POLS 207

Problem Set 1

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Problem 1: Thinking about Potential Outcomes

a) Consider binary treatment $D_i \in \{0, 1\}$, observed outcome variable Y_i. What is the meaning of Y_{1i} and Y_{0i}? Describe both in words, and choose an example to illustrate.

Y_{1i} represents the *potential* value that we would observe for a variable of interest Y if unit i receives the treatment (*i.e.* $D_i = 1$). Converseley, Y_{0i} represents the *potential* value that we would observe for a variable of interest Y if unit i **does not** receive the treatment (*i.e.* $D_i = 0$).

These potential outcomes can be exemplified by a variable of interest such as Y_i which represents the total fishing hours by vessel i. The binary treatment D_i represents a policy intervention (implementation of Individual Transferable Quotas: ITQ) randomly applied to the vessels. Y_{0i} represents the potential fishing hours of vessel i if it is not within the randomly selected group of vessels that operate under ITQ. However, if that same vessel were randomly selected to participate in the ITQ program, its fishing hours woulod be characterized by Y_{1i}. The implementation of ITQs provides fishers with a sense of ownership, which reduces "the race to fish". Therefore we would expect $Y_{1i} < Y_{0i}$.

b) What is the difference between Y_{1i} and "Y_i for a unit that actually received the treatment"? Explain the difference using the example you began in section a.

Y_{1i} is the treated potential outcome for a unit. However, this same unit also has an untreasted potential outcome Y_{0i}. Conditional on receiving the treatment, $Y_i = Y_{1i}$.

The observed outcome can be written as a function of potential outcomes and treatment status:

$$Y_i = D_iY_{1i} + (1 - D_i)Y_{0i}$$

The statement "a unit that actually received the treatment" implies $D_i = 1$. Therefore:

$$\begin{aligned}
Y_i &= D_iY_{1i} + (1 - D_i)Y_{0i} \\
&= (1)Y_{1i} + (1 - (1))Y_{0i} \\
&= (1)Y_{1i} + (0)Y_{0i} \\
&= (1)Y_{1i} + 0 \\
Y_i &= Y_{1i}
\end{aligned}$$

c) Define the average treatment effect (ATE) and average treatment effect among the treated (ATT) using potential outcomes notation. Describe in words what each quantity means.

ATE

The average treatment effect in the potential outcomes notation is given by:

$$ATE = \mathbb{E}[Y_{1i} - Y_{0i}]$$

For an individual unit i, the treatment effect τ_i is just the difference between its potential outcomes Y_{1i} and Y_{0i}. For the entire population of interest, the average treatment effect is given by the expectation of this τ_i value across all units. In other words,the ATE is given by the difference of potential outcomes for all i's, independent on the treatment status D_i.

ATT

The average treatment effect on the treated is given by:

$$ATT = \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1]$$

This estimand focuses *only* on treated units (*i.e.* units for which $D_i = 1$). It represents the expected difference in potential outcomes for *treated* units only.

d) When will the ATT and the ATE be the same? Prove it.

As stated above, ATE represents the expected differences in potential outcomes for all units, while ATT represents this same measure but for treated units only. If D_i is assigned at random, one can expect $ATE = ATE$, the assertion is that:

$$\mathbb{E}[Y_{1i} - Y_{0i}] = \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1]$$

The observed difference in means with selection bias is given by:

$$\mathbb{E}[Y_{1i} - Y_{0i}] = \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] + (\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0])$$

Where the first term is the ATT second therm on the right-hand side (the one in parentheses) is the bias term. Since Y_{0i} and D_i are independent, the last term on the right-hand side ($\mathbb{E}[Y_{0i}|D_i = 0]$) can be expressed as $\mathbb{E}[Y_{0i}|D_i = 1]$. This means that, if the potential untreated outcome values are independent of D_i, then the values of the "selected" or "unselected" are equal in expectation, and the terms are interchangable such that the equation becomes:

$$\begin{aligned}
\mathbb{E}[Y_{1i} - Y_{0i}] &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] + (\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 1]) \\
\mathbb{E}[Y_{1i} - Y_{0i}] &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] + (0)
\end{aligned}$$

The right-hand side of the above equation is now just the ATT. This now states that the $ATE = ATT$.

Problem 2: Potential Outcomes with Interference

a) You decide to conduct a get-out-the-vote experiment. For your treatment, you encourage a randomly assigned set of voters to put up yard signs supporting a particular candidate. Your hypothesis is that people who are asked to publicly declare their support for a candidate are more likely to remember to vote. Is this experiment likely to violate the SUTVA assumption? Why or why not?

Yes, this is likely to violate SUTVA if the untreated units can also see the signs. When treated individuals put up yard signs and declare their support for a candidate, this is likely to also incentivize neighbours to go vote. An estimation of the ATE would result in a downard biased estimate of the true effect of putting up the signs.

b) Now suppose you decide to conduct a different get-out-the-vote experiment. With this experiment, you send a randomly assigned set of voters a mailer. These mailers tell each treated individual whether their neighbors voted or not in the last election. Is this experiment likely to violate the SUTVA assumption? Why or why not?

This experiment is less likely to violate SUTVA. Since the mailer is received privately and individuals are chosen at random, it is unlikely that untreated units will be affected by the mailer that a neighbour might have received. Of course, communication between neighbours may still lead to a violation of SUTVA.

c) Imagine you have the following study population. For this population, calculate the ATE, ATT, and ATC. Then calculate the ATE for the subgroup of odd-numbered cases.

Table 1: Data for potential outcomes of 5 observational units.

i	D_i	Y_{1i}	Y_{0i}	τ
1	0	6	5	1
2	1	-1	3	-4
3	1	2	2	0
4	0	5	2	3
5	1	2	-3	5

ATE

$$\begin{aligned} ATE &= \mathbb{E}[Y_{1i} - Y_{0i}] \\ &= \mathbb{E}[\tau_i] \\ &= \frac{1 - 4 + 0 + 3 + 5}{5} \\ &= \frac{5}{5} \\ &= 1 \end{aligned}$$

```r
mean(data$tau) # The mean of tau
```

```
## [1] 1
```

ATT

$$
\begin{aligned}
ATT &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] \\
&= \mathbb{E}[\tau_i|D_i = 1] \\
&= \frac{-4+0+5}{3} \\
&= \frac{1}{3} \\
&= 0.333
\end{aligned}
$$

```
# Calculate the mean of tau for treated units only
data %>%
  filter(Di == 1) %$%
  mean(tau)
```

```
## [1] 0.3333333
```

ATC

$$
\begin{aligned}
ATC &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 0] \\
&= \mathbb{E}[\tau_i|D_i = 0] \\
&= \frac{1+3}{2} \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

```
# Calculate the mean of tau for untreated units only
data %>%
  filter(Di == 0) %$%
  mean(tau)
```

```
## [1] 2
```

d) Assume some arbitrary treatment. Now suppose that a small boy's treatment status depends on the treatment status of his sister, his mother and his father. Write out the full set of potential outcomes for this boy. How many such potential outcomes are there? How many τ_i (*e.g.* unit-level treatment effects) can be defined for this situation?

The number of potential outcomes (n_y) is given by the dicotomous treatment state ($D \in (0, 1)$) raised to the number of treatment (n) "switches":

$$n_y = 2^n$$

When the potential outcomes depend only on one D_i, the above becomes $n_y = 2^1 = 2$. In this case, however, the treatment status depends on the boy's as well as sister's, mother's and father's treatment status. Let $D_{i,s}, D_{i,m}, D_{i,f}$ denote the treatment status of the sister, mother, and father, respectively. The small boy's treatment status ($D_{i,b}$) is therefore given by $D_{i,s}, D_{i,m}, D_{i,f}$). Therefore, $Y_i(D_{i,b}, D_{i,s}, D_{i,m}, D_{i,f})$. All possible combinations are given in the table below (Table 2).

Table 2: Combination of treatment status.

$D_{i,b}$	$D_{i,s}$	$D_{i,m}$	$D_{i,f}$	Y_i
0	0	0	0	$Y_i = (0,0,0,0)$
1	0	0	0	$Y_i = (1,0,0,0)$
0	1	0	0	$Y_i = (0,1,0,0)$
1	1	0	0	$Y_i = (1,1,0,0)$
0	0	1	0	$Y_i = (0,0,1,0)$
1	0	1	0	$Y_i = (1,0,1,0)$
0	1	1	0	$Y_i = (0,1,1,0)$
1	1	1	0	$Y_i = (1,1,1,0)$
0	0	0	1	$Y_i = (0,0,0,1)$
1	0	0	1	$Y_i = (1,0,0,1)$
0	1	0	1	$Y_i = (0,1,0,1)$
1	1	0	1	$Y_i = (1,1,0,1)$
0	0	1	1	$Y_i = (0,0,1,1)$
1	0	1	1	$Y_i = (1,0,1,1)$
0	1	1	1	$Y_i = (0,1,1,1)$
1	1	1	1	$Y_i = (1,1,1,1)$

These sixteen possible different outcomes imply that there are many more possible τ_i unit-level treatment effects. Our treatment effect is defined as the difference between to potential outcomes. The number of these possible combinations can be characterized by the permutation without duplication:

$$P(n,r) = \frac{n!}{(n-r)!r!}$$

Where $n - 16$ and $r = 2$. This indicates that there are 120 possible combinations of Yi depending on what we term as “treated” and “control” for the calculation of our τ_i.

e) It turns out that the boy and his sister get treated, by construction, jointly. Otherwise the same conditions apply. Now how many potential outcomes exist for the boy? How many τ_i are defined for this new situation?

This new design suposes that $D_{b,i} = D_{s,i} \forall i$. Therefore, there are only three possible “switches”: 1) kids get treated, mother’s treatment status and father’s treatment status. This produces $n_y = 2^n = 2^3 = 8$ possible combinations. The different potential outcomes for the boy are shown in Table 3. In this case, there are 28 different possible τ_i values:

$$\begin{aligned} P(n,r) &= \frac{n!}{(n-r)!r!} \\ &= \frac{8!}{(8-2)!2!} \\ &= 28 \end{aligned}$$

Table 3: Combination of treatment status.

$D_{i,b}$	$D_{i,s}$	$D_{i,m}$	$D_{i,f}$	Y_i
0	0	0	0	$Y_i = (0,0,0,0)$
1	1	0	0	$Y_i = (1,1,0,0)$
0	0	1	0	$Y_i = (0,0,1,0)$
1	1	1	0	$Y_i = (1,1,1,0)$
0	0	0	1	$Y_i = (0,0,0,1)$
1	1	0	1	$Y_i = (1,1,0,1)$
0	0	1	1	$Y_i = (0,0,1,1)$
1	1	1	1	$Y_i = (1,1,1,1)$

Problem 3: Potential Outcomes and the Difference in Means Estimand

a) Show that the difference in means estimand (the expectation of the difference in means estimator) can be decomposed into the ATT and a bias term. Derive the decomposition, showing your work – do not simply state it. Also describe clearly and concisely the meaning of the bias term.

The difference in means estimand is given by:

$$\tau = \mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0]$$

We do not observe both possible realizations of $Y_i|D_i$, but we do observe the treated outcome of treated units and the untreated outcome from untreated units. Our assumption is that we can compare the treated and untreated units to infer something about the difference in means:

$$\tau = \mathbb{E}[Y_{1i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$$

The above states that the difference in means equals the treated potential outcome of treated units minus the untreated potential outcome of treated units. However, there might be a bias due to non-random treatment assignment. To account for the bias, we can write the differencce in means as the sum of the pre-existing differences observed between *untreated* outcomes of two groups plus the ATT:

$$\tau = \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] + \mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$$

A way to think about the equation above is that the bias term corrects for any pre-existing differences in the untreated potential outcomes of both treated and untreated groups by accounting for their differences (*i.e.* $\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$).

A short example

There are six observational units i, with potential treated and untreated outcomes Y_{1i} and Y_{0i}, respectively. These outcomes depend on the treatment status of each unit $D_i \in (0,1)$. I will assume a a known $\tau_i = 2$, and I will non-randomly select units $i \in (1,3)$ as my treated group:

```
# Indices
x <- 1:6
# Untreated potential outcomes
y0 <- x
# Treated potential outcomes
y1 <- y0 + 2
# Non-randomly treated units
t <- 1:3

# Put into a data.frame
dat <- tibble(x = c(x, x),
              y = c(y0, y1),
              Yi = c(rep("Y0i", 6), rep("Y1i", 6)),
              observed = c(F, F, F, T, T, T, T, T, T, F, F, F))
# Visualize schedule of potential outcomes
ggplot(dat, aes(x, y, group = x)) +
  geom_line(linetype = "dashed") +
  geom_point(aes(alpha = observed, color = Yi), size = 5) +
  theme_bw() +
  scale_y_continuous(breaks = c(1:15)) +
  scale_alpha_manual(values = c(0.3, 1)) +
  startR::ggtheme_plot() +
  labs(x = "Unit index (i)", y = quo(Y[i])) +
  scale_color_brewer(palette = "Set1")
```

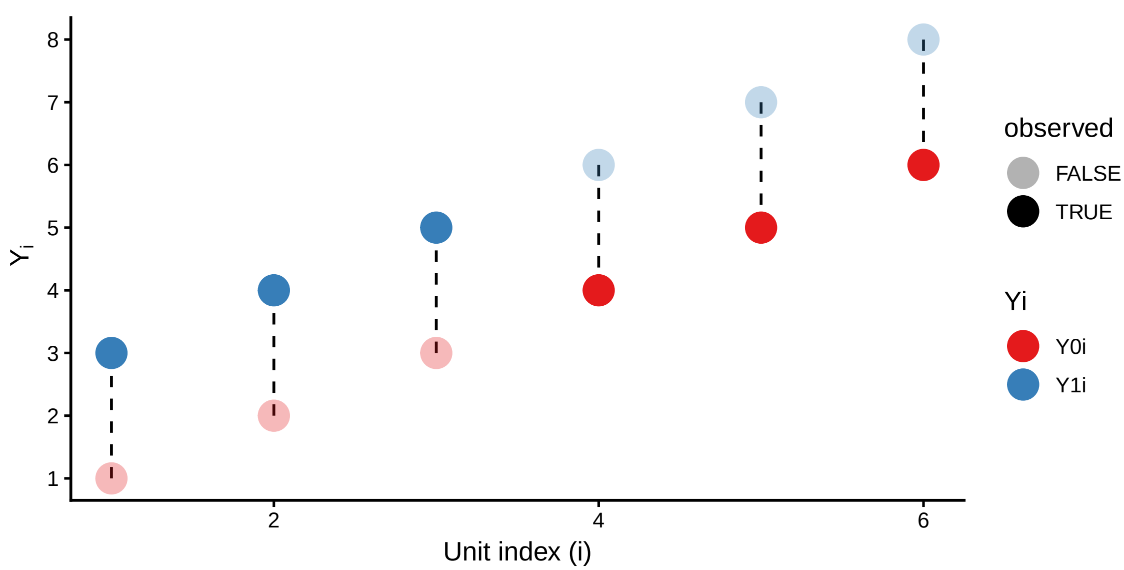

Figure 1: Schedule of potential outcomes. Blue circles represent treated potential outcomes (Y_{1i}) and red circles represent untreated potential outcomes (T_{0i})). The transparency of the circles indicates if the outcome is observable or not. The dashed lines connect each unit's potential outcomes.

From these, we can derive the expectations of each potential outcome (2 potential outcomes for every i) given a treatment status (2 potential treatment status), and the difference in means (*i.e.* $\mathbb{E}[Y_i|D_i = 1] - \mathbb{E}[Y_i|D_i = 0]$).

The expectation of treated outcome for treated units is $\mathbb{E}[Y_{1i}|D_i = 1] = \frac{3+4+5}{3} = 4$. The expectation of the untreated outcome for untreated units is $\mathbb{E}[Y_{0i}|D_i = 0] = \frac{4+5+6}{3} = 5$. These two are observable quantities.

The unobservable quantities are the expectation of untreated outcome for the treated units (*i.e.* $\mathbb{E}[Y_{0i}|D_i = 1] = \frac{1+2+3}{3} = 2$) as well as the expectation of the treated outcome for untreated units (*i.e.* $\mathbb{E}[Y_{1i}|D_i = 0] = \frac{6+7+8}{3} = 7$).

The difference in means is therefore:

$$\begin{aligned}
\tau &= ATT + \text{difference in untreated outcomes} \\
\tau &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 1] + (\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]) \\
&= 4 - 2 + (2 - 5) \\
&= 2 - 3 \\
&= -1
\end{aligned}$$

If we were tu compare only the observed treated outcome for treated units (4) and untreated outcome for untreated units (5), we would have arrived at the same value.

b) Show how the difference in means estimand can be decomposed into the average treatment effect on the controls (ATC) and a bias term. Again derive the decomposition and describe clearly the meaning of the bias term.

Following the same train of thought as above, we can show that the difference in means can be expressed as the *ATC* minus the difference in the *treated* potential outcomes of both treated and untreated groups of individuals ($\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$).

$$\tau = \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 0] + \mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$$

This accounts for the inherent differences in the treated outcomes of both groups. Using the values from the above figure, we would obtain the following:

$$\begin{aligned}
\tau &= ATC + \text{difference in treated outcomes} \\
\tau &= \mathbb{E}[Y_{1i} - Y_{0i}|D_i = 0] + \mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0] \\
&= 7 - 5 + (4 - 7) \\
&= 2 - 3 \\
&= -1
\end{aligned}$$

In this case, the bias terms corrects for the pre-existing differences in the potential treated outcomes of both treated and untreated groups ($\mathbb{E}[Y_{0i}|D_i = 1] - \mathbb{E}[Y_{0i}|D_i = 0]$).

c) *Harder* Now show how the difference in mean estimands can be decomposed into the ATE and a bias term. Again describe the meaning of the bias term.

Couldn't. I'll go to office hours next time.