# SPECTRAL THEOREM

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Abstract. We study the properties of operators over finite dimensional, complex inner product spaces and the conditions for spectral decomposition to be possible. We look into one of the spectral theorems, proving the diagonalizability of normal, self-adjoint and unitary operators. From this we get all square symmetric matrices with real entries are diagonalizable with real eigenvalues. We study the notion of stable subspaces, most important example being the eigenspaces of the operators. We look at the generalization of inner product maps to complex vector spaces and the properties that arise from the structure of orthonormal basis and orthogonal complements of subspaces.

## 1. Introduction

The motivation of is this paper is to introduce the conditions which assure the ability to diagonalize linear maps from a vector space to itself. In the study of operators we would like to find a nice basis under which the matrix of the transformation is diagonal. This is not always possible, even over the field of complex numbers. The variety of Spectral Theorems tells us about different types of linear maps for which we can always find such a basis. However, they just ensure the existence of such eigenbasis, they don't tell us how to get it. They also grant us the special property that the basis is orthonormal, so it makes calculations easier.

## 2. Polynomials

To diagonalize a matrix, we must first find the eigenvalues. This amounts to finding the roots of the characteristic polynomial of the matrix.

**Definition 2.1.** The *Characteristic Polynomial* of a Matrix $A$ is the polynomial with roots that correspond to the Eigenvalues of the Matrix. $\forall A \in \mathbb{M}_{n\times n}$

$$p(\lambda) = det(A - \lambda I_n) = 0 \tag{2.2}$$

In our study of operators, we will learn how to apply polynomials over them. We will also be required to decompose an operator into its factors. We must first look at the unique factorization of polynomials over 2 different fields, the Reals $\mathbb{R}$ and Complex numbers $\mathbb{C}$.

**Definition 2.3.** The *Algebraic Multiplicity* of an Eigenvalue is the number of times it factors into the characteristic polynomial. The *Geometric Multiplicity* is the dimension of the corresponding Eigen-space.

**Theorem 2.4.** *Every polynomial $p(x)$ over $\mathbb{R}$ of degree $n > 0$ can be represented by a unique product of linear and quadratic factors.*

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$$p(x) = \prod_{i=0}^{m}(x-\lambda_i)^{m_i}\prod_{j=0}^{M}(x^2+\alpha_j x+\beta_j)^{M_j} \tag{2.5}$$

*with distinct roots* $\lambda_i \in \mathbb{R}$*, coefficients* $\alpha_j, \beta_j \in \mathbb{R}$ *and algebraic multiplicities* $m_i, M_j \in \mathbb{N}$

However, we quickly see the problem that not all polynomials have roots over the Reals. In order to find all the eigenvalues of our matrices, we will focus polynomials over the complex numbers.

**Theorem 2.6.** ***Fundamental Theorem of Calculus****: Every polynomial p(z) over* $\mathbb{C}$ *of degree* $n > 0$ *can be represented by a unique product of* $n$ *linear factors.*

$$p(z) = \prod_{i=0}^{m}(z-\lambda_i)^{m_i} \tag{2.7}$$

*with disctinc roots* $\lambda_i, \in \mathbb{C}$ *and algebraic multiplicities* $m_i \in \mathbb{N}$

Even though the Fundamental Theorem of Calculus ensures the existence of $n$ roots over $\mathbb{C}$ for all polynomials of degree $n$, the algebraic multiplicity of some eigenvalues can be greater than one. This means we don't always get n distinct eigenvalues, so there may not exist $n$ linearly independent eigenvectors of the matrix.

**Example 2.8.** Consider the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\forall z, w \in \mathbb{C}$

$$T\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix} \tag{2.9}$$

Under the standard basis L has matrix representation: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We find the eigenvalue of the matrix is $\lambda = 0$ with algebraic multiplicity 2. The eigenspace is $E_\lambda(T) = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \middle| \; w \in \mathbb{C} \right\}$ and is only of dimension 1. Consequently, there is no basis of eigenvectors to diagonalize the matrix.

From this example we see matrices which have eigenvalues of algebraic multiplicity greater than one are not always diagonalizable over $\mathbb{C}$. This is the case when at least one eigenvalue has geometric multiplicity smaller than algebraic multiplicity.

## 3. Complex Inner Product Vector Spaces

**Definition 3.1.** A *Complex Inner Product Space* is a Vector Space $V$ over the field $\mathbb{F} = \mathbb{C}$ with an operation $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ which takes two vectors and assigns it a complex scalar. It satisfies the axioms: $\forall u, v, w \in V$ and $\forall \alpha \in \mathbb{C}$

1. Positive: $\langle u, u \rangle \geq 0$
2. Definite: $\langle u, u \rangle = 0 \iff u = 0_V$
3. Additive: $\langle v + u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$
4. Homogeneous: $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$
5. Conjugate Symmetric: $\langle v, w \rangle = \overline{\langle w, v \rangle}$

Property (3) and (4) together describe the Linearity of the inner product in the first slot. From this axioms we can study the properties in the second slot.

(a) Additive: $\langle v, w+u\rangle = \langle v, w\rangle + \langle v, u\rangle$
(b) Conjugate Homogeneous: $\langle v, \alpha w\rangle = \overline{\alpha}\langle v, w\rangle$

*Proof.* Take arbitrary $v, u, w \in V$ and $\alpha \in \mathbb{C}$ :

$$\begin{aligned} \textbf{(a)}\ \langle v, w+u\rangle &= \overline{\langle w+u, v\rangle} \\ &= \overline{\langle w, v\rangle} + \overline{\langle u, v\rangle} \\ &= \langle v, w\rangle + \langle v, u\rangle \end{aligned}$$

$$\begin{aligned} \textbf{(b)}\ \langle v, \alpha w\rangle &= \overline{\langle \alpha w, v\rangle} \\ &= \overline{\alpha}\overline{\langle w, v\rangle} \\ &= \overline{\alpha}\langle v, w\rangle \end{aligned}$$

□

Property (a) and (b) together describe the Conjugate Linearity of the inner product in the second slot. Combining with the Linearity in the first slot we say the inner product is $Sesqui - Linear$.

**Definition 3.2.** A $Norm$ over an Inner Product Space $V$ is a function $||\cdot|| : V \to \mathbb{R}$ which assigns a length to every vector in the space. $\forall v \in V$

$$||v|| = \sqrt{\langle v, v\rangle} \tag{3.3}$$

**Corollary 3.4.** *Every Inner Product Space has a norm induced by the Inner Product operation.*

**Definition 3.5.** Fixing a vector $u$ in an Inner Product Space $V$, we define the $Projection$ operation which takes any other vector $v$ in the space and gives us a new vector with magnitude of the component of $v$ in the direction of $u$ and pointing in the direction of $u$.

$$proj_u(v) = \frac{\langle v, u\rangle}{\langle u, u\rangle}u \tag{3.6}$$

To make $\mathbb{C}^n$ a Complex Inner Product Space we must define the inner product operation which respects all the axioms.

**Definition 3.7.** The $Hermitian\ form$ is the generalization of the inner product defined on $\mathbb{C}^n$ by: $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ where $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n)$

$$\langle \vec{x}, \vec{y}\rangle = \left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^{n} x_i\overline{y_i} \tag{3.8}$$

where we first conjugate the entries of the second vector and then take the real dot product between the vectors.

**Definition 3.9.** Two vectors $u$ and $v$ in an Inner Product Space $V$ are *orthogonal* if their inner product is zero. $u, v \in V$

$$u \perp v \iff \langle u, v\rangle = 0 \tag{3.10}$$

**Definition 3.11.** Given a subspace $U$ of an Inner Product Space $V$ we define the *Perpendicular Complement* of $U$ as the space of vectors $v$ in V orthogonal to every vector $u$ in U.

$$U^{\perp} = \left\{ \; v \in V \;\middle|\; \langle u, v\rangle = 0 \;,\; \forall u \in U \; \right\} \tag{3.12}$$

**Theorem 3.13.** *Every Inner Product Vector Space $V$ can be decomposed as the direct sum of any Subspace $U$ and it's orthogonal complement $U^{\perp}$*

$$V = U \oplus U^{\perp} \tag{3.14}$$

**Theorem 3.15.** *The dimension of an inner product space $V$ is the sum of the dimensions of any subspace $U$ and its orthogonal complement $U^{\perp}$.*

$$dim_{\mathbb{C}}\; V = dim_{\mathbb{C}}\; U + dim_{\mathbb{C}}\; U^{\perp} \tag{3.16}$$

This theorem and corollary will be useful when we decompose our vector space into the direct sum of the eigenspaces of the linear transformations we study.

**Definition 3.17.** An orthonormal basis $\{e_1, \dots, e_n\}$ is basis of an n-dimensional Inner Product Vector Space $V$ where each vector has norm 1 and is orthogonal to all others.

$$\langle e_i, e_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{3.18}$$

**Theorem 3.19.** *Any finite list of non-zero orthogonal vectors is linearly independent.*

*Proof.* Given a finite orthogonal list of non-zero vectors $\{v_1, \dots, v_n\}$ the vectors are linearly dependent if: $\exists \alpha_i \in \mathbb{C}$ where $a_i \neq 0$ for some $i \in [1, n]$ such that $\sum_{i=1}^{n} \alpha_i v_i = 0_V$. We then fix some $v_j$ and take inner product with it on both sides:

$$\begin{aligned} \left\langle \sum_{i=1}^{n} \alpha_i v_i \;,\; v_j \right\rangle &= \langle 0_V, v_j\rangle \\ \sum_{i=1}^{n} \alpha_i \langle v_i, v_j\rangle &= 0 && (\langle \cdot, \cdot\rangle \; \mathit{Linear}\; 1^{st}) \\ \alpha_j \langle v_j, v_j\rangle &= 0 && (\mathit{Orthogonality}) \\ \alpha_j ||v_j||^2 &= 0 \end{aligned}$$

Since the vectors are non-zero, by definiteness of the inner product $||v_j|| \neq 0$ so we get $\alpha_j = 0, \; \forall j \in [1, n]$ which means $v_1, \dots, v_n$ are linearly independent. □

**Theorem 3.20.** *Given an orthonormal basis $\{e_1, \dots, e_n\}$ of an Inner Product Space $V$ we can decompose any vector $v$ as them sum of the projections of $v$ into every $e_i$ basis vector.* $\forall v \in V$ :

$$v = \sum_{i=1}^{n} \langle v, e_i\rangle e_i \tag{3.21}$$

**Theorem 3.22.** *Given any list of linearly independent vectors in a Inner Product Space, the* ***Gram-Schmidt algorithm*** *lets us find a set of orthonormal vectors with the same span.*

**Lemma 3.23.** *Every finite dimensional Vector Space has a basis.*

**Corollary 3.24.** *Every finite dimensional Inner Product Space $V$ has an orthonormal basis.*

*Proof.* Given any Inner Product Space, **Lemma(3.23)** assures us the existence of a basis. Applying the Gram-Schmidt process we get an orthonormal basis. □

This corollary will be useful when we work on linear maps between Complex Inner Product Spaces, since we only need to know the effects on the orthonormal basis.

## 4. Operators

**Definition 4.1.** A map $L : V \to W$ between Vectors Spaces is linear if it preserves the structure of the domain and co-domain. The properties:

(1) Additive: preserve operation of vector addition
$\forall u, v \in V \quad L(u + v) = L(u) + L(v)$
(2) Homogeneous: preserve operation of scalar multiplication
$\forall v \in V,\ \forall \alpha \in \mathbb{C} \quad L(\alpha v) = \alpha L(v)$

**Theorem 4.2.** *The set of all Linear Maps $L : V \to W$ form a Vector Space.*

$$\mathcal{L}(V, W) = \left\{ \ L : V \to W \ \middle| \ L \text{ is linear} \ \right\} \tag{4.3}$$

**Definition 4.4.** An *Operator* is a Linear Map $T : V \to V$ from a Vector Space $V$ to itself.

**Theorem 4.5.** *The set of all Operators $T : V \to V$ form a Vector space, which is also a Ring.*

$$\mathcal{L}(V) = \mathcal{L}(V, V) = \left\{ \ T : V \to V \ \middle| \ T \text{ is linear} \ \right\} \tag{4.6}$$

Since operators have the same domain as co-domain, we can define the operation of composition between them.

**Definition 4.7.** The composition of two operators $T, S \in \mathcal{L}(V)$ over a vector space $V$ is also an operator over V. $(T \circ S) \in \mathcal{L}(V) \ : \ \forall v \in V$

$$(T \circ S)(v) = T(Sv) \tag{4.8}$$

**Definition 4.9.** We can raise an operator $T \in \mathcal{L}(V)$ to the $n^{th}$ power by composing the operator with itself $n$ times to get a new operator.

$$T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n} \in \mathcal{L}(V) \tag{4.10}$$

**Corollary 4.11.** *Given any operator $T : V \to V$ over a vector space $V$, raising the operator to the $0^{th}$ power gives the identity operator $I_V$ on V. $\forall T \in \mathcal{L}(V)$*

$$T^0 = I_V \tag{4.12}$$

The ring structure allows us to apply polynomials over operators.

**Definition 4.13.** Given a n-degree polynomial polynomial with coefficients $a_i \in \mathbb{C}$:

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

we apply the polynomial over an operator $T \in \mathcal{L}(V)$ to get the new operator:

$$p(T) = \sum_{k=0}^{n} a_k T^k \in \mathcal{L}(V) \tag{4.14}$$

**Definition 4.15.** An operator $T \in \mathcal{L}(V)$ has an eigenvalue $\lambda \in \mathbb{C}$ if and only if the operator $(T - \lambda id_V)$ has a non-trivial Nullspace.

**Definition 4.16.** The *spectrum* of an operator $T \in \mathcal{L}(V)$ is the set of all its Eigenvalues $\lambda$.

$$\sigma(T) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \exists v \neq 0_V \in V,\ Tv = \lambda v \right\} \tag{4.17}$$

**Definition 4.18.** Given an operator $T \in \mathcal{L}(V)$ an *Eigenspace* is a subspace of $V$ spanned by the eigenvectors $v \neq 0_V$ of an eigenvalue $\lambda \in \sigma(T)$.

$$E_\lambda(T) = \left\{ v \in V \;\middle|\; Tv = \lambda v \right\} = null(T - \lambda I_V) \tag{4.19}$$

**Theorem 4.20.** *Given an Operator $T \in \mathcal{L}(V)$ with $\lambda_1, \ldots, \lambda_m$ distinct eigenvalues on a complex finite-dimensional Vectors Space $V$ the following are equivalent:*

*(1) $T$ is diagonalizable over $\mathbb{C}$*
*(2) $V$ has a basis consisting of eigenvectors of $T$.*
*(3) $V = \bigoplus_{j=1}^{m} E_{\lambda_j}(T)$*
*(4) $dim_{\mathbb{C}}\ V = \sum_{j=1}^{m} dim_{\mathbb{C}}\ E_{\lambda_j}(T)$*

**Lemma 4.21.** *Any list of $n+1$ vectors in a n-dimensional vector space in linearly dependent.*

**Theorem 4.22.** *Every Operator $T \in \mathcal{L}(V)$ over a finite dimensional complex vector space $V$ has at least one eigenvalue $\lambda$.*

*Proof.* We select an arbitrary vector $v \neq 0_V \in V$ where $V$ is a n-dimensional complex space. We can apply our operator different amount of times on $v$ to construct $(v, Tv, T^2v, \ldots, T^nv)$ a list of $n+1$ vectors. By **Lemma(4.21)** the list of vectors must be linearly dependent, so $\exists a_i \in \mathbb{C}$ with at least one $a_i \neq 0$:

$$a_0v + a_1Tv + a_2T^2v + \ldots + a_nT^nv = 0_V$$

so we can construct the polynomial with the same coefficients:

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n = 0$$

and by the Fundamental Theorem of Calculus we can decompose it into the linear factors and then we apply it over the operator T to get:

$$p(T)(v) = c\prod_{j=0}^{M}(T - \lambda_j I_V)^{m_j}v = 0_V$$

which mean one of the operators is mapping a non-zero vector to the $0_V$ vector. $\exists \lambda_j \in \mathbb{C}$ so $(T - \lambda_j I_V)$ has a non-trivial Nullspace, so $\lambda_j$ is an eigenvalue of T. $\square$

## 5. Stable Subspaces

**Definition 5.1.** Given an operator $T \in \mathcal{L}(V)$ over a vector space $V$, a subspace $U$ is defined to be $T$-*stable* if $T|_U$,the restriction of $T$ to $U$ is an operator.

$$\forall u \in U,\ Tu \in U \iff T|_U \textit{ is an operator} \tag{5.2}$$

*Remark* 5.3. Some texts name subspaces as *invariant* under transformations.

**Theorem 5.4.** *The following subspaces are always stable under any operator:*

*(1)* $V$ *the vector space itself*
*(2)* $\{0_V\}$ *the set with only the zero vector*
*(3)* $null(T)$ *the Nullspace of the operator*
*(4)* $range(T)$ *the Range of the operator*

*Remark* 5.5. It $U$ is a $T$-stable subspace of $V$, it is NOT the case that every subspace $W$ of $U$ is $T$-stable.

**Theorem 5.6.** *Every Eigenspace of an operator* $T \in \mathcal{L}(V)$ *is* $T$*-stable.*

*Proof.* For an operator $T \in \mathcal{L}(V)$ fix any eigenvalue $\lambda$ with a corresponding eigenvector $v$. By definition $Tv = \lambda v$ and the Eigenspace is a subspace of $V$ so it's closed under scaling. Then $\forall v \in E_\lambda \to Tv \in E_\lambda$ so $\forall \lambda \in \sigma(T),\ E_\lambda$ is $T$-stable. □

**Theorem 5.7.** *If two operators* $S, T \in \mathcal{L}(V)$ *commute, then* $T$ *stabilizes the Eigenspaces of* $S$ *and vice-versa.*

*Proof.* Given an operator $T \in \mathcal{L}(V)$ fix an eigenvalue $\lambda$ with eigenvector $v$. Assume T commutes with the operator $S \in \mathcal{L}(V)$.

$$T(Sv) = (T \circ S)v = (S \circ T)v = S(Tv) = S(\lambda v) = \lambda(Sv)$$

So $\forall \lambda \in \sigma(T),\ \forall v \in E_\lambda(T) \to Sv \in E_\lambda(T)$ which means $E_\lambda(T)$ is S-Stable and a similar argument changing around T and S gives $\forall \lambda \in \sigma(S),\ E_\lambda(S)$ is $T$-stable □

**Corollary 5.8.** *If two operators* $S, T \in \mathcal{L}(V)$ *commute, then* $T$ *stabilizes every subspace* $U$ *of the Eigenspaces of* $S$ *and vice-versa.*

**Theorem 5.9.** *An operator* $T \in \mathcal{L}(V)$ *has an eingenvalue* $\lambda$ *if and only if* $V$ *has a 1-dimensional subspace* $U$ *which is* $T$*-stable.*

*Proof.* First assume $\exists \lambda \in \sigma(T)$ with $v \in E_\lambda(T)$. We define a 1-dimensional subspace $U = span(v)$. Take any arbitrary vector $w = \alpha v \in U$, where $\alpha \in \mathbb{C}$ then:

$$Tw = T(\alpha v) = \alpha(Tv) = \alpha(\lambda v) = (\alpha\lambda)v$$

where $\alpha\lambda \in \mathbb{C}$ so $Tw \in U \Rightarrow U$ is T-stable
Now to prove the other direction assume $T$ has a 1-dimensional subspace $U$. Then $\forall u \in U,\ Tu \in U$ which means $\exists \alpha \in \mathbb{C} : Tu = \alpha u \Rightarrow \alpha \in \sigma(T)$ □

**Corollary 5.10.** *Every finite dimensional complex vector space* $V$ *with an operator* $T \in \mathcal{L}(V)$ *has at least one 1-dimensional* $T$*-stable subspace* $U$ *spanned by an eigenvector* $v$ *of* $T$*.*

*Proof.* Consider an operator $T \in \mathcal{L}(V)$ over a finite dimensional complex vector space $V$. Then the result of **Theorem(4.22)** assures us the existence of some $\lambda \in \sigma(T)$. This is equivalent to the existence of a 1-dimensional subspace $U$ of $V$ which is $T$-stable by **Theorem(5.9)**. □

**Theorem 5.11.** *An operator $T \in \mathcal{L}(V)$ over a complex inner product space $V$ is diagonalizable if and only if there exist one-dimensional $T$-stable subspaces $U_1, \dots, U_n$ of $V$ such that $V$ is the direct sum of the subspaces.*

$$V = \bigoplus_{i=1}^{n} U_i \tag{5.12}$$

## 6. Linear Functionals

**Definition 6.1.** A *linear functional* is a linear map $\Phi : V \to \mathbb{F}$ from a vector space $V$ to a field $\mathbb{F}$.

**Definition 6.2.** The set of all linear functional from a vector space to a field form a Vector space, called the Dual Space $V^*$.

$$V^* = \left\{ \Phi : V \to \mathbb{F} \;\middle|\; \Phi \textit{ is linear} \right\} \tag{6.3}$$

We will focus on linear functions mapping from complex inner product spaces to the complex numbers. ($\mathbb{F} = \mathbb{C}$)

**Theorem 6.4.** ***Riesz Representation Theorem****: Every linear functional over an inner product space can be uniquely represented by inner product with a vector.*

$$\forall \Phi \in V^*,\ \exists! v \in V \ :\ \forall u \in V,\ \Phi(u) = \langle u, v\rangle \tag{6.5}$$

*Proof.* Given an inner product space $V$ and linear functional $\Phi : V \to \mathbb{C}$ we begin by constructing an orthonormal basis $\{e_1, \dots, e_n\}$ of $V$. We take an arbitrary vector $u \in V$ and $u = \sum_{i=1}^{n} \langle u, e_i\rangle e_i$ and we apply the linear functional:

$$\begin{aligned}
\Phi(u) &= \Phi\left(\sum_{i=1}^{n} \langle u, e_i\rangle e_i\right) \\
&= \sum_{i=1}^{n} \langle u, e_i\rangle \Phi(e_i) && (\Phi \textit{ Linear}) \\
&= \sum_{i=1}^{n} \langle u,\ \overline{\Phi(e_i)}\ e_i\rangle && (\langle\cdot,\cdot\rangle \textit{ Conjugate Homogenous } 2^{nd}) \\
&= \langle u,\ \sum_{i=1}^{n} \overline{\Phi(e_i)}\ e_i\rangle && (\langle\cdot,\cdot\rangle \textit{ Additive } 2^{nd}) \\
&= \langle u, v\rangle
\end{aligned}$$

By letting $v = \sum_{i=1}^{n} \overline{\Phi(e_i)}\ e_i \in V$ be our vector we proved the existence part.
Now assume there exist two $v_1, v_2 \in V$ which satisfy this so:

$$\begin{aligned}
\Phi(u) = \langle u, v_1\rangle &= \langle u, v_2\rangle \\
0 &= \langle u, v_2\rangle - \langle u, v_1\rangle \\
0 &= \langle u, v_2 - v_1\rangle
\end{aligned}$$

This is true for all $u \in V$ so we let $u = v_2 - v_1$ to get $||v_2 - v_1||^2 = 0$ which means $v_2 - v_1 = 0_V \ \Rightarrow\ v_2 = v_1$ proving the uniqueness part.

$\square$

## 7. Adjoint Map

**Definition 7.1.** Given a linear map $L : V \to W$ between two inner product spaces $V$ and $W$ with respective inner product $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ operations we can construct the *adjoint map* $L^* : W \to V$ which satisfies that $\forall v \in V$ and $\forall w \in W$ the **Adjoint Equation**:

$$\langle Lv, w \rangle_W = \langle v, L^* w \rangle_V \tag{7.2}$$

*Remark* 7.3. The adjoint operator, sometimes referred as the Hermitian adjoint, should not be confused with the Classical adjoint of a matrix which is used to compute it's inverse matrix.

As a direct consequence of Riesz Representation Theorem the adjoint map is well defined when our linear map is an operator $T \in \mathcal{L}(V)$ from an inner product space $V$ to itself.

**Theorem 7.4.** *For every operator $T \in \mathcal{L}(V)$ over an inner product space $V$ there exits a unique adjoint operator $T^* \in \mathcal{L}(V)$ which satisfies the adjoint equation.*

$$\forall T \in \mathcal{L}(V),\ \exists! T^* \in \mathcal{L}(V) :\ \langle Tv, w \rangle = \langle v, T^* w \rangle\ \forall v, w \in V \tag{7.5}$$

**Theorem 7.6.** *$T^* : V \to V$ is linear, hence $T^* \in \mathcal{L}(V)$*

*Proof.* Consider an operator $T \in \mathcal{L}(V)$ with adjoint $T^* : V \to V$ and $u, v, w \in V$ an inner product space and $\alpha \in \mathbb{C}$.

(1) Additivity

$$\begin{aligned} \langle v, T^*(w+u) \rangle &= \langle Tv, w+u \rangle \\ &= \langle Tv, w \rangle + \langle Tv, u \rangle \\ &= \langle v, T^* w \rangle + \langle v, T^* u \rangle \\ &= \langle v, T^* w + T^* u \rangle \end{aligned}$$

By the uniqueness of Riesz Representation Theorem $T^*(w+u) = T^*w + T^*u$

(2) Homogeneity

$$\begin{aligned} \langle v, T^*(\alpha w) \rangle &= \langle Tv, \alpha w \rangle \\ &= \bar{\alpha} \langle Tv, w \rangle \\ &= \bar{\alpha} \langle v, T^* w \rangle \\ &= \langle v, \alpha T^* w \rangle \end{aligned}$$

By the uniqueness of Riesz Representation Theorem $T^*(\alpha w) = \alpha T^* w$

$\square$

Next we list some common properties of the adjoint operators.

**Theorem 7.7.** *Given two operators $T, S \in \mathcal{L}(V)$ and $\alpha \in \mathbb{C}$ then:*

*(1)* $(T + S)^* = T^* + S^*$
*(2)* $(\alpha T)^* = \bar{\alpha} T^*$
*(3)* $(T^*)^* = T$
*(4)* $(T \circ S)^* = S^* \circ T^*$

**Theorem 7.8.** *Given an operator $T \in \mathcal{L}(V)$ over an inner product space $V$, a subspace $U$ of $V$ is $T$-stable if and only if $U^{\perp}$ is $T^*$-stable.*

*Proof.* Consider $T, T^* \in \mathcal{L}(V)$ over an inner product space $V$, with a subspace $U$. Assume $U$ is $T$-stable and take $v \in U$ and $w \in U^\perp$:

$$\begin{aligned} \langle v, w \rangle &= 0 \\ \langle Tv, w \rangle &= 0 && (T \textit{ stabilizes } U) \\ \langle v, T^* w \rangle &= 0 \end{aligned}$$

So $\forall w \in U^\perp \to T^* w \in U^\perp$ so $U^\perp$ is $T^*$-stable □

## 8. Operator Types

We now analyze the properties of different types of operators, including: Normal, Self-Adjoint and Unitary operators.

### 8.1. Normal.

**Definition 8.1.** An operator $U \in \mathcal{L}(V)$ is $Normal$ if it commutes with its adjoint.

$$U \circ U^* = U^* \circ U \tag{8.2}$$

### 8.2. Self-Adjoint.

**Definition 8.3.** An operator $T \in \mathcal{L}(V)$ is $Self\text{-}Adjoint$ or $Hermitian$ if it is equal to its adjoint.

$$T^* = T \tag{8.4}$$

**Corollary 8.5.** *Every Self-Adjoint operator is Normal.*

**Corollary 8.6.** *Every self-adjoint operator over a real vector space is Symmetric.*

$$A = A^t \tag{8.7}$$

**Theorem 8.8.** *Every eigenvalue $\lambda$ of a self-adjoint operator $T \in \mathcal{L}(V)$ is real.*

*Proof.* Assume $T \in \mathcal{L}(V)$ is self-adjoint. Take any eigenvector $v \neq 0_V \in E_\lambda(T)$

$$\begin{aligned} \lambda ||v||^2 &= \lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, T^* v \rangle \\ &= \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle = \bar{\lambda} ||v||^2 \end{aligned}$$

By choice of $v$ being non-zero and the exactness of the inner product $||v||^2 \neq 0$ and we can divide on both sides, giving $\lambda = \bar{\lambda}$ so $\lambda \in \mathbb{R}$ □

### 8.3. Unitary.

**Definition 8.9.** An operator $T \in \mathcal{L}(V)$ is $Unitary$ if the composition with its adjoint on both sides is equivalent to the identity operator on V.

$$T \circ T^* = T^* \circ T = I_V \tag{8.10}$$

**Corollary 8.11.** *Every unitary operator is Normal.*

**Corollary 8.12.** *The inverse of a Unitary Operator is the adjoint operator.*

$$T^{-1} = T^* \tag{8.13}$$

**Corollary 8.14.** *Every unitary operator over a Real vector space is $Orthogonal$.*

$$Q^t \circ Q = Q \circ Q^t = I \tag{8.15}$$

## 9. Spectral Theorem

We are ready to prove every normal operator is diagonalizable over finite dimensional complex inner product spaces. Then for free, we get the same result for Self-Adjoint and Unitary operators.

**Theorem 9.1.** ***Spectral Theorem for Normal Operators*** *Let $T \in \mathcal{L}(V)$ be a Normal operator on a finite dimensional, complex inner product space $V$. Then there exists an orthonormal basis $\{e_i\}$ for $V$ consisting of eigenvectors of $T$.*

*Proof.* We proof this by induction on the dimension of $V$, a finite dimensional complex inner product space. We let $dim_{\mathbb{C}}V = n$ and assume the operator $T \in \mathcal{L}(V)$ is Normal. First, by **Corollary(5.10)** we are assured the existence of at least one $T$-stable, 1-dimensional subspace $U$ of $V$ spanned by an eigenvector $w$ of $T$. Then since $T$ is normal, it commutes with $T^*$ so by **Corollary(5.8)** the subspace is also $T^*$-stable. Then restricting $T$ and $T^*$ to the subspace, $T|_U$ and $T^*|_U$ are operators. Therefore, $T|_U$ is a Normal operator over a 1-dimensional space. We can take any $w$ and normalize it by setting $v = \frac{w}{||w||}$. This vector $v$ is an orthonormal basis of U, hence proving the base case of a 1-dimensional space.

Next, we make the induction hypothesis that any $(n-1)$ dimensional, complex inner product space has an orthonormal basis consisting of eigenvectors of a normal operator over the space. Since we have $U$ we can construct the orthogonal complement $U^{\perp}$. By **Theorem(7.8)**, since $U$ is $T$-stable then $U^{\perp}$ is stabilized by $T^*$. Similarly, since $U$ is $T^*$-stable then $U^{\perp}$ is stabilized by $(T^*)^* = T$. Then restricting $T$ and $T^*$ to the orthogonal complement space, $T|_{U^{\perp}}$ and $T^*|_{U^{\perp}}$ are operators. By **Theorem(3.15)** we get:

$$dim_{\mathbb{C}}\ U^{\perp} = dim_{\mathbb{C}}\ V - dim_{\mathbb{C}}\ U = n - 1$$

Consequently, $T|_{U^{\perp}}$ is a Normal operator over a $(n-1)$ dimensional complex inner product space. We can then apply the induction hypothesis on $U^{\perp}$ and construct an orthonormal basis of eigenvectors $\{u_1, \cdots, u_{n-1}\}$ of $T$.

Final step, we construct a list containing the eigenvector $v \in U$ from the base case and all the $u_i \in U^{\perp}$ from the induction hypothesis. By definition of the orthogonal complement $v \perp u_i, \forall i \in [1, n-1]$ so all the vectors in the list are orthogonal to each other. Since all vectors have norm 1, they must be non-zero vectors by the definiteness of the inner product, so by **Theorem(3.19)** the orthogonality implies they are linearly independent. We then have $n$ linearly independent vectors in a n-dimensional space $V$, so they form a basis of $V$. More specifically, $\{u_1, \ldots, u_{n-1}, v\}$ is an orthonormal basis of eigenvectors of $T$. □

**Theorem 9.2.** ***Spectral Theorem for Unitary Operators*** *Let $T \in \mathcal{L}(V)$ be a Unitary operator on a finite dimensional, complex inner product space $V$. Then there exists an orthonormal basis $\{e_i\}$ for $V$ consisting of eigenvectors of $T$.*

*Proof.* By **Corollary(8.11)** every Unitary Operator is Normal, so **Theorem(9.1)** ensures the existence of the orthonormal eigenbasis. □

**Theorem 9.3.** ***Spectral Theorem for Self-Adjoint Operators*** *Let $T \in \mathcal{L}(V)$ be a Hermitian operator on a finite dimensional, complex inner product space $V$. Then there exists an orthonormal basis $\{e_i\}$ for $V$ consisting of eigenvectors of $T$.*

*Proof.* By **Corollary(8.5)** every Hermitian Operator is Normal, so **Theorem(9.1)** ensures the existence of the orthonormal eigenbasis. □

**Corollary 9.4.** *Every square symmetric matrix with real entries is diagonalizable and has real Eigenvalues.*

*Proof.* Square symmetric matrices with real entries represent a special case of self-adjoint operators over complex inner product spaces. Then **Theorem(9.3)** ensures the existence of the eigenvectors to construct the Unitary Matrix $U$ and **Theorem(8.8)** ensures the eigenvalues of the Diagonal Matrix $D$ are Real.

$$\begin{aligned}\forall A \in \mathbb{M}_{n\times n}(\mathbb{R}) : \ A^T = A \ &\Rightarrow \ \exists U, D \in \mathbb{M}_{n\times n}(\mathbb{R}) : \ A = UDU^* \\ &\Rightarrow \ \forall \lambda \in \sigma(A) \ \rightarrow \ \lambda \in \mathbb{R}\end{aligned}$$

□

## 10. Extensions

In this paper we have only examined the Spectral Theorems concerning finite-dimensional, complex inner product spaces. There exists a proof for finite dimensional, real inner product spaces showing the diagonalizability of self-adjoint operators. However, this proof is harder, since it requires specializing to $\mathbb{R}$ which is not Algebraically closed, so it is not the case that all operators have roots. Even harder to extend is Spectral Theorems for infinite dimensional vector spaces, which strips us of some of the assumptions we have built upon.

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