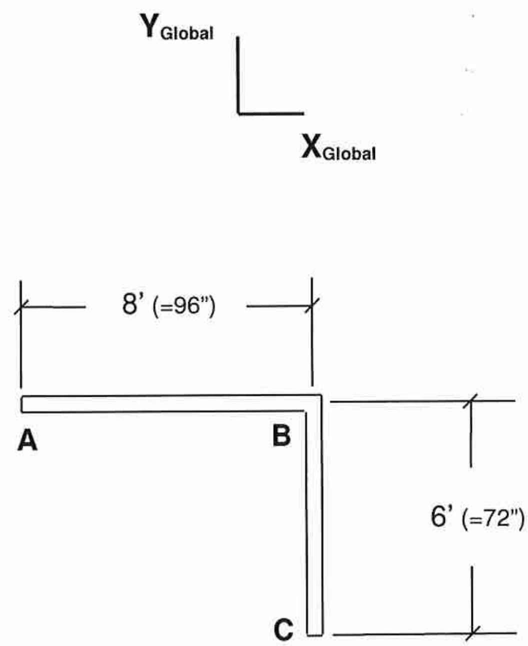


Figure 4.5.10

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

Choice of Redundants for This Frame

Following Method 2 on Page 4.216, an obvious choice here would be to use P_C, V_C, and M_C for the redundants.

If the fixed support at C were removed, the frame
1. would be statically determinate and
2. would still be able to support the loads without the entire frame having any rigid body motions.

P_C, V_C, and M_C will be chosen as the redundants

External Reactions
[Figure 4.5.10]

$$\sum F_x = 0$$

$$\Rightarrow \quad \left[P_A + \delta P_A\right] + \left[V_C + \delta V_C\right] + (245\ lb)\cos 30^\circ = 0$$

$$\Rightarrow \quad \left[P_A + \delta P_A\right] = -\left[V_C + \delta V_C\right] - 212.2$$

$$\Rightarrow \quad \boxed{P_A = -V_C - 212.2} \qquad \boxed{\delta P_A = -\delta V_C} \tag{4.5.20}$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \left[V_A + \delta V_A\right] + \left[P_C + \delta P_C\right] + (245\ lb)\sin 30^\circ = 0$$

$$\Rightarrow \quad \left[V_A + \delta V_A\right] = -\left[P_C + \delta P_C\right] - 122.5$$

$$\Rightarrow \quad \boxed{V_A = -P_C - 122.5} \qquad \boxed{\delta V_A = -\delta P_C} \tag{4.5.21}$$

$$\circlearrowleft + \quad \sum M_A = 0$$

$$\Rightarrow \quad \left\{ \begin{array}{l} \left[M_A + \delta M_A\right] + \left[M_C + \delta M_C\right] + \left[(245\ lb)\sin 30^\circ\right](96") \\ + 4{,}000\ in \cdot lb + \left[P_C + \delta P_C\right](96") + \left[V_C + \delta V_C\right](72") \end{array} \right\} = 0$$

$$\Rightarrow \quad \left[M_A + \delta M_A\right] = \left\{ \begin{array}{l} -(96)\left[P_C + \delta P_C\right] - (72)\left[V_C + \delta V_C\right] \\ -\left[M_C + \delta M_C\right] - 15{,}760 \end{array} \right\}$$

$$\Rightarrow \quad \boxed{M_A = -(96)P_C - (72)V_C - M_C - 15{,}760} \qquad \boxed{\delta M_A = -(96)(\delta P_C) - (72)(\delta V_C) - (\delta M_C)} \tag{4.5.22}$$

* Equations 4.5.20 - 4.5.22 are our 3 Static Equilibrium Equations.

* But there are 6 unknown external reactions.

* The Principle of Complementary Virtual Work will yield the other 3 equations to find the redundants P_C, V_C, and M_C.

* According to Equation 4.4.1, in order to use the Principle of Complementary Virtual Work, the internals P, δP, M, and δM must be obtained as functions of x.

* This will be accomplished by taking the necessary cuts at arbitrary x-locations.

4.220

* This frame consists of 2 straight beams.

* None of the beams have a concentrated load along its span.

* Thus a single cut must be made at an arbitrary x-location along each beam.

Why does this frame need 2 cuts?
Because there are 2 straight beams, none of them having any concentrated loads along their span.

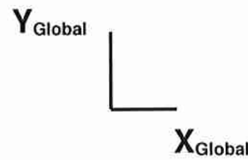
Cut #1

Cut #1 $\Rightarrow$ Cut Between B and C

* When Cut #1 is made, we must choose either the portion of the frame above the cut or the portion of the frame below the cut for analysis.

* Both portions will give the exact same result.

The Portion of the Frame Below Cut #1 is shown in Figure 4.5.11.



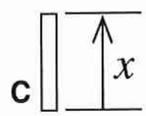


Figure 4.5.11

Sign Conventions

* Externals-----Positive if in the Positive Global Axes Directions

* Internals-----Positive based on the Outward Normal Sign Convention w.r.t. the **Temp Axes**

Temporary Axes

* All of the formulas in Chapter 1 are derived for standard $x-y$ axes along the length of the member.

* When using Chapter 1 formulas in problems where the axes are not the standard $x-y$ axes, it is best to set up a temporary set of $x_{\text{Temp}}-y_{\text{Temp}}$ axes for using the Chapter 1 formulas.

* The x_{Temp} axis must be lined up along the beam axis.

Internals for Cut #1
[Figure 4.5.11]

$$\sum F_{x_{\text{Temp}}}=0$$

$$\Rightarrow \qquad [P+\delta P]+\left[P_C+\delta P_C\right]=0$$

$$\Rightarrow \qquad [P+\delta P]=-\left[P_C+\delta P_C\right]$$

$$\Rightarrow \qquad \boxed{P=-P_C} \qquad \boxed{\delta P=-\delta P_C} \tag{4.5.23}$$

$$\sum F_{y_{\text{Temp}}}=0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \quad \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad [M + \delta M] + [M_C + \delta M_C] + [V_C + \delta V_C]x = 0$$

$$\Rightarrow \quad [M + \delta M] = -[V_C + \delta V_C]x - [M_C + \delta M_C]$$

$$\Rightarrow \quad \boxed{M = -(V_C)x - M_C} \qquad \boxed{\delta M = -(\delta V_C)x - (\delta M_C)} \qquad (4.5.24)$$

Cut #2

$$\boxed{\text{Cut \#2} \Rightarrow \text{Cut Between A and B}}$$

* When Cut #2 is made, we must choose either the portion of the frame to the left of the cut or the portion of the frame to the right of the cut for analysis.

* Both portions will give the exact same result.

The Right Portion of the Frame for Cut #2 is shown below in Figure 4.5.12.

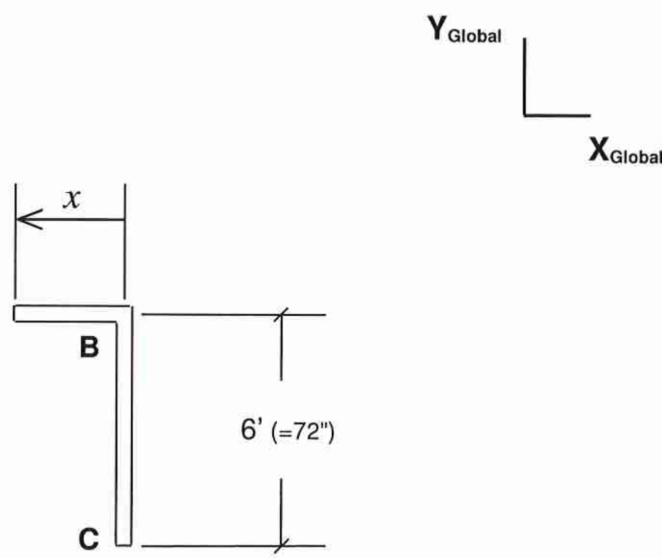


Figure 4.5.12

Internals for Cut #2
[Figure 4.5.12]

$$\sum F_{x_{\text{Temp}}} = 0$$

$$\Rightarrow \quad [P+\delta P]-\left[V_C+\delta V_C\right]-(245\ lb)\cos 30^\circ = 0$$

$$\Rightarrow \quad [P+\delta P]=\left[V_C+212.2\right]+\delta V_C$$

$$\Rightarrow \quad \boxed{P = V_C + 212.2} \qquad \boxed{\delta P = \delta V_C} \tag{4.5.25}$$

$$\sum F_{y_{\text{Temp}}} = 0$$

There is no need to write the equation to determine the internal V functions since the effects of shear deformation are being neglected.

$$\circlearrowleft + \ \sum M_{\text{Cut}} = 0$$

$$\Rightarrow \quad \left\{\begin{matrix} [M+\delta M]+\left[M_C+\delta M_C\right]+4{,}000\ in\cdot lb \\ +\left[(245\ lb)\sin 30^\circ\right]x+\left[P_C+\delta P_C\right]x+\left[V_C+\delta V_C\right](72") \end{matrix}\right\} = 0$$

$$\Rightarrow \quad [M+\delta M]=\left\{\begin{matrix} -\left[P_C+\delta P_C\right]x-(72)\left[V_C+\delta V_C\right] \\ -\left[M_C+\delta M_C\right]-(122.5)x-4{,}000 \end{matrix}\right\}$$

$$\Rightarrow \quad \begin{matrix} \boxed{M = -P_C x-(72)V_C-M_C-(122.5)x-4{,}000} \\ \boxed{\delta M = -\left(\delta P_C\right)x-(72)\left(\delta V_C\right)-\left(\delta M_C\right)} \end{matrix} \tag{4.5.26}$$

The redundants P_C, V_C, and M_C, will now be found by utilizing the Principle of Complementary Virtual Work.

COMPLEMENTARY VIRTUAL WORK

The Principle of Complementary Virtual Work (Equation 4.4.1) for a frame states,

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on}\\ \text{Original Structure}\\ \text{Caused by Actual}\\ \text{Loads with Actual}\\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}}\right]} = \sum_i \left\{ \begin{aligned} & \int_0^L \left[\frac{\left(P\big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ & + \int_0^L \left[\frac{\left(M_z\big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \\ & + \int_0^L \left[\frac{\left(V_y\big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{G_i (I_z)_i^2} (\delta V_y)_i \right] \left[\iint_A \left(\frac{(Q_z)_i}{b_i} \right)^2 dA \right] dx \end{aligned} \right\} \qquad (4.4.1)$$

For the beams in our frame, $P \neq 0$ and the effects of shear deformation are neglected.

These will cause Equation 4.4.1 above to reduce to

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on}\\ \text{Original Structure}\\ \text{Caused by Actual}\\ \text{Loads with Actual}\\ \text{Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}}\right]} = \sum_i \left\{ \begin{aligned} & \int_0^L \left[\frac{\left(P\big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i A_i} (\delta P)_i \right] dx \\ & + \int_0^L \left[\frac{\left(M_z\big|_{\substack{\text{In Beam Due}\\ \text{to Actual Loads}}} \right)_i}{E_i (I_z)_i} (\delta M_z)_i \right] dx \end{aligned} \right\} \qquad (4.5.27)$$

Plugging Equations 4.5.23 - 4.5.26 into Equation 4.5.27,

$$\Rightarrow \left\{\begin{array}{l} \overbrace{u_A}^{0}(\delta P_A) \\ +\overbrace{v_A}^{0}(\delta V_A) \\ +\overbrace{\theta_A}^{0}(\delta M_A) \\ +\overbrace{u_C}^{0}(\delta P_C) \\ +\overbrace{v_C}^{0}(\delta V_C) \\ +\overbrace{\theta_C}^{0}(\delta M_C) \end{array}\right\} = \left\{\begin{array}{l} \dfrac{1}{\left(15.7\,x10^6\,psi\right)\underbrace{\left[\pi\left(0.875"\right)^2\right]}_{A=2.405\ in^2}}\left\{\begin{array}{l} \overbrace{\int_0^{72}\left[-P_C\right]\left[-\left(\delta P_C\right)\right]dx}^{\left[\text{Cut \#1, Axial}\right]} \\ +\overbrace{\int_0^{96}\left[V_C+212.2\right]\left[\delta V_C\right]dx}^{\left[\text{Cut \#2, Axial}\right]} \end{array}\right\} \\ +\dfrac{1}{\left(15.7\,x10^6\,psi\right)\underbrace{\left[\dfrac{\pi\left(0.875"\right)^4}{4}\right]}_{I_z=0.4604\ in^4}}\left\{\begin{array}{l} \overbrace{\int_0^{72}\begin{bmatrix}-V_C\,x \\ -M_C\end{bmatrix}\begin{bmatrix}-\left(\delta V_C\right)x \\ -\left(\delta M_C\right)\end{bmatrix}dx}^{\left[\text{Cut \#1, Bending}\right]} \\ \overbrace{+\int_0^{96}\begin{bmatrix}-P_C\,x \\ -(72)V_C \\ -M_C \\ -(122.5)x \\ -4{,}000\end{bmatrix}\begin{bmatrix}-\left(\delta P_C\right)x \\ -(72)\left(\delta V_C\right) \\ -\left(\delta M_C\right)\end{bmatrix}dx}^{\left[\text{Cut \#2, Bending}\right]} \end{array}\right\} \end{array}\right\}$$

4.228

$$\Rightarrow \quad 0=\left\{\begin{array}{l}\left(2.648x10^{-8}\right)\left\{\overbrace{P_C\left(\delta P_C\right)\int_0^{72}dx}^{[\text{Cut \#1, Axial}]}+\overbrace{\left(V_C+212.2\right)\left(\delta V_C\right)\int_0^{96}dx}^{[\text{Cut \#2, Axial}]}\right\}\\[2ex] +\left(1.383x10^{-7}\right)\left\{\begin{array}{l}\overbrace{\left\{\begin{array}{l}V_C\left(\delta V_C\right)\int_0^{72}x^2dx\\ +\left[M_C\left(\delta V_C\right)+V_C\left(\delta M_C\right)\right]\int_0^{72}x\,dx\\ +M_C\left(\delta M_C\right)\int_0^{72}dx\end{array}\right\}}^{[\text{Cut \#1, Bending}]}\\[2ex] +\overbrace{\left\{\begin{array}{l}\left(P_C+122.5\right)\left(\delta P_C\right)\int_0^{96}x^2dx\\ +\left\{\left(P_C+122.5\right)\begin{bmatrix}(72)\left(\delta V_C\right)\\ +\left(\delta M_C\right)\end{bmatrix}+\begin{bmatrix}(72)V_C\\ +M_C\\ +4{,}000\end{bmatrix}\left(\delta P_C\right)\right\}\int_0^{96}x\,dx\\ +\left[(72)V_C+M_C+4{,}000\right]\left[(72)\left(\delta V_C\right)+\left(\delta M_C\right)\right]\int_0^{96}dx\end{array}\right\}}^{[\text{Cut \#2, Bending}]}\end{array}\right\}\end{array}\right\}$$

Performing the integrations and doing some algebraic simplifications

$$\Rightarrow \quad 0 = \left\{ \begin{array}{l} \overbrace{\left[(72)P_C\right](\delta P_C)}^{[\text{Cut \#1, Axial}]} + \overbrace{\left[(96)(V_C+212.2)\right](\delta V_C)}^{[\text{Cut \#2, Axial}]} \\ +(5.223)\left\{ \begin{array}{l} \overbrace{\left\{ \begin{array}{l} (124{,}400)V_C(\delta V_C)+(2{,}592)\left[M_C(\delta V_C)+V_C(\delta M_C)\right] \\ +(72)M_C(\delta M_C) \end{array} \right\}}^{[\text{Cut \#1, Bending}]} \\ \overbrace{+\left\{ \begin{array}{l} (294{,}900)(P_C+122.5)(\delta P_C) \\ +(4{,}608)\left\{ (P_C+122.5)\begin{bmatrix} (72)(\delta V_C) \\ +(\delta M_C) \end{bmatrix} + \begin{bmatrix} (72)V_C \\ +M_C \\ +4{,}000 \end{bmatrix}(\delta P_C) \right\} \\ +(96)\left[(72)V_C+M_C+4{,}000\right]\left[(72)(\delta V_C)+(\delta M_C)\right] \end{array} \right\}}^{[\text{Cut \#2, Bending}]} \end{array} \right\} \end{array} \right\}$$

Doing some additional algebraic simplifications and regrouping

$$\Rightarrow \quad 0 = \left\{ \begin{array}{l} \left\{ \overbrace{\left[(72)P_C\right]}^{\text{[Due To Internal Axial Forces]}} + \overbrace{\left[\begin{array}{l} \left(1.540\,x10^6\right)P_C + \left(1.733\,x10^6\right)V_C \\ +\left(2.407\,x10^4\right)M_C + 2.850\,x10^8 \end{array} \right]}^{\text{[Due To Internal Bending Moments]}} \right\}(\delta P_C) \\ + \left\{ \overbrace{\left[(96)V_C + 2.037\,x10^4\right]}^{\text{[Due To Internal Axial Forces]}} + \overbrace{\left[\begin{array}{l} \left(1.733\,x10^6\right)P_C + \left(3.249\,x10^6\right)V_C \\ +\left(4.964\,x10^4\right)M_C + 3.567\,x10^8 \end{array} \right]}^{\text{[Due To Internal Bending Moments]}} \right\}(\delta V_C) \\ + \overbrace{\left[\left(2.407\,x10^4\right)P_C + \left(4.964\,x10^4\right)V_C + (877.5)M_C + 4.954\,x10^6\right]}^{\text{[Due To Internal Bending Moments]}}(\delta M_C) \end{array} \right\}$$

We can now write the 3 equations below.

$$\overbrace{\left[(72)P_C\right]}^{\left[\text{Due To Internal Axial Forces}\right]} + \overbrace{\left[\begin{array}{l}\left(1.540\,x10^6\right)P_C + \left(1.733\,x10^6\right)V_C \\ +\left(2.407\,x10^4\right)M_C + 2.850\,x10^8\end{array}\right]}^{\left[\text{Due To Internal Bending Moments}\right]} = 0$$

$$\overbrace{\left[(96)V_C + 2.037\,x10^4\right]}^{\left[\text{Due To Internal Axial Forces}\right]} + \overbrace{\left[\begin{array}{l}\left(1.733\,x10^6\right)P_C + \left(3.249\,x10^6\right)V_C \\ +\left(4.964\,x10^4\right)M_C + 3.567\,x10^8\end{array}\right]}^{\left[\text{Due To Internal Bending Moments}\right]} = 0$$

$$\overbrace{\left[\left(2.407\,x10^4\right)P_C + \left(4.964\,x10^4\right)V_C + (877.5)M_C + 4.954\,x10^6\right]}^{\left[\text{Due To Internal Bending Moments}\right]} = 0$$

Performing some algebraic simplifications, we get the following 3 equations.

$$\overbrace{\left[\left(4.675\,x10^{-5}\right)P_C\right]}^{\left[\text{Due To Internal Axial Forces}\right]} + \overbrace{\left[P_C+(1.125)V_C+\left(1.563\,x10^{-2}\right)M_C+185.1\right]}^{\left[\text{Due To Internal Bending Moments}\right]}=0$$

$$\overbrace{\left[\left(5.540\,x10^{-5}\right)V_C+1.175\,x10^{-2}\right]}^{\left[\text{Due To Internal Axial Forces}\right]}+\overbrace{\left[P_C+(1.875)V_C+\left(2.864\,x10^{-2}\right)M_C+205.8\right]}^{\left[\text{Due To Internal Bending Moments}\right]}=0$$

$$\overbrace{\left[P_C+(2.062)V_C+\left(3.646\,x10^{-2}\right)M_C+205.8\right]}^{\left[\text{Due To Internal Bending Moments}\right]}=0$$

These are the 3 additional equations, beyond those of static equilibrium, that we have been seeking.

The solutions are

$$\boxed{\begin{aligned} P_C &= -149.3 \\ V_C &= -47.62 \\ M_C &= 1{,}143 \end{aligned}} \qquad (4.5.28)$$

In this particular frame problem, including the effect of axial deformation caused no change in the answer for the redundants (out to 4 significant digits).

The reactions at all connections now have to be found.
This is easily done by substituting Equation 4.5.28 into Equations 4.5.20 - 4.5.22.

Therefore, the reactions at all connections are

$$P_A = 164.6\ lb \leftarrow$$

$$V_A = 26.80\ lb \uparrow$$

$$M_A = 858.4\ in \cdot lb \circlearrowleft$$

$$P_C = 149.3\ lb \downarrow$$

$$V_C = 47.62\ lb \leftarrow$$

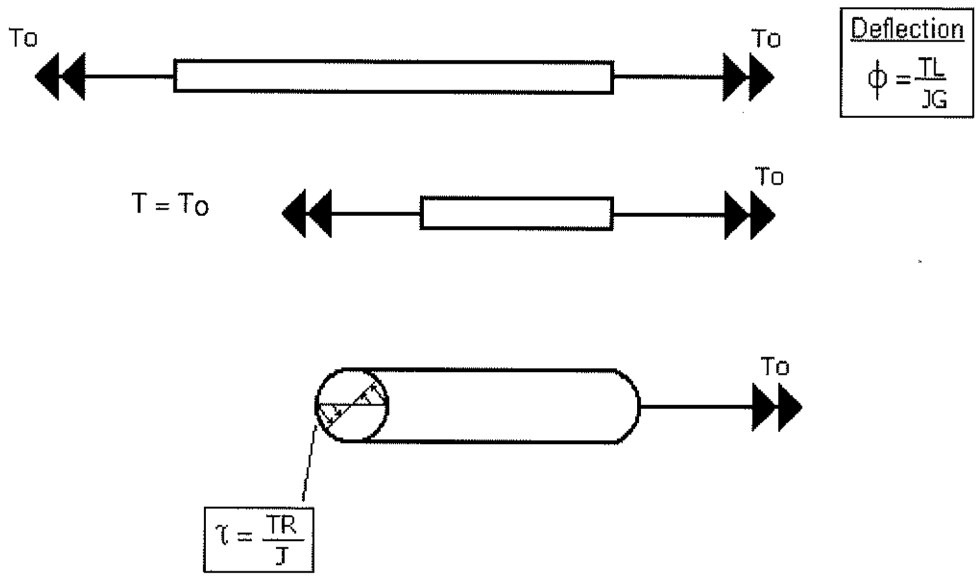
$$M_C = 1{,}143\ in \cdot lb \circlearrowleft$$

You should now have the ability to do Problem 4.12

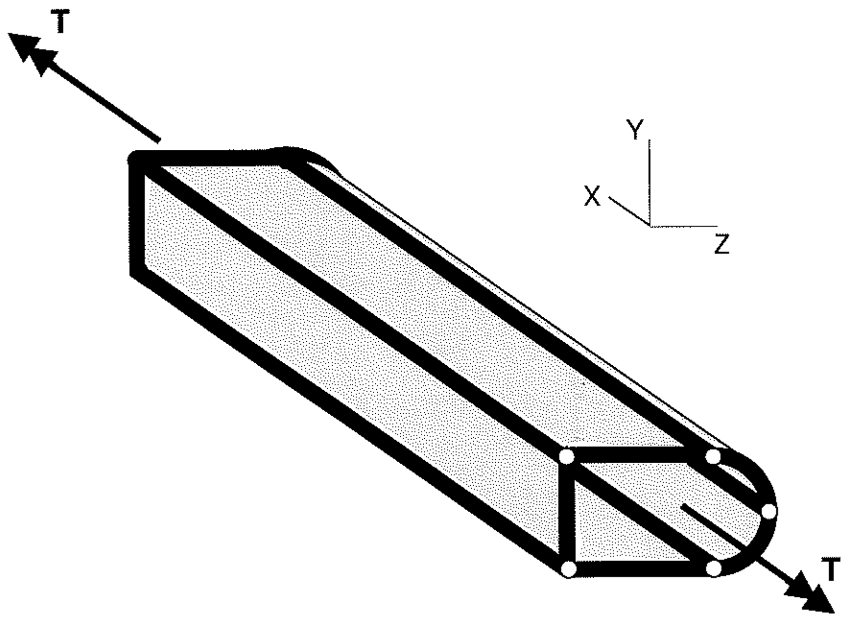
Homework: Do Problem 4.12

4.6 Torsion of Cylindrical Shafts

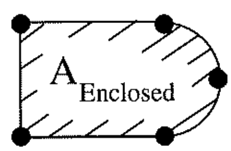
Previously we learned about *Torsion of Non-Idealized Circular Shafts* [see Page 1.17] as shown below.



We have also learned about *Torsion of Idealized Closed-Section Shafts* [see Page 2.7] as shown below.



$$T = 2\left(A_{\text{Enclosed}}\right)q$$



Let us now consider torsion of *Non-Idealized Shafts* which are *Not Necessarily Circular*.

Consider the untapered shaft of arbitrary cross-section under pure torsion below.

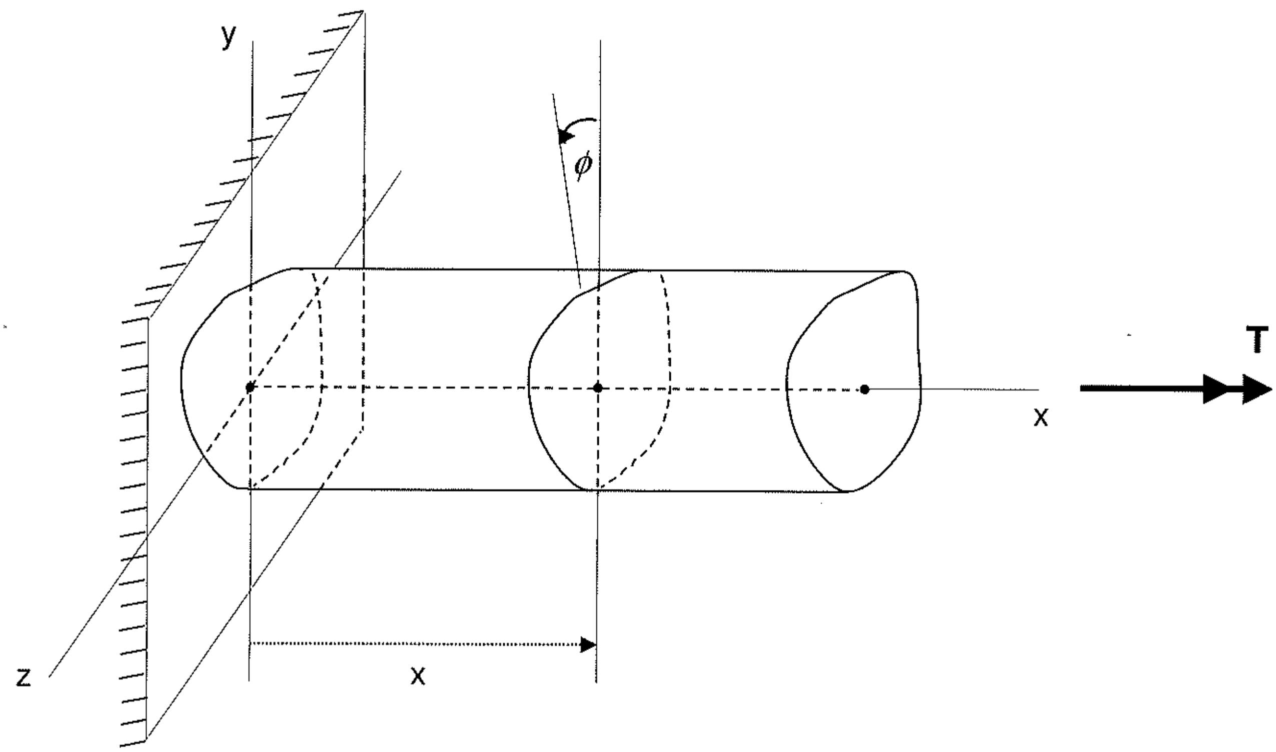


Figure 5.6.1

Assumptions

1. The x-axis passes through the centroid of the cross-section.
2. Every line on a particular cross-section rotates through the same angle ϕ about the x-axis; i.e., each cross-section rotates as a rigid body.
3. The angle of twist per unit length is constant; i.e.,

$$\boxed{\frac{d\phi}{dx} = \text{Constant} = C} \tag{4.6.1}$$

This is Saint Venant's Torsion Theory.

Separating and integrating Equation 4.6.1 $\Rightarrow$

$$\phi = Cx + C_1$$

Using the B.C.,

$$\phi = 0 \text{ at } x = 0$$

$\Rightarrow$ $$C_1 = 0$$

$\Rightarrow$ $$\boxed{\phi = Cx} \tag{4.6.2}$$

Figure 4.6.2 below shows an arbitrary point B of area dA on a cross-section at an arbitrary location x. It has coordinates (r, θ) in cylindrical coordinates and coordinates (y, z) in rectangular coordinates.

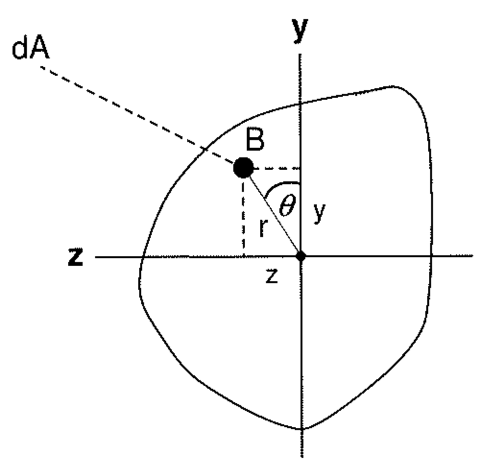


Figure 4.6.2

Figure 4.6.3 shows point B after it has rotated an angle ϕ to B′, and also defines the displacements v and w .

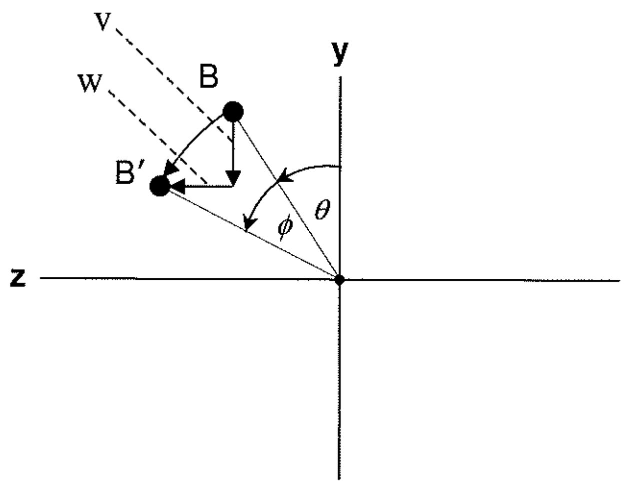


Figure 4.6.3

We will assume that the angle of deformation ϕ is small and hence, for geometric purposes, will approximate the deformation from B to B′ as a straight line as shown in Figure 4.6.4.

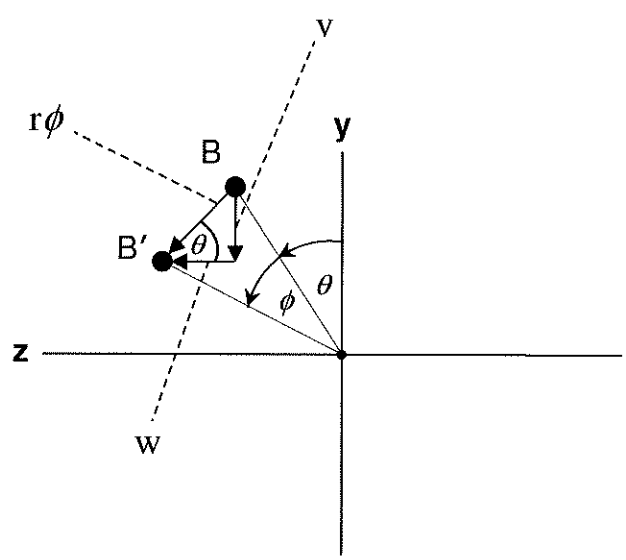


Figure 4.6.4

When this cross-section rotates an angle ϕ about the x-axis, the line of length r also rotates by the angle ϕ.

This causes point B to have a displacement in the y-z plane of an amount $r\phi$. Resolving this displacement into 2 components v and w parallel to the y and z axes respectively, we have [see Figure 4.6.4]

$$\mathrm{v} = -(r\phi)\sin\theta \qquad \mathrm{w} = (r\phi)\cos\theta$$

From Figure 4.6.2, the coordinates of Point B in terms of r and θ can be written as

$$y = r\cos\theta \qquad z = r\sin\theta$$

The 2 components of displacement of Point B in the y-z plane can now be written as

$$\mathrm{v} = -z\phi \qquad \mathrm{w} = y\phi \tag{4.6.3}$$

Warping of the Cross-Sections

When the cross-section of the shaft is not circular, it may not remain plane (flat) as it rotates; i.e., it may *warp*.

Thus point B in Figure 4.6.3 may have an additional displacement u in the x-direction. The displacement u is assumed to be given by

$$\mathrm{u} = \Psi(y,z)\left(\frac{d\phi}{dx}\right) \tag{4.6.4}$$

where

$$\Psi(y,z) = \text{the } \textit{Warping Function}$$

Ψ is assumed to be purely a function of the (y,z) location on a cross-section and not dependent on the x-location on the shaft.

Summarizing, and utilizing Equations 4.6.1 and 4.6.2, the displacements of any Point B on any particular cross-section of the shaft are given by

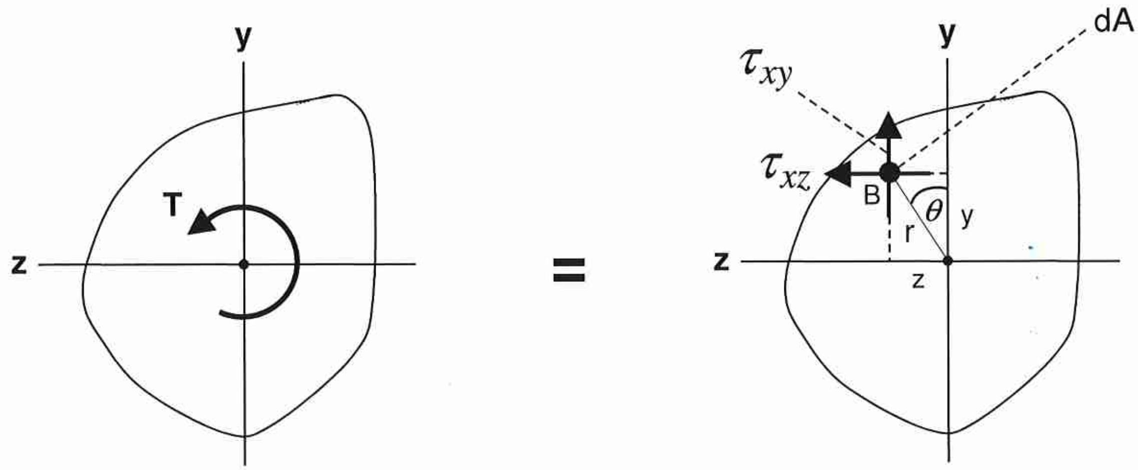
$$\boxed{\begin{aligned} u &= \Psi(y,z)\overbrace{\left(\frac{d\phi}{dx}\right)}^{C} \\ v &= -z\,\overbrace{\phi}^{Cx} \\ w &= y\,\overbrace{\phi}^{Cx} \end{aligned}} \tag{4.6.5}$$

This indicates that the displacement u is the same for every cross-section. Obviously, that would not be true for the cross-section at the wall. Hence, the function for u is only valid for cross-sections away from the wall.

Angle of Twist and Stresses

[The derivation below was provided by Dr. Ali Tamijani]

We know that torsion produces only shear stresses. They are shown at Point B in Figure 4.6.5 below.



Equivalent Interior Faces

Figure 4.6.5

From Solid Mechanics we know that

$$\tau_{xy} = G\,\gamma_{xy}$$

$$= G\left(\frac{\partial \mathrm{v}}{\partial x} + \frac{\partial \mathrm{u}}{\partial y}\right) \qquad \text{[see Equation 2.2.2 on Page 2.8]}$$

$$= G\left(-C\,z + C\frac{\partial \Psi}{\partial y}\right) \qquad \text{[see Equation 4.6.5]}$$

$$\boxed{\tau_{xy} = G\,C\left(\frac{\partial \Psi}{\partial y} - z\right)} \tag{4.6.6}$$

Similarly,

$$\tau_{xz} = G\,\gamma_{xz}$$

$$= G\left(\frac{\partial \mathrm{w}}{\partial x} + \frac{\partial \mathrm{u}}{\partial z}\right)$$

$$= G\left(C\,y + C\frac{\partial \Psi}{\partial z}\right)$$

$$\boxed{\tau_{xz} = G\,C\left(\frac{\partial \Psi}{\partial z} + y\right)} \qquad (4.6.7)$$

Moment Equivalence
Using Figure 4.6.5,

$$\sum\left(M_{\text{Origin}}\right)_{\text{Left Figure}} = \sum\left(M_{\text{Origin}}\right)_{\text{Right Figure}}$$

$$\Rightarrow \qquad T = \int_A \left[\,y\left(\tau_{xz}\,dA\right) - z\left(\tau_{xy}\,dA\right)\right]$$

Using Equations 4.6.6 and 4.6.7 and also factoring out dA,

$$\Rightarrow \qquad T = \int_A \left\{ y\left[G\,C\left(\frac{\partial \Psi}{\partial z} + y\right)\right] - z\left[G\,C\left(\frac{\partial \Psi}{\partial y} - z\right)\right]\right\} dA$$

$$\Rightarrow \qquad T = G\,C\int_A \left[\left(y^2 + z^2\right) + \left(y\frac{\partial \Psi}{\partial z} - z\frac{\partial \Psi}{\partial y}\right)\right] dA$$

Let

$$\boxed{J = \int_A \left[\left(y^2 + z^2\right) + \left(y\frac{\partial \Psi}{\partial z} - z\frac{\partial \Psi}{\partial y} \right) \right] dA} \qquad (4.6.8)$$

J is known as the *Torsion Constant*.

$\Rightarrow$

$$T = GCJ \qquad (4.6.9)$$

Substituting Equation 4.6.1,

$\Rightarrow$

$$T = GJ\frac{d\phi}{dx}$$

Since T, G, and J are not functions of x, separating and integrating gives

$$\boxed{\phi = \frac{TL}{JG}} \qquad (4.6.10)$$

Notes

* Equation 4.6.10 appears exactly the same as the formula for circular cross-sections [see Page 1.17].

* However, here J is not the Polar Moment of Inertia.

* It reduces to the Polar Moment of Inertia if there is no warping $(\Psi = 0)$, which is the case for circular cross-sections.

4.244

Solving for C in Equation 4.6.9, we get

$$C = \frac{T}{GJ} \tag{4.6.11}$$

Plugging Equation 4.6.11 into Equation 4.6.6

$$\Rightarrow \qquad \tau_{xy} = G\overbrace{\left(\frac{T}{GJ}\right)}^{C}\left(\frac{\partial\Psi}{\partial y} - z\right)$$

$$\Rightarrow \qquad \boxed{\tau_{xy} = \frac{T}{J}\left(\frac{\partial\Psi}{\partial y} - z\right)} \tag{4.6.12}$$

Similarly, plugging Equation 4.6.11 into Equation 4.6.7 yields

$$\boxed{\tau_{xz} = \frac{T}{J}\left(\frac{\partial\Psi}{\partial z} + y\right)} \tag{4.6.13}$$

All other stresses = 0

The resultant τ at any point on any cross-section would be

$$\tau = \frac{T}{J}\sqrt{\left(\frac{\partial\Psi}{\partial y} - z\right)^2 + \left(\frac{\partial\Psi}{\partial z} + y\right)^2}$$

For the case of a circular cross-section, Ψ would = 0 since there would be no warping.

Hence, for circular cross-sections, the resultant τ τ would be given by

$$\tau = \frac{T}{J}\sqrt{y^2 + z^2}$$

$$\Rightarrow \qquad \boxed{\tau = \frac{T\,r}{J}}$$

This is the same formula that we have been using for circular cross-sections
[see Page 1.17].

Complementary Virtual Work for Torsion

Equation 3.5.6 is repeated below.

$$\sum_i \underbrace{\left(b_i \Big| \begin{array}{l}\text{At Location i on Original Structure}\\ \text{Caused by Actual Loads}\\ \text{with Actual Constraints}\end{array} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\begin{array}{c}\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}\end{array}\right]} = \iiint_V \left(\begin{array}{l} \varepsilon_x\,\delta\sigma_x + \varepsilon_y\,\delta\sigma_y + \varepsilon_z\,\delta\sigma_z \\ \\ + \gamma_{xy}\,\delta\tau_{xy} + \gamma_{xz}\,\delta\tau_{xz} + \gamma_{yz}\,\delta\tau_{yz} \end{array} \right) dV \qquad (3.5.6)$$

Placing Equations 4.6.12 and 4.6.13 into Equation 3.5.6, we obtain

$$\underbrace{\sum_i \left(b_i \Bigg| \begin{array}{l}\text{At Location i on Original Structure}\\ \text{Caused by Actual Loads}\\ \text{with Actual Constraints}\end{array} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\begin{array}{c}\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}\end{array}\right]} = \iiint_V \left(\begin{array}{l} \overbrace{\left[\frac{T}{JG}\left(\frac{\partial \Psi}{\partial y} - z \right) \right]}^{\gamma_{xy}} \overbrace{\left[\frac{\delta T}{J}\left(\frac{\partial \Psi}{\partial y} - z \right) \right]}^{\delta\tau_{xy}} \\ + \overbrace{\left[\frac{T}{JG}\left(\frac{\partial \Psi}{\partial z} + y \right) \right]}^{\gamma_{xz}} \overbrace{\left[\frac{\delta T}{J}\left(\frac{\partial \Psi}{\partial z} + y \right) \right]}^{\delta\tau_{xz}} \end{array} \right) dV$$

Simplifying and remembering that Ψ is only a function of y and z, we get

$$\underbrace{\sum_i \left(b_i \Bigg| \begin{array}{l}\text{At Location i on Original Structure}\\ \text{Caused by Actual Loads}\\ \text{with Actual Constraints}\end{array} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\begin{array}{c}\text{Virtual}\\ \text{External}\\ \text{Force or}\\ \text{Couple}\end{array}\right]} = \int_L \frac{T}{J^2 G}(\delta T) \left\{ \iint_A \left[\begin{array}{l} \left(\frac{\partial \Psi}{\partial y} - z \right)^2 \\ + \left(\frac{\partial \Psi}{\partial z} + y \right)^2 \end{array} \right] dA \right\} dx \qquad (4.6.14)$$

Recall that Equation 4.6.8 gives J as

$$J = \iint_A \left[\left(y^2 + z^2 \right) + \left(y \frac{\partial \Psi(y,z)}{\partial z} - z \frac{\partial \Psi(y,z)}{\partial y} \right) \right] dA \qquad (4.6.8)$$

It can be shown that the value for J above can be put in the form given in Equation 4.6.15 below.
[The derivation of this formula is beyond the scope of this class]

$$\boxed{J = \iint_A \left[\left(\frac{\partial \Psi}{\partial y} - z \right)^2 + \left(\frac{\partial \Psi}{\partial z} + y \right)^2 \right] dA} \tag{4.6.15}$$

Plugging Equation 4.6.15 into Equation 4.6.14 results in

$$\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \int_L \frac{T}{JG} (\delta T)\, dx \tag{4.6.16}$$

Adding some additional subscripts as reminders, the *Principle of Complementary Virtual Work for Torsion* is

$$\boxed{\sum_i \underbrace{\left(b_i \Big|_{\substack{\text{At Location i on Original Structure} \\ \text{Caused by Actual Loads} \\ \text{with Actual Constraints}}} \right)}_{[\text{Displacement}]} \underbrace{\delta B_i}_{\left[\substack{\text{Virtual} \\ \text{External} \\ \text{Force or} \\ \text{Couple}}\right]} = \int_L \left(\frac{T \Big|_{\substack{\text{In Shaft Due} \\ \text{to Actual Loads}}}}{JG} \right) (\delta T)\, dx} \tag{4.6.17}$$

PART IV

IDEALIZED STRUCTURES