ECON 260A: Basic Renewable Resource Model

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1 Model

The canonical renewable resource economics model is an infinite horizon model with harvest, concave resource growth, and instantaneous utility that is concave in harvest:

$$\max_{h(t)} \int_0^\infty U(h(t))e^{-rt}dt \qquad \text{s.t.} \qquad \dot{x} = F(x) - h(x), \quad x(0) = a \tag{1}$$

Where $U(h)$ is increasing and concave: $U'(h) > 0$, $U''(h) < 0$.

The Hamiltonian is:

$$H = U(h(t))e^{-rt} + \lambda(t)[F(x(t)) - h(t)] \tag{2}$$

We could also write the Hamiltonian it is "current value" form, which is the undiscounted value of the Hamiltonian:

$$\tilde{H} = U(h(t)) + \tilde{\lambda}(t)[F(x(t)) - h(t)] \tag{3}$$

Note that the shadow value for the Hamitonian is different than the shadow value for the current value Hamiltonian, as follows:

$$\tilde{\lambda}(t) = \lambda(t)e^{rt} \tag{4}$$

I will work with $\tilde{H}$ to provide an example. It would be a good exercise to repeat the analysis with H. First, we want to maximize $\tilde{H}$ with respect to harvest:

$$\frac{d\tilde{H}}{dh(t)} = U' - \tilde{\lambda} = 0 \tag{5}$$

Note for future use that we can take the time-derivative of this, which gives:

$$U''\dot{h} - \dot{\tilde{\lambda}} = 0 \tag{6}$$

The derivative of $\tilde{H}$ with respect to the state must satisfy:

$$\dot{\tilde{\lambda}} - r\tilde{\lambda} = -\frac{d\tilde{H}}{dx} = -\tilde{\lambda}F' \tag{7}$$

Substituting Equation 6 into Equation 7, and making use of Equation 5, we get an "equation of motion" for the optimal control, $h(t)$:

$$\dot{h} = \frac{U'}{U''}[r - F'] \tag{8}$$

We have derived a system of two differential equations in h and x:

$$\begin{aligned} \dot{h} &= \frac{U'(h)}{U''(h)}[r - F'(x)] && (9) \\ \dot{x} &= F(x) - h && (10) \end{aligned}$$

These equations are amenable to phase plane analysis.

2 Phase Plane Analysis

We first derive the nullclines: all the places in (h,x) space where one of the variables is not changing. The $\dot{h} = 0$ isocline is the line where $F'(x) = r$. Note that this is independent of h so it is a vertical line. The $\dot{x} = 0$ isocline is the curve where $h = F(x)$, which is just the biological growth curve. The steady state occurs at the intersection of these nullclines: when $\dot{h} = \dot{x} = 0$, i.e. when:

$$\begin{aligned} F'(x^*) &= r \qquad \text{and} && (11) \\ h^* &= F(x^*) && (12) \end{aligned}$$

Where an asterisk indicates steady state.

Now that we have the nullclines (and have divided the space into *isosectors*), we can derive the directional arrows.

2.1 Right of $\dot{h} = 0$

Recall that $\dot{h} = \frac{U'(h)}{U''(h)}(r - F'(x))$. Invoking the assumptions on $U(h)$, we know that $\frac{U'}{U''} < 0$. When $\dot{h} = 0$, we know that $F' = r$. If we move to the right (if we increase x), then $r - F' > 0$, so $\dot{h} < 0$. This implies that if we move to the right of $\dot{h} = 0$, then h is decreasing. This also implies that if we move left of $\dot{h} = 0$, then h is increasing. This provides all of the "up/down" directional arrows in all isosectors.

2.2 Above $\dot{x} = 0$

Recall that $\dot{x} = F(x) - h$. When $\dot{x} = 0$, this implies that $F(x) = h$. If we move up (i.e. increase h), then $\dot{x} < 0$, which implies that x is declining. If we move down, then x is increasing. This provides all of the "left/right" directional arrows in all isosectors.

2.3 Optimal Policy

The optimal policy is a convergent "separatrix" that goes from isosectors 1 and 3 to the steady state. It is shown in bold blue in the figure. Note that the steady state is a saddle point - if you are off the optimal policy, you will diverge from the steady state.

Figure 1: Phase plane for the renewable resource model.