QUATERNIONS AND ROTATIONS

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ABSTRACT. We introduce the skew-field of quaternions $\mathbb{H}$ to study rotations in euclidean space. We generalize properties from complex numbers $\mathbb{C}$ to properties for quaternions $\mathbb{H}$. We study some of it's substructures, which give rise to group actions analogue to those of complex numbers $\mathbb{C}$ on euclidean 2-space $\mathbb{R}^2$, but generalized to 3 and 4 dimensions.

1. Introduction

During the 19^{th} Century, the Irish mathematician Hamilton was trying to generalize the construction of algebraic structures like the complex numbers $\mathbb{C}$ to higher dimensions. After much pondering, he realized the constructions would not work with only 2 imaginary directions i and j and hence added the third one k. The birth of the quaternion group gave rise to many applications in physics, mathematics and computer science. In our study of continuous symmetries, the quaternions are a useful tool for simplifying and studying rotations in $\mathbb{R}^3$. In order to understand this, we must first look at how complex numbers give us rotations in $\mathbb{R}^2$.

2. Complex Numbers

The complex numbers are the field constructed as the algebraic closure of the real numbers $\mathbb{R}$ by appending $i = \sqrt{-1}$.

Definition 2.1. Complex numbers form a vector space isomorphic to real 2-space.

$$\mathbb{C} = \left\{ \, a + ib \, \middle| \; a, b \; \in \mathbb{R} \, , \, i^2 = -1 \, \right\} \cong \mathbb{R}^2 \tag{2.2}$$

With the operation of complex conjugation negating only the imaginary part:

$$z = a + ib \, \mapsto \, z^* = a - ib. \tag{2.3}$$

Given $z = a + ib \in \mathbb{C}$ we can express its real and imaginary parts by:

$$Re[z] = a = \frac{z + z^*}{2} \quad , \quad Im[z] = b = \frac{z - z^*}{2i} \tag{2.4}$$

Conjugation also lets us define the magnitude of a complex number as:

$$|z| = |z^*| = \sqrt{zz^*} = \sqrt{a^2 + b^2} \tag{2.5}$$

where given $z, w \in \mathbb{C}$ they satisfy: $|zw| = |z||w|$

We construct multiplicative inverses by:

$$z^{-1} = \frac{z^*}{|z|^2} \tag{2.6}$$

Date: November 25, 2017.

Another way to visualize complex numbers is in polar form, as a magnitude and angle using Euler's identity:

$$z = a + bi = |z|(\cos\phi + i\sin\phi) = |z|e^{i\phi} \quad , \quad \tan\phi = b/a \tag{2.7}$$

Definition 2.8. The unit n-sphere in an (n+1)-dimensional normed vector space V is the subset of vectors v with unit norm.

$$S^n = \{\ v \in V\ \mid\ |v| = 1\ \} \tag{2.9}$$

Lemma 2.10. *A subset S of a group G with an operation $*$ is a subgroup if:*

$$s_1, s_2 \in S \rightarrow (s_1 * {s_2}^{-1}) \in S \tag{2.11}$$

Theorem 2.12. *The unit circle in $\mathbb{C}$ is a group under multiplication.*

$$S^1 = \{\ z \in \mathbb{C}\ \mid\ |z| = 1\ \} = \left\{\ e^{i\phi} \in \mathbb{C}\ \right\} \tag{2.13}$$

Proof. Take $z_1 = e^{i\phi_1}, z_2 = e^{i\phi_2} \in S^1$.
$|z_1 {z_2}^{-1}| = |e^{i\phi_1} \cdot e^{-i\phi_2}| = |e^{i(\phi_1 - \phi_2)}| = 1$
$\rightarrow z_1 {z_2}^{-1} \in S^1$ a subset of $\mathbb{C}$, so by **Lemma 2.10** it is a subgroup. □

Definition 2.14. We define the left action of the unit circle on the complex plane:

$$S^1 \curvearrowright \mathbb{C} : (e^{i\phi}, z) \mapsto e^{i\phi} z \tag{2.15}$$

which rotates z by an angle ϕ counterclockwise in the complex plane.

Theorem 2.16. *The rotations of n-dimensional Euclidean space $SO(n)$ form a group under multiplication.*

$$SO(n) = \left\{\ A \in \mathbb{M}_{nxn}(\mathbb{R})\ \middle|\ A^t = A^{-1}\, , \, det(A) = +1\ \right\} \tag{2.17}$$

Proof. Take $A, B \in SO(n)$.
$(i)\ (AB^{-1})^t = (B^{-1})^t A^t = (B^t)^{-1} A^{-1} = (AB^t)^{-t} = (AB^{-1})^{-1}$
$(ii)\ det(AB^{-1}) = det(A) det(B^{-1}) = det(B)^{-1} = \frac{1}{det(B)} = 1$
$\rightarrow AB^{-1} \in SO(n)$ a subset of $\mathbb{M}_{nxn}(\mathbb{R})$ and hence by **Lemma 2.10** it is a subgroup. □

Theorem 2.18. *We then establish the group isomorphism $\rho : S^1 \rightarrow SO(2)$:*

$$\rho(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{2.19}$$

so $S^1 \cong SO(2)$ and we get all rotations of $\mathbb{R}^2$ by left multiplication by unit complex numbers as desired.

3. Quaternions

We introduce the quaternion group $\mathbb{Q}_8$ and the quaternion vector space $\mathbb{H}$.

Definition 3.1. The quaternion group is defined as:

$$\mathbb{Q}_8 = \left\{\ \pm 1\ ,\ \pm i\ ,\ \pm j\ ,\ \pm k\ \middle|\ i^2 = j^2 = k^2 = ijk = -1\ \right\} \tag{3.2}$$

Definition 3.3. We can also construct the quaternion vector space isomorphic to real 4-space, with the quaternions $1, i, j, k$ as vectors over $\mathbb{R}$:

$$\mathbb{H} = \left\{\ a1 + bi + cj + dk\ \middle|\ a, b, c, d \in \mathbb{R}\, ,\, i^2 = j^2 = k^2 = ijk = -1\ \right\} \cong \mathbb{R}^4 \tag{3.4}$$

Consider the identification $i \mapsto e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $j \mapsto e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $k \mapsto e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ between the quaternions and the natural basis of $\mathbb{R}^3$. Then we can think of the quaternions as being composed of a real scalar part and a real 3-vector part. These correspond to the real and imaginary parts respectively.

$$\mathbb{H} \cong \mathbb{R} \oplus \mathbb{R}^3 = \left\{ p = a + \vec{v} \;\middle|\; a \in \mathbb{R}\,,\, \vec{v} \in \mathbb{R}^3 \right\} \tag{3.5}$$

Then we can define the operations in our vector space by taking arbitrary elements $p = a + \vec{v}$, $q = b + \vec{w} \in \mathbb{H}$ and a scalar $\lambda \in \mathbb{R}$:

(1) Addition: $p + q = (a + \vec{v}) + (b + \vec{w}) = (a + b) + (\vec{v} + \vec{w})$
(2) Scaling: $\lambda p = \lambda(a + \vec{v}) = \lambda a + \lambda\vec{v}$

We can put further structure on our vector space to turn it into a ring by defining the product of two quaternions:

(3) Product: $pq = (a + \vec{v})(b + \vec{w}) = ab + a\vec{w} + b\vec{v} + \vec{v}\vec{w}$

Where the product of the 2 vectors $\vec{v}$ and $\vec{w}$ can be expressed in terms of the dot product and cross-product:

$$\vec{v}\vec{w} = -\vec{v} \cdot \vec{w} + \vec{v} \times \vec{w} \tag{3.6}$$

If we write each vector in their components $\vec{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, $\vec{w} = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$:

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{3} v_i w_i \tag{3.7}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \sum_{i=1}^{3} \varepsilon_{ijk} e_i v_j w_k \tag{3.8}$$

Where the Levi-Civita symbol encodes the anti-symmetrization property of the cross-product: $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$

$$\varepsilon_{ijk} = \begin{cases} +1 & (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

Rearranging the scalar and vector parts together we can rewrite the quaternion product as:

$$pq = (a + \vec{v})(b + \vec{w}) = (ab - \vec{v} \cdot \vec{w}) + (a\vec{w} + b\vec{v} + \vec{v} \times \vec{w}) \tag{3.10}$$

Remark 3.11. Since scalar multiplication, the dot product and the scaling of vectors by real numbers are commutative operations but the cross product is anti-commutative, we note the quaternion product is not commutative.

$$\begin{aligned} qp &= (b + \vec{w})(a + \vec{v}) \\ &= (ba - \vec{w} \cdot \vec{v}) + (b\vec{v} + a\vec{w} + \vec{w} \times \vec{v}) \\ &= (ab - \vec{v} \cdot \vec{w}) + (a\vec{w} + b\vec{v} - \vec{v} \times \vec{w}) \neq pq \end{aligned} \tag{3.12}$$

Also noteworthy to mention, under this product operation the quaternions vector space forms an associative Algebra over the reals, called a Clifford Algebra.

4. Generalizations from Complex Numbers

In this section we want to generalize properties from complex numbers to quaternions. Just as we had complex conjugation in $\mathbb{C}$, we can generalize this map to $\mathbb{H}$:

Definition 4.1. Complex Conjugation acts on the basis vectors by:

$$1 \mapsto 1 \quad , \quad i \mapsto -i \quad , \quad j \mapsto -j \quad , \quad k \mapsto -k$$

Since the map is linear, it preserves the real part of a quaternions and flips the imaginary part. Given a quaternion $p \in \mathbb{H}$:

$$p = a + \vec{v} \mapsto p^* = a - \vec{v} \tag{4.2}$$

Given $p = a + \vec{v} \in \mathbb{H}$ we can express its real and imaginary parts by:

$$Re[p] = a = \frac{p + p^*}{2} \quad , \quad Im[p] = \vec{v} = \frac{p - p^*}{2} \tag{4.3}$$

Remark 4.4. When talking about Complex numbers we could just refer to the real coefficient of i as the imaginary part. However for quaternions, which have two extra imaginary dimensions, we can't refer to the imaginary part as simply a scalar. Instead the imaginary part is linear combination of i, j, k , which we can represent as a 3-vector. Then there is no need to have the division by i or j or k in the formula.

We look at the product of a quaternion $p \in \mathbb{H}$ and its complex conjugate p^*:

$$\begin{aligned} pp^* &= (a + \vec{v})(a - \vec{v}) \\ &= (aa - \vec{v} \cdot (-\vec{v})) + (a\vec{v} + a(-\vec{v}) + \vec{v} \times (-\vec{v})) \\ &= (a^2 - (-\vec{v} \cdot \vec{v})) + (a\vec{v} - a\vec{v} - \vec{v} \times \vec{v}) \\ &= a^2 + \vec{v} \cdot \vec{v} + \vec{0} \\ &= |a|^2 + |\vec{v}|^2 \end{aligned}$$

This gives the sum of the squares of the magnitude of the scalar part and the magnitude of the vector part. Seeing this we can define the length of a quaternion using complex conjugation:

Definition 4.5. Given a quaternion $p = a + \vec{v} \in \mathbb{H}$, the *Norm* is given by:

$$|p| = |p^*| = \sqrt{pp^*} = \sqrt{|a|^2 + |\vec{v}|^2} \tag{4.6}$$

Theorem 4.7. *The quaternion norm satisfies the property:*

$$\forall p, q \in \mathbb{H}, \quad |pq| = |p||q| \tag{4.8}$$

Proof. $|pq|^2 = (pq)(pq)^* = (pq)(q^*p^*) = p(qq^*)p^* = p|q|^2p^* = pp^*|q|^2 = |p|^2|q|^2$ □

Theorem 4.9. *For any quaternion $p \in \mathbb{H}$ we construct the multiplicative inverse:*

$$p^{-1} = \frac{p^*}{|p|^2} \tag{4.10}$$

Proof. We first check multiplication on the right:

$$p\left(\frac{p^*}{|p|^2}\right) = pp^*\left(\frac{1}{|p|^2}\right) = |p|^2\left(\frac{1}{|p|^2}\right) = 1$$

And similarly multiplication on the left gives:

$$\left(\frac{p^*}{|p|^2}\right)p = \left(\frac{1}{|p|^2}\right)p^*p = \left(\frac{1}{|p|^2}\right)|p|^2 = 1$$

□

5. Quaternion Substructures

We introduce two important substructures of $\mathbb{H}$, the subgroup of unit quaternions $\mathbb{H}_1$ and the subspace of purely imaginary quaternions $\mathbb{H}_p$.

Theorem 5.1. *The quaternions of unit norm form a group under the quaternion product isomorphic to the unit 3-sphere on $\mathbb{R}^4$.*

(5.2)
$$\mathbb{H}_1 = \{\, u \in \mathbb{H} \mid |u| = 1 \,\} = \left\{\, a + bi + cj + dk \;\middle|\; a^2 + b^2 + c^2 + d^2 = 1 \,\right\} \cong S^3$$

Proof. Take $u_1, u_2 \in \mathbb{H}_1 \to |u_1 u_2^{-1}| = |u_1 u_2^* / |u_2|^2\,| = |u_1||u_2^*| = |u_2| = 1$
$\to u_1 u_2^{-1} \in \mathbb{H}_1$ a subset of $\mathbb{H}$, so by **Lemma 2.10** it is a subgroup. □

For any given $u = u_0 + \vec{u} \in \mathbb{H}_1$, we have the condition: $|u|^2 = |u_0|^2 + |\vec{u}|^2 = 1$, so if we think of $z = u_0 + i|\vec{u}| \in \mathbb{C}$ it must be a point on the unit circle S^1. Then there must exist $\phi \in [0, 2\pi]$ such that $z = e^{i\phi} \to u_0 = \cos\phi$ and $|\vec{u}| = \sin\phi$. Substituting these we have:

$$u = u_0 + \vec{u}\frac{|\vec{u}|}{|\vec{u}|} = \cos\phi + \frac{\vec{u}}{|\vec{u}|}|\vec{u}| = \cos\phi + \hat{u}\sin\phi = e^{\hat{u}\phi} \tag{5.3}$$

Where $\hat{u} = \vec{u}/|\vec{u}|$ is the unit vector in the direction of $\vec{u}$ and $\phi = \arctan\left(|\vec{u}|/u_0\right)$ is the angle between the real and imaginary components of u. So unitary quaternions can be described by an angle ϕ and a axis $\hat{u}$, just as rotations in $\mathbb{R}^3$. If we want a quaternion with different magnitude, we can scale u by any $\lambda \in \mathbb{R}^+$ to get:

$$q = \lambda u \quad , \quad |q| = |\lambda||u| = \lambda \quad , \quad q = |q|e^{\hat{u}\phi} \tag{5.4}$$

So we can generate every quaternion in $\mathbb{H}$ by scaling the elements in $\mathbb{H}_1$. By finding their axis, angle and magnitude we can express quaternions in their polar form, just as we did for complex numbers.

Theorem 5.5. *Pure quaternions form a vector space isomorphic to real 3-space.*

$$\mathbb{H}_p = \{\, p \in \mathbb{H} \mid Re[p] = 0 \,\} = \{\, \vec{v} \in \mathbb{H} \,\} \cong \mathbb{R}^3 \tag{5.6}$$

Proof. (i) $Re[\,0 + \vec{0}\,] = 0 \to 0 \in \mathbb{H}_p$
(ii) $p_1, p_2 \in \mathbb{H}_p \to Re[p_1 + p_2] = Re[p_1] + Re[p_2] = 0 \to p_1 + p_2 \in \mathbb{H}_p$
(iii) $\lambda \in \mathbb{R} \to Re[\lambda p_1] = \lambda Re[p_1] = 0 \to \lambda p_1 \in \mathbb{H}_p$
Since $\mathbb{H}_p$ is a subset of $\mathbb{H}$ containing 0, closed under addition and closed under scaling, it's a subspace. □

The quaternion product reduces to the vector product from **Equation(3.6)** for pure quaternions: $\vec{v}, \vec{w} \in \mathbb{H}_p \to \vec{v}\vec{w} = -\vec{v}\cdot\vec{w} + \vec{v}\times\vec{w}$. Consider the unit quaternion $\hat{u} \in \mathbb{H}_1$, then:

$$\hat{u}^2 = \hat{u}\hat{u} = -\hat{u}\cdot\hat{u} + \hat{u}\times\hat{u} = -|\hat{u}|^2 = -1 \tag{5.7}$$

So we found every unit pure quaternion, including i, j, k, is a square root of (-1).

6. Rotations in Euclidean 3-Space

We begin be reviewing some trigonometric identities:

$$\cos 2\phi = \cos^2\phi - \sin^2\phi \quad , \quad \sin 2\phi = 2\sin\phi\cos\phi \tag{6.1}$$

Definition 6.2. The group of unit quaternions acts by conjugation on the vector space of pure quaternions:

$$\mathbb{H}_1 \curvearrowright \mathbb{H}_p : (e^{\hat{u}\phi}, \vec{v}) \mapsto e^{\hat{u}\phi}\vec{v}e^{-\hat{u}\phi} \tag{6.3}$$

which rotates $\vec{v}$ counterclockwise around the axis defined by $\hat{u}$ by an angle 2ϕ.

Proof. Fix $u = e^{\hat{u}\phi} \in \mathbb{H}_1$, since $u^{-1} = u^*/|u|^2 = u^*$ we can define the unit quaternion conjugation map $L_u : \mathbb{H}_p \to \mathbb{H}_p,\ L_u(\vec{v}) = u\vec{v}u^*$. First we want to show this map is Linear: $\forall \vec{v}, \vec{w} \in \mathbb{H}_p\ ,\ \lambda \in \mathbb{R}$

$$L_u(\lambda\vec{v} + \vec{w}) = u(\lambda\vec{v} + \vec{w})u^* = \lambda u\vec{v}u^* + u\vec{w}u^* = \lambda L_u(\vec{v}) + L_u(\vec{w}) \tag{6.4}$$

We note this map preserves the quaternion norm:

$$\forall \vec{v} \in \mathbb{H}_p \quad , \quad |L_u(\vec{v})| = |u\vec{v}u^*| = |e^{\hat{u}\phi}||\vec{v}||e^{-\hat{u}\phi}| = |\vec{v}| \tag{6.5}$$

So L_u is a linear isometry, so it can be a reflection or rotation. We consider it's action on $\hat{u}$, the unit axis of u:

$$\begin{aligned}
L_u(\hat{u}) &= u\hat{u}u^* = (\cos\phi + \hat{u}\sin\phi)\hat{u}(\cos\phi - \hat{u}\sin\phi)\\
&= (\hat{u}\cos\phi + \hat{u}^2\sin\phi)(\cos\phi - \hat{u}\sin\phi) = (\hat{u}\cos\phi - \sin\phi)(\cos\phi - \hat{u}\sin\phi)\\
&= \hat{u}\cos^2\phi - \hat{u}^2\cos\phi\sin\phi - \sin\phi\cos\phi + \hat{u}\sin^2\phi\\
&= \hat{u}(\cos^2\phi + \sin^2\phi) + \cos\phi\sin\phi - \sin\phi\cos\phi = \hat{u}
\end{aligned}$$

By homogeneity of L_u we conclude it stabilizes the 1-dimensional subspace $\mathbb{R}\hat{u}$. Since L_u preserves orthogonality, it preserves the 2-dimensional subspace $\hat{u}^\perp \subset \mathbb{H}_p$ which is the orthogonal complement of $\hat{u}$. Take any unit vector $\hat{v} \in \hat{u}^\perp$ and let $\hat{w} = \hat{u} \times \hat{v}$ which is also unit and contained in $\hat{u}^\perp$ by construction. Since $\hat{u}, \hat{v}, \hat{w}$ are all orthogonal their quaternion product reduces to the cross-product: $\hat{u}\hat{v} = -\hat{u}\cdot\hat{v} + \hat{u}\times\hat{v} = \hat{w}$, $\hat{v}\hat{w} = \hat{u}$, $\hat{w}\hat{u} = \hat{v}$ and they satisfy the same cyclic identities of i, j, k. Since $\hat{w} \perp \hat{v}$ they are linearly independent and form a basis of $\hat{u}^\perp$, so we consider the action of L_u on $\vec{v}, \vec{w}$:

$$\begin{aligned}
L_u(\hat{v}) &= u\hat{v}u^* = (\cos\phi + \hat{u}\sin\phi)\hat{v}(\cos\phi - \hat{u}\sin\phi)\\
&= (\hat{v}\cos\phi + \hat{u}\hat{v}\sin\phi)(\cos\phi - \hat{u}\sin\phi)\\
&= (\hat{v}\cos\phi + \hat{w}\sin\phi)(\cos\phi - \hat{u}\sin\phi)\\
&= \hat{v}\cos^2\phi - \hat{v}\hat{u}\cos\phi\sin\phi + \hat{w}\sin\phi\cos\phi - \hat{w}\hat{u}\sin^2\phi\\
&= \hat{v}\cos^2\phi - (-\hat{w})\cos\phi\sin\phi + \hat{w}\sin\phi\cos\phi - \hat{v}\sin^2\phi\\
&= \hat{v}(\cos^2\phi - \sin^2\phi) + \hat{w}(2\sin\phi\cos\phi)\\
&= \hat{v}(\cos 2\phi) + \hat{w}(\sin 2\phi)\\
L_u(\hat{w}) &= u\hat{w}u^* = (\cos\phi + \hat{u}\sin\phi)\hat{w}(\cos\phi - \hat{u}\sin\phi)\\
&= (\hat{w}\cos\phi + \hat{u}\hat{w}\sin\phi)(\cos\phi - \hat{u}\sin\phi)\\
&= (\hat{w}\cos\phi + (-\hat{v})\sin\phi)(\cos\phi - \hat{u}\sin\phi)\\
&= \hat{w}\cos^2\phi - \hat{w}\hat{u}\cos\phi\sin\phi - \hat{v}\sin\phi\cos\phi + \hat{v}\hat{u}\sin^2\phi\\
&= \hat{w}\cos^2\phi - \hat{v}\cos\phi\sin\phi - \hat{v}\sin\phi\cos\phi + (-\hat{w})\sin^2\phi\\
&= \hat{v}(-2\sin\phi\cos\phi) + \hat{w}(\cos^2\phi - \sin^2\phi)\\
&= \hat{v}(-\sin 2\phi) + \hat{w}(\cos 2\phi)
\end{aligned}$$

Then in the $\hat{v}, \hat{w}$ basis, we have: $L_u = \begin{pmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{pmatrix}$, $det(L_u) = +1$, so it is a counterclockwise rotation by angle 2ϕ in $\hat{u}^\perp$. Since $\hat{u}, \hat{v}, \hat{w}$ are orthonormal and $dim_{\mathbb{R}}(\mathbb{H}_p) = 3$, they form a basis and we get the direct sum decomposition $\mathbb{H}_p \cong \mathbb{R}\hat{u} \oplus \hat{u}^{\perp}$. Then for any $\vec{p} \in \mathbb{H}_p$ we can decompose it into the parallel $\vec{p}_{\|}$ and orthogonal $\vec{p}_{\perp}$ components to $\hat{u}$, so $\vec{p} = \vec{p}_{\|} + \vec{p}^{\perp}$. Then using the Linearity of L_u we have:

$$L_u(\vec{p}) = L_u(\vec{p}_{\|} + \vec{p}^{\perp}) = L_u(\vec{p}_{\|}) + L_u(\vec{p}^{\perp}) \tag{6.6}$$

Where L_u fixes $\vec{p}_{\|}$ and rotates $\vec{p}^{\perp}$ by 2ϕ around $\hat{u}$ so it has the net effect of rotating $\vec{p}$ counterclockwise around $\hat{u}$ by an angle 2ϕ. We use the identification $\mathbb{H}_p \cong \mathbb{R}^3$ so we can think of the rotation as happening in real 3-space. Since we can vary over all $u \in \mathbb{H}_p$, we can rotate around any axis by any angle. We look at the map $L_{(-u)}$:

$$L_{(-u)}(\vec{v}) = (-u)\vec{v}(-u)^* = (-u)\vec{v}(-u^*) = u\vec{v}u^* = L_u(\vec{v}) \tag{6.7}$$

So both unit quaternions u and $(-u)$ give us the same rotations, so conjugation by unit quaternions gives us every rotation in $\mathbb{R}^3$ twice. □

Theorem 6.8. *The map $\rho : \mathbb{H}_1 \to SO(3)$ is a surjective group homomorphism:*

$$\rho(u) = L_u \tag{6.9}$$

Proof. We check the map is a group homomorphism: $\forall u_1, u_2 \in \mathbb{H}_1$:

$$\begin{aligned}\rho(u_1u_2)(\vec{v}) &= L_{(u_1u_2)}(\vec{v}) = (u_1u_2)\vec{v}(u_1u_2)^* = (u_1u_2)\vec{v}(u_2^*u_1^*)\\ &= u_1(u_2\vec{v}u_2^*)u_1^* = u_1(L_{u_2}(\vec{v}))u_1^* = L_{u_1}(L_{u_2}(\vec{v}))\\ &= (L_{u_1} \circ L_{u_2})(\vec{v}) = (\rho(u_1) \circ \rho(u_2))(\vec{v})\end{aligned}$$

Since this is true $\forall \vec{v} \in \mathbb{H}_p$ then $\rho(u_1u_2) = \rho(u_1) \circ \rho(u_2)$ and ρ is a group homomorphism. Since any rotation in $SO(3)$ can be described by an angle and axis, we can always find two unit quaternions with that axis and half the angle, so ρ is surjective and $Image(\rho) = SO(3)$. Last we look at the kernel:

$$\begin{aligned}\mathrm{Ker}(\rho) &= \{\, u \in \mathbb{H}_1 \mid L_u = I_3 \,\} = \{\, u \mid L_u(\vec{v}) = \vec{v} \quad \forall \vec{v} \in \mathbb{H}_p,\ |u| = 1 \,\}\\ &= \{\, u \mid u\vec{v}u^* = \vec{v} \quad \forall \vec{v} \in \mathbb{H}_p,\ |u| = 1 \,\}\\ &= \{\, u \mid u\vec{v}u^*u = \vec{v}u \quad \forall \vec{v} \in \mathbb{H}_p,\ |u| = 1 \,\}\\ &= \left\{\, u \;\middle|\; u\vec{v}|u|^2 = \vec{v}u \quad \forall \vec{v} \in \mathbb{H}_p,\ |u| = 1 \,\right\}\\ &= \{\, u \mid u\vec{v} = \vec{v}u \quad \forall \vec{v} \in \mathbb{H}_p,\ |u| = 1 \,\}\\ &= \{\, u \mid u \in Z(\mathbb{H}_p),\ |u| = 1 \,\} = \{\, u \mid u \in \mathbb{R},\ |u| = 1 \,\} = \{\, \pm 1 \,\}\end{aligned}$$

Where $Z(\mathbb{H}_p) = \mathbb{R}$ is the center of the pure quaternions, which are all the real scalars. We then get the exact sequence:

$$1 \to \{\, \pm 1 \,\} \to \mathbb{H}_1 \to SO(3) \to 1 \tag{6.10}$$

Then by the first isomorphism theorem:

$$\mathbb{H}_1/Ker(\rho) \cong Image(\rho) \to \mathbb{H}_1/\{\, \pm 1 \,\} \cong SO(3) \tag{6.11}$$

And we get the unit quaternions $\mathbb{H}_1$ form a double cover of the rotation group $SO(3)$, since we have 2 copies of each rotation in $\mathbb{R}^3$. □

Since we made the identification earlier $\mathbb{H}_1 \cong S^3$ we get:

$$SO(3) \cong \mathbb{H}_1/\{\, \pm 1 \,\} \cong S^3/\{\, \pm 1 \,\} \cong \mathbb{RP}^3 \tag{6.12}$$

Where $\mathbb{RP}^3$ real projective 3-space is obtained by identifying the antipodal points in the unit 3-sphere S^3 in $\mathbb{R}^4$. This is space of all 1-dimensional subspaces in $\mathbb{R}^4$, which is equivalent to the set of all lines through the origin.

7. Rotations in Euclidean 4-Space

Just as we used the unit quaternion conjugation map L_u in **Section 6** to get all rotation in $\mathbb{R}^3$, we can also use the unit quaternion group to generate all rotation in $\mathbb{R}^4$.

Definition 7.1. The group of unit quaternions direct product with itself acts by left and right multiplication on the vector space of quaternions:

$$\mathbb{H}_1 \times \mathbb{H}_1 \curvearrowright \mathbb{H} : (e^{\hat{u}_1\phi_1}, e^{\hat{u}_2\phi_2}, q) \mapsto e^{\hat{u}_1\phi_1} q e^{-\hat{u}_2\phi_2} \tag{7.2}$$

where we make the identification $\mathbb{H} \cong \mathbb{R}^4$ so we can think of the rotations as happening in real 4-space. We have the exact sequence:

$$1 \to \{\, \pm 1 \,\} \to \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4) \to 1 \tag{7.3}$$

Then by the first group isomorphism theorem:

$$\mathbb{H}_1 \times \mathbb{H}_1 / \{\, \pm 1 \,\} \cong SO(4) \tag{7.4}$$

And we get the group $\mathbb{H}_1 \times \mathbb{H}_1$ forms a double cover of the rotation group $SO(4)$, since we have 2 copies of each rotation in $\mathbb{R}^4$.

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