1 Optimal Control Lecture

1.1 Lagrangian methods applied to discrete-time dynamic allocation problem

Let $t = 0, 1, ..., T$, x_t=state variable in t (x describes the state of the system; e.g., stock of a resource), y_t=control variable in t (e.g., harvest), $V(x_t, y_t, t)$ is the net return in period t (e.g., profits or utility; note that the form of $V()$ depends on time, which is termed non-autonomous), and $F(x_T)$=salvage value in terminal time T. The equation of motion is given by:

$$x_{t+1} - x_t = f(x_t, y_t) \tag{1}$$

How does the state of the system change given the state in t (x_t) and the control in t (y_t)? The equation of motion is defined for $t = 0, 1, ..., T - 1$.

The optimization problem is given by:

$$\max_{y_t} \sum_{t=0}^{T-1} V(x_t, y_t, t) + F(x_T) \quad \text{s.t.} \quad x_{t+1} - x_t = f(x_t, y_t), \quad x_0 = a \tag{2}$$

In words, the control variable (y) is set through time to determine (to control) the state of the system (x) in such a way that the value of the system (the V) is maximized.

The Lagrangian function is given by:

$$\mathcal{L} = \max_{y_t} \sum_{t=0}^{T-1} \{V(x_t, y_t, t) + \lambda_{t+1}(x_t + f(x_t, y_t) - x_{t+1})\} + F(x_T) \tag{3}$$

Assuming an interior solution:

$$\frac{\partial \mathcal{L}}{\partial y_t} = \frac{\partial V}{\partial y_t} + \lambda_{t+1} \frac{\partial f}{\partial y_t} = 0 \qquad t = 0, 1, ..., T - 1 \tag{4}$$

There is no y_T because once y_{T-1} is chosen, x_T is defined by the equation of motion. Interpretation: the first term is simply the effect of y_t on the current return. In a static context, the second term is missing. Here, λ_{t+1} is the shadow price on the constraint. How much does the value of the optimal program increase if the $t+1$ constraint is relaxed (i.e., we have another unit of x_{t+1})? This is a dynamic version of the envelope theorem, where the effect is on the value of the *program* from $t + 1$ to T. $\frac{\partial f}{\partial y_t}$ gives the effect of the control variable on x_{t+1} (from the equation of motion, $\frac{dx_{t+1}}{dy_t} = \frac{\partial f}{\partial y_t}$). Thus, $\lambda_{t+1} \frac{\partial f}{\partial y_t}$ is the value of the effect of y_t on the future state of the system.

$$\frac{\partial \mathcal{L}}{\partial x_t} = \frac{\partial V}{\partial x_t} + \lambda_{t+1}(1 + \frac{\partial f}{\partial x_t}) - \lambda_t = 0 \qquad t = 1, ..., T - 1 \tag{5}$$

There is no term for $t = 0$ because $x_0 = a$. Interpretation: rewrite as $(\frac{\partial V}{\partial x_t} + \lambda_{t+1} \frac{\partial f}{\partial x_t}) + \lambda_{t+1} = \lambda_t$. Ignore the term in parentheses for a moment. If the state variable is optimally allocated over time, then the shadow prices should be equal. If an extra unit of x were more valuable in a given period, then it should be moved to that period. The term in parentheses measures the additional value of a unit of x in period t in terms of its effect on current net returns and the future state of the system. λ_t must also reflect this additional value.

$$\frac{\partial \mathcal{L}}{\partial x_T} = -\lambda_T + \frac{dF}{dx_T} = 0 \tag{6}$$

The value of a unit of stock in T should equal its marginal salvage value. Think of being able to sell any remaining x in T at a price $\frac{dF}{dx_T}$. Then, you should be indifferent between selling left over units of x in T and having consumed these units earlier.

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} = x_t + f(x_t, y_t) - x_{t+1} = 0 \qquad t = 0, 1, ..., T-1 \tag{7}$$

$$x_0 = a \tag{8}$$

Equations 6 and 8 are called boundary conditions. Such conditions are needed so that there are an equal number of equations and unknowns (there are $3T + 1$ unknown values of y, x, and λ). Different kinds of boundary conditions define different types of problems. This is a fixed-time, free-state problem. That is, T is given, but x_T is freely chosen. Can also have free-time, fixed-state problems. That is, you have to end up at a certain value of x_T, but there is no restriction on T. And, other variations.

Also, infinite-time horizon problems, where the interest is in whether a steady state is reached and how it is reached. Many nonrenewable resource problems are cast as free-time, fixed-state problems (e.g., mineral extraction) and renewable resource problems are cast as infinite horizon problems. Consider an infinite-horizon problem now:

$$\max_{y_t} \sum_{t=0}^{\infty} V(x_t, y_t) \quad \text{s.t.} \quad x_{t+1} - x_t = f(x_t, y_t), \quad x_0 = a \tag{9}$$

$$\frac{\partial V}{\partial y_t} + \lambda_{t+1} \frac{\partial f}{\partial y_t} = 0 \qquad t = 0, 1, ... \tag{10}$$

$$\frac{\partial V}{\partial x_t} + \lambda_{t+1}(1 + \frac{\partial f}{\partial x_t}) - \lambda_t = 0 \qquad t = 1, 2, ... \tag{11}$$

$$x_t + f(x_t, y_t) - x_{t+1} = 0 \qquad t = 0, 1, ... \tag{12}$$

This is an autonomous problem because V and f do not depend on t. In a steady-state, $x_t = x^*$, $y_t = y^*$, and $\lambda_t = \lambda^*$ for some $t \geqslant 0$. This gives three equations in three unknowns:

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 \tag{13}$$

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0 \tag{14}$$

$$f(x, y) = 0 \tag{15}$$

An example is:

$V = 3x + 3y$
$f(x, y) = 2x - 0.5x^2 - y$

$x^* = 3, y^* = 1.5, \lambda^* = 3$

These are the steady-state values. But, what if we had started with some value of x other than 3? How would we adjust to the steady state? Two types of approaches: asymptotic approach ($x_t \to x^*$ as $t \to \infty$) and Most Rapid Approach Path ($x_t \to x^*$ as quickly as possible). Which of these two applies depends on the structure of the problem (see Spence and Starrett, 1975 IER for details). MRAP is optimal when the Hamiltonian is linear in the control, implying a "bang-bang" solution: the control is put at an extreme value until the steady-state is reached. Asymptotic approaches require finding a solution to a system of differential equations. I will post material on Gauchospace showing how this works.

Optional problem: $V_t = 3x_t + 3y_t$, $f(x_t, y_t) = 2x_t - 0.5x_t^2 - y_t$, $x_0 = 0.5$, $y_t \geq 0$. This problem has a MRAP solution.

1.2 Discrete-time optimal control problem

Define the discrete-time Hamiltonian as:

$$H(x_t, y_t, \lambda_{t+1}, t) = V(x_t, y_t, t) + \lambda_{t+1} f(x_t, y_t) \tag{16}$$

Then, the set of first-order conditions from the original problem can be written:

$$\frac{\partial H}{\partial y_t} = 0 \qquad t = 0, 1, ..., T-1 \tag{17}$$

$$\lambda_{t+1} - \lambda_t = -\frac{\partial H}{\partial x_t} \qquad t = 1, ..., T-1 \tag{18}$$

$$\lambda_T = \frac{dF}{dx_T} \tag{19}$$

$$x_{t+1} - x_t = \frac{\partial H}{\partial \lambda_{t+1}} \qquad t = 0, 1, ..., T-1 \tag{20}$$

$$x_0 = a \tag{21}$$

This simply shows that there is a direct correspondence between optimal control and the Lagrangian method. The main advantage of optimal control is that it simplifies the solution of continuous-time problems.

1.3 Continuous time optimal control problem

Interpretation of the Hamiltonian: $H(\cdot) = V(\cdot) + \lambda(t) f(\cdot)$. Dynamic version of the envelope theorem gives $\lambda(t) = \frac{\partial J(x,t)}{\partial x}$ where:

$$J(x, t) = \max_{y(t)} \int_t^T V(x(t), y(t), t) dt \tag{22}$$

J is the maximized value of the program from t to T, and thus $\lambda(t)$ gives the marginal value of the state variable in time t. The Hamiltonian can be interpreted as the rate of increase in the total value of assets: the flow of net returns in t plus the increase in the value of the state variable x ($\lambda f = \lambda \dot{x}$). In a fisheries model, for example, V might be the current net return from harvest and λf would be the value of the growth in the fish stock. The conditions for an optimum are analogous to what we found for the discrete-time problem (see materials posted on Gauchospace for more details):

$$\frac{\partial H(\cdot)}{\partial y(t)} = 0 \qquad \text{(maximum condition)} \tag{23}$$

$$\dot{\lambda}(t) = -\frac{\partial H(\cdot)}{\partial x(t)} \qquad \text{(adjoint equation)} \tag{24}$$

$$\lambda(T) = \frac{dF}{dx(T)} \tag{25}$$

Adding the equation of motion and the initial condition gives:

$$\dot{x}(t) = \frac{\partial H(\cdot)}{\partial \lambda(t)} \tag{26}$$

$$x(0) = a \tag{27}$$

In a free-time problem, we must choose T to maximize the Lagrangian for the problem. This yields $H(T) = 0$. Intuition: if $H(T) > 0$ were positive, then T should be increased and vice-versa. $H(T) = 0$ and $\lambda(T) = \frac{dF}{dx(T)}$ are referred to as the transversality conditions, which together with equations 23 and 24 are referred to as the maximum principle.

You always need a condition on the time and the final state. Free-time, free-state: $H(T) = 0$, $\lambda(T) = \frac{dF}{dx(T)}$. Free-time, fixed-state: $H(T) = 0$, $x(T) = b$ ($\lambda(T) = \frac{dF}{dx(T)}$ is not needed to pin down $x(T)$). Fixed-time, free-state: T is given so $H(T) = 0$ does not apply, $\lambda(T) = \frac{dF}{dx(T)}$; fixed-time, fixed-state: T and $x(T)$ are given. Note that the steady-state in an infinite horizon problem replaces the transversality condition.

A simple example helps to clarify the solution to an optimal control problem:

$$\max \int_0^1 (x+y)dt \qquad \text{s.t.} \qquad \dot{x} = 1 - y^2 \qquad x(0) = 1 \tag{28}$$

$H(x, y, \lambda) = x + y + \lambda(1 - y^2)$
$H_y = 1 - 2\lambda y = 0$, $\dot{\lambda} = -H_x = -1$, $\lambda(1) = 0$ (no salvage value)
$\int \dot{\lambda} dt = \int -1dt$ yields $\lambda(t) = -t + c$. Using the boundary condition, $\lambda(1) = -1 + c = 0$ or $c = 1$. Thus, $\lambda(t) = 1 - t$. Substituting gives $y(t) = \frac{1}{2\lambda(t)} = \frac{1}{2(1-t)}$. Finally, $\dot{x} = 1 - y^2 = 1 - [\frac{1}{4(1-t)^2}]$. Integrating gives $x(t) = t - [\frac{1}{4(1-t)}] + c$, where $x(0) = 1$ implies $1 = 0 - \frac{1}{4} + c$ or $c = 5/4$. Thus, $x(t) = t - [\frac{1}{4(1-t)}] + \frac{5}{4}$ and we have solved for $x(t)$, $y(t)$, and $\lambda(t)$.

1.4 Continuous time optimal control problem with discounting

One way in which the objective function $V(x(t), y(t), t)$ can depend on t is through discounting: $V(x(t), y(t), t) = V(x(t), y(t))e^{-rt}$. In the context of a natural resource problem, managers may have available an alternative investment yielding an annual return r. This leads them to place less weight on future returns. Consider a fixed-time, free-state continuous time problem with discounting:

$$\max \int_0^T V(x(t),y(t))e^{-rt}dt + F(x(T))e^{-rT} \qquad \dot{x} = f(x(t),y(t)) \qquad x(0)=a \tag{29}$$

The Hamiltonian is defined $H = V(x(t),y(t))e^{-rt} + \lambda(t)f(x(t),y(t))$ and the necessary conditions include:

$$\frac{\partial H}{\partial y(t)} = \frac{\partial V(\cdot)}{\partial y(t)}e^{-rt} + \lambda(t)\frac{\partial f(\cdot)}{\partial y(t)} = 0 \tag{30}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x(t)} = -\frac{\partial V(\cdot)}{\partial x(t)}e^{-rt} - \lambda(t)\frac{\partial f(\cdot)}{\partial x(t)} \tag{31}$$

$\lambda(t)$ is the present-value shadow price. Namely, it tells you the present value of one more unit of x in t. As above, $\lambda(t) = \frac{\partial J(x,t)}{\partial x}$ where, in this case,

$$J(x,t) = \max_{y(t)} \int_t^T V(x(t),y(t))e^{-rt}dt \tag{32}$$

That is, net returns are discounted to time zero and, thus, $\lambda(t)$ gives the present value of another unit of $x(t)$.

Alternatively, write the current-value Hamiltonian as $\tilde{H} = He^{rt} = V(\cdot) + \mu(t)f(\cdot)$ where $\mu(t) = e^{rt}\lambda(t)$. Now, all values are in current (undiscounted) terms. In particular, $\mu(t)$ is the value of another unit of $x(t)$ at time t rather than from the perspective of time zero. The first-order conditions, from above, for the current-value problem are:

$$\frac{\partial \tilde{H}}{\partial y(t)} = \frac{\partial V(\cdot)}{\partial y(t)} + \mu(t)\frac{\partial f(\cdot)}{\partial y(t)} = 0 \tag{33}$$

$$\dot{\mu(t)} - r\mu(t) = -\frac{\partial \tilde{H}}{\partial x(t)} = -\frac{\partial V(\cdot)}{\partial x(t)} - \mu(t)\frac{\partial f(\cdot)}{\partial x(t)} \tag{34}$$

which can be confirmed by manipulating $\mu(t) = e^{rt}\lambda(t)$. Often, you'll see problems with discounting written as current-value Hamiltonians because the simplifies the notation a bit.