Homework Challenge #1

ECON 260A

Villaseñor-Derbez, J.C.

Compiled on Wed Oct 17 15:00:58 2018 by JC

The stock of a renewable resource has the following equation of motion:

$$x_{t+1} = f(y_t)$$

Where $y_t \equiv x_t - h_t$ is the residual stock in period t, determined by the difference between stock size at the begining of that period (h_t) and the respective harvest h_t. There is no cost of fishing, and there is a price of $p = 1$ such that payoff in period t is simply h_t. The discount factor is given by δ.

Since $p = 1$ and we have no costs, payoff $\pi(h_t) = h_t$. Our harvest is not influenced by stock size, therefore the period-t Dynamic Optimization Problem is given by:

$$\max_{h_t} \sum_{t=0}^{T} \delta^t h_t \text{ s.t. } x_{t+1} = f(y_t)$$

1 Period-t dynamic programming equation

At every period, our value is determined by our payoff, equivalent to h_t, and the discounted value of leaving some resources to harvest in the future (*i.e.* x_{t+1}):

$$V_t(x_t) = \max_{h_t} \left[h_t + \delta V_{t+1}(x_{t+1}) \right]$$

Since stock size next period is given by a growth function $f(y_t)$ that depends on what's left from the previous extraction ($x_t - h_t$), then the above becomes:

$$V_t(x_t) = \max_{h_t} \left[h_t + \delta V_{t+1}(f(x_t - h_t)) \right]$$

2 Backward induction

2.1 Period-T

On the last period, the DPE would be given by:

$$V_T(x_T) = \max_{h_T} \left[h_T + \delta V_{T+1}(x_{T+1}) \right]$$

The discounted term only includes stock size, because there are no harvests in period $T + 1$. Since this is a fixed-horizon problem ending at $t = T$ and there is no salvage value, there is no incentive to leave resources for further time periods (*i.e.* $T + 1$), so $V_{T+1}(x_T) = 0$. Therefore, the above becomes:

$$V_T(x_T) = \max_{h_T} [h_t + 0]$$

This means that in period $t = T$, it would be optimal to extract whatever is left of the resource ($h_T^* = x_T$), thus making the period-T DPE:

$$V_T(x_T) = x_T$$

2.2 Period-$T - 1$

On the previous period, the DPE would be given by:

$$V_{T-1}(x_{T-1}) = \max_{h_{T-1}} [h_{T-1} + \delta V_T(x_T)]$$

Since we know that $V_T(x_T) = x_T$, then the above becomes:

$$V_{T-1}(x_{T-1}) = \max_{h_{T-1}} [h_{T-1} + \delta x_T]$$

And in order to have stock size x_T, it means that the previous period we had stock size x_{T-1} and harvested h_{T-1} which then grew to x_T (i.e. $x_T = f(x_{T-1} - h_{T-1})$). Therefore:

$$V_{T-1}(x_{T-1}) = \max_{h_{T-1}} [h_{T-1} + \delta f(x_{T-1} - h_{T-1})]$$

By taking the first derivative, we get that:

$$f'(x_{T-1} - h_{T-1}) = \frac{1}{\delta}$$

The intuition is that if I get more returns from harvesting now than from leaving some to grow and harvest it next period, I will keep fishing. Whatever's left after harvest from $T - 1$ is the residual stock (y_t), in this case denoted by $y_{T-1}^* = x_{T-1} - h_{T-1}^*$. The period-$T - 1$ DPE then becomes:

$$\begin{aligned} V_{T-1}(x_{T-1}) &= h_{T-1}^* + \delta f(y_{T-1}^*) \\ &= x_{T-1} - y_{T-1}^* + \delta f(y_{T-1}^*) \end{aligned}$$

This says that in period $T - 1$ my value is given by this period's harvest ($h_{T-1}^* = x_{T-1} - y_{T-1}^*$) + the discounted value of leaving something (y_{T-1}^*) for the next period. I can re-write the above equation by stating a constant $A \equiv -y_{T-1}^* + \delta f(y_{T-1}^*)$ so that it becomes:

$$V_{T-1}(x_{T-1}) = x_{T-1} + A$$

Period-$T - 2$

If we go back to the third-to-last period, the DPE is given by:

$$\begin{aligned} V_{T-2}(x_{T-2}) &= \max_{h_{T-2}} [h_{T-2} + \delta V_{T-1}(x_{T-1})] \\ &= \max_{h_{T-2}} [h_{T-2} + \delta(x_{T-1} + A)] \\ &= \max_{h_{T-2}} [h_{T-2} + \delta f((x_{T-2} - h_{T-2}) + A)] \end{aligned}$$

2

First order conditions give:

$$1 = \delta f'(x_{T-2} - h_{T-2})$$

$$\frac{1}{\delta} = f'(x_{T-2} - h_{T-2})$$

$$1 + r = f'(x_{T-2} - h_{T-2})$$

Again, this indicates that I will leave remaing stock y^*_{T-2} after taking h^*_{T-2} from x_{T-2}. My DPE becomes:

$$V_{T-2}(x_{T-2}) = x_{T-2} + B$$

Up to a constant, where $B = \delta \left[f(y^*_{T-2} + A) \right] - y^*_{T-2}$, this equation has the same form as the one for period $T - 1$. Therefore we can say that our period-t value function is:

$$V_t(x_t) = x_t + \theta_t$$

Based on the last two first order conditions taken for the DPE at $T - 1$ and $T - 2$, it becomes evident that it is optimum for one to harvest down to:

$$\delta f'(x_t - h_t) - 1 = 0$$

3   Period-t policy function and p

Assuming a linear payoff function, returns are given by the product of a constant price p and my harvest h_t:

$$\pi(h_t) = ph_t$$

In this case p is just a scaling parameter that should not affect my optimum policy function (provided that p is constant through time). Again, we can use backward induction to show this.

3.1   Period-T

At period T, my DPE is:

$$V_t(x_t) = \max_{h_t} \left[ph_t + \delta V_{t+1}(x_{t+1}) \right]$$

$$= \max_{h_t} \left[ph_t + 0 \right]$$

Which has first order condition of $h^*_T = xT$ leading to a value function with the form

$$V_T(x_T) = px_T$$

.

3.2 Period-$T-1$:

$$V_{T-1}(x_{T-1}) = \max_{h_{T-1}} \left[ph_{T-1} + \delta V_T(x_T) \right]$$
$$= \max_{h_{T-1}} \left[ph_{T-1} + \delta p x_T \right]$$

factor out p

$$= p \left[\max_{h_{T-1}} \left[h_{T-1} + \delta x_T \right] \right]$$
$$= p \left[\max_{h_{T-1}} \left[h_{T-1} + \delta f(x_{T-1} - h_{T-1}) \right] \right]$$

Our first order contition is the same as in the previous exercise. With p being independent of harvest and stock size, it is just a constant so the first order condition is still the same, but multiplied by p:

$$0 = 1 - \delta f'(y_t^*)$$
$$\frac{1}{\delta} = f'(y_t^*)$$

Yielding a value function of

$$V_{T-1}(x_{T-1}) = ph_{T-1}^* + \delta f'(y_{T-1}^*)$$
$$= p(x_{T-1} - y_{T-1}^*) + \delta f'(y_{T-1}^*)$$
$$= px_{T-1} + \gamma$$

Where $\gamma = -py_{T-1}^* + \delta f'(y_{T-1}^*)$

Going back to $T-2$ I should see this hold:

$$V_{T-2}(x_{T-2}) = \max_{h_{T-2}} \left[ph_{T-2} + \delta V_{T-1}(x_{T-1}) \right]$$
$$= \max_{h_{T-2}} \left[ph_{T-2} + \delta(px_{T-1} + \gamma) \right]$$

factor out p

$$= p \left[\max_{h_{T-2}} \left[h_{T-2} + \delta \left(x_{T-1} + \frac{\gamma}{p} \right) \right] \right]$$
$$= p \left[\max_{h_{T-2}} \left[h_{T-2} + \delta \left(f(x_{T-2} - h_{T-2}) + \frac{\gamma}{p} \right) \right] \right]$$

Maximizng the above does not include p (it's only present as a constant $\frac{\gamma}{p}$), which indicates that the policy function is independent of a *constant* price p. If price were a function of h_t or x_t this would likely need a different optimization approach.

4 Non-linear payoff function

Assuming a period-t payoff function of the form $\pi_t(h_t) = \alpha h_t - \beta h_t^2$ (with $\alpha > 0$ and $\beta > 0$) and *guessing* that the value function is a linear function of the state (*i.e.* $V_t(x_t) = \phi x_t + \theta_t$), my DPE would then become:

$$V_t(x_t) = \max_{h_t} \left[\pi_t(h_t) + \delta V_{t+1}(x_{t+1}) \right]$$
$$= \max_{h_t} \left[\pi_t(h_t) + \delta(\phi x_t + \theta_{t+1}) \right]$$
$$= \max_{h_t} \left[\pi_t(h_t) + \delta(\phi f(x_t - h_t) + \theta_{t+1}) \right]$$

Which would have first order conditions of:

$$\pi_t'(h_t) - \delta\phi f'(x_t - h_t) = 0$$
$$\pi_t'(h_t) = \delta\phi f'(x_t - h_t)$$
$$\frac{\alpha + 2\beta h_t}{\delta\phi} = f'(x_t - h_t)$$

At optimal harvest h_t^* this is:

$$\frac{\alpha + 2\beta h_t^*}{\delta\phi} = f'(x_t - h_t^*)$$
$$\frac{\alpha + 2\beta h_t^*}{\delta\phi} = f'(y_t^*)$$
$$\Omega h_t^* = f'(y_t^*)$$

This form shows that the grwoth rate of the optimal residual stock $f'(y_t^*)$ is not a constant as before, but that it depends on the optimal level of harvest h^*t multiplied by a constant $\Omega = \frac{\alpha+2\beta}{\delta\phi}$. In this case, the state is linear in the DPE.

5 Numerical solution to stochastic evolution in stock

In a deterministic scenario, the current payoff function allows us derive the policy and value functions with a numerical approach. Figure 1 shows how every time I take a step back (indicated by colors) the policy and value function converge. The policy function takes about 11 steps to visually converge, while the value function takes much more.
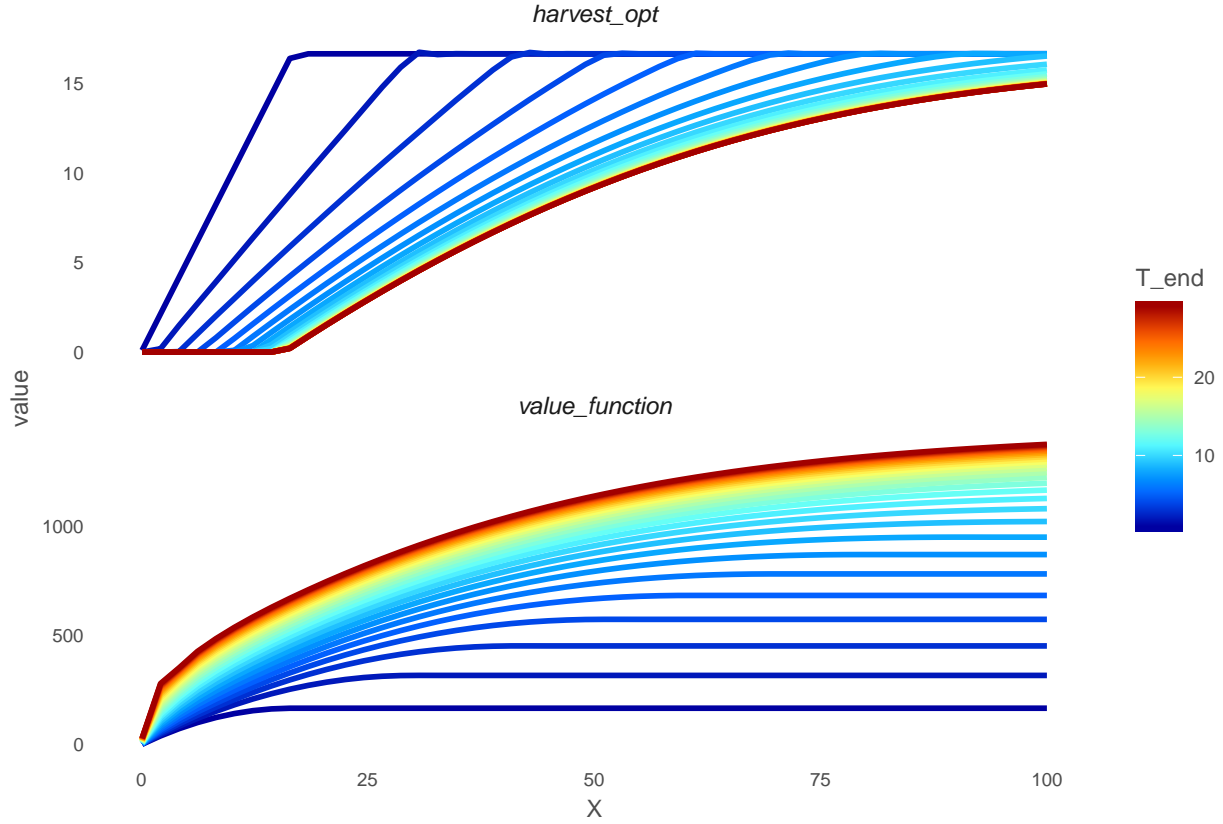
Figure 1: Numerical solution of a tereministic case, showing convergence of the policy function and value functions, top and bottom respectively.

Now, this might change once we account for the stochastic growth in our stock $(x_{t+1} = zfx_t)$, determined by a stochastic term:

$$z = \begin{cases} \text{if } p < 0.5 \text{ then } 1 + \theta \\ \text{if } p > 0.5 \text{ then } 1 - \theta \end{cases} \quad \text{and} \, p = B(n = 1, p = 0.5)$$

My initial thought was that since each state has equal probability of occurring, on average they would cancel out such that the optimal policy function would be the average of the two states. However, 2 shows how the optimal deterministic policy function is the same as when $z > 1$ for low values of X. On the opposite side, the optimal deterministic policy function is the same as when $z < 1$ for high values of X. This suggests that the timing of positive or negative stochastic events has different influences depending on the state of the resource and the time period in which we are.
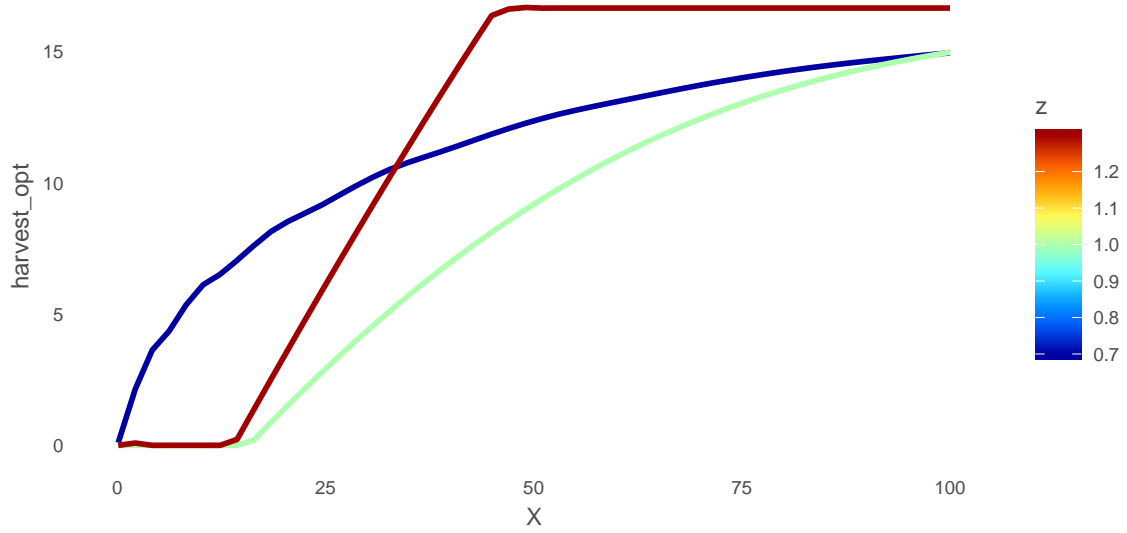
Figure 2: Comparison of running three different scenarios, where z = {0.7, 1, and 1.3} throughout the entire simulation.

To inspect the relationship between θ, r, δ and α I will do the following:

- Define a space for parameters, where the default value used before is centered
- Hold all other parameters constant at the default values
- Run all possible combinations of Xgrid and the parameter of interest
- Create surface plots and h(x) plots for visual inspection

Figure 3 shows the relationship between θ (represented by z) and the harvest policy. We see that, all else constant, values of $z > 1$ result in higher levels of harvest for lower stock sizes for population sizes $X > 15$, while values of $z < 1$ lead to overall lower optimal harvest across a range of X. Figure 4 shows the relationship between the intrinsic population growth rate (r) and the harvest rule. The differences are not as observable as before, but it is evident that higher values of r lead to larger harvests, even for smaller stock sizes as compared to low r values. Figure 5 shows the relationship between δ and the policy function. Higher discount factors lead to lower optimum harvests across all stock sizes. These differences are particularly large toward small population sizes, where a high discount factor may result in effectively no harvest for small populations. Finally, Figure 6 shows the relationship between α and the policy function. The relationship seems to be quite uniform across a range of small (*i.e.* $X_t < K/2$) stock sizes. However, as the stock grows, larger values of α lead to larger optimum harvests.
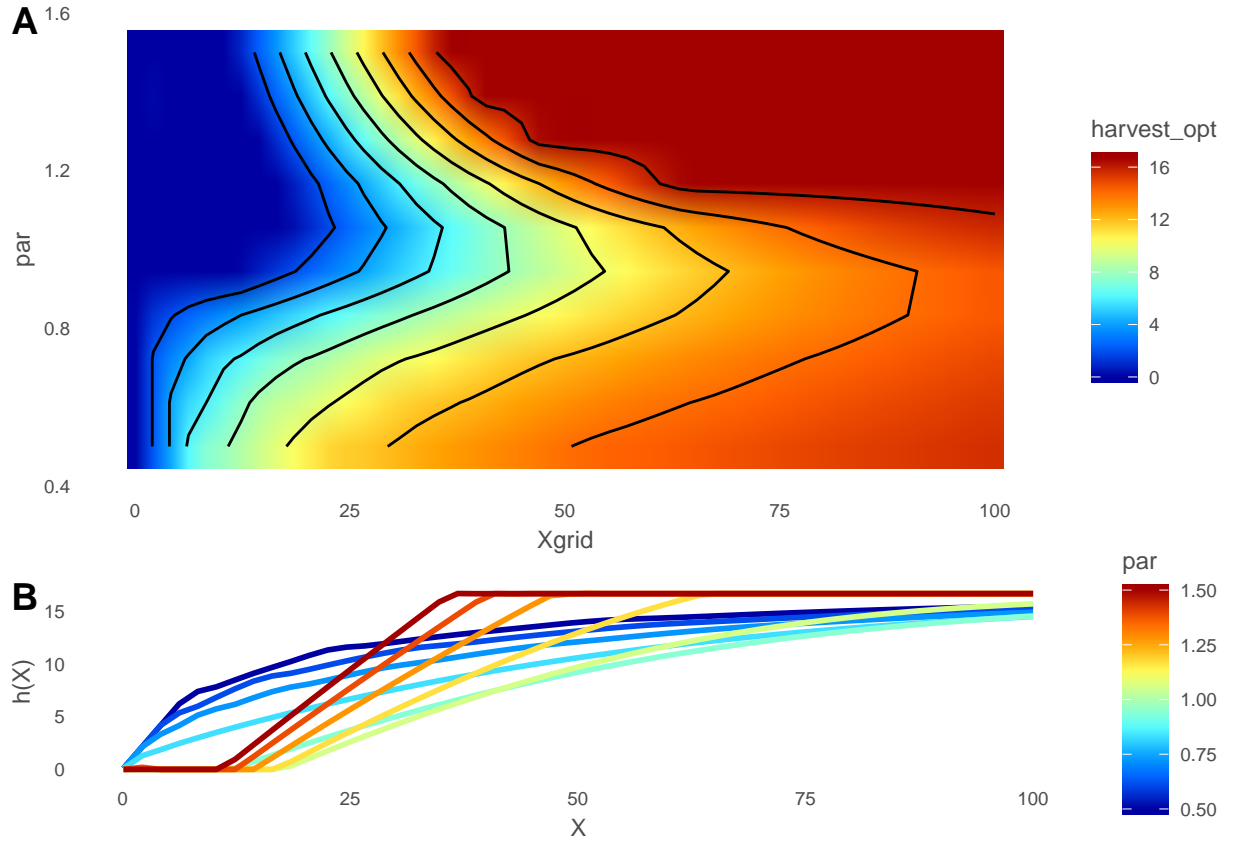
Figure 3: Relationship between z and the policy function. A shows the surface of h* for a combination of X and z. B shows the h(X) with colors for different parameter values.

Figure 4: Relationship between r and the policy function. A shows the surface of h* for a combination of X and r. B shows the h(X) with colors for different parameter values.
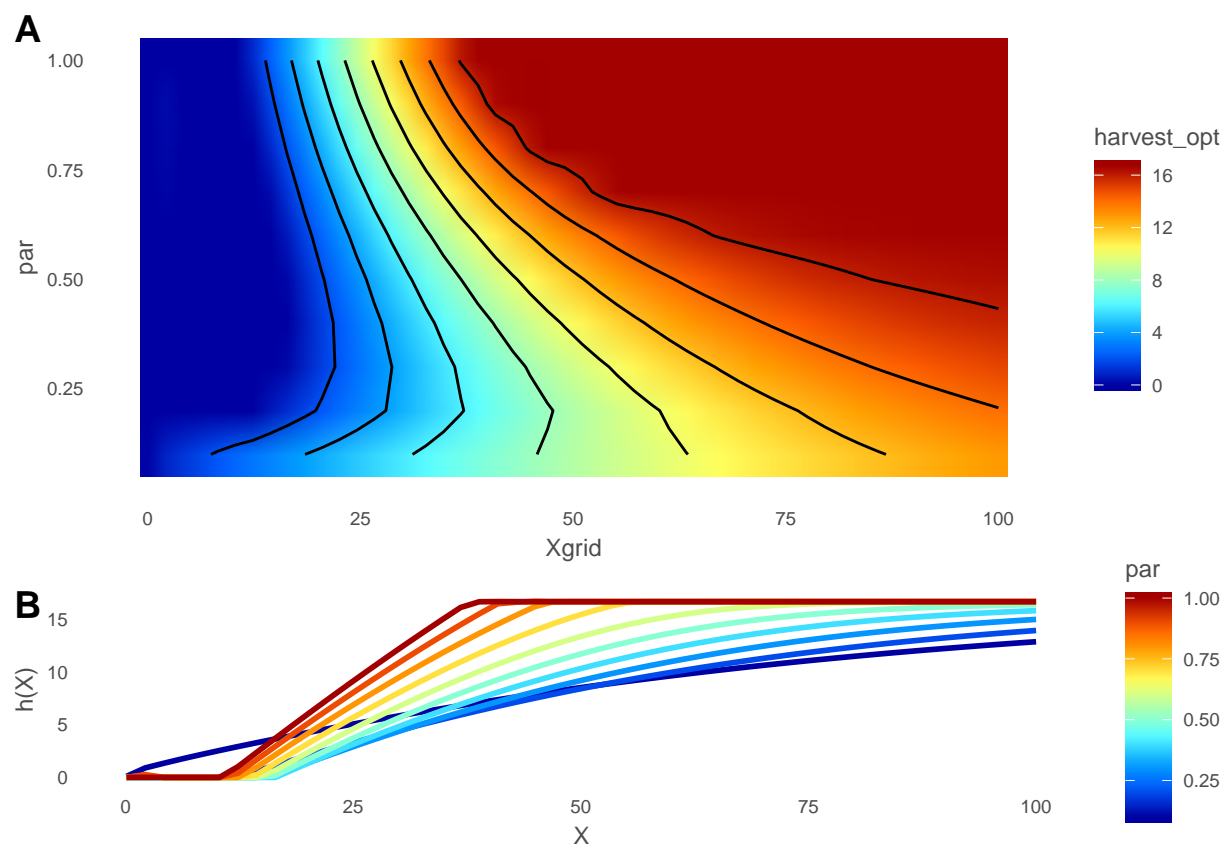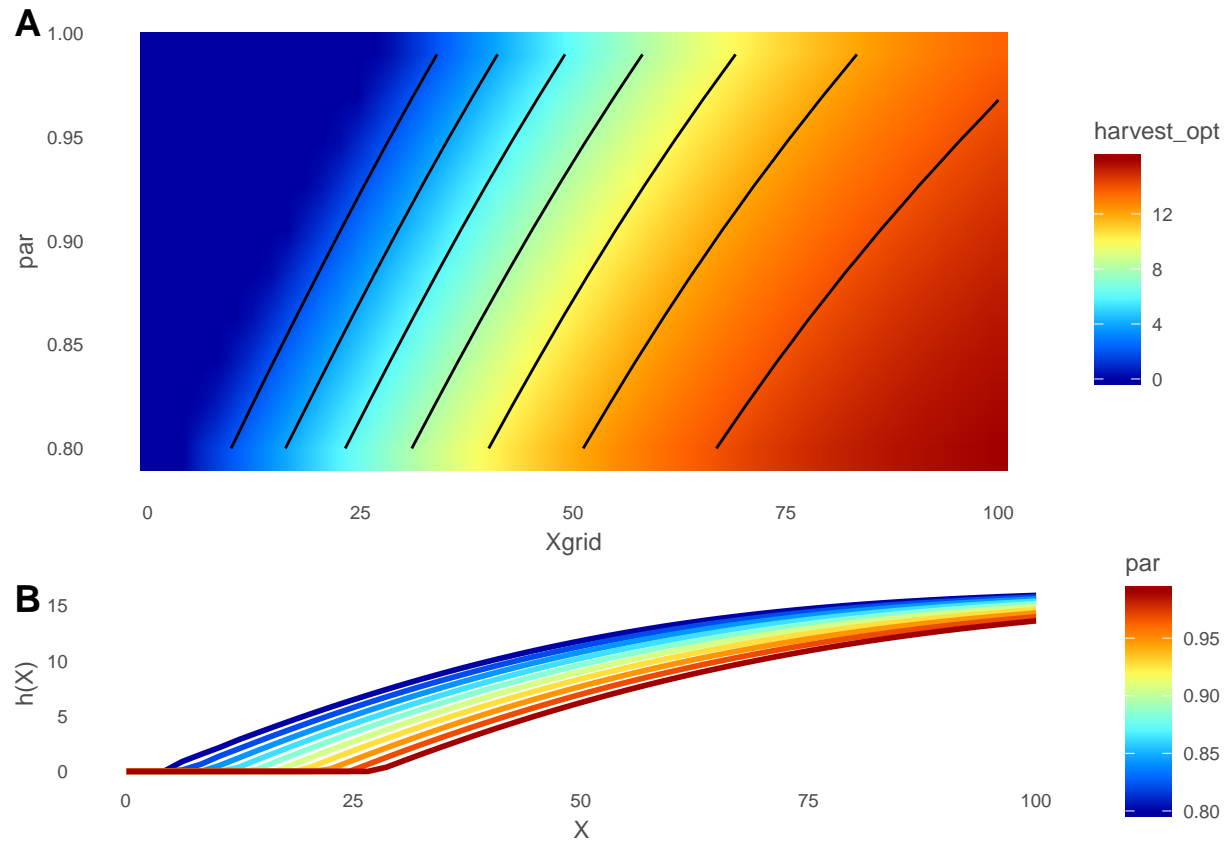
Figure 5: Relationship between delta and the policy function. A shows the surface of h* for a combination of X and delta. B shows the h(X) with colors for different parameter values.
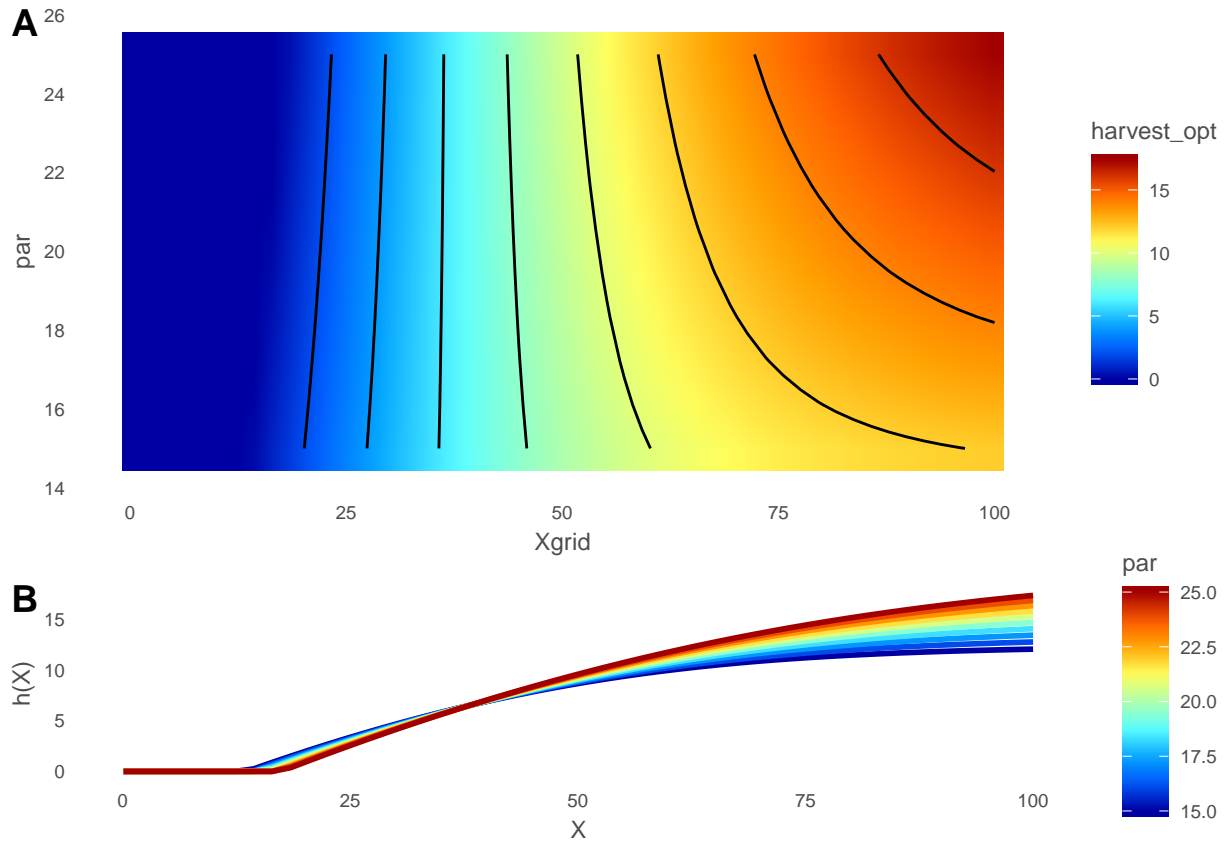
Figure 6: Relationship between a and the policy function. A shows the surface of h* for a combination of X and a. B shows the h(X) with colors for different parameter values.
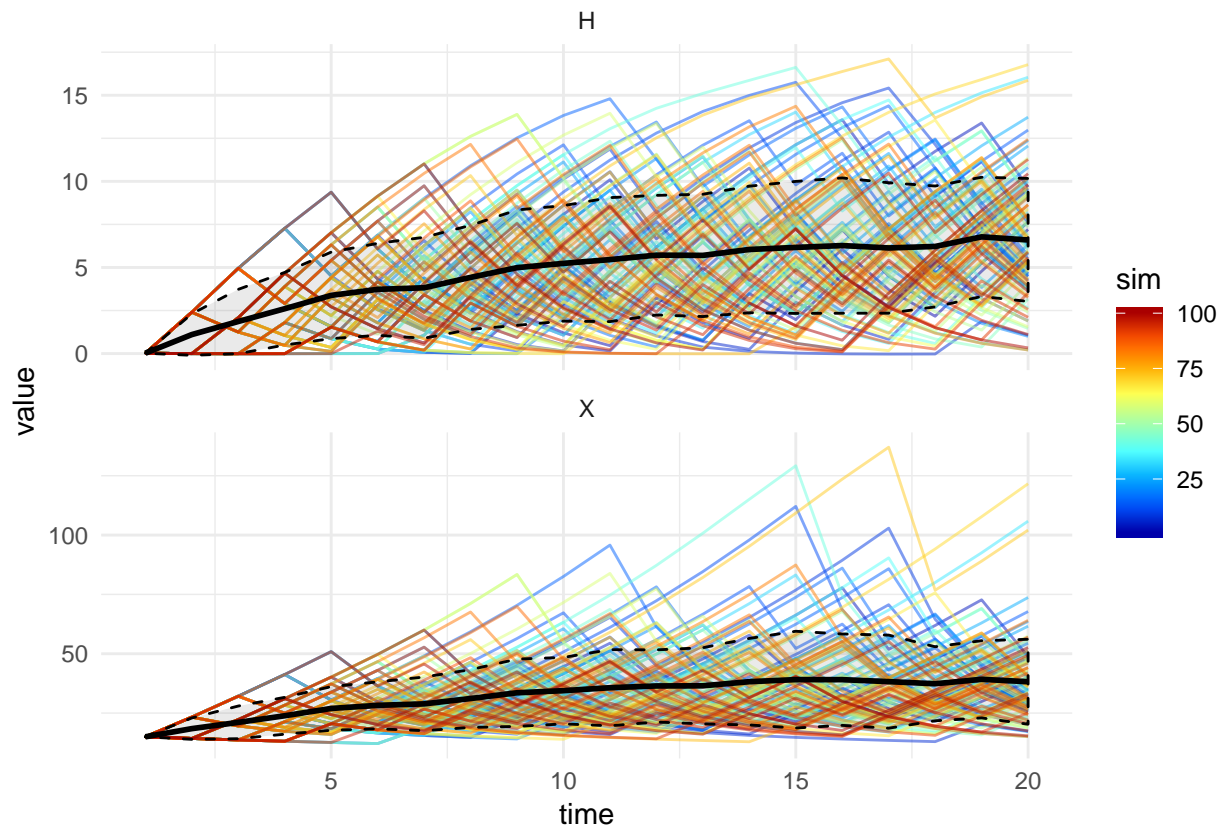
6   Project into the future

Figure 7: 10 simulations for 20 periods. Each simulation is plotted in the background with a different color. The solid dark line represents the mean, and the dashed lines the 1 SD space around the mean.