Homework Challenge #2

ECON 260A

Villaseñor-Derbez, J.C.

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Problem

An exhaustible resource is costlessly extracted in continuous time by competitive firms (assume all other assumptions of the Hotelling model apply). The market inverse demand curve is $p(y) = a - by(t)$ where $p(t)$ and $y(t)$ denote the price and quantity in time t and a and b are finite parameters. The initial stock of the resource is given by $x(0) = x_0$, the interest rate is r, and the time horizon is $0 < t < T$.

Characterize the dynamic market equilibrium

The flow of benefits ($U(y(t))$) is given by the area under the inverse demand curve:

$$
\begin{aligned}
U(y(t)) &= \int_0^{y(t)} p(s)\, ds \\
&= \int_0^{y(t)} (a + bs)\, ds \\
&= as + \frac{1}{2}bs^2 \Big|_0^{y(t)} \\
&= ay(t) + \frac{1}{2}by(t)^2 - \left((a \times 0) + \left(\frac{1}{2}b \times 0^2\right)\right) \\
&= ay(t) + \frac{1}{2}by(t)^2
\end{aligned}
$$

We know the initial conditions, and that there is no growth in the resource, so the optimization problem then becomes:

$$
\max_{y(t)} \int_{t=0}^{T} \left[ay(t) + \frac{1}{2}by(t)^2\right] e^{-rt}\, dt
$$

$$
\text{s.t. } \dot{\text{x}}(\text{t}) = -\text{y}(\text{t});\ \text{x}(0) = \text{x}_0
$$

The current-value Hamiltonian is given by:

$$
\begin{aligned}
\tilde{H} &= U(y(t)) - \mu(t)y(t) \\
&= ay(t) + \frac{1}{2}by(t)^2 - \mu(t)y(t)
\end{aligned}
$$

The maximum principle gives:

$$
\begin{aligned}
\tilde{H}_y = U'(y(t)) - \mu(t) = 0 \\
a + by(t) - \mu(t) = 0 \\
p(y(t)) - \mu(t) = 0
\end{aligned}
$$

With a free-time problem, $H(T) = 0$ and the free-state condition is $\mu(T)x(T) = 0$. As in the lecture notes, the adjoint equation is $-\tilde{H}_x = \dot{\mu}(t) - r\mu(t) = 0$. We then apply Leibniz rule to the maximum condition above and obtain:

$$
\begin{aligned}
\tilde{H}_y = 0 \\
p(y(t)) - \mu(t) = 0 \\
p(t) - \mu(t) = 0 \\
p(t) = \mu(t)
\end{aligned}
$$

And since the adjoint equation was $\frac{\dot{\mu}(t)}{\mu(t)} = r$, then $\frac{\dot{p}(t)}{p(t)} = r$ and so Hotelling's rule can be satisfied and prices grow at the discount rate: $p(t) = p(0)e^{rt}$.

Our free-time condition stated above implies that no resources should be left in time $t = T$, and so the sum (integral) of all extractions should equal the initial stock size: $x_0 = \int_0^T y(t)\,dt$.

Implicit T function and its marginal changes[1]

We know that at time $t = T$ there are no resources left (free-state condition), so $y(T) = 0$. We are also told that $p(t) = a - by(t)$, where I can solve for y and obtain my downward sloping demand curve:

$$
y(t) = \frac{a - p(t)}{b}
$$

Also, since a is my choke price I can have:

$$
p(T) = p(0)e^{rt} = a
$$

I can take the condition from above stating that initial stock size must be equal to the area under all extractions $y(t)$ and substitute my downard sloping demand curve into it:

$$
\begin{aligned}
x_0 &= \int_0^T y(t)\,dt \\
x_0 &= \int_0^T \frac{a - p(t)}{b}\,dt \\
x_0 &= \frac{1}{b}\left(aT - \frac{p_0}{r}(e^{rt} - 1)\right) \\
x_0 b &= aT - \frac{p_0}{r}(e^{rt} - 1) \\
x_0 b - aT + \frac{a}{r} &= \frac{a}{r}e^{-rT}
\end{aligned}
$$

[1] I did not do the marginal changes because I was unsure how to, and it only became clear after I saw it in class today. It is now evident, but I will refrain from including these as part of my assignment because I would not have been able to turn them in. However, it is now clear clear how increases in b, X_0, interest rate, and choke price cause a increases-decrease in T.