Homework Challenge #3

ECON 260A

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Set up

Let x_{it} be the stock of an invasive species in patch i at the beginning of time period t, and h_{it} is the control in patch i during time period t. The timing is as follows: The stock is observed in each patch, some level of control is undertaken in each patch, the remaining stock grows, and then moves across space. Movement from patch i to patch j is given by the constant D_{ij}. So the equation of motion is:

$$x_{it+1} = \sum_{j=1}^{N} D_{ji} g(e_{jt})$$

where e_{jt} is the residual stock in patch j and N is the number of patches. If the stock at the beginning of the period in patch i is x_i and the control is h_i (leaving residual stock e_i), then the total control cost during that period in patch i is $\int_{e_i}^{x_i} \theta_i c(s)\, ds$, where the downward-sloping function $\theta_i c(s)$ is the marginal control cost when the stock is s (the parameter θ_i is a constant). After control takes place, but before growth and spread occur, the residual stock imposes a patch-specific marginal damage of k_i, so the total damage in patch i during period t is given by $k_i e_i$.

Dynamics and the steady state of this system with myopic landowner

From the problem set up, we know that "*After control takes place, but before growth and spread occur, the residual stock imposes a patch-specific marginal damage of k_i, so the total damage in patch i during period t is given by $k_i e_i$*". Myopic landowners would not care about the future growth, dispersal, and damages of an invasive species. Instead, they would only care about damages and control costs in this time step.

Total costs for owner of property i at time t are given by:

$$\phi(x_{it}, e_{it}) = \int_{e_{it}}^{x_{it}} \theta c(s)\, ds + k_i e_{it}$$

Our current "damage function" assumes a constant marginal impact from remaining stock size. Therefore, our total costs can be shown by the graphical representation of the problem in Figure 1.

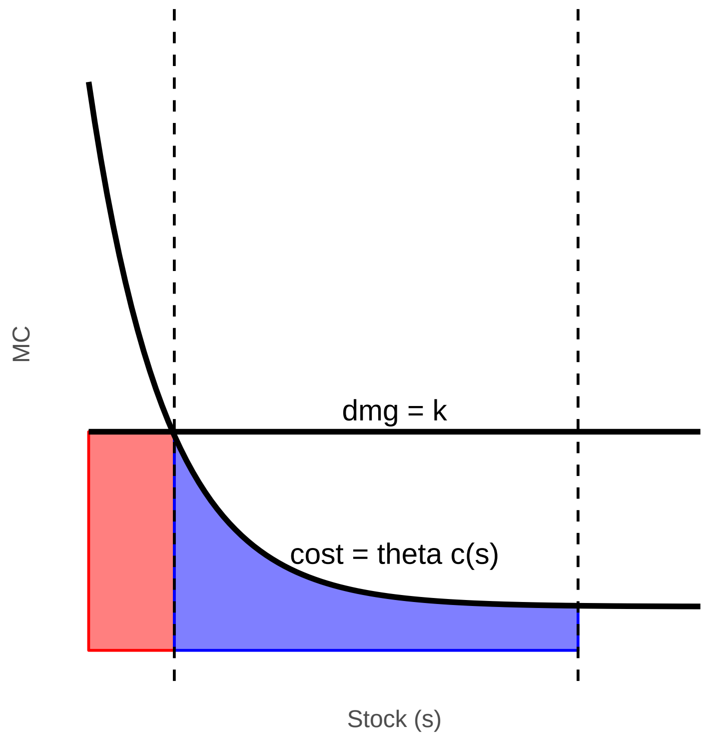

Figure 1: Illustration of marginal costs of control and damage. The area under each curve show the total cost of control (blue) and damage (red).

The myopic landowner would minimize present costs of control and damage, without accounting for the growth and dispersal of the stock and the future damaged that would come with it. Therefore, the myopic landowner seeks to minimize total costs (shaded area in Fig. 1) by chosing an optimal h:

$$\min_e \phi(x_i, e_i)$$
$$\min_e \left[\int_{e_i}^{x_i} \theta_i c(s)\, ds + k_i e_i \right]$$
$$\min_e \left[\int_{e_i}^{x_i} \theta_i c(s)\, ds + k_i (x_i - h_i) \right]$$

Here we would take first-order conditions and find the optimal h. But by visually inspecting Figure 1, we know that each individual landowner would control up to a point where marginal control costs are equal to marginal damage costs. With this in mind, and knowing that $e_{it} = x_{it} - h_{it}$, we drop the t subindeces to turn this into a simple problem, and denote e^* as the one-period equilibrium stock size left after harvesting and for which marginal costs of control are equal to the marginal costs of damages:

$$\theta_i c(e^*) = k_i$$
$$c(e^*) = \frac{k_i}{\theta_i}$$

If we had a functional form for $c(s)$ we would substitute e^* into it and then just solve for e^* in the above. Intuitevely $h^* = x - e^*$: that is, the optimal h^* would be whatever brings the stock from x down to e^*.

Evidently, this approach does not account for the growth and dispersal of the remaining stock, as well as the future damages of the immigrating stock left from other landowner's suboptimal management.

Central planner determines control $\forall\, t$ and i

The central planner, on the other hand, would account for present reductions in damages as well as future ones. Therefore, the planner must balance between present costs of managing and future costs of damage.

Period t dynamic programming equation

The social planer would seek to minimize the sum of present value of total costs by optimally selecting the spatial and temporal control. I denote vectors as $\bar{x_t}$, so that $\bar{x_t} = x_{1t}, x_{2t}, ..., x_{N-1t}, x_{Nt}$. Then, the period-$t$ dynamic programming equation is given by:

$$V_t(\bar{x}_t) = \min_{e_{it}} \sum_{i=1}^{N} \phi(x_{it}, e_{it}) + \delta V_{t+1}(\bar{x}_{it+1})$$

Where $\phi(x_{it}, e_{it}) = \int_{e_{it}}^{x_{it}} \theta c(s)\, ds + k e_{it}$

Subject to the equation of motion given by:

$$\mathrm{s.t.\ x_{it+1}} = \sum_{\mathrm{j=1}}^{\mathrm{N}} \mathrm{D_{ji} g(e_{jt})}$$

My state variable is x_{it}, and my control is $e_{i,t}$. In this formulation, the planner doesn't explicitly choose how much to harvest, but rather how much to leave in every parcel i at every time period t.

Fixed-time, no salvage value

With a fixed-time problem and assuming no salvage value, the problem relies only in the present costs (from control and damages) but does not account for the dispersal, growth, and future damages because $V_{T+1}(\bar{x}_{T+1}) = 0$. Therefore, the DPE becomes:

$$\begin{aligned}
V_T(\bar{x}_T) &= \min_{e_{iT}} \left[\sum_{i=1}^{N} \phi(x_{iT}, e_{iT}) + \delta V_{T+1}(\bar{x}_{T+1}) \right] \\
&= \min_{e_{iT}} \left[\sum_{i=1}^{N} \phi(x_{iT}, e_{iT}) + 0 \right] \\
&= \min_{e_{iT}} \left[\sum_{i=1}^{N} \phi(x_{iT}, e_{iT}) \right] \\
&= \min_{e_{iT}} \left[\sum_{i=1}^{N} \int_{e_{iT}}^{x_{iT}} \theta c(s)\, ds + k_i e_{iT} \right]
\end{aligned}$$

Taking first order conditions we obtain:

$$k_i - \theta_i c(e_{iT}) = 0$$
$$k_i = \theta_i c(e_{iT})$$

Assume $c(\dot{})$is invertible

$$e_{iT}^* = c^{-1}\left(\frac{k_i}{\theta_i}\right)$$

We can substitute that into our value function and obtain:

$$V_T(\bar{x}_T) = \sum_{i=1}^{N} \int_{e_{iT}}^{x_{iT}} \theta c(s)\ ds + k_i e_{it}^*$$

The oplicy function implies that the optimal harvest is:

$$h_{iT}^*(x_{iT}) = x_{iT} - e_{iT}^*$$
$$h_{iT}^*(x_{iT}) = x_{iT} - c^{-1}\left(\frac{k_i}{\theta_i}\right)$$

Backward induction

The period-T solution is shown above, so we start at period $T-1$

The DPE is given by:

$$V_{T-1}(\bar{x}_{T-1}) = \min_{e_{iT-1}} \left[\sum_{i=1}^{N} \left(\int_{e_{iT-1}}^{x_{iT-1}} \theta c(s)\ ds + k_i e_{iT-1}\right) + \delta V_T(\bar{x}_T)\right]$$

Substitute the $V_T(\dot{})$ value function:

$$V_{T-1}(\bar{x}_{T-1}) = \min_{e_{iT-1}} \left[\sum_{i=1}^{N} \left(\int_{e_{iT-1}}^{x_{iT-1}} \theta c(s)\ ds + k_i e_{iT-1}\right) + \delta \sum_{i=1}^{N} \left[\left(F(x_{iT}) - F\left(c^{-1}\left(\frac{k_i}{\theta_i}\right)\right)\right) + c^{-1}\left(\frac{k_i}{\theta_i}\right)\right]\right]$$

Taking first order conditions we obtain:

$$0 = k_i - \theta_i c(e_{iT-1}) + \delta \left[\sum_{j=1}^{N} \theta_j c(x_{jT}) g'(e_{iT-1}) D_{ij}\right]$$

solve for k_i

$$k_i = \theta_i c(e_{iT-1}) - \delta \left[\sum_{j=1}^{N} \theta_j c(x_{jT}) g'(e_{iT-1}) D_{ij}\right]$$

This formulation shows the discounted future values of the remaining stock that has moved according to the dispersal given by matrix D_{ij} must equal the margtinal damages for the optimum e_i^* ($g'()$ represents).

We can then substitute the value function derived for T into the value function of $T-1$ and obtain:

$$
\begin{aligned}
V_{T-1}(\bar{x}_{T-1}) &= \sum_{i=1}^{N}\left(\int_{e_i^*}^{x_{iT-1}} \theta c(s)\ ds + k_i e_i^*\right) + \delta V_T(\bar{x}_T) \\
&= \sum_{i=1}^{N}\left(\int_{e_i^*}^{x_{iT-1}} \theta c(s)\ ds + k_i e_i^*\right) + \delta \sum_{i=1}^{N}\left(\int_{e_{iT}}^{x_{iT}} \theta c(s)\ ds + k_i e_{iT}^*\right) \\
&= \sum_{i=1}^{N}\left(\int_{e_i^*}^{x_{iT-1}} \theta c(s)\ ds + k_i e_i^*\right) + \delta \sum_{i=1}^{N}\left(\int_{e_{iT}}^{\sum_j D_{ij} g(e_j^*)} \theta c(s)\ ds + k_i e_{iT}^*\right)
\end{aligned}
$$

The last term on the RHS can be set to Ω_{T-1} which is a constant that depends on T. Therefore, our value function can be:

$$
V_{T-1}(\bar{x}_{T-1}) = \sum_{i=1}^{N}\left(\int_{e_i^*}^{x_{iT-1}} \theta c(s)\ ds + k_i e_i^*\right) + \Omega_{T-1}
$$

And where e_i^* would be defined as the value that makes the following true:

$$
k_i = \theta_i c(e_{iT-1}) - \delta \left[\sum_{j=1}^{N} \theta_j c\left(\sum_{i=1}^{N} D_{ij} g(e_j^*)\right) g'(e_{iT-1}^*) D_{ij}\right]
$$

Moving to period $T-2$ we have that:

$$
V_{T-2}(\bar{x}_{T-1}) = \min_{e_{iT-2}} \left[\sum_{i=1}^{N}\left(\int_{e_{iT-2}}^{x_{iT-2}} \theta c(s)\ ds + k_i e_{iT-2}\right) + \delta V_{T-1}(\bar{x}_{T-1})\right]
$$

Taking first order conditions we would get the same as before but for $T-2$:

$$
0 = k_i - \theta_i c(e_{iT-2}) + \delta \left[\sum_{j=1}^{N} \theta_j c(x_{jT-1}) g'(e_{iT-2}) D_{ij}\right]
$$

This would imply that, up to a constant Ω_t, the function has converged and we obtain that:

$$
V_t^*(\bar{x}_t) = \sum_{i=1}^{N}\left(\int_{e_i^*}^{x_{it}} \theta c(s)\ ds + k_i e_i^*\right) + \Omega_t
$$

With

$$
k_i = \theta_i c(e_i^*) - \delta \left[\sum_{j=1}^{N} \theta_j c\left(\sum_{i=1}^{N} D_{ij} g(e_j^*)\right) g'(e_{iT-1}^*) D_{ij}\right]
$$

And h_{it}^* is just what ever brings x_{it} down to $e*_i$

Dependence of control in i on xi, ki, ϕ_i D_{ij} and D_{ji}