ECON 260A: Phase Plane Analysis

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1 What is Phase Plane Analysis?

Phase plane analysis is a graphical method for "solving" systems of differential equations. The word "solving" is in quotes because phase plane analysis does not actually allow us to obtain a specific numerical solution to a system of differential equations, but it does give us a very good idea of how the system will behave over time.

These notes develop the concept of phase plane analysis by example.

1.1 A predator prey model

Much of the science of ecology is devoted to trying to better understand the dynamics of species populations in an ecosystem. We can get a handle on food web dynamics by first looking at a simple model with two species: one predator and one prey. The most widely cited model of predator-prey dynamics is called the Lotka Volterra Predator Prey model. The model consists of a system of two differential equations, as follows:

$$\dot{X} = X(A - BY - \lambda X) \tag{1}$$
$$\dot{Y} = Y(CX - D - \mu Y) \tag{2}$$

where $X(t)$ is the population of the prey at time t, $Y(t)$ is the population of the predator at time t, and A, B, C, D, λ, and μ are parameters of this dynamical system.

This fairly complex system of differential equations describes how the populations of predator and prey will change over time in response to each other. We would like to see if we can say anything about how this system will behave over time.

1.1.1 Steady States and Isoclines

A steady state of a dynamic system is a place (in this case a population of X and a population of Y) with the following property: If the system somehow finds itself in steady state it will remain at the steady state in perpetuity. There are two very important facts that you should know about steady states:

- Dynamic systems often have more than one steady state and
- Not all steady states are *stable* steady states.

A *stable* steady state is a steady state with the following property: If the system is at a stable steady state and is perturbed slightly off of that steady state, then the system will return to the steady state.

So we know that a steady state is one where the variables (X and Y in this case) do not change over time. To find the steady states, we simply set equations 2 and 2 equal to zero and solve. We have the following:

1. $\dot{X} = 0$ implies that either:
    - $X = 0$ or
    - $A - BY - \lambda X = 0$
2. $\dot{Y} = 0$ implies that either:
    - $Y = 0$ or
    - $CX - D - \mu Y = 0$

The equations for which $\dot{X} = 0$ or $\dot{Y} = 0$ above are called "nullclines" or "isoclines". A steady state exists at any point for which $\dot{X} = 0$ AND $\dot{Y} = 0$. In this problem, there are potentially 3 steady states (depending on parameter values). They are:

- (SS #1:) $X = 0$, $Y = 0$
- (SS #2:) $X = A/\lambda$, $Y = 0$
- (SS #3:) $X = \frac{\frac{A\mu}{B}+D}{C+\frac{\mu\lambda}{B}}$, $Y = \frac{\frac{CA}{\lambda}-D}{\mu+\frac{BC}{\lambda}}$

1.1.2 Predicting model dynamics with a phase plane

Now that we know what the nullclines and the steady states look like, we are in a position to evaluate the dynamics of this system from any point in the phase plane. Remember that any point in the phase plane represents some population of the prey (X) and some population of the predator (Y). What we would like to do is to be able to predict, given any current position, where in the phase plane we will move next. To do so, we need to generate the "directional arrows" for this problem.

Any point in the phase plane has two directional arrows associated with it. The first directional arrow (which either points left or right) tells us, at that point, whether the prey population (X) is increasing or decreasing, that is, whether $\dot{X} > 0$ (right) or $\dot{X} < 0$ (left). The second directional arrow (which either points up or down) tells us whether the predator population (Y) is increasing or decreasing.

How many directional arrows are there? The answer is that each "isosector" has one directional arrow that points up or down and one directional arrow that points left or right. An isosector is a space in the phase plane that is bounded by nullclines.

How do I determine which way my directional arrows point? This is the tricky part. Remember that along a nullcline for which $\dot{X} = 0$ there is no movement in the left or right direction. If we move to the left of that nullcline, (that is, we reduce X) then we need to determine whether $\dot{X}$ is positive (in which case the directional arrow points right) or negative (in which case the directional arrow points left) at that point. By crossing the nullcline in the other direction, the directional arrow will switch directions. The same procedure can be undertaken to determine the direction of the up/down directional arrows in each isosector.

In our example, remember that along the nullcline for $\dot{X} = 0$, the following equation holds:

$$X(A - BY - \lambda X) = 0 \tag{3}$$

If we increase the value of Y (that is, we move upward from any nullcline for which $\dot{X} = 0$), we must necessarily be decreasing the left hand side of equation 3, and therefore $\dot{X} < 0$. In other words, any directional arrow *above* an $\dot{X} = 0$ nullcline must point to the left. Consequently, any directional arrow *below* an $\dot{X} = 0$ nullcline must point to the right.

Similarly, we can examine what happens when we move the right of any $\dot{Y} = 0$ nullcline. Along $\dot{Y} = 0$ nullclines, the following equation holds:

$$Y(CX - D - \mu Y) = 0 \tag{4}$$

By increasing X (that is moving to the right), the left hand side must get larger, and therefore $\dot{Y} > 0$. So, any directional arrow *to the right* of a $\dot{Y} = 0$ nullcline must point up. And any directional arrow *to the left* of a $\dot{Y} = 0$ nullcline must point down.

After all the directional arrows have been drawn, we are in a position to predict the behavior of this system over time because the movement of the system has to conform to the directional arrows. Typically, the system will converge to a steady state. But other dynamics are possible such as "limit cycles" in which the system stays on a circular route and never approaches a steady state.

1.2 A recipe for phase plane analysis

In summary, we offer the following recipe for conducting phase plane analysis.

1. Write the two ordinary differential equations that govern the values of each of the two variables (call them X and Y) over time.

2. Find and plot the **nullclines**: graphs of the set of points in the phase plane for which $\dot{X} = 0$ and $\dot{Y} = 0$. Typically there will be more than one nullcline for each case.

3. Graphically find the **steady states**: the locations where one nullcline for $\dot{X} = 0$ and one nullcline for $\dot{Y} = 0$ cross.

4. Label the **isosectors**: the spaces in the phase plane that are bordered by the nullclines. Label them I, II, III, etc.

5. In each isosector, find the **directional arrows**: one arrow that points up or down (indicating whether $\dot{Y} \gtrless 0$ in that isosector) and one arrow that points left or right (indicating whether $\dot{X} \gtrless 0$ in that isosector).

6. Draw a sample **trajectory** of the system through time, conforming to the directional arrows.

2 An open access renewable resource

Let $X(t)$ be the population of a renewable resource at time t and $E(t)$ be the "effort" toward harvesting that resource at time t. Harvest is proportional to effort and the population of the resource: $H(t) = qE(t)X(t)$, where q is a constant that is sometimes called "catchability". The growth rate of the resource population depends on the current size of the population and on the harvest rate, as follows:

$$\dot{X} = rX(1 - X/K) - qEX \tag{5}$$

This is an open access resource, so entry and exit (i.e. changes in effort) depend on the profits made in the industry, as follows:

$$\dot{E} = \alpha(pqEX - cE) \tag{6}$$

where p is the price per unit population (e.g. price per pound of fish), and c is the cost per unit effort (e.g. the cost of fishing for a day).

Question: Conduct a phase plane analysis of this system.

Figure 1: The Lotka Volterra phase plane with parameters: $A = 100$, $B = 4$, $\lambda = 2$, $C = 15$, $D = 5$, $\mu = 8$