Let Ω be a Lipschitz domain in $\mathbb{R}^m$ and let $\sigma \in L^\infty(\Omega)$ with $\operatorname{essinf}(\sigma) > 0$. In the pioneering paper [1] Calderón proposed the following question: is it possible to recover the function σ from boundary measurements of solutions of the differential equation $-\operatorname{div}(\sigma\nabla u) = 0$? As it was shown later, the answer to the question is affirmative if the Neumann-to-Dirichlet operator $\Lambda(\sigma)$ is known. That is, if $u|_{\partial\Omega}$ is known for each u solution of

$$\begin{cases} -\operatorname{div}(\sigma\nabla u) = 0, \\ \sigma\frac{\partial u}{\partial n} = g, \end{cases} \tag{1}$$

with g varying over $L^2_0(\partial\Omega)$. The problem is, however, ill-posed in the sense of Hadamard, which makes it difficult to design reconstruction numerical schemes. In [2] it was shown that if σ is restricted to a suitable finite dimensional space of $L^\infty(\Omega)$ then the reconstruction can be obtained as the solution of a convex optimization problem.

Suppose that Ω is divided into J disjoint pixels (connected open sets with Lipschitz boundary) $P_1, \ldots, P_J$, $\overline{\Omega} = \bigcup_{j=1}^J \overline{P}_j$. Moreover, assume the following:

Assumption 1. *The pixels are enumerated according to their distance to the boundary so that for each $j = 1, \ldots, J$ the set $\overline{\Omega} \setminus \left(\overline{P}_{j+1} \cup \cdots \cup \overline{P}_J\right)$ is connected and contains a non-empty open subset of $\partial\Omega$.*

We identify a vector $\sigma \in \mathbb{R}^n_+$ with a conductivity $\sigma(x) = \sum_j \sigma(j)\chi_{P_j}(x)$ and will usually restrict ourselves to σ belonging to the compact set $K = [a, b]^J$ for some fixed $b > a > 0$. Let $\Lambda(\sigma) \in L(L^2_0(\partial\Omega))$ be the Neumann-to-Dirichlet map associated to the differential expression $-\operatorname{div}(\sigma\nabla u) = 0$. One of the main results in [2] stating that σ can be characterized as the corner of certain convex set is shown below.

Theorem 2. *There exists $c \in \mathbb{R}^J_+$ such that for all $\sigma \in K$ the optimization problem*

$$\begin{aligned} &\min \;\; c\tau \\ &\text{subject to} \;\; \tau \in K \\ &\qquad\qquad\quad \Lambda(\tau) \leqslant \Lambda(\sigma) \end{aligned} \tag{2}$$

has σ as the unique minimizer.

In this work we study the structure of the feasible set $\{\tau \in K : \Lambda(\tau) \leqslant \Lambda(\sigma)\}$ for a given σ and then use that to conclude information about the weight vector c. We start by recalling well-known facts about Calderón's map $\sigma \mapsto \Lambda(\sigma)$.

Lemma 3. *Let $\Lambda : \mathbb{R}^J_+ \to L(L^2_0(\partial\Omega))$ be the Calderón's map. The following holds:*

1. *$\Lambda(\sigma)$ is a self-adjoint positive semi-definite compact operator for each $\sigma \in \mathbb{R}^J_+$.*
2. *Λ is monotonically decreasing. That is, if $\tau \geqslant \sigma$ component-wise then $\Lambda(\tau) \leqslant \Lambda(\sigma)$. In particular $\Lambda'(\sigma)(e_j) \leqslant 0$ for $j = 1, \ldots, J$.*
3. *Λ is convex, and in particular for $\tau, \sigma \in \mathbb{R}^J_+$ it holds that $\Lambda(\tau) - \Lambda(\sigma) \geqslant \Lambda'(\sigma)(\tau - \sigma)$.*
4. *For any $\sigma \in \mathbb{R}^J_+$, $j \in \{1, 2, \ldots, J\}$ and constants $d_{j+1}, \ldots, d_J \in \mathbb{R}$ the operator*

$$\Lambda'(\sigma)\left(e_j + d_{j+1}e_{j+1} + \cdots + d_Je_J\right)$$

is not positive semi-definite.

Proof. See [2]. $\square$

Construction of Weights

In this section we explore properties of the feasible set and their implications in terms of the weight vector c.

Lemma 4. *Let $\sigma, \tau \in \mathbb{R}^J_+$ such that $\Lambda(\tau) \leqslant \Lambda(\sigma)$. If $\sigma \neq \tau$ and $j \in \{1, 2, \dots, J\}$ is the first component where they differ then $\tau(j) > \sigma(j)$.*

Proof. Suppose by contradiction that $\delta = \tau(j) - \sigma(j) < 0$. By convexity of Λ in the form of part 3 in Lemma 3 we have

$$\begin{aligned}\Lambda(\tau) - \Lambda(\sigma) &\geqslant \Lambda'(\sigma)(\tau - \sigma)\\ &= \Lambda'(\sigma)\left(\delta e_j + \sum_{k>j}(\tau(k) - \sigma(k))e_k\right)\\ &\nleqslant 0\end{aligned}$$

where the last assertion follows from a localized potentials result as in part 4 of Lemma 3. Therefore, we have reached a contradiction $\Lambda(\tau) - \Lambda(\sigma) \nleqslant 0$ from what follows $\tau(j) \geqslant \sigma(j)$. ☐

Corollary 5. *Let $\sigma \in K$ and $j \in \{1, 2, \dots, J\}$. If $\tau \in K$ is defined by*

$$\tau(k) = \begin{cases} \sigma(k) & \text{if } k \leqslant j,\\ b & \text{otherwise,}\end{cases}$$

then $\Lambda(\tau) \leqslant \Lambda(\sigma)$ and $\Lambda(\tau - \delta e_j) \nleqslant \Lambda(\sigma)$ for any $0 < \delta < \sigma(j)$.

Proof. The first part follows by monotonicity (Lemma 3 part 2) of the map Λ since $\tau \geqslant \sigma$ while the second follows directly from Lemma 4. ☐

Corollary 5 says that, given the measurement $\Lambda(\sigma)$, one possible algorithm for recovering σ is to start with the conductivity $\tau = b\mathbb{1}$ and decrease one component at a time until τ is out of the feasibility region $\Lambda(\tau) \leqslant \Lambda(\sigma)$. It is important to note that care should be taken when approximating $\Lambda(\tau)$ by a finite-dimensional projection as the boundary conditions used to show part 4 in Lemma 3 may have arbitrarily large oscillations (see Figure 1). The result can be improved as shown below.

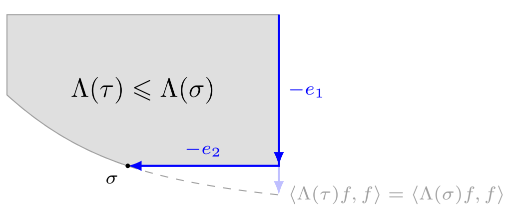

Figure 1: Two dimensional example of the feasible set K_σ.

We say that P_j is adjacent to P_k if $j \neq k$ and $\overline{P}_j \cap \overline{P}_k$ is a non-empty open relative set in $\partial P_j \cup \partial P_k$. This relation defines a graph G with vertices $P_1, \dots, P_J$ with a distinguished set $\partial G := \{P_j : P_j \text{ touches } \partial\Omega\}$.

Corollary 6. *Let $\sigma \in K$ and let $d : \{1, \dots, J\} \to \mathbb{N}$ defined by $d(j) = 1 + \mathrm{dist}_G(P_j, \partial G)$. Then, for each $1 \leqslant l \leqslant \max_j d(j) =: m$, any minimizer τ of*

$$\begin{aligned} \min \quad & \sum_{j \in d^{-1}(\{l\})} \tau(j) \\ \text{subject to} \quad & \tau \in K \\ & \Lambda(\tau) \leqslant \Lambda(\sigma) \\ & \tau(k) = \sigma(k) \text{ if } d(k) < l \end{aligned}$$

satisfies $\tau(j) = \sigma(j)$ for each j with $d(j) = l$.

Proof. Let $1 \leqslant l \leqslant m$ and fix j such that $d(j) = l$. If necessary relabel $P_1, \dots, P_J$ in such a way that P_j comes as the l-th element and Assumption 1 is satisfied. Then it follows that $d(k) = k$ for $k < l$ and therefore, any τ in the feasible satisfies $\tau(k) = \sigma(k)$ for $k < l$. Therefore, Lemma 4 implies that $\tau(j) \geqslant \sigma(j)$ for each j with $d(j) = l$, and the assertion follows. $\square$

Corollary 6, as Corollary 5, provides an algorithm for recovering σ by descending a group of coordinates at a time according to their distance to the boundary.

Example 7. Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$ and $P_j = \left\{ x \in \Omega : 0 < \|x\| < 1 \text{ and } \frac{2\pi(j-1)}{6} < \arg(x) < \frac{2\pi j}{6} \right\}$ for $j = 1, \dots, J$ with $J = 6$. Note that $d(j) = 1$ for each $j = 1, \dots, J$ (i.e. each pixel P_j touches the boundary) and therefore Corollary 6 implies that the weight vector c can be chosen as $c = \mathbb{1}$ An interior-point method for solving problem 2 can be implemented as described below. For each $\tau \in K$ let $\hat{\Lambda}(\tau)$ be a Galerkin projection of $\Lambda(\tau)$ with respect to the set $\{\sin(k\theta), \cos(k\theta)\}_{k=1}^{n_b}$ for some $n_b \in \mathbb{N}$. We consider the barrier function given by

$$B(\tau) = -\sum_j \log(b - \tau_j) - \sum_j \log(\tau_j - a) - \log(\det(Y - \hat{\Lambda}(\tau)))$$

where $Y = \hat{\Lambda}(\sigma)$ is the measurement associated to the unknown conductivity σ. The first two terms in the definition of B guarantee that τ stays within $K = [a, b]^J$ while the last term guarantees that $\hat{\Lambda}(\tau) \leqslant \hat{\Lambda}(\sigma)$. For each $t > 0$ we consider the (in principle unbounded) optimization problem

$$\min_{\tau \in \mathbb{R}^J} \quad t \sum_j \tau(j) + B(\tau). \tag{3}$$

We take $t \to \infty$, and for each fixed t we approximate the solution to (3) by a simple gradient-descent scheme. That is, if f_t is the objective function in (3), we update the iterates $\{\tau_n\}_n$ following the rule $\tau_{n+1} = \tau_n - \alpha \nabla f_t(\tau_n)$ where the step-size depends on t and possibly on n. A straightforward computations yields

$$\partial_j f_t(\tau) = -\frac{1}{\tau(j) - b} - \frac{1}{\tau(j) - a} - \mathrm{tr}\left((Y - \hat{\Lambda}(\tau))^{-1} \hat{\Lambda}'(\tau)(e_j) \right)$$

where $\Lambda'(\tau)$ (and therefore $\hat{\Lambda}'(\tau)$) can be computed using the fact that

$$\int_{\partial\Omega} g \Lambda'(\tau)(d) h \,\mathrm{d}s = -\int_\Omega d(x) \nabla u^g(x) \cdot \nabla u^h(x) \,\mathrm{d}x$$

where u^g and u^h solve $-\mathrm{div}(\tau \nabla u) = 0$ with Neumann boundary conditions g and h respectively. A FreeFEM++ [3] implementation can be found in this repository. If $\{t_n\}_n$ is chosen geometrically increasing by $t_0 = 0$, $t_1 = 1.5$ and $t_{n+1} = 1.01 t_n$ and for each n ten gradient descent steps are applied, then the error by epoch decreases as shown in Figure 2.

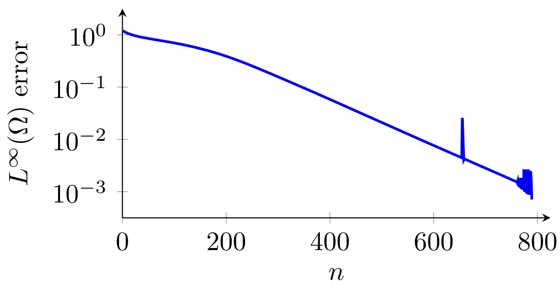

Figure 2: Error by iteration for problem (3).

If the functionals $c_1, \ldots, c_m$ are defined as $c_l(j) = 1$ if $d(j) = l$ and 0 otherwise,

$$\mathrm{face}_{c_m}\left(\mathrm{face}_{c_{m-1}}\left(\ldots\left(\mathrm{face}_{c_1}(K_\sigma)\right)\right)\right) = \{\sigma\}$$

where $K_\sigma = \{\tau \in K : \Lambda(\tau) \leqslant \Lambda(\sigma)\}$. The next conjecture is based on the fact, that if K' is a polytope and $c_1, \ldots, c_m$ are functionals, studying the normal fan to K' yields an $\varepsilon > 0$ such that

$$\mathrm{face}_{c_m}(\mathrm{face}_{c_{m-1}}(\ldots(\mathrm{face}_{c_1}(K')))) = \mathrm{face}_{c_1+\varepsilon c_2+\cdots+\varepsilon^{m-1}c_m}(K')\,.$$

Conjecture 8. There exists an $\varepsilon > 0$ such that for each $\sigma \in K$ the unique minimizer of

$$\begin{aligned} \min \quad & \sum_j \varepsilon^{d(j)-1}\tau(j) \\ \text{subject to} \quad & \tau \in K \\ & \Lambda(\tau) \leqslant \Lambda(\sigma) \end{aligned}$$

is σ. In other words, the weight vector can be chosen as $c = \sum_l \varepsilon^{l-1} c_l$ where $c_l(j) = 1$ if $d(j) = l$ and 0 otherwise (see Figure 3).

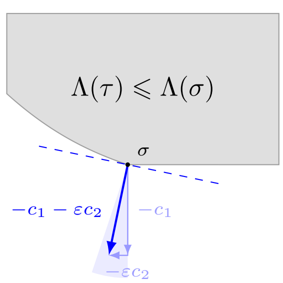

Figure 3: Normal cone of K_σ at σ.

We now go back to Conjecture 8 and prove some facts pointing in that direction.

Lemma 9. *Suppose that $d(j) = 1$ for each $j = 1, 2, \ldots, J-1$. Then there exists an $\varepsilon > 0$ such that the weight vector can be chosen as $c = (1, \ldots, 1, \varepsilon)$. That is, if $\sigma \in K$, $\tau \in \mathbb{R}^J$ and $\Lambda(\tau) \leqslant \Lambda(\sigma)$ then $c\tau > c\sigma$.*

Proof. Let σ and τ as in the statement and note that, as in the proof of Corollary 6, we have $\tau(j) \geqslant \sigma(j)$ for $j = 1, \dots, J-1$. If $\tau(J) \geqslant \sigma(J)$ it follows trivially that $c\tau \geqslant c\sigma$ and therefore we assume $\tau(J) < \sigma(J)$. Let $R > \sup\limits_{\substack{j<J \\ \sigma \in K}} \dfrac{\|\Lambda'(\sigma)(e_j)\|}{\|\Lambda'(\sigma)(e_J)\|}$, choose $\varepsilon = \frac{1}{R}$ and define $c = (1, \dots, 1, \varepsilon)$. If we suppose by contradiction that $c\tau < c\sigma$ then,

$$\begin{aligned} 0 &> \|\Lambda'(\sigma)(e_J)\| Rc(\tau - \sigma) \\ &= \sum_{j<J} \|\Lambda'(\sigma)(e_J)\| R(\tau(j) - \sigma(j)) + \|\Lambda'(\sigma)(e_J)\|(\tau(J) - \sigma(J)) \\ &\geqslant \sum_{j} \|\Lambda'(\sigma)(e_j)\|(\tau(j) - \sigma(j)). \end{aligned} \tag{4}$$

If $f \in L^2_0(\partial\Omega)$ is an eigenvector associated to the smallest eigenvalue $-\|\Lambda'(\sigma)(e_J)\|$ of $\Lambda'(\sigma)(e_J)$ with $\|f\| = 1$ then, by convexity,

$$\begin{aligned} \langle (\Lambda(\tau) - \Lambda(\sigma))f, f\rangle &\geqslant \langle \Lambda'(\sigma)(\tau - \sigma)f, f\rangle \\ &= \sum_{j<J} (\tau(j) - \sigma(j))\langle \Lambda'(\sigma)(e_j)f, f\rangle - (\tau(J) - \sigma(J))\|\Lambda'(\sigma)(e_J)\| \\ &\geqslant -\sum_{j<J} (\tau(j) - \sigma(j))\|\Lambda'(\sigma)(e_j)\| - (\tau(J) - \sigma(J))\|\Lambda'(\sigma)(e_J)\| \\ &> 0 \end{aligned}$$

where the last inequality follows from (4). Finally, this implies that $\Lambda(\tau) - \Lambda(\sigma)$ is not negative semi-definite contradicting $\Lambda(\tau) \leqslant \Lambda(\sigma)$. ∎

Lemma 10. *Let $\sigma \neq \tau \in K$ with $\Lambda(\tau) \leqslant \Lambda(\sigma)$. Then there exists $p \in (0, 1]$ such that $\left(\sum_l \varepsilon^{l-1} c_l\right)(\tau - \sigma) > 0$ for any $\varepsilon \in (0, p]$.*

Proof. Take $j \geqslant 1$ as in Lemma 4 and let $l' = d(j)$. By an argument similar to that of the proof of Corollary 6, $\tau(j') \geqslant \sigma(j')$ for each j' with $d(j') = l'$. Therefore, if $\delta := \sum_{d(j')=l'} \tau(j') - \sigma(j') > 0$, then

$$\begin{aligned} \left(\sum_l \varepsilon^{l-1} c_l\right)(\tau - \sigma) &= \varepsilon^{l'-1}\delta + \varepsilon^{l'} \sum_{l>l'} \varepsilon^{l-l'-1} c_l(\tau - \sigma) \\ &\geqslant \varepsilon^{l'-1}\delta - 2bJ\varepsilon^{l'} \\ &= \varepsilon^{l'-1}\left(\delta - 2bJ\varepsilon\right). \end{aligned}$$

The result follows by setting $p = \dfrac{\delta}{3bJ}$. ∎

TODO: Arreglar la prueba o eliminar.

Proposition 11. *Fix $\sigma \in K$ and for each $\varepsilon > 0$ let $\tau_\varepsilon \in K$ such that $\Lambda(\tau_\varepsilon) \leqslant \Lambda(\sigma)$ and $c_\varepsilon \tau_\varepsilon \leqslant c_\varepsilon \sigma$. Then $\tau_\varepsilon \to \sigma$ as $\varepsilon \to 0^+$.*

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and $x_n := \tau_{\varepsilon_n} - \sigma$ for τ_{ε_n} as in the statement. Since K is compact, $(x_n)_n$ contains a convergent subsequence which for simplicity we assume is $(x_n)_n$ itself. Denote $\lim\limits_{n\to\infty} x_n$ by x and suppose by contradiction that $x \neq 0$.

By Lemma 10 there exists $p > 0$ sufficiently small such that $x \in H := \{y \in \mathbb{R}^J : c_p y > 0\}$. Since H is open, there exists $r > 0$ such that $P := \overline{B}_r(x) \subseteq H$ with respect to the ℓ_1 metric. P is a polytope and therefore has a finite vertex set $V := \mathrm{vert}(P)$. Again by Lemma 10 for each $v \in V$ there exists p_v such that $c_\varepsilon v > 0$ as long as $\varepsilon \in (0, p_v)$. If $q = \min_{v \in V} p_v$ then for each $v \in V$ we have $c_\varepsilon v > 0$ if $\varepsilon \in (0, q)$. Moreover, the same inequality holds for $y \in P = \mathrm{conv}(V)$.

Let $N \in \mathbb{N}$ such that $\|x_n - x\|_1 < r$ and $\varepsilon_n < q$ for $n \geqslant N$. By the previous discussion $c_{\varepsilon_n} x_n > 0$ since $x_n \in P$, which is a contradiction with the hypothesis $c_{\varepsilon_n} x_n \leqslant 0$. Therefore $x = 0$ or in other terms, $\lim_{n\to\infty} \tau^{\varepsilon_n} = \sigma$. Finally, as any subsequence of $(\tau_\varepsilon)_{\varepsilon>0}$ contains a subsequence converging to σ it follows that $(\tau_\varepsilon)_\varepsilon$ itself converges to σ. $\square$

References

[1] Alberto Calderón. "On an Inverse Boundary Value Problem". In: *Atas (Brazilian Mathematical Society)* (1980), pp. 65–73.

[2] Bastian Harrach. "The Calderón problem with finitely many unknowns is equivalent to convex semidefinite optimization". In: *SIAM J. Math. Anal.* 55.5 (2023), pp. 5666–5684. ISSN: 0036-1410,1095-7154. DOI: 10.1137/23M1544854. URL: https://doi.org/10.1137/23M1544854.

[3] F. Hecht. "New development in FreeFem++". In: *J. Numer. Math.* 20.3-4 (2012), pp. 251–265. ISSN: 1570-2820. URL: https://freefem.org/.