GEOMETRIC ALGEBRAS ROTATIONS

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ABSTRACT. We introduce the notion of multivectors, generated by taking the outer product of vectors in $\mathbb{R}^n$, which generalizes the cross-product in $\mathbb{R}^n$. The linear combinations of these higher grade objects generate a Geometric Algebra, also known as Clifford Algebra. On this space we can define the geometric product, which lets us define multiplicative inverses of vectors. We construct rotors in our space, which acts as operators on our elements which lets us generalize rotations in n-dimensional Euclidean space.

CONTENTS

1. INTRODUCTION

Many problems in physics can be simplified by the use of symmetries. Objects like spheres are of special interest, due to their many continuous symmetries they posses, like rotations and reflections. While we normally work in $\mathbb{R}^3$ or some subset of it, we would like to generalize transformations to more generalized spaces. For instance, we would like to study rotations in higher dimensional spaces $\mathbb{R}^n$ or spaces with different signatures, like 4-dimensional space-time with signature $(-1, +1, +1, +1)$. In order to do this we will need to generalize the complex numbers algebra and the quaternions algebra. We first review the aforementioned algebraic structures and how they generate rotations in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$.

2. COMPLEX NUMBERS AND QUATERNIONS

Definition 2.1. *Complex numbers* form a 2-dimensional algebra over the reals $\mathbb{R}$:

$$\mathbb{C} = \left\{ z = a + ib \;\middle|\; a, b \in \mathbb{R}, \, i^2 = -1 \right\} \tag{2.1}$$

Theorem 2.2. *The unit complex numbers form a group under multiplication:*

$$\mathbb{C}_1 = \left\{ e^{i\phi} \in \mathbb{C} \right\} \tag{2.2}$$

Definition 2.3. *Quaternions* form a 4-dimensional algebra over the reals $\mathbb{R}$:

$$\mathbb{H} = \left\{ q = a1 + bi + cj + dk \;\middle|\; a, b, c, d \in \mathbb{R}\, ,\, i^2 = j^2 = k^2 = ijk = -1 \right\} \tag{2.3}$$

Theorem 2.4. *The unit quaternions form a group under multiplication:*

$$\mathbb{H}_1 = \left\{ e^{\hat{u}\phi} \in \mathbb{H} \;\middle|\; \hat{u} \in \mathbb{R}^3, |\hat{u}| = 1 \right\} \tag{2.4}$$

and so does the direct product with itself $\mathbb{H}_1 \times \mathbb{H}_1$

Theorem 2.5. *The pure quaternions form a vector space under addition:*

$$\mathbb{H}_p = \left\{ \vec{v} \in \mathbb{H} \;\middle|\; \vec{v} \in \mathbb{R}^3 \right\} \tag{2.5}$$

We can define the action of these groups on their corresponding algebraic structure, isomorphic to some Euclidean Space $\mathbb{R}^n$ as a vector space. Then we can construct a surjective group homomorphism from our group to the corresponding rotation group $SO(n)$.

Definition 2.6. The rotations of n-dimensional Euclidean space $\mathbb{R}^n$ form a group under multiplication:

$$SO(n) = \left\{ A \in \mathbb{M}_{nxn}(\mathbb{R}) \;\middle|\; A^t = A^{-1}\, ,\, det(A) = +1 \right\} \tag{2.6}$$

Group	**Euclidean Space**	**Action**	**Map**	**Rotation Group**
$\mathbb{C}_1$	$\mathbb{C} \cong \mathbb{R}^2$	Left Multiplication	$z \mapsto e^{i\phi} z$	$SO(2)$
$\mathbb{H}_1$	$\mathbb{H}_p \cong \mathbb{R}^3$	Conjugation	$\vec{v} \mapsto e^{\hat{u}\phi} \vec{v} e^{-\hat{u}\phi}$	$SO(3)$
$\mathbb{H}_1 \times \mathbb{H}_1$	$\mathbb{H} \cong \mathbb{R}^4$	Left & Right Multiplication	$q \mapsto e^{\hat{u}_1\phi_1} q e^{-\hat{u}_2\phi_2}$	$SO(4)$

The most natural step to generalize this to $\mathbb{R}^5$ would be to look at the next structure, the Octonions $\mathbb{O}$. However, we immediately run intro trouble since the octonion product is non-associative, so we can't talk about a unit group $\mathbb{O}_1$ in it. We conclude, this is not the right approach to generalize the rotations in higher dimensions. We then look to develop new tools from which we can build the complex numbers, quaternions and can easily extend to higher dimensional structures which generate rotations. This is the motivation which leads us to study of geometric algebras.

3. Outer Product

We take a closer look back at complex numbers to see which properties we can generalize from them. Take any $z = |z|e^{i\phi}, w = |w|e^{i\theta} \in \mathbb{C}$ then we look at the product $zw^* = |z||w|e^{i(\phi-\theta)}$ which has a real and imaginary part:

$$Re[zw^*] = |z||w|\cos(\phi - \theta) \quad , \quad Im[zw^*] = |z||w|\sin(\phi - \theta) \tag{3.1}$$

We thinking of $z, w \in \mathbb{R}^3$ as vectors we notice the similarity with the formulas:

$$z \cdot w = |z||w|\cos(\phi - \theta) \quad , \quad |z \times w| = |z||w|\sin(\phi - \theta) \tag{3.2}$$

We see the first term represents a scalar projection and the second term represents the area element spanned by the vectors. We should then begin thinking of the real and imaginary components of our complex number as two different objects. We focus on the imaginary part of the complex product, which corresponds to an oriented area element, since it also has a sign. We also want to generalize the cross product to higher dimensions, so we look at its properties:

Definition 3.1. For vectors $a, b, c \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$ the cross product $\times$ satisfies:

$$\begin{array}{lll} (1) & a \times b = -b \times a & Anti-Symmetric \\ (2) & (\lambda a \times b) = \lambda(a \times b) & Homogenous \\ (3) & (a+b) \times c = a \times c + b \times c & Distributive \end{array}$$

Remark 3.2. The cross product is non-associative: $a \times (b \times c) \neq (a \times b) \times c$, but we would like to construct a product operation which is associative.

Definition 3.3. For vectors $a, b, c \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ the outer product $\wedge$ is then defined by the properties:

$$\begin{array}{lll} (1) & a \wedge b = -b \wedge a & Anti-Symmetric \\ (2) & (\lambda a \wedge b) = \lambda(a \wedge b) & Homogenous \\ (3) & (a+b) \wedge c = a \wedge c + b \wedge c & Distributive \\ (4) & a \wedge (b \wedge c) = (a \wedge b) \wedge c & Associative \end{array}$$

Corollary 3.3.1. *The outer product of any vector with itself is zero.*

$$\forall a \in \mathbb{R}^n \quad a \wedge a = 0 \tag{3.3}$$

Proof. Let $a \in \mathbb{R}^n$, then by the anti-symmetric property of the outer product

$$a \wedge a = -a \wedge a \;\rightarrow\; a \wedge a = 0$$

□

Where the geometric interpretation comes from the complex product we mentioned above. The product $a \wedge b$ is the oriented area constructed by sliding b along a. The area is given by the parallelogram spanned by a and b and the orientation is given by following the perimeter along a and then along b. We see the anti-symmetry arises from the orientation of the plane being reverted for $b \wedge a$.

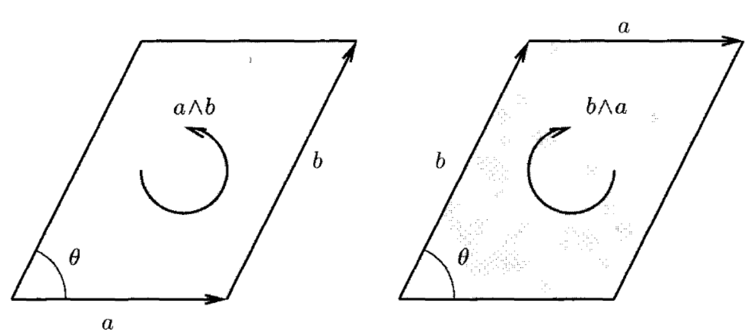

FIGURE 1.

Example 3.4. Take $a, b \in \mathbb{R}^2$ with an orthonormal basis $\{ e_1, e_2 \}$:

$$\begin{aligned} a \wedge b &= (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \\ &= a_1 e_1 \wedge b_1 e_1 + a_1 e_1 \wedge b_2 e_2 + a_2 e_2 \wedge b_1 e_1 + a_2 e_2 \wedge b_2 e_2 \\ &= a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \\ &= a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) \\ &= a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (-e_1 \wedge e_2) \\ &= (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} (e_1 \wedge e_2) = det(a, b)(e_1 \wedge e_2) \end{aligned}$$

Where $(e_1 \wedge e_2)$ is the unit oriented area element in $\mathbb{R}^2$.

Remark 3.5. The plane $(e_1 \wedge e_2)$ is not elements of the vector space $\mathbb{R}^3$, which only contains vectors. It belongs in its own vector space which has its own operations of addition and scaling.

Example 3.6. Take $a, b \in \mathbb{R}^3$ with an orthonormal basis $\{ e_1, e_2, e_3 \}$:

$$\begin{aligned} a \wedge b &= \left(\sum_{i=1}^{3} a_i e_i \right) \wedge \left(\sum_{j=1}^{3} b_j e_j \right) = \sum_{i,j=1}^{3} (a_i e_i) \wedge (b_j e_j) = \sum_{i,j=1}^{3} (a_i b_j) (e_i \wedge e_j) \\ &= \sum_{i \neq j}^{3} (a_i b_j) (e_i \wedge e_j) \\ &= a_1 b_2 (e_1 \wedge e_2) + a_1 b_3 (e_1 \wedge e_3) + a_2 b_1 (e_2 \wedge e_1) \\ &\quad + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) + a_3 b_2 (e_3 \wedge e_2) \\ &= a_1 b_2 (e_1 \wedge e_2) + a_1 b_3 (e_1 \wedge e_3) + a_2 b_1 (-e_1 \wedge e_2) \\ &\quad + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (-e_1 \wedge e_3) + a_3 b_2 (-e_2 \wedge e_3) \\ &= (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) + (a_3 b_1 - a_1 b_3)(e_3 \wedge e_1) + (a_2 b_3 - a_3 b_2)(e_2 \wedge e_3) \\ &= \begin{vmatrix} (e_2 \wedge e_3) & (e_3 \wedge e_1) & (e_1 \wedge e_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = det(\, (\, (e_2 \wedge e_3) \, , \, (e_3 \wedge e_1) \, , \, (e_1 \wedge e_2) \,) \, \, , a, b) \end{aligned}$$

We have a linear combination of the unit planes $(e_2 \wedge e_3)$, $(e_3 \wedge e_1)$, $(e_1 \wedge e_2)$ in $\mathbb{R}^3$. We notice the coefficients in the wedge product of the 2 vectors in $\mathbb{R}^3$ matches those of the cross-product, but the outer product can be generalized to higher dimensions. We will use the Levi-Civita symbol to simplify our notation when we take 3 vectors:

$$\varepsilon_{ijk} = \begin{cases} +1 & (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

Example 3.7. Take $a, b, c \in \mathbb{R}^3$ with an orthonormal basis $\{ e_1, e_2, e_3 \}$:

$$a \wedge b \wedge c = \left(\sum_{i=1}^{3} a_i e_i \right) \wedge \left(\sum_{j=1}^{3} b_j e_j \right) \wedge \left(\sum_{k=1}^{3} c_k e_k \right) = \sum_{i,j,k=1}^{3} (a_i e_i) \wedge (b_j e_j) \wedge (c_k e_k)$$

$$\begin{aligned}
&= \sum_{i,j,k=1}^{3} (a_i b_j c_k)(e_i \wedge e_j \wedge e_k) = \sum_{i,j,k=1}^{3} (a_i b_j c_k)(\varepsilon_{ijk}(e_1 \wedge e_2 \wedge e_3)) \\
&= \left(\sum_{i,j,k=1}^{3} \varepsilon_{ijk} a_i b_j c_k \right)(e_1 \wedge e_2 \wedge e_3) \\
&= (a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1)(e_1 \wedge e_2 \wedge e_3) \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (e_1 \wedge e_2 \wedge e_3) = det(a,b,c)(e_1 \wedge e_2 \wedge e_3)
\end{aligned}$$

Where $(e_1 \wedge e_2 \wedge e_3)$ is an oriented volume with orientation given by the right hand rule.

We show the outer product encodes linear dependence of the vectors.

Theorem 3.8. *A set of k linearly dependent vectors $u_1, \dots, u_k \in \mathbb{R}^n$ has outer product equal to zero.*

$$u_1 \wedge u_2 \wedge \cdots \wedge u_k = 0 \tag{3.5}$$

Proof. Without loss of generality we assume u_1 can be written as a linear combination of the other vectors, so by the distributive and homogeneous property:

$$u_1 \wedge u_2 \wedge \cdots \wedge u_k = \left(\sum_{i=2}^{k} \lambda_i u_i \right) \wedge u_2 \wedge \cdots \wedge u_k = \sum_{i=2}^{k} \lambda_i (u_i \wedge u_2 \wedge \cdots \wedge u_k) = 0$$

Since we have repeated indices, the anti-symmetrization tells us each term in the sum is 0 since we can always rearrange the elements in each product at the cost of a $(-)$ sign to get $u_i \wedge u_i = 0$. □

Theorem 3.9. *For any set of k vectors $u_1, \dots, u_k \in \mathbb{R}^n$ with orthonormal basis $e_1, \dots, e_n$, the outer product simplifies:*

$$u_1 \wedge u_2 \cdots \wedge u_k = \begin{cases} det(u_1, \dots, u_k)(e_1 \wedge e_2 \cdots \wedge e_k) & k = n \\ 0 & k > n \end{cases} \tag{3.6}$$

where $(e_1 \wedge e_2 \cdots \wedge e_k)$ is the unit oriented k-volume in $\mathbb{R}^k$.

4. Geometric Algebra

We introduce the notion of *k-blades* as a generalization of oriented areas and volumes of higher dimensions.

Definition 4.1. Given an orthonormal basis $\beta = \{ e_1, \dots, e_n \}$ of $\mathbb{R}^n$ and a $k \leq n \in \mathbb{N}$ we construct a k-blade by taking the outer product of k number of vectors in our basis:

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \quad , \quad e_{i_1}, \dots, e_{i_k} \in \beta \tag{4.1}$$

Example 4.2. In the previous section we saw the the oriented area element $(e_1 \wedge e_2)$ and oriented volume element $(e_1 \wedge e_2 \wedge e_3)$ in $\mathbb{R}^3$, also known as bivector and trivector respectively.

Since we build k-blades from wedging vectors, they are also referred to as *k-vectors* or *k-volumes*. For each k, we have a new vector space spanned by the corresponding basis of k-blades $\{ (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) \}_{i_1,\ldots,i_k}^{n}$.

Definition 4.3. The grade of any linear combination of k-vectors is k.

Definition 4.4. The space spanned by a a basis of k-blades in $\mathbb{R}^n$ with basis $\beta = \{ e_1, \ldots, e_n \}$ contains all elements of grade k.

$$\Lambda^k(\mathbb{R}^n) = \left\{ \sum_{i_1,\ldots,i_k} \lambda_{i_1,\ldots,i_k}(e_{i_1} \wedge \cdots \wedge e_{i_k}) \;\middle|\; \lambda_{i_1,\ldots,i_k} \in \mathbb{R}\,,\; e_{i_j} \in \beta \right\} \tag{4.2}$$

Example 4.5. We take the 3-dimensional Euclidean space $\mathbb{R}^3$ with basis $\{ e_1, e_2, e_3 \}$ which generates the following k-blade subspaces:

$$\begin{aligned}
\Lambda^0(\mathbb{R}^3) &= \{ \lambda \in \mathbb{R} \} = \mathbb{R} && (scalars)\\
\Lambda^1(\mathbb{R}^3) &= \{ \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \} = \mathbb{R}^3 && (vectors)\\
\Lambda^2(\mathbb{R}^3) &= \{ \lambda_1(e_2 \wedge e_3) + \lambda_2(e_3 \wedge e_1) + \lambda_3(e_1 \wedge e_2) \} && (bivectors)\\
\Lambda^3(\mathbb{R}^3) &= \{ \lambda(e_1 \wedge e_2 \wedge e_3) \} && (trivectors)
\end{aligned}$$

taking the direct sum of these spaces we generate the Geometric Algebra of $\mathbb{R}^3$

$$\mathbb{G}(\mathbb{R}^3) = \mathbb{R} \oplus \mathbb{R}^3 \oplus \Lambda^2(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3) \tag{4.3}$$

which is the vector space that contains all linear combinations of scalars, vectors, oriented planes and oriented volumes in $\mathbb{R}^3$. In order to upgrade this to an algebra we must introduce a product operation on it, the geometric product.

5. Geometric Product

We introduce a new product operation on our vector space $G(\mathbb{R}^n)$ to make it into a ring.

Definition 5.1. The geometric product is defined in terms of the inner and outer product, given multivectors $C, D \in \mathbb{G}(\mathbb{R}^n)$:

$$CD = C \rfloor D + C \wedge D \tag{5.1}$$

Definition 5.2. The inner product is a generalization of the inner product to multivectors that satisfies:

$$\begin{aligned}
\text{scalars} \quad & \lambda_1 \rfloor \lambda_2 = \lambda_1 \lambda_2\\
\text{vectors \& scalar} \quad & a \rfloor \lambda = 0\\
\text{scalar \& vector} \quad & \lambda \rfloor a = \lambda a\\
\text{vectors} \quad & ab = a \cdot b\\
\text{vector, multivector} \quad & a \rfloor (b \wedge C) = (a \rfloor b) \wedge C - b \wedge (a \rfloor C)
\end{aligned}$$

Where it reduces to the dot product for vectors and is defined recursively for multivectors. It also satisfies distributive property: $(C \wedge D) \rfloor E = C \rfloor (D \rfloor E)$

Definition 5.3. For any set of k vectors $u_1, \dots, u_k \in \mathbb{R}^n$ the outer product:

$$u_1 \wedge u_2 \wedge \dots \wedge a_k = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) u_{\sigma(1)} u_{\sigma(2)} \dots u_{\sigma(k)} \tag{5.2}$$

where σ are the permutations of the symmetry group on k letters S_k and the anti-symmetrization comes from the the *sgn* function:

$$sgn(\sigma) = \begin{cases} +1 & \sigma \text{ is } \textit{even} \\ -1 & \sigma \text{ is } \textit{odd} \end{cases} \tag{5.3}$$

We can look at the inner product and outer product as the symmetric product of two vectors as the symmetric and anti-symmetric parts of the geometric product correspondingly:

$$a \cdot b = \frac{1}{2}(ab - ba) \quad , \quad a \wedge b = \frac{1}{2}(ab - ba) \tag{5.4}$$

Since we defined our inner and outer products to be associative, the geometric product also satisfies associativity. Then we have an associative ring which is also a vector space, so it is an associative algebra, known as Clifford Algebra. We look at the square of a vector $a \in \mathbb{R}^n$, which is the geometric product with itself:

$$a^2 = aa = a \cdot a + a \wedge a = |a|^2 \in \mathbb{R} \tag{5.5}$$

gives a scalar, so we construct the inverse by:

$$a^{-1} = \frac{a}{a^2} \in \mathbb{R}^n \tag{5.6}$$

For a unit vector $e_i \in \mathbb{R}^n$:

$$e_i^2 = 1 \quad , \quad e_i^{-1} = e_i \tag{5.7}$$

For two orthogonal vectors $e_i, e_j \in \mathbb{R}^n$ the geometric product reduces to the outer product, so they anti commute:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_1 \tag{5.8}$$

Definition 5.4. The blade of highest grade $I \in \mathbb{G}(\mathbb{R}^n)$ called the *Pseudo-Scalar* is unique up to sign. It satisfies the property $I^2 = -1$

Example 5.5. In $\mathbb{G}(\mathbb{R}^3)$, the pseudo-scalar $I = e_1 \wedge e_2 \wedge e_3 = e_1 e_2 e_3$ of grade 3:

$$\begin{aligned} I^2 &= (e_1e_2e_3)(e_1e_2e_3) = e_1e_2(e_3e_1)e_2e_3 = e_1e_2(-e_1e_3)e_2e_3 \\ &= -e_1(e_2e_1)e_3e_2e_3 = -e_1(-e_1e_2)e_3e_2e_3 = (-1)^2(e_1e_1)(e_2e_3)e_2e_3 \\ &= e_1^2(-e_3e_2)(e_2e_3) = -e_3(e_2e_2)e_3 = -e_3e_2^2e_3 = -e_3e_3 = -e_3^2 = -1 \end{aligned}$$

6. Generalized Complex Numbers

We look at the geometric algebra $\mathbb{G}(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}^2 \oplus \Lambda^2(\mathbb{R}^2)$ of two-dimensional real space, more specifically the even subalgebra $\mathbb{G}(\mathbb{R}^2)^+ = \mathbb{R} \oplus \Lambda^2(\mathbb{R}^2)$ of all elements of even grade. This is all linear combinations of scalars and a single oriented area element.

$$\mathbb{G}(\mathbb{R}^2)^+ = \{ Z = \lambda_0 + I\lambda_1 \mid \lambda_j \in \mathbb{R} \} \cong \mathbb{C} \tag{6.1}$$

Where $I = (e_1 \wedge e_2) = (e_1 e_2)$ is the highest degree blade. Then it's the pseudoscalar, so it satisfies $I^2 = 1$. We look at the effect of left multiplication by I:

$$IZ = I(\lambda_0 + I\lambda_1) = I\lambda_0 + I^2\lambda_1 = -\lambda_1 + I\lambda_0 \tag{6.2}$$

Similarly, using the fact that I commutes with all scalars, right multiply by I:

$$ZI = (\lambda_0 + I\lambda_1)I = \lambda_0 I + I^2\lambda_1 = -\lambda_1 + I\lambda_0 \tag{6.3}$$

They both give rotations by $\pi/2$ counterclockwise and we showed I commutes with every element in $\mathbb{G}(\mathbb{R}^2)^+$, just like $i \in \mathbb{C}$. We note however I anti-commutes with all elements in $\mathbb{R}^2$. Take $a = (\lambda_1 e_1 + \lambda_2 e_2) \in \mathbb{R}^2$:

$$\begin{aligned} aI &= (\lambda_1 e_1 + \lambda_2 e_2)(e_1 e_2) = \lambda_1(e_1 e_1 e_2) + \lambda_2(e_2 e_1 e_2) \\ &= \lambda_1 e_1(e_1 e_2) + \lambda_2(e_2 e_1)e_2 \lambda_1 e_1(-e_2 e_1) + \lambda_2(-e_1 e_2)e_2 = \\ &= -\lambda_1(e_1 e_2)e_1 - \lambda_2(e_1 e_2)e_2 = -(e_1 e_2)(\lambda_1 e_1 + \lambda_2 e_2) = -Ia \end{aligned}$$

More generally for rotations by an arbitrary angle θ, we have left multiplication by the exponential:

$$Z \mapsto Ze^{I\theta} \tag{6.4}$$

7. Generalized Quaternions

We now look at the geometric algebra $\mathbb{G}(\mathbb{R}^3) = \mathbb{R} \oplus \mathbb{R}^3 \oplus \Lambda^2(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3)$ of three-dimensional real space, more specifically the even subalgebra $\mathbb{G}(\mathbb{R}^3)^+ = \mathbb{R} \oplus \Lambda^2(\mathbb{R}^3)$ of all elements of even grade. Again we have linear combinations of scalars and oriented area elements, but now their exist three linearly independent planes.

$$\mathbb{G}(\mathbb{R}^2)^+ = \{\, Q = \lambda_0 + \lambda_1(e_2 e_3) + \lambda_2(e_3 e_1) + \lambda_3(e_1 e_2) \;|\; \lambda_i \in \mathbb{R} \,\} \cong \mathbb{H} \tag{7.1}$$

Since $e_1 e_2$ is no longer the blade of highest grade in $\mathbb{G}(\mathbb{R}^3)$, we must check it satisfies being a square root of (-1). We check in general if $i \neq j$:

$$(e_i e_j)^2 = (e_i e_j)(e_i e_j) = e_i(e_j e_i)e_j = e_i(-e_i e_j)e_j = -(e_i e_i)(e_j e_j) = -e_i^2 e_j^2 = -1$$

We then check the commutation relationship:

$$(e_2 e_3)(e_3 e_1) = e_2(e_3 e_3)e_1 = e_2 e_3^2 e_1 = e_2 e_1 = -(e_1 e_2)$$

Where we get a negative$(-)$ sign, and hence we realize the proper identification with quaternions is: $(e_2 e_3) \mapsto -i$, $(e_3 e_1) \mapsto -j$, $(e_1 e_2) \mapsto -k$ which means the quaternions form a left-handed oriented basis.

8. Reflections

We begin by looking at the orthogonal decomposition of vectors. Take vectors $a, b \in \mathbb{G}(\mathbb{R}^n)$:

$$a = a1 = a(bb^{-1}) = (ab)b^{-1} = (a \cdot b + a \wedge b)b^{-1} = (a \cdot b)b^{-1} + (a \wedge b)b^{-1} = a_{\|b} + a_{\perp b}$$

Where $a_{\|b}$ is the projection of a onto b and $a_{\perp b}$ the rejection of a from b. This can be generalized for higher grade objects, for a multivector $C \in \mathbb{G}(\mathbb{R}^n)$:

$$a = (a\rfloor C)\, C^{-1} + (a \wedge C)\, C^{-1} = a_{\|C} + a_{\perp C} \tag{8.1}$$

We take vectors $u, w \in \mathbb{G}(\mathbb{R}^n)$ where u is a unit vector, so $u^{-1} = u$. Then we can decompose $w = w_{\|u} + w_{\perp u}$ We then look at conjugation by u:

$$u^{-1}wu = uwu = (uw)u = (u \cdot w + u \wedge w) = (w \cdot u - w \wedge u)u^{-1}$$

$$(w \cdot u)u^{-1} - (w \wedge u)u^{-1} = w_{\parallel u} + w_{\perp u}$$

Where the projection along u is preserved but the rejection is reversed in sign, which corresponds to reflection of w along u.

9. Rotations

We look at $\mathbb{G}(\mathbb{R}^3)$, where given any 2 orthogonal unit vectors e_1, e_2, they span a plane $e_1e_2 = I$ which is the pseudo-scalar in the subspace they span. Then given any vector $a \in \mathbb{G}(\mathbb{R}^3)$ we can decompose by: $a = a_{\parallel I} + a_{\perp I}$. We rotate a to a' by an angle θ counterclockwise along I. We can decompose again: $a' = a'_{\parallel I} + a'_{\perp I}$.

$$a'_{\parallel I} = e^{I\theta} a_{\parallel I} \quad , \quad a'_{\perp} = a_{\perp}$$

Since $a'_{\parallel I}$ is in I it gets rotated, but $a'_{\perp I}$ is preserved since it's orthogonal.

$$a' = e^{I\theta} a_{\parallel I} + a_{\perp} = e^{I\theta/2} e^{I\theta/2} a_{\parallel I} + e^{I\theta/2} e^{-I\theta/2} a_{\perp}$$

Where we use the fact that I anti-commutes with every vectors in I, but commutes with those orthogonal to I.

$$a' = e^{I\theta/2} a_{\parallel I} e^{-I\theta/2} + e^{I\theta/2} a_{\perp} e^{-I\theta/2} = e^{I\theta/2}(a_{\parallel I} + a_{\perp I})e^{-I\theta/2} = e^{I\theta/2} a e^{-I\theta/2}$$

Where we retrieve the quaternions conjugation formula for rotations. Alternatively, we can construct rotations in a more general way, by using reflections.

Lemma 9.1. *Any rotation can be decomposed into two consecutive reflections.*

Take vectors $u_1, u_2, w \in \mathbb{G}(\mathbb{R}^n)$ with u_1, u_2 unit vectors so $u_1^{-1} = u_1, u_2^{-1} = u_2$. We first reflect w along u_1 by conjugating by it, followed by reflection along u_2, conjugating again.

$$w \mapsto u_1 w u_1 \mapsto u_2(u_1 w u_1)u_2 = (u_2u_1)w(u_1u_2) = RwR^*$$

So we get that conjugation by the rotor $R = u_1u_2$ gives rotations. So rotations in higher dimensions can be generalized by conjugation.

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