ISOMETRIES OF THE EUCLIDEAN PLANE $\mathbb{R}^2$

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ABSTRACT. We study and classify the different isometries of the Euclidean plane by looking at the number of points they fix and if they preserve orientation. We take a look at the conservation of vector lengths and what other properties arise as a consequence of this invariance. Focusing on linear isometries provides an insight to the relation equating vector length preservation with orthogonal matrices.

1. ISOMETRY

The word isometry derives from the Greek prefix *iso-* meaning "equal" and *metria* meaning "a measuring". This gives the definition of isometry "equality of measure". In mathematics an isometry is any transformation which preserves distance between points in Euclidean n-space. For the distance we refer to the Euclidean metric, which is derived from the dot product.

Definition 1.1. The distance between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Euclidean Plane is given by:

$$|P_1, P_2| = \sqrt{(P_1 - P_2) \cdot (P_1 - P_2)} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Definition 1.2. A transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of the Euclidean Plane iff $\forall P_1, P_2 \in \mathbb{R}^2$:

$$|P_1, P_2| = |f(P_1), f(P_2)|$$

This statement is true because isometries preserve the dot product operation between the domain and codomain. That is: $\forall \vec{u}, \vec{v} \in \mathbb{R}^2$

$$\vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle = \langle f(\vec{u}), f(\vec{v}) \rangle = f(\vec{u}) \cdot f(\vec{v})$$

This leads to the preservation of magnitude of all vectors belonging to the Euclidean plane. The operation to define vector magnitudes in a vector space is called the vector norm. $\forall \vec{u} \in \mathbb{R}^2$

$$||\vec{v}]] = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\langle f(\vec{v}), f(\vec{v}) \rangle} = ||f(\vec{v})||$$

Another consequence of magnitude conservation is the preservation of angle measurements up to orientation. Isometries are also referred to as Rigid Body transformations. This name is given because objects undergoing these transformations preserve their shape and size. We will look at isometries of $\mathbb{R}^2$, which are split into two categories.

Proper transformations preserve the orientation, including:

- Translation: by a vector $\vec{v}$
- Rotation: by an angle θ around a point P
- Identity Map: maps every point to itself

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Improper transformations revert the orientation, these include:

- Reflection: across a line L
- Glide Reflection: across a line L followed by translation parallel to L

Remark 1.3. Dilation transformations are not isometries, because distance between pair of points is not preserved, except for the trivial case of scaling by a factor of one.

2. Linear Isometries

We now want to focus mainly on the isometries which also have the special property of being linear transformations.

Definition 2.1. A map $L : V \to W$ is linear if it has the properties:

(1) Additive: preserve operation of addition
$\forall u, v \in V \quad L(u + v) = L(u) + L(v)$
(2) Homogeneous: preserve operation of scalar multiplication
$\forall v \in V,\ \forall c \in \mathbb{R} \quad L(cv) = cL(v)$

Using the first property we can prove all linear functions fix the origin.

Theorem 2.2. *If L is a linear map, then it maps the zero vector in the domain to the zero vector in the codomain.*

$$f : \mathbb{R}^n \to \mathbb{R}^m \textit{ is linear} \Rightarrow \ L(\vec{0}_n) = \vec{0}_m$$

Proof. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. So,

$$\begin{aligned} \forall \vec{v} \in \mathbb{R}^n \quad \vec{v} - \vec{v} &= \vec{0}_n \\ L(\vec{v} - \vec{v}) &= L(\vec{0}_n) \\ L(\vec{v}) - L(\vec{v}) &= L(\vec{0}_n) && \text{(Additive property)} \\ \vec{0}_m &= L(\vec{0}_n) \end{aligned}$$

□

Using the contrapositive of Theorem 3.2 we can easily show which isometries are not linear transformations.For those which the theorem holds true, we still have to prove Theorem 3.1 holds true. Linear maps have an interesting property that they can be represented by a unique matrix.

Theorem 2.3. *A map $L : \mathbb{R}^n \to \mathbb{R}^m$ between finite vector spaces is a linear map iff there exists a unique $m \times n$ representative matrix made of column vectors which are the image of the standard basis vectors under L.*
$L : \mathbb{R}^n \to \mathbb{R}^m$ *linear* $\iff \exists! A \in \mathbb{M}_{m\times n}(\mathbb{R})$ *st* : $\forall \vec{x} \in \mathbb{R}^n \quad L(\vec{x}) = A\vec{x}$

More specifically, linear isometries are represented by orthogonal matrices.

Definition 2.4. An $m \times n$ matrix Q is orthonormal iff the product of the transpose with the original matrix is the $n \times n$ identity matrix.
$Q \in \mathbb{M}_{m\times n}(\mathbb{R})$ is orthonormal $\iff Q^T Q = I_n$

Definition 2.5. An matrix Q is orthogonal if it's orthonormal and squared.

$$Q^T Q = I_n \quad \text{and} \quad Q \in \mathbb{M}_{n\times n}(\mathbb{R})$$

Definition 2.6. Multiplying a $1 \times n$ matrix x^T with any $n \times 1$ column vector y is equivalent to the dot product operation between column vectors x and y.

$$\forall x, y \in \mathbb{R}^n \quad x^T y = x \cdot y$$

Theorem 2.7. *Every linear isometry $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ has a unique orthogonal 2×2 matrix Q which represents it.*
$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$ *linear isometry* $\iff \exists! Q \in \mathbb{M}_{2\times 2}(\mathbb{R})$ *st* : $\forall \vec{p} \in \mathbb{R}^2 \quad \mathcal{L}(\vec{p}) = Q\vec{p}$

Proof. We want to show the equivalence between transformations preserving vector lengths and representation by orthogonal matrices. Consider an arbitrary vector $v \in \mathbb{R}^n$ and linear isometry $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$ with representative matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$.

(1) Vector length preservation implies orthogonal matrix representation.

$$\begin{aligned}
||v|| &= ||Av|| && \text{(Vector Length Preservation)}\\
||v||^2 &= ||Av||^2\\
v \cdot v &= Av \cdot Av\\
v^T v &= (Av)^T Av\\
v^T v &= (v^T A^T) Av\\
v^T v &= v^T (A^T A) v
\end{aligned}$$

This is true for all vectors $v \in \mathbb{R}^n$ and matrices are unique, so the product of the transpose and the original matrix must equal the identity matrix.

$$||v|| = ||Av|| \ \Rightarrow \ A^T A = I_n$$

Therefore, the matrix A is orthogonal.

(2) Orthogonal matrices preserve vector lengths.

$$\begin{aligned}
||Av||^2 &= Av \cdot Av\\
&= (Av)^T Av\\
&= (v^T A^T) Av\\
&= v^T (A^T A) v\\
&= v^T I_n v && \text{(Orthogonal Definition)}\\
&= v^T v\\
&= v \cdot v\\
||Av||^2 &= ||v||^2
\end{aligned}$$

This is true for all vectors $v \in \mathbb{R}^n$

$$A^T A = I_n \ \Rightarrow \ ||v|| = ||Av||$$

So vector length is preserved by Matrix A.

(3) Combining proofs (1) and (2):

$$A^T A = I_n \iff ||v|| = ||Av||$$

Therefore length preservation and orthogonal matrix representation are equivalent properties for linear functions.

□

3. Properties of Linear Isometries

We study some of the following properties of linear isometries by looking at properties of orthogonal matrices:

(1) The columns and rows of an orthogonal matrix form an orthonormal set. Which means they are unit vectors orthogonal to each other. These vectors are linearly independent and span $\mathbb{R}^n$, so they are a basis of it.

Example 3.1. The Standard Basis$\{e_1, ..., e_n\}$ forms the Identity Matrix(I_n)

The equivalence between the image of the column vectors and the image under the linear transformation tells us the image of all linear isometries $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{R}^n$. Therefore, all linear isometries are surjective.

(2) All orthogonal matrices are invertible. The inverse of any orthogonal matrix is the transpose of the matrix.

Proof. Consider the orthogonal Matrix $Q \in \mathbb{M}_{n\times n}(\mathbb{R})$

$$\begin{aligned} Q^T Q &= I_n && \text{(Orthogonal Definition)} \\ QQ^T &= I_n \\ &\Rightarrow Q^T = Q^{-1} \end{aligned}$$

□

This tells us all linear isometries have a corresponding inverse functions, which is easy to find.

(3) The product of orthogonal matrices is orthogonal.

Proof. Consider the orthogonal Matrices $A, B \in \mathbb{M}_{n\times n}(\mathbb{R})$:

$$\begin{aligned} (AB)^T(AB) &= (B^T A^T)AB \\ &= B^T(A^T A)B \\ &= B^T I_n B && \text{(Orthogonal Definition)} \\ &= B^T B \\ &= I_n \end{aligned}$$

□

The equivalence between matrix multiplication and function composition tells us any composition of linear isometries is a linear isometry too.

Lemma 3.2. *The determinant of the transpose of a square matrix is equal to the determinant of the matrix.* $\forall A \in \mathbb{M}_{n\times n}(\mathbb{R}) \quad det(A^T) = det(A)$

(4) The determinant of an orthogonal matrix is equal to -1 or 1.

Proof. Consider the Orthogonal Matrix $Q \in \mathbb{M}_{n\times n}(\mathbb{R})$

$$\begin{aligned}
Q^TQ &= I_n \\
det(Q^TQ) &= det(I_n) \\
det(Q^T)\cdot det(Q) &= 1 \\
det(Q)\cdot det(Q) &= 1 && \text{(Lemma 3.2)} \\
(det(Q))^2 &= 1 \\
det(Q) &= \pm 1
\end{aligned}$$ ☐

Linear isometries with determinants +1 preserve orientation. The remaining isometries with determinant -1 revert orientation.

(5) All eigenvalues of orthogonal matrices are either 1 or -1.

Proof. Consider the Orthogonal Matrix $Q \in \mathbb{M}_{n\times n}(\mathbb{R})$ and eigenvector $\vec{v}$ with eigenvalue λ.

$$\begin{aligned}
Q\vec{v} &= \lambda\vec{v} \\
||Q\vec{v}|| &= ||\lambda\vec{v}|| \\
||\vec{v}|| &= ||\lambda\vec{v}|| && \text{(Length preservation)} \\
&\Rightarrow\ ||\lambda|| = 1 \\
&\Rightarrow\ \lambda = \pm 1
\end{aligned}$$ ☐

This is a consequence of the preservation of vector lengths. This reaffirms that linear isometries can revert orientation, but not scale a vector's magnitude.

There are some other interesting properties obtained by the preservation of length definition:

Theorem 3.3. *Every isometry is an injective map.*

Proof. Consider an isometry $f : \mathbb{R}^n \to \mathbb{R}^n, \quad \forall P_1, P_2 \in \mathbb{R}^n$

$$\begin{aligned}
Assume : f(P_1) = f(P_2)\ \Rightarrow\ |f(P_1), f(P_2)| &= 0 \\
|P_1, P_2| &= 0 \qquad \text{(Isometry Definition)} \\
\Rightarrow\ P_1 &= P_2
\end{aligned}$$ ☐

Corollary 3.4. *Every linear isometry is a bijective map. Therefore, they also have inverses.*

Proof. By property(1) of orthogonal matrices we get linear isometries are surjective maps. From Theorem 3.3 we prove they are injective maps. Therefore, linear isometries are one-to-one correspondences. ☐

Corollary 3.5. *Every isometry has a trivial kernel space.*

Proof. Consider an isometry $f : \mathbb{R}^n \to \mathbb{R}^n, \quad \vec{v} \in \mathbb{R}^2$

$$\begin{aligned} Assume : f(\vec{v}) &= \vec{0}_n \\ \langle f(\vec{v}), f(\vec{v}) \rangle &= 0 \\ \langle \vec{v}, \vec{v} \rangle &= 0 && \text{(Isometry Definition)} \\ &\Rightarrow \vec{v} = \vec{0}_n \end{aligned}$$

□

Therefore all linear isometries map only the zero vector to the zero vector. Every other vector in the plane therefore has an image under any linear isometry.

We know address our attention to study the specific types of isometries of the plane.

4. Translation

Translations fix no points in the plane. Translations keep the orientation.

Definition 4.1. A translation of the plane is any transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which moves a point $p = (x, y)$ by a vector $\vec{v} = (a, b)$.

$$T(p) = p + \vec{v} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

Theorem 4.2. *Translation is an isometry of the plane.*

Proof. Consider a translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by a vector $\vec{v}$. $\forall p, q \in \mathbb{R}^2$

$$\begin{aligned} |T(p), T(q)| &= (p + \vec{v}) - (q + \vec{v}) \\ &= (p - q) \\ &= |p, q| \end{aligned}$$

□

Theorem 4.3. *Translation is not a linear transformation.*

Proof. Consider a translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by a vector $\vec{v}$. We consider the non-trivial case, where $\vec{v} \neq \vec{0}$.

$$\begin{aligned} T(\vec{p}) &= \vec{p} + \vec{v} \\ T(\vec{0}) &= \vec{0} + \vec{v} \\ &= \vec{v} \\ T(\vec{0}) &\neq \vec{0} \end{aligned}$$

□

Translation doesn't fix any points, therefore it can't be a linear transformation. However it can be useful to translate the origin when you want to reflect around an arbitrary point which is not the origin. Also, when you want to reflect along a line which doesn't go through the origin you can translate the point to be the origin.

Remark 4.4. Trivial translation by the vector $\vec{v} = \vec{0}$ is the identity map.

5. Rotation

Rotations fix only the rotocenter, which is the point you rotate around. We only focus on rotations about the origin, since those transformations fix the origin and are linear. Rotations maintain the orientation.

Definition 5.1. The matrix ρ_θ is associated to counterclockwise rotation around the origin by an angle θ from the positive x-axis.

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Theorem 5.2. *Rotation is an isometry of the plane.*

Proof. Consider an arbitrary vector $\vec{v} = (x, y) \in \mathbb{R}^2$

$$\rho_\theta \vec{v} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

$$\begin{aligned} ||\rho_\theta \vec{v}||^2 &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\ &= (x\cos\theta)^2 - 2xy\cos\theta\sin\theta + (y\sin\theta)^2 + (x\sin\theta)^2 + 2xy\cos\theta\sin\theta + (y\cos\theta)^2 \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\ &= x^2 + y^2 \\ &= ||\vec{v}||^2 \end{aligned}$$

□

Theorem 5.3. *All rotations in the plane have determinant* $+1$

Proof.

$$\begin{aligned} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} &= (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= +1 \end{aligned}$$

□

Excluding the identity matrix, all orthogonal 2×2 matrices with determinant $+1$ represent a rotation transformation in the plane.

Remark 5.4. Trivial rotation by any angle $\theta = n2\pi$ where $n \in \mathbb{Z}$ is the identity map.

6. Identity Transformation

The identity map fixes all points in the plane. It keeps orientation.

Definition 6.1. The identity matrix I_2 is associated to the identity map.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Corollary 6.2. *By definition the determinant of the identity matrix is +1.*

Corollary 6.3. *The identity transformation is an isometry of the plane.*

$$||I\vec{v}|| = ||\vec{v}||$$

Definition 6.4. A symmetric matrix is a square matrix which is equal to it's transpose.

$$A \in \mathbb{M}_{n\times n}(\mathbb{R}) \text{ and } A^T = A$$

Theorem 6.5. *If a matrix is orthogonal and symmetric, then the matrix, the inverse and the transpose are all equal.*

$$A = A^T = A^{-1}$$

Corollary 6.6. *An orthogonal symmetric matrix is it's own inverse.*

$$A \in \mathbb{M}_{n\times n}(\mathbb{R}) \quad AA = I_n$$

Corollary 6.7. *The identity matrix is an orthogonal symmetric matrix.* $I^T = I$

Remark 6.8. The identity map is equivalent to a trivial rotation or translation, because they all preserve orientation.

7. Reflection

Reflections fix all points along the line of reflection. We only focus on reflections along lines going through the origin, since those transformations fix the origin and are linear. Reflections revert orientation.

Definition 7.1. The matrix R_θ is associated to reflection along the line L passing through the origin, forming an angle θ with the positive x-axis.

$$R_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Theorem 7.2. *Rotation is an isometry of the plane.*

Proof. Consider an arbitrary vector $\vec{v} = (x, y) \in \mathbb{R}^2$

$$R_\theta \vec{v} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos 2\theta + y\sin 2\theta \\ x\sin 2\theta - y\cos 2\theta \end{pmatrix}$$

$$\begin{aligned} ||R_\theta \vec{v}||^2 &= (x\cos 2\theta + y\sin 2\theta)^2 + (x\sin 2\theta - y\cos 2\theta)^2 \\ &= (x\cos 2\theta)^2 + 2xy\cos 2\theta \sin 2\theta + (y\sin 2\theta)^2 + (x\sin 2\theta)^2 - 2xy\cos 2\theta\sin 2\theta + (y\cos 2\theta)^2 \\ &= x^2(\cos^2 2\theta + \sin^2 2\theta) + y^2(\cos^2 2\theta + \sin^2 2\theta) \\ &= x^2 + y^2 \\ &= ||\vec{v}||^2 \end{aligned}$$

□

Theorem 7.3. *All reflections in the plane have determinant* -1

Proof.

$$\begin{aligned} \begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{vmatrix} &= (\cos 2\theta)(-\cos 2\theta) - (\sin 2\theta)(\sin 2\theta) \\ &= -\cos^2 2\theta - \sin^2 2\theta \\ &= -1 \end{aligned}$$

□

All orthogonal 2×2 matrices with determinant -1 represent a reflection transformation in the plane.

Theorem 7.4. *The reflection matrix is an orthogonal symmetric matrix.* $R_\theta^T = R_\theta$

Applying a reflection along the same line twice is equivalent to the identity transformation.

Theorem 7.5. *We can reflect along any line going through the origin at an angle θ by rotating the plane to be along the x-axis, reflecting along the x-axis and then undoing the rotation.*

Proof.

$$\begin{aligned}
R_\theta &= \rho_\theta R_0 \rho_{-\theta} \\
&= \rho_\theta R_0 \rho_\theta^T && \text{(Orthogonal Matrix Property)} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos 2(0) & \sin 2(0) \\ \sin 2(0) & -\cos 2(0) \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} (\cos\theta)(\cos\theta) + (-\sin\theta)(\sin\theta) & (\cos\theta)(\sin\theta) + (-\sin\theta)(-\cos\theta) \\ (\sin\theta)(\cos\theta) + (\cos\theta)(\sin\theta) & (\sin\theta)(\sin\theta) + (\cos\theta)(-\cos\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}
\end{aligned}$$

□

Theorem 7.6. *All rotations are compositions of 2 reflections. We reflect along line through origin at θ and along line through origin at 2θ in any order.*

Example 7.7. Rotation by π counterclockwise is equivalent to reflection along the x-axis composed with reflection along the y-axis.

$$\begin{aligned}
\rho_\pi &= R_0 R_{\pi/2} \\
\begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} &= \begin{pmatrix} \cos 2(0) & \sin 2(0) \\ \sin 2(0) & -\cos 2(0) \end{pmatrix} \begin{pmatrix} \cos 2(\pi/2) & \sin 2(\pi/2) \\ \sin 2(\pi/2) & -\cos 2(\pi/2) \end{pmatrix} \\
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

Theorem 7.8. *A composition of n number of reflections is equivalent to one rotation or reflection. If n is even we get a rotation. If n is odd we get a reflection.*

8. Glide Reflections

Glide reflections fix no points in the plane. They revert orientation.

Definition 8.1. A glide reflection is a composition of a translation with a reflection transformation.

Lemma 8.2. *Any composition of isometries is an isometry.*

Corollary 8.3. *By Lemma 8.2 and Definition 8.1 we get Glide reflection is an isometry of the plane.*

Corollary 8.4. *Glide reflection is not a linear isometry, because it's a composition with a translation, which is not a linear map.*

9. Applications

9.1. **Congruent Shapes.** Isometries of the plane give rise to congruent shapes.

Definition 9.1. Two shapes are said to be congruent iff there exists an isometry between them.

This is very important for Geometry, since we can relate different objects and their properties. We can find symmetries of polygons and apply it to find lengths of lines and angles between them.

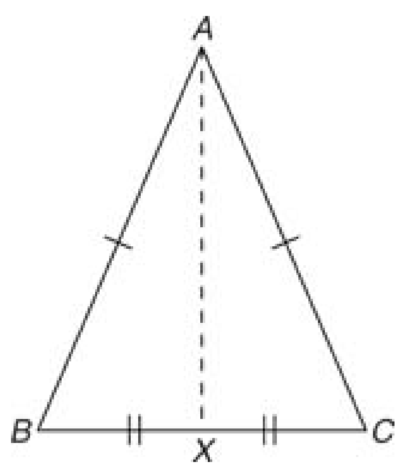

Figure 1. Diagram of an isosceles triangle.

Example 9.2. From Figure(1) we can see there is a line of symmetry which is the perpendicular bisector of the line BC and going through A. The existence of the isometry tells us the small triangle on the left side is congruent to the small triangle on the right side. From this we learn that the angle ABC and ACB are equal in magnitude. However we see the orientation of the second triangle is reversed, so the angles are in reversed orientations. Also the pairs of lines AC,AB and BX,CX have the same length.

9.2. **Complex Numbers.**

Definition 9.3. A complex number $z = a + bi$ can be denoted by a 2×1 matrix:

$$z = a + bi \equiv \begin{pmatrix} a \\ b \end{pmatrix}$$

By relating the Euclidean plane $\mathbb{R}^2$ to the complex plane $\mathbb{C}$ we can use this representation to define operations on complex numbers as linear isometries.

Example 9.4. We can define complex conjugation as a reflection along the real line($\theta = 0$). That is $\forall z = a + bi \in \mathbb{C}$:

$$\begin{aligned} \bar{z} &\equiv \rho_0(z) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a \\ -b \end{pmatrix} \\ &= a - bi \end{aligned}$$

Example 9.5. We can define multiplication of unit complex numbers as conjugation of both rotations along the origin by the arguments.
Consider $z_1 = e^{i\theta}, z_2 = e^{i\varphi} \in \mathbb{C}$:

$$\begin{aligned} z_1 \cdot z_2 &= R_{arg(z_2)} R_{arg(z_1)} \\ &= R_\varphi R_\theta \\ &= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} (\cos\varphi)(\cos\theta) + (-\sin\varphi)(\sin\theta) & (\cos\varphi)(-\sin\theta) + (-\sin\theta)(\cos\theta) \\ (\sin\varphi)(\cos\theta) + (\cos\varphi)(\sin\theta) & (\sin\varphi)(-\sin\theta) + (\cos\varphi)(\cos\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{pmatrix} \\ &= e^{i(\varphi + \theta)} \end{aligned}$$

10. Extension to $\mathbb{R}^n$

The isometries of $\mathbb{R}^2$ form a group $O(2)$ which is closed under composition and inverses. This means they have the following properties:

(1) There exists an identity map I
 $\exists I \in O(2)\ st\ \forall F \in O(2)\ F \circ I = I \circ F = F$
(2) Any composition of isometries is an isometry
 $\forall F, G \in O(2) \ \Rightarrow\ F \circ G, G \circ F \in O(2)$
(3) Any inverse of an isometry is an isometry
 $\forall F \in O(2) \ \Rightarrow\ \exists F^{-1} \in O(2)\ st:$
 (a) $F \circ F^{-1} = I$
 (b) $F^{-1} \circ F = I$

Remark 10.1. The composition of isometries has the associative property but not the commutative property. This means: $\forall F, G, H \in O(2)$

(4) $(F \circ G) \circ H = F \circ (G \circ H)$
(5) $F \circ G \neq G \circ F$

The group O(2) is just a subgroup of O(N) which contains all the isometries of $\mathbb{R}^n$. As we go into larger dimensions like $\mathbb{R}^3$ we start getting more possible combinations of the basic isometries to produce more isometry categories with properties to study. Beyond this paper there is much more to explore about the preservation of the inner product operation between vector spaces and vector norms. Isometries of the plane are useful tools for geometry and present an interesting connection between linear isometries and orthogonal matrices.

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