1 Econ 260A. Fall 2018. Homework Challenge 2 Suggested Answers

1. An exhaustible resource is costlessly extracted in continuous time by competitive firms (assume all the other assumptions of the Hotelling model apply). The market inverse demand curve is $p(y) = a - by$ where $p(t)$ and $y(t)$ denote the price and quantity in time t and a and b are finite parameters. The initial stock of the resource is given by $x(0) = x_0$, the interest rate is r, and the time horizon is $0 \leq t \leq T$.

(a) Use optimal control theory to characterize the dynamic market equilibrium. Clearly state all of the first-order and transversality conditions that must hold.

The competitive market equilibrium is found by solving the social planner's problem:

$$\max \int_0^T U(y(t))e^{-rt}dt \qquad s.t. \qquad \dot{x}(t) = -y(t), x(0) = x_0 \tag{1}$$

where:

$$U(y(t)) = \int_0^{y(t)} (a - bs)ds \tag{2}$$

The current value Hamiltonian is $\tilde{H} = U(y(t)) - \mu(t)y(t)$ and the maximum principle gives:

$$\tilde{H}_y = U'(y(t)) - \mu(t) = p(t) - \mu(t) = a - by(t) - \mu(t) = 0 \tag{3}$$

$$\dot{\mu}(t) - r\mu(t) = -\tilde{H}_x = 0 \tag{4}$$

$$\dot{x}(t) = \tilde{H}_\mu = -y(t) \tag{5}$$

As discussed in class, the transversality conditions require $\tilde{H}(T) = 0$ and $\mu(T)x(T) = 0$, which implies $y(T) = 0$ and $x(T) = 0$. By manipulating the first-order conditions, we get Hotelling's rule: $p(t) = p(0)e^{rt}$.

(b) Use your results from part (a) to derive an implicit function for T in terms of the parameters of the problem (a, b, x_0, r). Show how T changes with marginal changes in each of the parameters and discuss the economic intuition for each comparative static result.

For the particular functional form of $p(y)$ and using $y(T) = 0$, we get $p(0) = ae^{-rT}$ and $p(t) = ae^{-r(T-t)}$. Using this result we get an expression for $y(t)$:

$$y(t) = \frac{a(1 - e^{-r(T-t)})}{b} \tag{6}$$

The condition $x(T) = 0$ means that the initial stock x_0 must be used up by T, or:

$$x_0 = \int_0^T y(t)dt = \int_0^T \frac{a(1 - e^{-r(T-t)})}{b}dt \tag{7}$$

Solving the integral gives:

$$F(a,b,r,x_0,T) = \frac{a}{b}\left[T - \frac{1}{r}(1-e^{-rT})\right] - x_0 \equiv 0 \tag{8}$$

By the Implicit Function Theorem,

$$\frac{dT}{dz} = -\frac{F_z}{F_T} \tag{9}$$

where z is one of the parameters. The denominator in this expression is $F_T = \frac{a}{b}[1-e^{-rT}] > 0$ and so

$$sign\left[\frac{dT}{dz}\right] = -sign[F_z] \tag{10}$$

Thus, we have

$$sign\left[\frac{dT}{da}\right] = -sign\left[\frac{1}{b}\left(T - \frac{1}{r}(1-e^{-rT})\right)\right] < 0 \tag{11}$$

One can verify that the term in large parentheses is positive by showing that:

$$\frac{rT}{1-e^{-rT}} > 1 \tag{12}$$

for any $rT > 0$. A higher intercept on the demand curve implies that prices have to be higher because, otherwise, the resource would be exhausted before the choke price is hit. A new price path is steeper and, thus, cannot cross the original price path, implying that T must occur sooner. Next, we have

$$sign\left[\frac{dT}{db}\right] = sign\left[\frac{a}{b^2}\left(T - \frac{1}{r}(1-e^{-rT})\right)\right] > 0 \tag{13}$$

A steeper demand curve implies that prices have to fall because, otherwise, there will be resource left when the choke price is hit. The new price path is flatter and, thus, cannot cross the original price path, implying that T must occur later. Next, we have

$$sign\left[\frac{dT}{dx_0}\right] = sign[1] > 0 \tag{14}$$

An increase in the initial stock implies that prices have to fall because, otherwise, the choke price would be hit before the resource is exhausted. As with the steeper demand curve, T has to occur later. Finally, we have

$$sign\left[\frac{dT}{dr}\right] = -sign\left[\frac{a}{b}\left(\frac{1}{r^2}(1-e^{-rT}) - \frac{T}{r}e^{-rT}\right)\right] < 0 \tag{15}$$

The term in large parentheses is positive if $rT > ln(1+rT)$, which holds for any $rT > 0$. An increase in the interest rate implies the initial price has to fall because, otherwise, the more rapid increase in prices will mean that the choke price is hit before all of the resource is exhausted. However, while the initial price is lower, the greater rate of price growth implies that the new price path will cross the old price path, implying that T will occur sooner.

The price paths have to cross because, otherwise, the new prices would be too low and the resource would be exhausted before the choke price is hit.